Complex Analysis

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

$$z = a + bi$$

$$\bar{z} = a - bi \qquad \text{(reflection about } x\text{-axis)}$$

$$|z| = \sqrt{a^2 + b^2}$$

Polar form:- $z = a + bi = r(\cos\theta + i\sin\theta)$

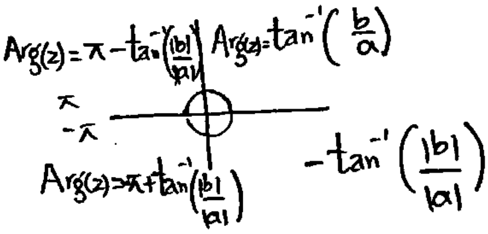
where $r = |z|$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\downarrow$

$\arg(z)$

Principal Argument $(\mathrm{Arg}(z))$

$$-\pi < \mathrm{Arg}(z) \le \pi$$



$$\arg(z) = \mathrm{Arg}(z) + 2n\pi, \quad n = 0, \pm 1, \pm 2, \dots$$

De Moivre's Theorem:-

1) If $n \in \mathbb{Z}$, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

2) If $n \in \mathbb{Q}$, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Properties

$\bar{\bar{z}} = z$, $z + \bar{z} = 2\,\mathrm{Re}(z)$, $z - \bar{z} = 2\,\mathrm{Im}(z)\,i$

z is real $\Leftrightarrow$ $z = \bar{z}$.

z is zero or pure imaginary $\Leftrightarrow$ $z = -\bar{z}$.

$z\bar{z} = |z|^2$, $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $|z_1 z_2| = |z_1||z_2|$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

$$\big||z_1|-|z_2|\big| \le |z_1 \pm z_2| \le |z_1| + |z_2| \quad \text{(Triangle inequality)}$$

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right) \quad \text{(Parallelogram law)}$$

$$\arg(z_1z_2) = \arg(z_1)+\arg(z_2), \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2)$$

$$\operatorname{Arg}(z_1z_2) \neq \operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)$$

If $\operatorname{Re}(z_1)>0$ and $\operatorname{Re}(z_2)>0$, then $\operatorname{Arg}(z_1z_2)=\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)$

$$\boxed{\text{No order relation in } \mathbb{C}.}$$

Roots

The n nth roots of a nonzero complex number $z=r(\cos\theta+i\sin\theta)$ are given by

$$\omega_k = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$$

where $k=0,1,2,\dots,n-1$

or any n consecutive integers.

Geometrically ω_k can be interpreted as the vertices of a regular polygon with n sides that is inscribed within a circle of radius $r^{1/n}$ centered at origin.

Circle

A circle with center z_0 and radius r is

$$|z-z_0| = r$$

$|z|=r$

Open disc $\quad |z-z_0|<r$

Closed disc $\quad |z-z_0|\le r$

Punctured disc $\quad 0<|z-z_0|<r.$

Annulus (circular) $r_1 < |z - z_0| < r_2$, $r_1 < r_2$.

Neighborhood (open)

$S \subseteq \mathbb{C}$ is nbd of z_0 if there exists an open disc centered at z_0 which lies entirely in S. i.e.,

$S \subseteq \mathbb{C}$ is nbd of z_0 if for some $r > 0$

$$\{z : |z - z_0| < r\} \subseteq S.$$

Interior Point

If S is nbd of z_0, then z_0 is called interior point of S.

Open Set S is open if it is nbd of each of its points. i.e.,

each $x \in S$ is interior point of S.

Connected Set

If any pair of points $z_1, z_2 \in S$ can be connected by a polygonal line that consists of a finite number of line segments joined end to end that lies entirely in the set, then S is connected.

Domain

Open and connected set.

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

Boundary Point

z_0 is a boundary point of S if every nbd of $z_0 \in S$ contains atleast one point of S and atleast one point not in S.

Boundary of a set

The collection of all boundary points.

Exterior Point

Interior point of S^c is called the exterior point of S.

Region

A set of points in $\mathbb{C}$ with all, some, or none of its boundary points.

Closed Set

A region that contains all of its boundary points is said to be closed set.

Bounded Set

A set $S \subseteq \mathbb{C}$ is bounded if for some real number $k > 0$,

$$|z| \le k, \qquad \forall\, z \in S.$$

Complex Exponential Function

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

$|e^z| = e^x.$

$\arg(e^z) = y + 2n\pi \quad , \quad n \in \mathbb{Z}.$

$\overline{(e^z)} = e^{\bar{z}}$

$e^0 = 1 \quad , \quad e^{z_1+z_2} = e^{z_1} \cdot e^{z_2} \quad , \quad e^{z_1 - z_2} = \dfrac{e^{z_1}}{e^{z_2}}$

$(e^z)^n = e^{nz}$

e^z is periodic with period $2\pi i$.

Sine and Cosine

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z = \frac{1}{\sin z}$$

Properties

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}$$

$$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$$

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College Jhang

$$\sin z = 0 \iff z = n\pi, \quad n \in \mathbb{Z}.$$

$$\cos z = 0 \iff z = (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z}.$$

$\sin z$, $\cos z$, $\operatorname{cosec} z$, $\sec z$ have period 2π.

$\tan z$, $\cot z$ have period π.

Real trigonometric sine and cosine functions are bounded, but complex sine and cosine functions are unbounded.

Complex Logarithmic Function

$$\log z = \ln|z| + i\arg(z).$$

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

i) $\log(z_1 z_2) = \log(z_1) + \log(z_2)$

ii) $\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$

iii) $\log(z^n) = n\log(z).$

iv) $e^{w} = z \iff w = \log(z) = \ln|z| + i\arg(z).$

Example Solve $e^{w} = -2$

$$w = \log(-2) = \ln|-2| + i(\pi + 2n\pi), \quad n \in \mathbb{Z}.$$

$$= \ln(2) + (2n+1)\pi i.$$

Principal value of $\log(z)$

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z).$$

Example

$$\operatorname{Log}(1+i) = \ln|\sqrt{2}| + i\left(\frac{\pi}{4}\right) = \frac{1}{2}\ln 2 + i\frac{\pi}{4}.$$

Limit

$\lim_{z \to z_0} f(z) = L$, if for every $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(z) - L| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Continuity

f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided the limit exists.

Necessary condition

$\boxed{\text{Differentiability implies C-R- equations.}}$ (but not converse)

If $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at $z = x+iy$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

i.e.,

$\boxed{\text{not C-R-equation} \Rightarrow \text{not differentiable}}$

Sufficient condition

If $u(x,y)$ are $v(x,y)$ are continuous and have continuous first-order partial derivatives in some nbd of z_0, and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

then $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at z_0.

and

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

or

$$f'(z) = u_x + i v_x$$

or

$$f'(z) = v_y - i u_y.$$

Polar form of C-R- equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Analytic Function

$f(z) = u+iv$ is analytic at z if $f(z)$ is differentiable at z, as well as in some nbd of z.

Holomorphic or Regular Function

A function that is analytic throughout a domain D

Entire function

f is entire if f is analytic in whole $\mathbb{C}$.

Result

If f is analytic in D, then it is infinitely many times differentiable in D.

Harmonic Function

$\phi(x,y)$ is called harmonic if

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

★ If $f(z) = u(x,y) + iv(x,y)$ is analytic, then $u(x,y)$ and $v(x,y)$ both are harmonic.

Harmonic Conjugate

$v(x,y)$ is harmonic conjugate of $u(x,y)$ if

$$f(z) = u(x,y) + iv(x,y) \quad \text{is analytic.}$$

★ Harmonic conjugacy is not a symmetric property.
(If v is harmonic conjugate of u, then not necessarily u is harmonic conjugate of v).

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

Cauchy's Theorem and Formula

Theorem (Cauchy)

If f is analytic in a simply connected domain D, then for every simple closed contour C in D,

$$\int_c f(z)\,dz = 0$$

Morera's Theorem (Converse of Cauchy's Theorem)

If $f(z)$ is continuous in D and $\int_c f(z)dz = 0$ taken around any closed contour in D, then $f(z)$ is analytic in D.

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

Result

i) If C is closed and z_0 in not in interior of C, then $\int_c \frac{1}{(z-z_0)^n}dz = 0.$

ii) If C is closed and z_0 is interior point of C, then

$$\int_c \frac{1}{(z-z_0)^n}dz = \begin{cases} 2\pi i & , n=1 \\ 0 & , n\neq 1 \end{cases}$$

Cauchy's Integral Formula

If f is analytic in simply connected domain D and C is any simple closed contour lying entirely in D, then for any z_0 with in C,

$$\int_c \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}\frac{d^n}{dz^n}\left[f(z_0)\right]$$

Liouville's Theorem

An entire and bounded function is constant.

or

A non constant entire function is unbounded.

For example

e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are non constant entire functions, so are unbounded.

ML- Inequality

If $f(z)$ is continuous on a contour C such that $|f(z)| \leq M$ for all z on C and L is the length of the contour, then

$$\left| \int_C f(z)dz \right| \leq ML.$$

Cauchy Inequality

If $f(z)$ is analytic on and within a circle C given by

$$|z-a| = R$$

and if the function is bounded by M, i.e.,

$$|f(z)| \leq M \quad , \quad \forall z \in C$$

then

$$|f^{(n)}(a)| \leq \frac{M \cdot n!}{R^n}.$$

Power Series

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

A series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

is called a power series. The co-efficients a_n's are complex constants and z_0 is called the center of the power series.

If we substitute $z-z_0 = w$, then series takes the form $\sum_{n=0}^{\infty} a_n w^n$.

Absolute Convergence

$\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent if the series $\sum_{n=0}^{\infty} |a_n| \, |z|^n$ is convergent.

Conditionally Convergent

$\sum_{n=0}^{\infty} a_n z^n$ is conditionally convergent if $\sum_{n=0}^{\infty} a_n z^n$ is convergent, but $\sum_{n=0}^{\infty} |a_n| \, |z|^n$ is not convergent.

* Absolute convergence $\Rightarrow$ convergence.

Radius of Convergence

A non-negative number R is called radius of convergence of $\sum a_n (z-z_0)^n$ if the series converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$.

Circle of Convergence

In above definition $|z-z_0| = R$ is called the circle of convergence.

Formula for radius of convergence

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

Cauchy's Root Test

Let $\sum_{n=0}^{\infty} u_n$ be a series of complex numbers and $\lim_{n\to\infty} |u_n|^{1/n} = l$. Then the series is

i) convergent if $l<1$.

ii) divergent if $l>1$.

iii) test fails if $l=1$.

Ratio Test

Let $\sum_{n=0}^{\infty} u_n$ be a sequence of complex numbers and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$. The the series is

i) convergent if $l<1$

ii) divergent if $l>1$

iii) test fails if $l=1$.

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

Points to remember

i) If $\lim_{n\to\infty} u_n \neq 0$, then the series diverges.

ii) Geometric series never converges on the boundary.

iii) If $\sum_{n=0}^{\infty} a_n z^n$ converges for some point $z=z_0$, then the power series converges for every point $z=z_1$ such that $|z_1|<|z_0|$.

iv) If $\sum_{n=0}^{\infty} a_n z^n$ diverges for $z=z_0$, then it diverges for every point $z=z_1$ such that $|z_1|>|z_0|$.

v) The sum function of power series is analytic with in its circle of convergence.

Taylor's Theorem

Let $f(z)$ be an analytic function inside a circle C with center at a, then for all z in C,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-z_0)^n = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

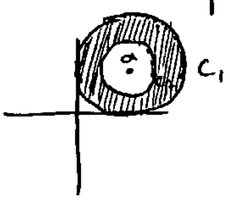
Maclaurin's Theorem

Let $f(z)$ be an analytic function inside a circle C with center at origin, then for all z in C,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z)^n = f(0) + f'(0)\,z + \frac{f''(0)}{2!}z^2 + \cdots$$

Laurent's Theorem:-

Let $f(z)$ be analytic inside and on the boundary of the ring shaped region bounded by two concentric circles C_1 and C_2 with centre at a and radii r_1 and r_2 $(r_2 < r_1)$ respectively, then for all z in R,



$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where
$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw, \quad n = 0, 1, 2, \ldots$$

and
$$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\,dw, \quad n = 1, 2, 3, \ldots$$

Here $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called regular part and $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called the principal part of Lauran's expansion of $f(z)$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad \text{valid for all } z.$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \text{valid for all } z.$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad \text{valid for all } z.$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \qquad \text{valid for all } z.$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad \text{valid for all } z.$$

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \cdots \qquad \text{valid for all } z.$$

$$\tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} - \frac{17z^7}{315} + \cdots \qquad \text{valid for all } z.$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad \text{valid for all } |z| < 1.$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \qquad \text{valid for } |z| < 1$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \qquad \text{valid for } |z| < 1.$$

$$\log(1-z) = -z + \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} + \cdots \qquad \text{valid for } |z| < 1.$$

Zeros

A number z_0 is zero of $f(z)$ if $f(z_0) = 0$.

Zero of order (multiplicity) n

An analytic function f has a zero of order n at $z = z_0$ if

$$f(z_0) = 0,\ f'(z_0) = 0,\ f''(z_0) = 0,\ \dots,\ f^{(n-1)}(z_0) = 0,\ \text{but}\ f^{(n)}(z_0) \neq 0.$$

A zero of order 1 is called a simple zero.

Theorem:-

A function f that is analytic in some disc $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if

$$f(z) = (z - z_0)^n \phi(z)$$

where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

Theorem

Let $f(z) \not\equiv 0$ ($f(z) = 0$, for all z in D) be an analytic function in a domain D, then every zero of $f(z)$ in D is isolated.

or

Let z_0 be zero of $f(z)$, where $f(z) \not\equiv 0$ be analytic in D, then there exists a positive number r s.t. $f(z)$ has no zero in $|z - z_0| < r$.

Singularities

Singularity

$z = z_0$ is called singularity of $f(z)$ if f fails to be analytic at z_0.

Isolated Singularity

We call $z = z_0$ an isolated singularity of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$.

For example $\tan z$ has isolated singularities at $\pm \frac{\pi}{2}$, $\pm \frac{3\pi}{2}$, etc.

Three types of isolated singularities are

i) Removable singularity

ii) Pole

iii) Isolated Essential Singularity.

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

Removable Singularity

If $f(z)$ is not defined at $z = a$, but $\lim\limits_{z \to a} f(z)$ exists, then $z = a$ is called a removable singularity. In such case we define $f(z)$ at $z = a$ as equal to $\lim\limits_{z \to a} f(z)$, and $f(z)$ will then be analytic at a.

(or)

If the principal part of $f(z)$ has no term (i.e., there is no negative power of $z - a$ in the expansion of $f(z)$), then

$z = a$ is called removable singularity.

For example.

$$f(z) = \frac{\sin z}{z}$$ is not defined at $z = 0$,

but $\lim_{z \to a} \frac{\sin z}{z} = 1$. So $z = 0$ is removable singularity of $f(z)$.

(or)

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

Since there is no negative power of $(z-0)$. Hence $z = 0$ is a removable singularity of $f(z)$.

<u>Pole</u>

If the principal part has only finitely many terms, it is of the form

$$\frac{b_1}{z-z_0}, \frac{b_2}{(z-z_0)^2}, \cdots, \frac{b_m}{(z-z_0)^m}$$

Then the singularity of $f(z)$ at $z = z_0$ is called a pole, and m is called its order.

Poles of the first order are known as simple poles.

<u>Theorem:-</u>

A function f analytic in a punctured disk $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ iff

$$f(z) = \frac{\phi(z)}{(z-z_0)^n}$$

where ϕ is analytic at $z = z_0$ and $\phi(z) \neq 0$.

Isolated Essential Singularity

If the principal part contains an infinity many terms, then $z=z_0$ is called an essential singularity.

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

For example

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

has an isolated essential singularity at $z=0$.

Also

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots$$

has an isolated essential singularity at $z=0$.

Alternatively

i) $z=z_0$ is removable singularity of $f(z)$ if and only if $\lim_{z\to z_0} f(z)$ exists finitely.

ii) $z=z_0$ is pole of $f(z)$ if and only if $\lim_{z\to z_0} f(z) = \infty$.

iii) $z=z_0$ is essential singularity of $f(z)$ if and only if $\lim_{z\to z_0} f(z)$ does not exist finitely or infinitely.

Theorem:- Let $f(z)$ be analytic at $z=z_0$ and have a zero of nth order at $z=z_0$. Then $\frac{1}{f(z)}$ has a pole of nth order at $z=z_0$.

Example Find Laurent series about the indicated singularity. Name the singularity and give the region of convergence.

(i) $\dfrac{e^{2z}}{(z-1)^3}$, $z=1$ (ii) $(z-3)\sin\dfrac{1}{z+2}$, $z=-2$

(iii) $\dfrac{z-\sin z}{z^3}$, $z=0$ (iv) $\dfrac{z}{(z+1)(z+2)}$, $z=-2$

(v) $\dfrac{1}{z^2(z-3)^2}$, $z=3$

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

Sol:-

(i) Let $u = z-1$, then $z = u+1$

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2u+2}}{u^3} = \frac{e^2}{u^3}\left(1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\cdots\right)$$

$$= \frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{z-1}+\frac{8e^2}{3!}+\frac{2e^2}{3}(z-1)+\cdots$$

$\Rightarrow$ $z=1$ is a pole of order 3.

The series converges for all values of $z \neq 1$.

(ii) Let $u = z+2$, then $z = u-2$

$$(z-3)\sin\left(\frac{1}{z+2}\right) = (u-5)\sin\left(\frac{1}{u}\right) = (u-5)\left\{\frac{1}{u}-\frac{1}{3!u^3}+\frac{1}{5!u^5}-\cdots\right\}$$

$$= 1-\frac{5}{u}-\frac{1}{3!u^2}+\frac{5}{3!u^3}+\frac{1}{5!u^4}-\cdots$$

$$= 1-\frac{5}{z+2}-\frac{1}{3!(z+2)^2}+\frac{5}{3!(z+2)^3}+\frac{1}{5!(z+2)^4}-\cdots$$

$\Rightarrow$ $z=-2$ is an essential singularity.

The series converges for all values of $z \neq -2$.

(iii) $$\frac{z-\sin z}{z^3} = \frac{1}{z^3}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\right\}$$

$$= \frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\cdots$$

$\Rightarrow$ $z=0$ is a removable singularity.

The series converges for all values of z.

(iv) Let $u = z+2$, then $z = u-2$

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)u} = \frac{2-u}{u}\cdot\frac{1}{1-u} = \frac{2-u}{u}\left(1+u+u^2+u^3+\cdots\right)$$

$$= \frac{2}{u} + 1 + u + u^2 + \cdots$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots$$

$\Rightarrow$ $z = -2$ is a simple pole.

The series converges for all values of z such that $0<|z+2|<1$.

(v) Let $u = z-3$, then $z = u+3$

$$\frac{1}{z^2(z-3)^2} = \frac{1}{(u+3)^2u^2} = \frac{1}{9u^2\left(1+\frac{u}{3}\right)^2}$$

$$= \frac{1}{9u^2}\left\{1+(-2)\left(\frac{u}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3+\cdots\right\}$$

$$= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots$$

$$= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4}{243}(z-3) + \cdots$$

$\Rightarrow$ $z = 3$ is a pole of order 2.

The series converges for all values of z such that $0 < |z-3| < 3$.

Residue

The coefficient b_1 of $\frac{1}{z-z_0}$ in the Laurent series expansion of $f(z)$ is called the residue of f at $z=z_0$. We denote it by

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

or $$b_1 = \operatorname{Res}\left[f(z), z_0\right].$$

Residue at a Pole

If f has a pole of order n at $z=z_0$, then

$$\operatorname{Res}\left(f(z), z_0\right) = \frac{1}{(n-1)!} \lim_{z\to z_0} \left[(z-z_0)^n f(z)\right]^{(n-1)}$$

In particular if $z=z_0$ is a simple pole, then

$$\operatorname{Res}\left(f(z), z_0\right) = \lim_{z\to z_0} \left[(z-z_0) f(z)\right]$$

(or)

A second formula for the residue at a simple pole is

$$\operatorname{Res}\left(f(z), z_0\right) = \operatorname{Res}\left(\frac{p(z)}{q(z)}, z_0\right) = \frac{p(z_0)}{q'(z_0)}.$$

with $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0, so that $f(z) = \frac{p(z)}{q(z)}$ has a simple pole at z_0.

Example (Residue at a Simple Pole).

$$f(z) = \frac{9z+i}{z^3+z} = \frac{9z+i}{z(z^2+1)} = \frac{9z+i}{z(z+i)(z-i)}$$

has a simple pole at i. So

$$\text{Res}\left(f(z),\ i\right) = \lim_{z\to i}(z-i)\frac{9z+i}{z(z+i)(z-i)} = \frac{10i}{-2} = -5i$$

(or)

$$\text{Res}\left(f(z),\ i\right) = \frac{p(i)}{q'(i)}$$

where $p(z) = 9z+i$, $q(z) = z^3+z$, $q'(z) = 3z^2+1$

$p(i) = 10i$, $q'(i) = -3+1 = -2$

So

$$\text{Res}\left(f(z),\ i\right) = \frac{10i}{-2} = -5i.$$

Example (Residue at a Pole of Higher Order)

$f(z) = \dfrac{50z}{z^3+2z^2-7z+4}$ has a pole of second order at $z=1$, because $z^3+2z^2-7z+4 = (z+4)(z-1)^2$. So

$$\text{Res}\left(f(z),\ 1\right) = \frac{1}{(2-1)!}\lim_{z\to 1}\left[(z-1)^2 f(z)\right]'$$

$$= \lim_{z\to 1}\left[\frac{50z}{z+4}\right]'$$

$$= \frac{200}{5^2} = 8.$$

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College (Jhang)

Residue Theorem

Let $f(z)$ be analytic inside a simple closed contour C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_n$ inside C. Then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_k\right).$$

Example

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)} dz$, where

(i) the contour C is the rectangle fined by

$$x=0, \quad x=4, \quad y=-1, \quad y=1.$$

(ii) and the contour C is the circle $|z|=2$.