Question 1: **Prove that** $\displaystyle\int_0^\infty \frac{\cos^2 x}{(1+x^2)^2}\,\mathbf{d}x = \frac{\pi}{8}(1+3\mathrm{e}^{-2})$

Solution: Let, $\displaystyle I = \int_0^\infty \frac{\cos^2 x}{(1+x^2)^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos^2 x}{(1+x^2)^2}\,\mathrm{d}x = \frac{1}{4}\int_{-\infty}^\infty \frac{1+\cos(2x)}{(1+x^2)^2}\,\mathrm{d}x = \frac{1}{4}\mathrm{Re}\int_{-\infty}^\infty \frac{1+\mathrm{e}^{2ix}}{(1+x^2)^2}\,\mathrm{d}x$

$$= \frac{1}{4}\mathrm{Re}\int_C \frac{1+\mathrm{e}^{2iz}}{(1+z^2)^2}\,\mathrm{d}z$$

Poles of $f(z) = \frac{1+\mathrm{e}^{2iz}}{(1+z^2)^2}$ are $z = \pm i$. Only the pole $z = i$ lies in upper half of the complex plane and it has order two.

$$R_1(f,i) = \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z-i)^2\left(1+\mathrm{e}^{2iz}\right)}{(z+i)^2(z-i)^2}\right] = \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1+\mathrm{e}^{2iz}}{(z+i)^2}\right] = \lim_{z\to i}\frac{(z+i)^2\left(2i\mathrm{e}^{2iz}\right) - 2(z+i)\left(1+\mathrm{e}^{2iz}\right)}{(z+i)^4}$$

$$= \lim_{z\to i}\frac{(z+i)\left(2i\mathrm{e}^{2iz}\right) - 2\left(1+\mathrm{e}^{2iz}\right)}{(z+i)^3} = \frac{(2i)(2i\mathrm{e}^{-2}) - 2(1+\mathrm{e}^{-2})}{(2i)^3} = \frac{-4\mathrm{e}^{-2} - 2(1+\mathrm{e}^{-2})}{-8i}$$

$$= \frac{-2\mathrm{e}^{-2} - (1+\mathrm{e}^{-2})}{-4i} = \frac{1+3\mathrm{e}^{-2}}{4i}$$

$$I = \frac{1}{4}\mathrm{Re}\left[2\pi i\sum_n R_n\right] = \frac{1}{4}\mathrm{Re}\left[2\pi i\left(\frac{1+3\mathrm{e}^{-2}}{4i}\right)\right] = \frac{\pi}{8}(1+3\mathrm{e}^{-2})$$ Hence proved.

Question 2: **Prove that** $\displaystyle\int_0^\infty \frac{\boldsymbol{x}\,\mathbf{sin}\,\boldsymbol{x}\,\mathbf{d}\boldsymbol{x}}{(\boldsymbol{x}^2+\boldsymbol{a}^2)^2} = \frac{\boldsymbol{\pi}\mathbf{e}^{-\boldsymbol{a}}}{\mathbf{4}\boldsymbol{a}}$, **where** $\boldsymbol{a} > \mathbf{0}$

Solution: Let, $\displaystyle I = \int_0^\infty \frac{x\sin x}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{1}{2}\mathrm{Im}\int_{-\infty}^\infty \frac{x\,\mathrm{e}^{ix}}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{1}{2}\mathrm{Im}\int_C \frac{z\,\mathrm{e}^{iz}}{(z^2+a^2)^2}\,\mathrm{d}z$

Poles of $f(z) = \frac{z\,\mathrm{e}^{iz}}{(z^2+a^2)^2}$ are $z = \pm ai$. Only the pole $z = i$ lies in upper half plane due to the fact that $a > 0$ (given) and it has order two.

$$R_1(f,ai) = \lim_{z\to ai}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z-ai)^2\left(z\,\mathrm{e}^{iz}\right)}{(z+ai)^2(z-ai)^2}\right] = \lim_{z\to ai}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z\,\mathrm{e}^{iz}}{(z+ai)^2}\right] = \lim_{z\to ai}\frac{(z+ai)^2(iz\mathrm{e}^{iz}+\mathrm{e}^{iz}) - 2(z+ai)\left(z\mathrm{e}^{iz}\right)}{(z+ai)^4} =$$

$$= \lim_{z\to ai}\frac{(z+ai)(iz\mathrm{e}^{iz}+\mathrm{e}^{iz}) - 2z\mathrm{e}^{iz}}{(z+ai)^3} = \frac{(2ai)(-a\mathrm{e}^{-a}+\mathrm{e}^{-a}) - 2ai\mathrm{e}^{-a}}{(2ai)^3} = \frac{(-2a^2i + 2ai - 2ai)\mathrm{e}^{-a}}{-8a^3i}$$

$$= \frac{-2a^2i\,\mathrm{e}^{-a}}{-8a^3i} = \frac{\mathrm{e}^{-a}}{4a}$$

$$I = \frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_n\right] = \frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{\mathrm{e}^{-a}}{4a}\right)\right] = \frac{\pi\mathrm{e}^{-a}}{4a}$$ Hence proved.

Question 3: **Prove that** $\displaystyle\int_{-\infty}^\infty \frac{\mathbf{sin}\,\boldsymbol{x}\;\mathbf{d}\boldsymbol{x}}{\boldsymbol{x}^2 - \mathbf{2}\boldsymbol{x} + \mathbf{5}} = \frac{\boldsymbol{\pi}}{\mathbf{2e}^2}\,\mathbf{sin}\,\mathbf{1}$

Solution: Let, $\displaystyle I = \int_{-\infty}^\infty \frac{\sin x\;\mathrm{d}x}{x^2-2x+5} = \mathrm{Im}\int_{-\infty}^\infty \frac{\mathrm{e}^{ix}\;\mathrm{d}x}{x^2-2x+5} = \mathrm{Im}\int_C \frac{\mathrm{e}^{iz}\;\mathrm{d}z}{z^2-2z+5}$

Poles of $f(z)=\frac{e^{iz}}{z^2-2z+5}$ are $z=\frac{2\pm\sqrt{4-20}}{2}=\frac{2\pm\sqrt{-16}}{2}=1\pm 2i$. Only the pole $z=1+2i$ lies in upper half plane and it has order one.

$$R_1(f,1+2i)=\lim_{z\to 1+2i}\frac{(z-1-2i)e^{iz}}{(z-1-2i)(z-1+2i)}=\lim_{z\to 1+2i}\frac{e^{iz}}{z-1+2i}=\frac{e^{i(1+2i)}}{1+2i-1+2i}=\frac{e^{-2+i}}{4i}=\frac{e^{-2}e^{i}}{4i}$$
$$=\frac{e^{-2}}{4i}(\cos 1+i\sin 1)$$

$$I=\operatorname{Im}\left[2\pi i\sum_n R_n\right]=\operatorname{Im}\left[2\pi i\left(\frac{e^{-2}}{4i}(\cos 1+i\sin 1)\right)\right]=\operatorname{Im}\left[\frac{\pi}{2e^2}(\cos 1+i\sin 1)\right]=\frac{\pi}{2e^2}\sin 1 \qquad \text{Hence proved.}$$

Question 4: **Prove that** $\displaystyle\int_0^\infty \frac{\cos(ax)}{x^4+4}\,dx=\frac{\pi}{8}[\cos a+\sin a]e^{-a},\qquad a>0$

Solution: Let, $I=\displaystyle\int_0^\infty \frac{\cos(ax)}{x^4+4}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos(ax)}{x^4+4}dx=\frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty}\frac{e^{iax}}{x^4+4}dx=\frac{1}{2}\operatorname{Re}\int_C\frac{e^{iaz}}{z^4+4}dz$

Poles of $f(z)=\frac{e^{iaz}}{z^4+4}$ are given by $z^4=-4=4e^{(2k\pi+\pi)i}\ \Rightarrow\ z_k=\sqrt{2}e^{\frac{(2k+1)\pi i}{4}},\ k=0,1,2,3.$

Here, $z_0=\sqrt{2}e^{\frac{\pi i}{4}}=\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=1+i,\quad z_1=\sqrt{2}e^{\frac{3\pi i}{4}}=-1+i,\quad z_2=\sqrt{2}e^{\frac{5\pi i}{4}}=-1-i,\quad z_1=\sqrt{2}e^{\frac{3\pi i}{4}}=1-i$

Only the poles z_0 and z_1 lie in upper half plane and they have order one.

$$R_1(f,z_0)=\lim_{z\to z_0}\frac{(z-z_0)e^{iaz}}{z^4+4}\qquad \left(\frac{0}{0}\right)\text{ form}$$
$$=\lim_{z\to z_0}\frac{iae^{iaz}(z-z_0)+e^{iaz}}{4z^3}=\frac{e^{iaz_0}}{4z_0^{\,3}}=\frac{z_0e^{iaz_0}}{4z_0^{\,4}}=\frac{z_0e^{iaz_0}}{4(-4)}=\frac{z_0e^{iaz_0}}{-16}\qquad \because z_0^{\,4}=-4$$

Similarly, $R_2(f,z_1)=\frac{z_1e^{iaz_1}}{-16}$

$$\text{Sum of residues}=\sum_n R_n=-\frac{1}{16}\left(z_0e^{iaz_0}+z_1e^{iaz_1}\right)=-\frac{1}{16}\left[(1+i)e^{ia(1+i)}+(-1+i)e^{ia(-1+i)}\right]$$
$$=-\frac{1}{16}\left[(1+i)e^{ia}+(-1+i)e^{-ia}\right]e^{-a}=-\frac{1}{16}\left[e^{ia}+ie^{ia}-e^{-ia}+ie^{-ia}\right]e^{-a}$$
$$=-\frac{1}{16}\left[(e^{ia}-e^{-ia})+i(e^{ia}+e^{-ia})\right]e^{-a}=-\frac{1}{16}[2i\sin a+2i\cos a]e^{-a}=-\frac{i}{8}[\sin a+\cos a]e^{-a}$$

$$I=\frac{1}{2}\operatorname{Re}\left[2\pi i\sum_n R_n\right]=\operatorname{Re}\left[2\pi i\left(-\frac{i}{8}[\sin a+\cos a]e^{-a}\right)\right]=\frac{\pi}{8}[\cos a+\sin a]e^{-a}\qquad \text{Hence proved.}$$

Question 5: **Prove that** $\displaystyle\int_0^\infty \frac{x^3\sin x\,dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{2(a^2-b^2)}\left[a^2e^{-a}-b^2e^{-b}\right],\qquad a,b>0$

Solution: Let, $I = \int\limits_{0}^{\infty} \frac{x^3 \sin x \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int\limits_{-\infty}^{\infty} \frac{x^3 \sin x \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\mathrm{Im}\int\limits_{-\infty}^{\infty} \frac{x^3 e^{ix} dx}{(x^2+a^2)(x^2+b^2)}$

$= \frac{1}{2}\mathrm{Im}\int\limits_{C} \frac{z^3 e^{iz} dz}{(z^2+a^2)(z^2+b^2)}$

Poles of $f(z) = \frac{z^3 e^{iz}}{(z^2+a^2)(z^2+b^2)}$ are $z = \pm ai, \pm bi$. Only the poles $z = ai, bi$ lie in upper half plane due to the fact that $a, b > 0$ (given) and they have order one.

$$R_1(f, ia) = \lim_{z\to ia} \frac{(z-ia)z^3 e^{iz}}{(z+ia)(z-ia)(z+b^2)} = \lim_{z\to ia} \frac{z^3 e^{iz}}{(z+ia)(z^2+b^2)} = \frac{-ia^3 e^{-a}}{2ia(b^2-a^2)} = \frac{a^2 e^{-a}}{2(a^2-b^2)}$$

$$R_2(f, ib) = \lim_{z\to ib} \frac{(z-ib)z^3 e^{iz}}{(z+a^2)(z+ib)(z-ib)} = \lim_{z\to ib} \frac{z^3 e^{iz}}{(z+a^2)(z+ib)} = \frac{-ib^3 e^{-b}}{(-b^2+a^2)(2ib)} = \frac{-b^2 e^{-b}}{2(a^2-b^2)}$$

$$I = \frac{1}{2}\mathrm{Im}\left[2\pi i \sum_n R_n\right] = \frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{a^2 e^{-a}}{2(a^2-b^2)} - \frac{b^2 e^{-b}}{2(a^2-b^2)}\right)\right] = \frac{\pi}{2(a^2-b^2)}[a^2 e^{-a} - b^2 e^{-b}]$$ Hence proved.

Question 6: **Prove that** $\int\limits_{0}^{\infty} \frac{x^3 \sin(mx)}{x^4+a^4} dx = \frac{\pi}{2} e^{\frac{-ma}{\sqrt{2}}} \cos\left(\frac{ma}{\sqrt{2}}\right), \quad m, a > 0$

Solution: Let, $I = \int\limits_{0}^{\infty} \frac{x^3 \sin(mx)}{x^4+a^4} dx = \frac{1}{2}\int\limits_{-\infty}^{-\infty} \frac{x^3 \sin(mx)}{x^4+a^4} dx = \frac{1}{2}\mathrm{Im}\int\limits_{C} \frac{x^3 e^{imx}}{x^4+a^4} dx = \frac{1}{2}\mathrm{Im}\int\limits_{C} \frac{z^3 e^{imz}}{z^4+a^4} dz$

Poles of $f(z) = \frac{z^3 e^{imz}}{z^4+a^4}$ are given by $z^4 = -a^4 = a^4 e^{(2k\pi+\pi)i} \Rightarrow z_k = ae^{\frac{(2k+1)\pi i}{4}}, \; k = 0, 1, 2, 3.$

Here, $z_0 = ae^{\frac{\pi i}{4}} = \frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}i, \quad z_1 = ae^{\frac{3\pi i}{4}} = -\frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}i, \quad z_2 = ae^{\frac{\pi i}{4}} = -\frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}}i, \quad z_1 = ae^{\frac{\pi i}{4}} = \frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}}i$

Only the poles z_0 and z_1 lie in upper half plane due to the fact that $a > 0$ (given) and they have order one.

$$R_1(f, z_0) = \lim_{z\to z_0} \frac{(z-z_0)z^3 e^{imz}}{z^4+a^4} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to z_0} \frac{z^3 e^{imz} + (z-z_0)(3z^2)e^{imz} + (z-z_0)z^3\left(ime^{imz}\right)}{4z^3} = \frac{z_0^3 e^{imz_0}}{4z_0^3} = \frac{e^{imz_0}}{4}$$

Similarly, $R_2(f, z_1) = \frac{e^{imz_1}}{4}$

Sum of residues $= \sum_n R_n = \frac{1}{4}[e^{imz_0} + e^{imz_1}] = \frac{1}{4}\left[e^{ima\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)} + e^{ima\left(\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}\right] = \frac{1}{4}\left[e^{\frac{ima}{\sqrt{2}}} \cdot e^{\frac{-ma}{\sqrt{2}}} + e^{\frac{-ima}{\sqrt{2}}} \cdot e^{\frac{-ma}{\sqrt{2}}}\right]$

$$= \frac{1}{4}\left[e^{\frac{-ma}{\sqrt{2}}}\left(e^{\frac{ima}{\sqrt{2}}} + e^{\frac{-ima}{\sqrt{2}}}\right)\right] = \frac{1}{4}\left[2e^{\frac{-ma}{\sqrt{2}}}\cos\left(\frac{ma}{\sqrt{2}}\right)\right] = \frac{1}{2}e^{\frac{-ma}{\sqrt{2}}}\cos\left(\frac{ma}{\sqrt{2}}\right)$$

$$I = \frac{1}{2}\mathrm{Im}\left[2\pi i \sum_n R_n\right] = \frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{1}{2}e^{\frac{-ma}{\sqrt{2}}}\cos\left(\frac{ma}{\sqrt{2}}\right)\right)\right] = \frac{\pi}{2}e^{\frac{-ma}{\sqrt{2}}}\cos\left(\frac{ma}{\sqrt{2}}\right)$$ Hence proved.

Question 7: **Prove that** $\displaystyle\int_0^\infty \frac{\boldsymbol{x\sin(mx)}}{\boldsymbol{x^4+a^4}}\,\mathbf{d}\boldsymbol{x} = \frac{\boldsymbol{\pi}}{\boldsymbol{4b^2}}\mathbf{e}^{-\boldsymbol{mb}}\,\mathbf{sin}(\boldsymbol{mb})$, **where**, $\boldsymbol{b = \frac{a}{\sqrt{2}}}$, $\boldsymbol{m, a > 0}$

Solution: Let, $$I = \int_0^\infty \frac{x\sin(mx)}{x^4+a^4}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin(mx)}{x^4+a^4}\,\mathrm{d}x = \frac{1}{2}\mathrm{Im}\int_C \frac{x\mathrm{e}^{imx}}{x^4+a^4} = \frac{1}{2}\mathrm{Im}\int_C \frac{z\mathrm{e}^{imz}}{z^4+a^4}\,\mathrm{d}z$$

Poles of $f(z) = \frac{z\mathrm{e}^{imz}}{z^4+a^4}$ are given by $z^4 = -a^4 = a^4\mathrm{e}^{(2k\pi+\pi)i} \Rightarrow z_k = a\mathrm{e}^{\frac{(2k+1)\pi i}{4}}$, $k = 0, 1, 2, 3.$

Here, $z_0 = a\mathrm{e}^{\frac{\pi i}{4}} = \frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}i$, $z_1 = a\mathrm{e}^{\frac{3\pi i}{4}} = -\frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}i$, $z_2 = a\mathrm{e}^{\frac{\pi i}{4}} = -\frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}}i$, $z_1 = a\mathrm{e}^{\frac{\pi i}{4}} = \frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}}i$

Only the poles z_0 and z_1 lie in upper half plane due to the fact that $a > 0$ (given) and they have order one.

$$R_1(f, z_0) = \lim_{z\to z_0}\frac{(z-z_0)z\mathrm{e}^{imz}}{z^4+a^4} \quad \left(\frac{0}{0}\right) \text{ form}$$
$$= \lim_{z\to z_0}\frac{z\mathrm{e}^{imz} + (z-z_0)\mathrm{e}^{imz} + (z-z_0)z\left(im\mathrm{e}^{imz}\right)}{4z^3} = \frac{z_0\mathrm{e}^{imz_0}}{4z_0^3} = \frac{z_0^2\mathrm{e}^{imz_0}}{4z_0^4} = \frac{z_0^2\mathrm{e}^{imz_0}}{-4a^4} \quad \because z_0^4 = -a^4$$

Similarly, $R_2(f, z_1) = \dfrac{z_1^2\mathrm{e}^{imz_1}}{-4a^4}$

Sum of residues $= \sum_n R_n = \dfrac{-1}{4a^4}\left[{z_0}^2\mathrm{e}^{imz_0} + {z_1}^2\mathrm{e}^{imz_1}\right]$
$$= \frac{-1}{4a^4}\left[\left(\frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}i\right)^2 i\mathrm{e}^{ima\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)} + \left(-\frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}i\right)^2 \mathrm{e}^{ima\left(\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}\right]$$
$$= \frac{-1}{4a^4}\left[a^2 i\mathrm{e}^{\frac{ima}{\sqrt{2}}}\cdot\mathrm{e}^{\frac{-ma}{\sqrt{2}}} - a^2 i\mathrm{e}^{\frac{-ima}{\sqrt{2}}}\cdot\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\right] = \frac{-i}{4a^2}\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\left[\mathrm{e}^{\frac{ima}{\sqrt{2}}} - \mathrm{e}^{\frac{-ima}{\sqrt{2}}}\right] = \frac{-i}{4a^2}\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\left[2i\sin\left(\frac{ma}{\sqrt{2}}\right)\right]$$
$$= \frac{1}{2a^2}\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\sin\left(\frac{ma}{\sqrt{2}}\right)$$

$$I = \frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_n\right] = \frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{1}{2a^2}\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\sin\left(\frac{ma}{\sqrt{2}}\right)\right)\right] = \frac{\pi}{2a^2}\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\sin\left(\frac{ma}{\sqrt{2}}\right)$$
$$= \frac{\pi}{4b^2}\mathrm{e}^{-mb}\sin(mb), \quad \text{where, } b = \frac{a}{\sqrt{2}} \qquad \text{Hence proved.}$$

Question 8: **Prove that** $\displaystyle\int_0^\infty \frac{\cos^2 x}{(1+x^2)^2}\,\mathbf{d}\boldsymbol{x} = \frac{\pi}{8}(1+3\mathrm{e}^{-2})$

Solution : Question 1 is repeated here in Iqbal's book. See solution of question 1.

Question 9: **Prove that** $\displaystyle\int_0^\infty \frac{\boldsymbol{\cos(ax)}}{\boldsymbol{(x^2+b^2)^2}}\,\mathbf{d}\boldsymbol{x} = \frac{\boldsymbol{\pi(1+ab)}\mathbf{e}^{-\boldsymbol{ab}}}{\boldsymbol{4b^3}}$, $\boldsymbol{b > 0}$

Solution: Let, $$I = \int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos(ax)}{(x^2+b^2)^2}\,\mathrm{d}x = \frac{1}{2}\mathrm{Re}\int_C \frac{\mathrm{e}^{iax}}{(x^2+b^2)^2}\,\mathrm{d}x = \frac{1}{2}\mathrm{Re}\int_C \frac{\mathrm{e}^{iaz}}{(z^2+b^2)^2}\,\mathrm{d}z$$

Poles of $f(z)=\frac{e^{iaz}}{(z^2+b^2)^2}$ are $z=\pm bi$. Only the pole $z=\ bi$ lies in upper half plane due to the fact that $b>0$ (given) and it has order two.

$$R_2\,(f\,,bi)=\lim_{z\to ib}\frac{d}{dz}\left[\frac{(z-ib)^2e^{iaz}}{(z+ib)^2(z-ib)^2}\right]=\lim_{z\to ib}\frac{d}{dz}\left[\frac{e^{iaz}}{(z+ib)^2}\right]=\lim_{z\to ib}\frac{(z+ib)^2iae^{iaz}-2\,(z+ib)e^{iaz}}{(z+ib)^4}$$

$$=\lim_{z\to ib}\frac{[(z+ib)ia-2]e^{iaz}}{(z+ib)^3}=\frac{[(2ib)ia-2]e^{-ab}}{(2ib)^3}=\frac{[-2ab-2]e^{-ab}}{-8ib^3}=\frac{[ab+1]e^{-ab}}{4ib^3}$$

$$I=\frac{1}{2}\mathrm{Re}\left[2\pi i\sum_n R_n\right]=\frac{1}{2}\mathrm{Re}\left[2\pi i\left(\frac{[ab+1]e^{-ab}}{4ib^3}\right)\right]=\frac{\pi(1+ab)e^{-ab}}{4b^3}$$ Hence proved.

Question 10: **Prove that** $$\int_0^\infty \frac{x(x^2+1)\sin x}{x^4+x^2+1}dx=\frac{\pi e^{-\sqrt{3}/2}}{2\sqrt{3}}\left[\sin\frac{1}{2}+\sqrt{3}\cos\frac{1}{2}\right]$$

Solution: Let, $$I=\int_0^\infty\frac{x(x^2+1)\sin x}{x^4+x^2+1}dx=\frac{1}{2}\int_{-\infty}^\infty\frac{x(x^2+1)\sin x}{x^4+x^2+1}dx=\frac{1}{2}\mathrm{Im}\int_C\frac{x(x^2+1)e^{ix}}{x^4+x^2+1}dx$$

$$=\frac{1}{2}\mathrm{Im}\int_C\frac{z(z^2+1)e^{iz}}{z^4+z^2+1}dz$$

Poles of $f(z)=\frac{z(z^2+1)e^{iz}}{z^4+z^2+1}$ are given by $z^4+z^2+1=0$. But $(z^2-1)(z^4+z^2+1)=z^6-1$. Thus the roots of the polynomial z^4+z^2+1 are the roots of polynomial z^6-1 other than $z=\pm 1$

Now $z^6-1=0\ \Rightarrow\ z^6=1=e^{2k\pi i}\ \Rightarrow\ z_k=e^{\frac{k\pi i}{3}},\quad k=0,1,2,3,4,5$

Here, $z_0=1,\quad z_1=\frac{1}{2}+\frac{\sqrt{3}i}{2},\quad z_2=-\frac{1}{2}+\frac{\sqrt{3}i}{2},\quad z_3=-1,\quad z_4=-\frac{1}{2}-\frac{\sqrt{3}i}{2},\quad z_5=\frac{1}{2}-\frac{\sqrt{3}i}{2}$

Poles of $f(z)$, which lie in upper half plane, are z_1 and z_2 which are simple poles. Let α is one of the pole of $f(z)$

$$R\,(f,\alpha)=\lim_{z\to\alpha}(z-\alpha)f(z)=\lim_{z\to\alpha}\frac{(z-\alpha)z(z^2+1)e^{iz}}{z^4+z^2+1}\qquad\left(\frac{0}{0}\right)\ \text{form}$$

$$=\lim_{z\to\alpha}\left[\frac{(z-\alpha)\frac{d}{dz}\left[z(z^2+1)e^{iz}\right]+z(z^2+1)e^{iz}}{4z^3+2z}\right]=\frac{\alpha(\alpha^2+1)e^{i\alpha}}{4\alpha^3+2\alpha}=\frac{(\alpha^2+1)e^{\alpha i}}{4\alpha^2+2}$$

Sum of residues at $z=z_1$ and $z=z_2$ is given by

$$R_1+R_2=\frac{e^{i\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}\left(\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2+1\right)}{4\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2+2}+\frac{e^{i\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}\left(\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2+1\right)}{4\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2+2}$$

$$=\frac{e^{i\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}\left(\frac{1}{4}-\frac{3}{4}+\frac{\sqrt{3}i}{2}+1\right)}{4\left(\frac{1}{4}-\frac{3}{4}+\frac{\sqrt{3}i}{2}\right)+2}+\frac{e^{i\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}\left(\frac{1}{4}-\frac{3}{4}-\frac{\sqrt{3}i}{2}+1\right)}{4\left(\frac{1}{4}-\frac{3}{4}-\frac{\sqrt{3}i}{2}\right)+2}$$

$$R_{1+}R_2 = \frac{e^{\frac{i}{2}}e^{\frac{-\sqrt{3}}{2}}\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}{-2+2\sqrt{3}i+2} + \frac{e^{-\frac{i}{2}}e^{\frac{-\sqrt{3}}{2}}\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)}{-2-2\sqrt{3}i+2} = \frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}i}\left[e^{\frac{i}{2}}\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right) - e^{\frac{-i}{2}}\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right]$$

$$= \frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}i}\left[\left(\frac{e^{\frac{i}{2}}-e^{\frac{-i}{2}}}{2}\right)+\sqrt{3}i\left(\frac{e^{\frac{i}{2}}+e^{\frac{-i}{2}}}{2}\right)\right] = \frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}i}\left[i\sin\frac{1}{2}+\sqrt{3}i\cos\frac{1}{2}\right] = \frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}}\left[\sin\frac{1}{2}+\sqrt{3}\cos\frac{1}{2}\right]$$

$$I = \frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_n\right] = \frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}}\left[\sin\frac{1}{2}+\sqrt{3}\cos\frac{1}{2}\right]\right)\right] = \frac{\pi e^{-\sqrt{3}/2}}{2\sqrt{3}}\left[\sin\frac{1}{2}+\sqrt{3}\cos\frac{1}{2}\right]$$ **Hence proved.**

Question 11: **Prove that** $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(nx)}{1+x^2}dx = \begin{cases}\pi e^{-n} & \text{for } n>0\\ \pi & \text{for } n=0\end{cases}$

Solution: Let, $\displaystyle I = \int_{-\infty}^{\infty}\frac{\cos(nx)}{1+x^2}dx = \mathrm{Re}\int_C\frac{e^{inx}}{1+x^2}dx = \mathrm{Re}\int_C\frac{e^{inz}}{1+z^2}dz$

Poles of $f(z) = \frac{e^{inz}}{1+z^2}$ are $z=\pm i$. Only the pole $z = i$ lies in upper half plane and it has order one.

$$R_1(f,bi) = \lim_{z\to i}\frac{(z-i)e^{inz}}{(z+i)(z-i)} = \lim_{z\to i}\frac{e^{inz}}{z+i} = \frac{e^{-n}}{2i}$$

$$I = \mathrm{Re}\left[2\pi i\sum_n R_n\right] = \mathrm{Re}\left[2\pi i\left(\frac{e^{-n}}{2i}\right)\right] = \pi e^{-n} = \begin{cases}\pi e^{-n} & \text{for } n>0\\ \pi & \text{for } n=0\end{cases}$$ **Hence proved**.