Question 1: **Prove that** $\displaystyle\int_0^{2\pi}\frac{\cos 3\theta d\theta}{5-4\cos\theta}=\frac{\pi}{12}$

Solution: Let, $\displaystyle I=\int_0^{2\pi}\frac{\cos 3\theta\, d\theta}{5-4\cos\theta}=\text{Re}\int_0^{2\pi}\frac{e^{3i\theta}\, dz}{5-4\cos\theta},$

Put, $e^{i\theta}=z \Rightarrow d\theta=\dfrac{dz}{ie^{i\theta}}=\dfrac{dz}{iz},\quad \cos\theta=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)=\dfrac{z^2+1}{2z}$

$$I=\text{Re}\int_C\frac{z^3}{5-2(z+\frac{1}{z})}\cdot\frac{dz}{iz}=\text{Re}\left[-\frac{1}{i}\int_C\frac{z^3dz}{2z^2-5z+2}\right],$$ where, C: $|z|=1$ (unit circle with centre at origin)

Poles of $f(z)=\dfrac{z^3}{2z^2-5z+2}$ are given by $2z^2-5z+2=0\Rightarrow z=\dfrac{5\pm\sqrt{25-16}}{4}=\dfrac{5\pm 3}{4}=2,\dfrac{1}{2}.$

Only the pole $z=\frac{1}{2}$ lies inside C: $|z|=1$ and it has order one.

$$R_1\left(f,\frac{1}{2}\right)=\lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)z^3}{2(z-2)\left(z-\frac{1}{2}\right)}=\frac{\frac{1}{8}}{2\left(\frac{1}{2}-2\right)}=\frac{\frac{1}{8}}{2\left(-\frac{3}{2}\right)}=-\frac{1}{24}$$

By Cauchy's residue theorem

$$I=\text{Re}\left[-\frac{1}{i}\left(2\pi i\sum_i R_i\right)\right]=\text{Re}\left[-\frac{1}{i}\left(2\pi i\left(-\frac{1}{24}\right)\right)\right]=\frac{\pi}{12}$$ **Hence proved**.

Question 2: **Prove that** $\displaystyle\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=\frac{2\pi}{\sqrt{3}}$

Solution: Let, $\displaystyle I=\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$

Put, $e^{i\theta}=z \Rightarrow d\theta=\dfrac{dz}{ie^{i\theta}}=\dfrac{dz}{iz},\quad \cos\theta=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)=\dfrac{z^2+1}{2z}$

$$I=\int_C\frac{\frac{dz}{iz}}{2+\frac{1}{2}\left(\frac{z^2+1}{z}\right)}=\int_C\frac{2dz}{i(4z+z^2+1)}=\frac{2}{i}\int_C\frac{dz}{z^2+4z+1},$$ where, C: $|z|=1$ (unit circle with centre at origin)

Poles of $f(z)=\dfrac{1}{z^2+4z+1}$ are given by $z^2+4z+1=0\Longrightarrow z=\dfrac{-4\pm\sqrt{16-4}}{2}=-2\pm\sqrt{3}.$

As, $|-2+\sqrt{3}|<1$ and $|-2-\sqrt{3}|>1$. So, only the pole $z=-2+\sqrt{3}$ lies inside C: $|z|=1$ and it has order one.

$$R_1(f,-2+\sqrt{3})=\lim_{z\to-2+\sqrt{3}}\frac{z-(-2+\sqrt{3})}{\left(z-(-2+\sqrt{3})\right)\left(z-(-2-\sqrt{3})\right)}=\frac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})}=\frac{1}{2\sqrt{3}}$$

By Cauchy's residue theorem

$$I = \frac{2}{i}(2\pi i)\sum_i R_i = \frac{2}{i}(2\pi i)\left(\frac{1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$ **Hence proved.**

Question 3: **Prove that** $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \qquad (a > 0,\ a^2 < 1)$

Solution: Let, $\displaystyle I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}$

Put , $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}, \qquad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I = \int_C \frac{\frac{dz}{iz}}{1 + a\left(\frac{z^2+1}{2z}\right)} = \frac{2}{i}\int_C \frac{dz}{az^2 + 2z + a},$$ where, C: $|z| = 1$ (unit circle with centre at origin)

Poles of $f(z) = \frac{1}{az^2 + 2z + a}$ are given by $az^2 + 2z + a = 0 \Rightarrow z = \frac{-2 \pm \sqrt{4 - 4a^2}}{2a} = \frac{-1 \pm \sqrt{1-a^2}}{a}$

Let, $\alpha = \frac{-1 + \sqrt{1-a^2}}{a}$ and $\beta = \frac{-1 - \sqrt{1-a^2}}{a} \Rightarrow \alpha\beta = \frac{1 - (1 - a^2)}{a^2} = 1 \Rightarrow |\alpha\beta| = 1$

Now, $1 - a^2 > 0 \Rightarrow a^2 < 1 \Rightarrow |a| < 1 \Rightarrow |\beta| = \left|\frac{-1 - \sqrt{1-a^2}}{a}\right| = \frac{|1 + \sqrt{1-a^2}|}{|a|} > 1 \Rightarrow |\alpha| = \frac{1}{|\beta|} < 1$

$\Rightarrow$ Only the pole $z = \alpha = \frac{-1 + \sqrt{1-a^2}}{a}$ lies inside C: $|z| = 1$ and it has order one.

$$R_1(f, \alpha) = \lim_{z \to \alpha} \frac{z - \alpha}{a(z - \alpha)(z - \beta)} = \frac{1}{a(\alpha - \beta)} = \frac{1}{a\left[\frac{-1 + \sqrt{1-a^2}}{a} - \left(\frac{-1 - \sqrt{1-a^2}}{a}\right)\right]} = \frac{1}{2\sqrt{1-a^2}}$$

By Cauchy's residue theorem

$$I = \frac{2}{i}(2\pi i)\sum_i R_i = \frac{2}{i}(2\pi i)\frac{1}{2\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$$ **Hence proved.**

Question 4: **Prove that** $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = \frac{\pi}{2}$

Solution: Let, $\displaystyle I = \int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta}$

Put , $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}, \qquad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I=\int_C \frac{\frac{dz}{iz}}{5+3\left(\frac{z^2+1}{2z}\right)}=\frac{2}{i}\int_C \frac{dz}{3z^2+10z+3},\qquad \text{where, } C\text{: } |z|=1 \text{ (unit circle with centre at origin)}$$

Poels of $f(z)=\dfrac{1}{3z^2+10z+3}$ are given by $3z^2+10z+3=0 \Rightarrow z=\dfrac{-10\pm\sqrt{64}}{6}=\dfrac{-10\pm 8}{6}=-\dfrac{1}{3},-3$

Only the pole $z=-\dfrac{1}{3}$ lie inside $C\colon |z|=1$ and it has order one.

$$R_1\left(f,-\frac{1}{3}\right)=\lim_{z\to-\frac{1}{3}}\frac{z+\frac{1}{3}}{3\left(z+\frac{1}{3}\right)(z+3)}=\frac{1}{3\left(-\frac{1}{3}+3\right)}=\frac{1}{8}$$

By Cauchy's residue theorem

$$I=\frac{2}{i}\left(2\pi i\sum_i R_i\right)=\frac{2}{i}(2\pi i)\left(\frac{1}{8}\right)=\frac{\pi}{2}$$ **Hence Proved.**

Question 5: **Prove that** $\displaystyle\int_0^{2\pi}\frac{\cos^2 3\theta\, d\theta}{5-4\cos\theta}=\frac{3\pi}{8}$

Solution: Let, $$I=\int_0^{2\pi}\frac{\cos^2 3\theta\, d\theta}{5-4\cos\theta}=\frac{1}{2}\int_0^{2\pi}\frac{2\cos^2 3\theta\, d\theta}{5-4\cos\theta}=\frac{1}{2}\int_0^{2\pi}\frac{(1+\cos 6\theta)d\theta}{5-4\cos\theta}=\mathrm{Re}\left[\frac{1}{2}\int_0^{2\pi}\frac{(1+e^{6i\theta})d\theta}{5-4\cos\theta}\right]$$

Put, $e^{i\theta}=z \;\Rightarrow\; d\theta=\dfrac{dz}{ie^{i\theta}}=\dfrac{dz}{iz},\qquad \cos\theta=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)=\dfrac{z^2+1}{2z}$

$$I=\mathrm{Re}\left[\frac{1}{2}\int_C\frac{(1+z^6)\frac{dz}{iz}}{5-4\left(\frac{z^2+1}{2z}\right)}\right]=\mathrm{Re}\left[-\frac{1}{2i}\int_C\frac{(1+z^6)dz}{2z^2-5z+2}\right],\quad \text{where, } C\text{: } |z|=1 \text{ (unit circle with centre at origin)}$$

Poles of $f(z)=\dfrac{1+z^6}{2z^2-5z+2}$ are given by $2z^2-5z+2=0 \;\Rightarrow\; z=\dfrac{5\pm\sqrt{25-16}}{4}=\dfrac{5\pm 3}{4}=2,\dfrac{1}{2}$

Only the pole $z=\dfrac{1}{2}$ lies inside $C\colon |z|=1$ and it has order one.

$$R_1\left(f,\frac{1}{2}\right)=\lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)(1+z^6)}{2\left(z-\frac{1}{2}\right)(z-2)}=\lim_{z\to\frac{1}{2}}\frac{(1+z^6)}{2(z-2)}=\frac{1+\frac{1}{64}}{2\left(\frac{1}{2}-2\right)}=\frac{\frac{65}{64}}{2\left(\frac{-3}{2}\right)}=\frac{-65}{192}$$

By Cauchy's residue theorem

$$\mathrm{I}=\mathrm{Re}\left[\frac{1}{-2i}\left(2\pi i\sum_i R_i\right)\right]=\mathrm{Re}\left[-\pi\left(\frac{-65}{192}\right)\right]=\frac{65\pi}{192}$$ **Hence Proved.**

Question 6: **If $a^2>b^2+c^2$, then proved that** $\displaystyle\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta+c\sin\theta}=\frac{2\pi}{\sqrt{a^2-b^2-c^2}}$

Solution: Let, $I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta}$

Put , $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2-1}{2iz}$

$$I = \int_C \frac{\frac{dz}{iz}}{a + b\left(\frac{z^2+1}{2z}\right) + c\left(\frac{z^2-1}{2iz}\right)},$$ where, $C: |z| = 1$ (unit circle with centre at origin)

$$= \int_C \frac{\frac{dz}{iz}}{\frac{2aiz + z^2bi + bi + cz^2 - c}{2iz}} = 2\int_C \frac{dz}{2aiz + z^2bi + bi + cz^2 - c}$$

$$= 2\int_C \frac{dz}{(c+bi)z^2 + 2iaz - (c-bi)}$$

Poles of $f(z) = \frac{1}{(c+bi)z^2 + 2iaz - (c-bi)}$ are given by $(c+bi)z^2 + 2iaz - (c-bi) = 0$

$$z = \frac{-2ia \pm \sqrt{(2ia)^2 + 4(c+bi)(c-bi)}}{2(c+bi)} = \frac{-2ai \pm \sqrt{-4a^2 + 4c^2 + 4b^2}}{2(c+bi)} = \frac{-ai \pm \sqrt{-a^2 + c^2 + b^2}}{c+bi}$$

$$= \frac{\left(-a \pm \sqrt{a^2 - c^2 - b^2}\right)i}{c+bi}$$

Let, $\alpha = \frac{\left(-a + \sqrt{a^2 - c^2 - b^2}\right)i}{c+bi}$ and $\beta = \frac{\left(-a - \sqrt{a^2 - c^2 - b^2}\right)i}{c+bi}$

$$\Rightarrow \alpha\beta = \frac{-\left[a^2 - (a^2 - c^2 - b^2)\right]}{(c+bi)^2} = \frac{-(c^2+b^2)}{(c+bi)^2} \Rightarrow |\alpha\beta| = \frac{|-(c^2+b^2)|}{|(c+bi)^2|} = \frac{c^2+b^2}{|c+bi|^2} = \frac{c^2+b^2}{c^2+b^2} = 1$$

Now, $a^2 > b^2 + c^2 \Rightarrow a > \sqrt{c^2+b^2} \Rightarrow a + \sqrt{a^2 - c^2 - b^2} > \sqrt{c^2+b^2} \Rightarrow |\beta| = \left|\frac{a + \sqrt{a^2 - c^2 - b^2}}{\sqrt{c^2+b^2}}\right| > 1$

$\Rightarrow |\alpha| = \frac{1}{|\beta|} < 1$. Only the pole $z = \alpha = \frac{\left(-a + \sqrt{a^2 - c^2 - b^2}\right)i}{c+bi}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f,\alpha) = \lim_{z\to\alpha} \frac{z-\alpha}{(c+bi)(z-\alpha)(z-\beta)} = \frac{1}{(c+bi)(\alpha-\beta)}$$

$$= \frac{1}{(c+bi)\left[\frac{\left(-a + \sqrt{a^2 - c^2 - b^2}\right)i}{(c+bi)} - \left(\frac{\left(-a - \sqrt{a^2 - c^2 - b^2}\right)i}{(c+bi)}\right)\right]} = \frac{1}{2i\sqrt{a^2 - b^2 - c^2}}$$

By Cauchy's residue theorem

$I = 2(2\pi i)\sum_i R_i = 2(2\pi i)\left[\frac{1}{2i\sqrt{a^2 - b^2 - c^2}}\right] = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}$ **Hence Proved**.

Question 7: **Prove that** $\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{(5 - 3\cos\theta)^4} = \frac{135\,\pi}{16384}$

Solution: Let, $I=\int_0^{2\pi}\frac{\cos 3\theta\, d\theta}{(5-3\cos\theta)^4}=\text{Re}\int_0^{2\pi}\frac{e^{3i\theta}\ d\theta}{(5-3\cos\theta)^4}$

Put , $e^{i\theta}=z \ \Rightarrow\ d\theta=\frac{dz}{ie^{i\theta}}=\frac{dz}{iz},\qquad \cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$

$$I=\text{Re}\int_C\frac{z^3\frac{dz}{iz}}{\left(5-3\frac{z^2+1}{2z}\right)^4}$$ where, C: $|z|=1$ (unit circle with centre at origin)

$$=\text{Re}\left[\frac{1}{i}\int_C\frac{z^2dz}{\left(\frac{-3z^2+10z-3}{2z}\right)^4}\right]=\text{Re}\left[\frac{16}{i}\int_C\frac{z^6\,dz}{(3z^2-10z+3)^4}\right]=\text{Re}\left[\frac{16}{i}\int_C\frac{z^6\,dz}{(3z^2-9z-z+3)^4}\right]$$

$$=\text{Re}\left[\frac{16}{i}\int_C\frac{z^6\,dz}{[3z(z-3)-(z-3)]^4}\right]=\text{Re}\left[\frac{16}{i}\int_C\frac{z^6\,dz}{(z-3)^4(3z-1)^4}\right]$$

Poles of $f(z)=\frac{z^6}{(z-3)^4(1-3z)^4}$ are $z=3,\frac{1}{3}$. Only the pole $z=\frac{1}{3}$ lies inside C: $|z|=1$ and it has order four.

$$R_1\left(f,\frac{1}{3}\right)=\lim_{z\to\frac{1}{3}}\frac{1}{3!}\frac{d^3}{dz^3}\left[\frac{z^6\left(z-\frac{1}{3}\right)^4}{(3)^4\left(z-\frac{1}{3}\right)^4(z-3)^4}\right]=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d^3}{dz^3}\left[\frac{z^6}{(z-3)^4}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d^2}{dz^2}\left[\frac{(z-3)^4(6z^5)-z^6\cdot 4\cdot(z-3)^3}{(z-3)^8}\right]=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d^2}{dz^2}\left[\frac{(z-3)(6z^5)-4z^6}{(z-3)^5}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d^2}{dz^2}\left[\frac{2z^6-18z^5}{(z-3)^5}\right]=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d}{dz}\left[\frac{(z-3)^5(12z^5-90z^4)-5(z-3)^4(2z^6-18z^5)}{(z-3)^{10}}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d}{dz}\left[\frac{(z-3)(12z^5-90z^4)-5(2z^6-18z^5)}{(z-3)^6}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d}{dz}\left[\frac{12z^6-90z^5-36z^5+270z^4-10z^6+90z^5}{(z-3)^6}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d}{dz}\left[\frac{2z^6-36z^5+270z^4}{(z-3)^6}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!(3)^4}\left[\frac{(z-3)^6(12z^5-180z^4+1080z^3)-6(z-3)^5(2z^6-36z^5+270z^4)}{(z-3)^{12}}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!(3)^4}\left[\frac{(z-3)(12z^5-180z^4+1080z^3)-6(2z^6-36z^5+270z^4)}{(z-3)^7}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!(3)^4}\left[\frac{12z^6-180z^5+1080z^4-36z^5+540z^4-3240z^3-12z^6+216z^5-1620z^4}{(z-3)^7}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\left[\frac{-3240z^3}{(z-3)^7}\right]=\frac{1}{3!\,(3)^4}\left[\frac{-3240\left(\frac{1}{27}\right)}{\left(\frac{1}{3}-3\right)^7}\right]=\frac{1}{6(3)^4}\left[\frac{-120}{\left(-\frac{8}{3}\right)^7}\right]=\frac{120}{6(3)^4}\left(\frac{3}{8}\right)^7=\frac{20(3)^3}{(8)^7}$$

By Cauchy's residue theorem

$$I=\operatorname{Re}\left[\frac{16}{i}(2\pi i)\sum_i R_i\right]=\operatorname{Re}\left[\frac{16}{i}(2\pi i)\left(\frac{20(3)^3}{(8)^7}\right)\right]=\frac{135\pi}{16384}$$ **Hence proved**.

Alternate solution: Let, $I=\int_0^{2\pi}\frac{\cos 3\theta\, d\theta}{(5-3\cos\theta)^4}=\operatorname{Re}\int_0^{2\pi}\frac{e^{-3i\theta}\, d\theta}{(5-3\cos\theta)^4}$

Put , $e^{i\theta}=z \Rightarrow d\theta=\frac{dz}{ie^{i\theta}}=\frac{dz}{iz}$, $\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$

$$I=\operatorname{Re}\int_C\frac{z^{-3}\frac{dz}{iz}}{\left(5-3\frac{z^2+1}{2z}\right)^4}$$ where, $C: |z|=1$ (unit circle with centre at origin)

$$=\operatorname{Re}\left[\frac{1}{i}\int_C\frac{z^{-4}dz}{\left(\frac{-3z^2+10z-3}{2z}\right)^4}\right]=\operatorname{Re}\left[\frac{16}{i}\int_C\frac{dz}{(3z^2-10z+3)^4}\right]=\operatorname{Re}\left[\frac{16}{i}\int_C\frac{dz}{(3z^2-9z-z+3)^4}\right]$$

$$=\operatorname{Re}\left[\frac{16}{i}\int_C\frac{dz}{[3z(z-3)-(z-3)]^4}\right]=\operatorname{Re}\left[\frac{16}{i}\int_C\frac{dz}{(z-3)^4(3z-1)^4}\right]$$

Poles of $f(z)=\frac{1}{(z-3)^4(1-3z)^4}$ are $z=3,\frac{1}{3}$. Only the pole $z=\frac{1}{3}$ lies inside $C: |z|=1$ and it has order four.

$$R_1\left(f,\frac{1}{3}\right)=\lim_{z\to\frac{1}{3}}\frac{1}{3!}\frac{d^3}{dz^3}\left[\frac{\left(z-\frac{1}{3}\right)^4}{(3)^4\left(z-\frac{1}{3}\right)^4(z-3)^4}\right]=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d^3}{dz^3}\left[\frac{1}{(z-3)^4}\right]=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d^2}{dz^2}[-4(z-3)^{-5}]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d}{dz}[20(z-3)^{-6}]=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}[-120(z-3)^{-7}]=\frac{1}{3!\,(3)^4}\left[-120\left(\frac{1}{3}-3\right)^{-7}\right]$$

$$=\frac{1}{6(3)^4}\left[-120\left(-\frac{8}{3}\right)^{-7}\right]=\frac{1}{6(3)^4}\left[-120\left(-\frac{3}{8}\right)^7\right]=\frac{20(3)^3}{(8)^7}\frac{120}{6(3)^4}\left(\frac{3}{8}\right)^7=\frac{20(3)^3}{(8)^7}$$

By Cauchy's residue theorem

$$I=\operatorname{Re}\left[\frac{16}{i}(2\pi i)\sum_i R_i\right]=\operatorname{Re}\left[\frac{16}{i}(2\pi i)\left(\frac{20(3)^3}{(8)^7}\right)\right]=\frac{135\pi}{16384}$$ **Hence proved**.

Question 8: **Prove that** $\int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos(n\theta)\, d\theta}{5-4\cos\theta}=\frac{2\pi}{3}\left(\frac{7}{4}\right)^n$, $n=0,1,2,3,\cdots$

Solution: Let, $I=\int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos(n\theta)\, d\theta}{5-4\cos\theta}=\operatorname{Re}\int_0^{2\pi}\frac{(1+2\cos\theta)^n e^{ni\theta}d\theta}{5-4\cos\theta}$

Put , $e^{i\theta}=z \Rightarrow d\theta=\frac{dz}{ie^{i\theta}}=\frac{dz}{iz}$, $\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$

$$I=\operatorname{Re}\left[\int_C \frac{\left[1+2\left(\frac{z^2+1}{2z}\right)\right]^n z^n \frac{dz}{iz}}{5-4\left(\frac{z^2+1}{2z}\right)}\right],\quad \text{where, } C:\ |z|=1 \text{ (unit circle with centre at origin)}$$

$$=\operatorname{Re}\left[\frac{2}{i}\int_C \frac{(z^2+z+1)^n dz}{-4z^2+10z-4}\right]=\operatorname{Re}\left[\frac{-1}{i}\int_C \frac{(z^2+z+1)^n dz}{2z^2-5z+2}\right]=\operatorname{Re}\left[\frac{-1}{i}\int_C \frac{(z^2+z+1)^n dz}{(z-2)(2z-1)}\right]$$

Poles of $f(z)=\dfrac{(z^2+z+1)^n}{(z-2)(2z-1)}$ are $z=\frac{1}{2},2$. Only the pole $z=\frac{1}{2}$ lies inside $C:|z|=1$and it has order one.

$$R_1\left(f,\frac{1}{2}\right)=\lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)(z^2+z+1)^n}{2(z-2)\left(z-\frac{1}{2}\right)}=\lim_{z\to\frac{1}{2}}\frac{(z^2+z+1)^n}{2(z-2)}=\frac{\left(\frac{1}{4}+\frac{1}{2}+1\right)^n}{2\left(\frac{1}{2}-2\right)}=\frac{1}{-3}\left(\frac{7}{4}\right)^n$$

By Cauchy's residue theorem

$$\mathrm{I}=\operatorname{Re}\left[\left(-\frac{1}{i}\right)\left(2\pi i\sum_i R_i\right)\right]=\operatorname{Re}\left[\left(-\frac{1}{i}\right)2\pi i\left(\frac{1}{-3}\left(\frac{7}{4}\right)^n\right)\right]=\frac{2\pi}{3}\left(\frac{7}{4}\right)^n$$ **Hence proved**.

Question 9: **Prove that** $\displaystyle\int_0^{2\pi}\cot\left(\frac{\theta-\alpha}{2}\right)d\theta=\begin{cases}2\pi i, & \text{If } \operatorname{Im}(\alpha)>0\\ -2\pi i, & \text{If } \operatorname{Im}(\alpha)<0\\ \text{diverges}, & \text{If } \operatorname{Im}(\alpha)=0\end{cases}$

Solution: Let, $\displaystyle I=\int_0^{2\pi}\cot\left(\frac{\theta-\alpha}{2}\right)d\theta=i\int_0^{2\pi}\left[\frac{e^{i\left(\frac{\theta-\alpha}{2}\right)}+e^{-i\left(\frac{\theta-\alpha}{2}\right)}}{e^{i\left(\frac{\theta-\alpha}{2}\right)}-e^{-i\left(\frac{\theta-\alpha}{2}\right)}}\right]d\theta=i\int_0^{2\pi}\left[\frac{e^{i(\theta-\alpha)}+1}{e^{i(\theta-\alpha)}-1}\right]d\theta=i\int_0^{2\pi}\left[\frac{e^{i\theta}+e^{\alpha i}}{e^{i\theta}-e^{\alpha i}}\right]d\theta$

Put, $e^{i\theta}=z \Rightarrow d\theta=\dfrac{dz}{ie^{i\theta}}=\dfrac{dz}{iz}$

$$I=i\int_C\left[\frac{z+e^{\alpha i}}{z-e^{\alpha i}}\right]\frac{dz}{iz}=\int_C\frac{z+e^{\alpha i}}{z(z-e^{\alpha i})}dz,\qquad \text{where, } C:\ |z|=1 \text{ (unit circle with centre at origin)}$$

Poles of $f(z)=\dfrac{z+e^{\alpha i}}{z(z-e^{\alpha i})}$ are $z=0,\ e^{i\alpha}$.

Let, $\alpha=a+bi \quad\Rightarrow\quad |e^{i\alpha}|=|e^{i(a+bi)}|=|e^{-b+ai}|=|e^{-b}e^{ai}|=e^{-b}|e^{ai}|=e^{-b} \qquad \because |e^{ai}|=1$

There are three cases:

Case I: If $b=\operatorname{Im}(\alpha)>0 \Rightarrow |e^{i\alpha}|=e^{-b}<1 \Rightarrow$ the pole $z=e^{i\alpha}$ lies inside $C:|z|=1$ and it has order one.

Case II: If $b=\operatorname{Im}(\alpha)<0 \Rightarrow |e^{i\alpha}|=e^{-b}>1 \Rightarrow$ the pole $z=e^{i\alpha}$ lies outside $C:|z|=1$.

Case III: If $b=\operatorname{Im}(\alpha)=0 \Rightarrow |e^{i\alpha}|=e^{-b}=1 \Rightarrow$ the pole $z=e^{i\alpha}$ lies on $C:|z|=1$.

Furthermore, the pole $z=0$ lies inside $C:|z|=1$ in any case and it has order 1.

$$R(f,0)=\lim_{z\to 0}\frac{z(z+e^{\alpha i})}{z(z-e^{\alpha i})}=\lim_{z\to 0}\frac{z+e^{\alpha i}}{z-e^{\alpha i}}=\frac{e^{\alpha i}}{-e^{\alpha i}}=-1$$

$$R_2(f, e^{i\alpha}) = \lim_{z \to e^{i\alpha}} \frac{(z - e^{\alpha i})(z + e^{\alpha i})}{z(z - e^{\alpha i})} = \lim_{z \to e^{i\alpha}} \frac{z + e^{\alpha i}}{z} = \frac{2e^{2i\alpha}}{e^{i\alpha}} = 2$$

By Cauchy's residue theorem

$$I = 2\pi i \sum_i R_i = \begin{cases} 2\pi i(-1+2) = 2\pi i, & \text{If } \operatorname{Im}(\alpha) > 0 \\ 2\pi i(-1) = -2\pi i, & \text{If } \operatorname{Im}(\alpha) < 0 \\ \text{diverges}, & \text{If } \operatorname{Im}(\alpha) = 0 \end{cases}$$ **Hence Proved.**

Note: $\int_C f(z)\mathrm{d}z$ diverges, if any of the pole (or poles) of $f(z)$ lies (or lie) on C.

Question 10: Prove that $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta) \cos\theta \, d\theta = \pi$

Solution: Let, $I = \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta) \cos\theta \, d\theta = \operatorname{Re}\left[\int_0^{2\pi} e^{\cos\theta} e^{i\sin\theta} \cos\theta \, d\theta\right]$

$\because e^{i\sin\theta} = \cos(\sin\theta) + i\sin(\sin\theta) \Rightarrow \operatorname{Re}[e^{i\sin\theta}] = \cos(\sin\theta)$

Put, $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$I = \operatorname{Re}\left[\int_C e^z \left(\frac{z^2+1}{2z}\right)\frac{dz}{iz}\right] = \operatorname{Re}\left[\frac{1}{2i}\int_C e^z \left(\frac{z^2+1}{z^2}\right) dz\right]$, where, C: $|z| = 1$ (unit circle with centre at origin)

Pole of $f(z) = e^z\left(\frac{z^2+1}{z^2}\right)$ is $z = 0$, which lie inside C: $|z| = 1$ and it has order two.

$$R_1(f, 0) = \lim_{z \to 0} \frac{d}{dz}\left[\frac{z^2 e^z (z^2+1)}{z^2}\right] = \lim_{z \to 0} \frac{d}{dz}[e^z(z^2+1)] = \lim_{z \to 0}[e^z(z^2+1) + e^z(2z)] = [1+0] = 1$$

By Cauchy's residue theorem

$$I = \operatorname{Re}\left[\left(\frac{1}{2i}\right)(2\pi i)\sum_i R_i\right] = \operatorname{Re}\left[\left(\frac{1}{2i}\right)2\pi i \times 1\right] = \pi$$ **Hence proved.**

Question 11: If $|a| < 1$, then prove that $\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \frac{2\pi}{1 - a^2}$

Solution: Let, $I = \int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta}$

Put, $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I=\int_C \frac{\frac{dz}{iz}}{1+a^2-2a\left(\frac{z^2+1}{2z}\right)} \qquad C:\ |z|=1 \text{ (unit circle with centre at origin)}$$

$$=\frac{1}{i}\int_C \frac{dz}{za^2-az^2+z-a}=\frac{1}{i}\int_C \frac{dz}{-az(z-a)+(z-a)}=\frac{1}{i}\int_C \frac{dz}{(z-a)(1-az)}$$

Poles of $f(z)=\frac{z}{(z-a)(1-az)}$ are $z=\frac{1}{a},a$.

As, $|a|<1$ (given), therefore, only the pole $z=a$ lies inside $C:|z|=1$ and it has order one.

$$R_1(f,a)=\lim_{z\to a}\frac{z-a}{(z-a)(1-az)}=\lim_{z\to a}\frac{1}{1-az}=\frac{1}{1-a^2}$$

By Cauchy's residue theorem

$$I=\frac{1}{i}(2\pi i)\sum_i R_i=\frac{1}{i}\left(2\pi i\left(\frac{1}{1-a^2}\right)\right)=\frac{2\pi}{1-a^2}$$ **Hence proved.**

Question 12: Prove that $\displaystyle\int_{-\pi}^{\pi}\frac{a\cos\theta\, d\theta}{a+\cos\theta}=2\pi a\left(1-\frac{a}{\sqrt{a^2-1}}\right), \qquad a>1$

Solution: Let, $\displaystyle I=\int_{-\pi}^{\pi}\frac{a\cos\theta\, d\theta}{a+\cos\theta}=2\int_0^{\pi}\frac{a\cos\theta\, d\theta}{a+\cos\theta}$ $\quad\because$ if f is even then $\displaystyle\int_{-a}^{a}f(\theta)d\theta=2\int_0^a f(\theta)d\theta$

$\displaystyle\Rightarrow I=2\int_0^{\pi}\frac{a\cos\theta\, d\theta}{a+\cos\theta}=2\cdot\frac{1}{2}\int_0^{2\pi}\frac{a\cos\theta\, d\theta}{a+\cos\theta}$ $\quad\because$ if $f(2b-\theta)=f(\theta)$ then $\displaystyle\int_0^{2b}f(\theta)d\theta=2\int_0^b f(\theta)d\theta$

$$I=a\int_0^{2\pi}\frac{(a+\cos\theta-a)d\theta}{a+\cos\theta}=a\int_0^{2\pi}d\theta-a^2\int_0^{2\pi}\frac{d\theta}{a+\cos\theta}=2\pi a-a^2\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} \qquad \to (A)$$

Let $\displaystyle I_1=\int_0^{2\pi}\frac{d\theta}{a+\cos\theta}$

Put, $e^{i\theta}=z \Rightarrow d\theta=\frac{dz}{ie^{i\theta}}=\frac{dz}{iz}$, $\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$

$$I_1=\int_C\frac{\frac{dz}{iz}}{a+\left(\frac{z^2+1}{2z}\right)}=\frac{2}{i}\int_C\frac{dz}{z^2+2az+1}, \qquad \text{where,} \qquad C:|z|=1 \text{ (unit circle with centre at origin)}$$

Poles of $f(z)=\frac{1}{z^2+2az+1}$ are given by $z^2+2az+1=0\Rightarrow z=\frac{-2a\pm\sqrt{4a^2-4}}{2}=-a\pm\sqrt{a^2-1}$.

Let, $\alpha=-a+\sqrt{a^2-1}$ and $\beta=-a-\sqrt{a^2-1} \Rightarrow \alpha\beta=a^2-(a^2-1)=1 \Rightarrow |\alpha\beta|=1$

Now, $a>1 \Rightarrow a^2>1 \Rightarrow |\beta|=\left|-a-\sqrt{a^2-1}\right|=\left|a+\sqrt{a^2-1}\right|>1 \Rightarrow |\alpha|=\frac{1}{|\beta|}<1$

$\Rightarrow$ Only the pole $z = \alpha = -a + \sqrt{a^2 - 1}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f, \alpha) = \lim_{z \to \alpha} \frac{z - \alpha}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha - \beta} = \frac{1}{-a + \sqrt{a^2-1} - \left(-a - \sqrt{a^2-1}\right)} = \frac{1}{2\sqrt{a^2-1}}$$

By Cauchy's residue theorem

$$I_1 = \frac{2}{i}\left[2\pi i \sum_i R_i\right] = \frac{2}{i}\left[2\pi i\left(\frac{1}{2\sqrt{a^2-1}}\right)\right] = \frac{2\pi}{\sqrt{a^2-1}}$$

Put the value of I_1 in equation (A)

$$I = 2\pi a - \frac{2\pi a^2}{\sqrt{a^2-1}} = 2\pi a\left(1 - \frac{a}{\sqrt{a^2-1}}\right)$$ **Hence proved.**

Question 13: **prove that** $\displaystyle\int_0^{2\pi} e^{-\cos\theta}\cos(n\theta + \sin\theta)\,d\theta = (-1)^n \frac{2\pi}{n!}$ **(n being positive integer)**

Solution: Let, $$I = \int_0^{2\pi} e^{-\cos\theta}\cos(n\theta + \sin\theta)\,d\theta = \text{Re}\left[\int_0^{2\pi} e^{-\cos\theta} e^{-i(n\theta + \sin\theta)}\,d\theta\right]$$

$$= \text{Re}\left[\int_0^{2\pi} e^{-(\cos\theta + i\sin\theta)} e^{-in\theta}\,d\theta\right] = \text{Re}\left[\int_0^{2\pi} e^{-e^{i\theta}} e^{-in\theta}\,d\theta\right]$$

Put, $e^{i\theta} = z \Rightarrow d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}$

$$I = \text{Re}\left[\int_C e^{-z} z^{-n}\frac{dz}{iz}\right] = \text{Re}\left[\frac{1}{i}\int_C \frac{e^{-z}}{z^{n+1}}\,dz\right],$$ where, $C: |z| = 1$ (unit circle with centre at origin)

Pole of $f(z) = \dfrac{e^{-z}}{z^{n+1}}$ is $z = 0$, which lies inside $C: |z| = 1$ and it has order $n+1$.

$$R_1(f, 0) = \lim_{z \to 0} \frac{1}{n!}\frac{d^n}{dz^n}\left[\frac{z^{n+1}e^{-z}}{z^{n+1}}\right] = \lim_{z \to 0} \frac{1}{n!}\frac{d^n}{dz^n}(e^{-z}) = \frac{1}{n!}(-1)^n$$

By Cauchy's residue theorem

$$I = \text{Re}\left[\frac{1}{i}\left(2\pi i \sum_i R_i\right)\right] = \text{Re}\left[\frac{1}{i}\left(2\pi i\left(\frac{1}{n!}(-1)^n\right)\right)\right] = (-1)^n\frac{2\pi}{n!}$$ **Hence proved.**

Question 14: **Prove that** $\displaystyle\int_0^{2\pi} (\cos\theta)^{2n}\,d\theta = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}$, **where n is a positive integer.**

Solution: Let, $$I = \int_0^{2\pi} (\cos\theta)^{2n}\,d\theta$$

Put , $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I = \int_C \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^{2n} \frac{dz}{iz} = \frac{1}{(2^{2n})i}\int_C \frac{(z^2+1)^{2n}}{z^{2n+1}}dz, \quad \text{where,} \quad C: |z| = 1 \text{ (unit circle with centre at origin)}$$

Pole of $f(z) = \frac{(z^2+1)^{2n}}{z^{2n+1}}$ is $z = 0$, which lies inside $C: |z| = 1$ and it has order $2n+1$.

$$f(z) = \frac{1}{z}\left(z + \frac{1}{z}\right)^{2n} = \frac{1}{z}\left[\binom{2n}{0}\cdot z^{2n} + \binom{2n}{1}\cdot z^{2n-1}\cdot\frac{1}{z} + \cdots + \binom{2n}{n}\cdot z^{2n-n}\cdot\frac{1}{z^n} + \cdots + \binom{2n}{2n}\cdot z^{2n-2n}\cdot\frac{1}{z^{2n}}\right]$$

Residue of $f(z)$ at $z = 0$ is the coefficient of $\frac{1}{z}$.

$$R_1(f,0) = \binom{2n}{n} = \frac{(2n)!}{n!\,(2n-n)!} = \frac{(2n)!}{n!\,n!} = \frac{(2n)!}{(n!)^2}$$

By Cauchy's residue theorem

$$I = \frac{1}{(2^{2n})i}\left(2\pi i \sum_i R_i\right) = \frac{1}{(2^{2n})i}\left[2\pi i\left(\frac{(2n)!}{(n!)^2}\right)\right] = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}$$ **Hence proved**.

Question 15: **Prove that** $\int_0^{\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \frac{\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$, **where,** $a > b > 0$.

Solution: Let, $I = \int_0^{\pi} \frac{\sin^2\theta d\theta}{a + b\cos\theta} = \frac{1}{2}\int_0^{\pi} \frac{(1 - \cos 2\theta)d\theta}{a + b\cos\theta}$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{2\pi} \frac{(1-\cos 2\theta)d\theta}{a + b\cos\theta} \qquad \because \text{ if } f(2B-\theta) = f(\theta) \text{ then } \int_0^{2B} f(\theta)d\theta = 2\int_0^{B} f(\theta)d\theta$$

$$I = \frac{1}{4}\text{Re}\left[\int_0^{2\pi} \frac{(1 - e^{2i\theta})d\theta}{a + b\cos\theta}\right],$$

Put , $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I = \text{Re}\left[\frac{1}{4}\int_C \frac{(1-z^2)\frac{dz}{iz}}{a + b\left(\frac{z^2+1}{2z}\right)}\right] = \text{Re}\left[\frac{1}{2i}\int_C \frac{(1-z^2)dz}{bz^2 + 2az + b}\right], \quad \text{where,} \quad C: |z| = 1 \text{ (unit circle with centre at origin)}$$

Poles of $f(z) = \frac{1-z^2}{bz^2 + 2az + b}$ are given by $bz^2 + 2az + b = 0 \Rightarrow z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$.

Let, $\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}$ and $\beta = \frac{-a - \sqrt{a^2 - b^2}}{b} \Rightarrow \alpha\beta = \frac{a^2 - (a^2 - b^2)}{b^2} = 1 \Rightarrow |\alpha\beta| = 1$

Now, $a > b > 0 \Rightarrow a^2 - b^2 > 0 \Rightarrow |\beta| = \left|\frac{-a - \sqrt{a^2 - b^2}}{b}\right| = \left|\frac{a + \sqrt{a^2 - b^2}}{b}\right| > 1 \Rightarrow |\alpha| = \frac{1}{|\beta|} < 1$

$\Rightarrow$ Only the pole $z = \alpha = \dfrac{-a + \sqrt{a^2 - b^2}}{b}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f, \alpha) = \lim_{z \to \alpha} \frac{(z - \alpha)(1 - z^2)}{b(z - \alpha)(z - \beta)} = \frac{1 - \alpha^2}{b(\alpha - \beta)} = \frac{1 - \left(\dfrac{-a + \sqrt{a^2 - b^2}}{b}\right)^2}{b\left[\dfrac{-a + \sqrt{a^2 - b^2}}{b} - \left(\dfrac{-a - \sqrt{a^2 - b^2}}{b}\right)\right]}$$

$$= \frac{b^2 - a^2 - (a^2 - b^2) + 2a\sqrt{a^2 - b^2}}{2b^2\sqrt{a^2 - b^2}} = \frac{2b^2 - 2a^2 + 2a\sqrt{a^2 - b^2}}{2b^2\sqrt{a^2 - b^2}} = \frac{-(a^2 - b^2) + a\sqrt{a^2 - b^2}}{b^2\sqrt{a^2 - b^2}}$$

$$= \frac{\sqrt{a^2 - b^2}\left(-\sqrt{a^2 - b^2} + a\right)}{b^2\sqrt{a^2 - b^2}} = \frac{a - \sqrt{a^2 - b^2}}{b^2}$$

By Cauchy's residue theorem

$$I = \text{Re}\left[\frac{1}{2i}\left(2\pi i \sum_i R_i\right)\right] = \text{Re}\left[\frac{1}{2i} . 2\pi i\left(\frac{a - \sqrt{a^2 - b^2}}{b^2}\right)\right] = \frac{\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right) \qquad \textbf{Hence proved.}$$

Note: In statement of the question 15 (in Iqbal's book), it is given that $|a| > |b| > 0$, which is wrong. As if $a = -5$ and $b = 3$, then $|\beta| = \left|\frac{-a - \sqrt{a^2 - b^2}}{b}\right| = \left|\frac{a + \sqrt{a^2 - b^2}}{b}\right| = \left|\frac{-5 + \sqrt{25 - 9}}{3}\right| = \frac{1}{3} < 1$ and we cannot reach the required result in this case. To get the required result, we must have $|\beta| > 1$, which can be attained if we take $a > b > 0$.

Question 16: **Prove that** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\alpha + \sin^2\theta} = \frac{\pi}{2\sqrt{\alpha^2 + \alpha}}$, **where $\alpha \geq 0$**

Solution: Let, $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\alpha + \sin^2\theta} = \frac{1}{2}\int_0^{\pi} \frac{d\theta}{\alpha + \sin^2\theta}$ $\quad \because$ if $f(2a - \theta) = f(\theta)$ then $\displaystyle\int_0^{2a} f(\theta)d\theta = 2\int_0^{a} f(\theta)d\theta$

$$= \frac{1}{2}\int_0^{\pi} \frac{2d\theta}{2\alpha + 2\sin^2\theta} = \frac{1}{2}\int_0^{\pi} \frac{2d\theta}{2\alpha + 1 - \cos 2\theta}$$

Put, $2\theta = \varphi$, $\theta = \dfrac{\varphi}{2} \Rightarrow d\theta = \dfrac{d\varphi}{2}$, $\quad \theta = 0 \Rightarrow \varphi = 0$ and $\theta = \pi \Rightarrow \varphi = 2\pi$

$$I = \frac{1}{2}\int_0^{2\pi} \frac{2\left(\frac{d\varphi}{2}\right)}{2\alpha + 1 - \cos\varphi} = \frac{1}{2}\int_0^{2\pi} \frac{d\varphi}{2\alpha + 1 - \cos\varphi}$$

Put, $e^{i\varphi} = z \Rightarrow d\varphi = \dfrac{dz}{ie^{i\varphi}} = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = \dfrac{z^2 + 1}{2z}$

$$I=\frac{1}{2}\int_C \frac{\frac{dz}{iz}}{2\alpha+1-\frac{z^2+1}{2z}},\qquad \text{where,}\quad C:\ |z|=1 \text{ (unit circle with centre at origin)}$$

$$=\frac{1}{2}\int_C \frac{\frac{dz}{iz}}{\frac{2(2\alpha+1)z-z^2-1}{2z}}=-\frac{1}{i}\int_C \frac{dz}{z^2-(2+4\alpha)z+1}$$

Poles of $f(z)=\frac{1}{z^2-(2+4\alpha)z+1}$ are given by $z^2-(2+4\alpha)z+1=0$

$$\Rightarrow\quad z=\frac{2+4\alpha\pm\sqrt{(2+4\alpha)^2-4}}{2}\quad\Rightarrow\quad z=\frac{2+4\alpha\pm\sqrt{4+16\alpha^2+16\alpha-4}}{2}=1+2\alpha\pm 2\sqrt{\alpha^2+\alpha}$$

Let, $\alpha'=1+2\alpha+2\sqrt{\alpha^2+\alpha}$ and $\beta'=1+2\alpha-2\sqrt{\alpha^2+\alpha}$ $\Rightarrow \alpha'\beta'=(1+2\alpha)^2-4(\alpha^2+\alpha)=1\Rightarrow |\alpha'\beta'|=1$

Now, $\alpha\geq 0$ (given) $\Rightarrow$ $|\alpha'|=\left|1+2\alpha+2\sqrt{\alpha^2+\alpha}\right|>1$ $\Rightarrow$ $|\beta'|=\frac{1}{|\alpha'|}<1$

$\Rightarrow$ Only the pole $z=\beta'=1+2\alpha-2\sqrt{\alpha^2+\alpha}$ lies inside $C:|z|=1$ and it has order one.

$$R_1(f,\beta')=\lim_{z\to\beta'}\frac{z-\beta'}{(z-\alpha')(z-\beta')}=\frac{1}{\beta'-\alpha'}=\frac{1}{1+2\alpha-2\sqrt{\alpha^2+\alpha}-\left(1+2\alpha+2\sqrt{\alpha^2+\alpha}\right)}=\frac{1}{-4\sqrt{\alpha^2+\alpha}}$$

By Cauchy's residue theorem

$$I=-\frac{1}{i}\left(2\pi i\sum_i R_i\right)=-\frac{1}{i}\left[2\pi i\left(\frac{1}{-4\sqrt{\alpha^2+\alpha}}\right)\right]=\frac{\pi}{2\sqrt{\alpha^2+\alpha}}$$

Hence proved.

Question 17: **Prove that** $\int_0^{\frac{\pi}{2}}\frac{d\theta}{\alpha^2\cos^2\theta+\beta^2\sin^2\theta}=\frac{\pi}{2\alpha\beta}$, **where** $\beta\geq\alpha>0$

Solution: **Case I**: **When** $\beta>\alpha>0$: Let,

$$I=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\alpha^2\cos^2\theta+\beta^2\sin^2\theta}=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\alpha^2\left(\frac{1+\cos 2\theta}{2}\right)+\beta^2\left(\frac{1-\cos 2\theta}{2}\right)}$$

$$=2\int_0^{\frac{\pi}{2}}\frac{d\theta}{\alpha^2(1+\cos 2\theta)+\beta^2(1-\cos 2\theta)}$$

Put, $2\theta=\varphi$, $\theta=\frac{\varphi}{2}\Rightarrow d\theta=\frac{d\varphi}{2}$, $\theta=0\ \Rightarrow\ \varphi=0$, $\theta=\frac{\pi}{2}\ \Rightarrow\ \varphi=\pi$

$$I=2\int_0^{\pi}\frac{\frac{d\varphi}{2}}{\alpha^2(1+\cos\varphi)+\beta^2(1-\cos\varphi)}=\int_0^{\pi}\frac{d\varphi}{\alpha^2(1+\cos\varphi)+\beta^2(1-\cos\varphi)}$$

$$=\frac{1}{2}\int_0^{2\pi}\frac{d\varphi}{\alpha^2(1+\cos\varphi)+\beta^2(1-\cos\varphi)}\qquad \because \text{if } f(2b-\varphi)=f(\varphi) \text{ then } \int_0^{2b}f(\varphi)d\varphi=2\int_0^{b}f(\varphi)d\varphi$$

Put , $e^{i\varphi} = z \Rightarrow d\varphi = \frac{dz}{ie^{i\varphi}} = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I = \frac{1}{2}\int_C \frac{\frac{dz}{iz}}{\alpha^2\left(1 + \frac{z^2+1}{2z}\right) + \beta^2\left[1 - \left(\frac{z^2+1}{2z}\right)\right]}\ , \qquad \text{where,}\quad C:\ |z| = 1 \text{ (unit circle with centre at origin)}$$

$$= \frac{1}{i}\int_C \frac{dz}{2\alpha^2 z + \alpha^2 z^2 + \alpha^2 + 2\beta^2 z - \beta^2 z^2 - \beta^2} = \frac{1}{i}\int_C \frac{dz}{(\alpha^2-\beta^2)z^2 + 2(\alpha^2+\beta^2)z + \alpha^2 - \beta^2}$$

Poles of $f(z) = \frac{1}{(\alpha^2-\beta^2)z^2 + 2(\alpha^2+\beta^2)z + \alpha^2-\beta^2}$ are given by $(\alpha^2-\beta^2)z^2 + 2(\alpha^2+\beta^2)z + \alpha^2 - \beta^2 = 0$

$$\Rightarrow z = \frac{-2\alpha^2 - 2\beta^2 \pm \sqrt{4(\alpha^2+\beta^2)^2 - 4(\alpha^2-\beta^2)(\alpha^2-\beta^2)}}{2(\alpha^2-\beta^2)}$$

$$= \frac{-2\alpha^2 - 2\beta^2 \pm \sqrt{4\alpha^4 + 4\beta^4 + 8\alpha^2\beta^2 - 4\alpha^4 - 4\beta^4 + 8\alpha^2\beta^2}}{2(\alpha^2-\beta^2)} = \frac{-2\alpha^2 - 2\beta^2 \pm 4\alpha\beta}{2(\alpha^2-\beta^2)}$$

$$= \frac{-\alpha^2 - \beta^2 \pm 2\alpha\beta}{\alpha^2-\beta^2}.$$

Let, $\alpha' = \frac{-\alpha^2-\beta^2+2\alpha\beta}{\alpha^2-\beta^2} = \frac{-(\alpha-\beta)^2}{(\alpha+\beta)(\alpha-\beta)} = \frac{-(\alpha-\beta)}{\alpha+\beta}$

and $\beta' = \frac{-\alpha^2-\beta^2-2\alpha\beta}{\alpha^2-\beta^2} = \frac{-(\alpha+\beta)^2}{(\alpha+\beta)(\alpha-\beta)} = \frac{-(\alpha+\beta)}{\alpha-\beta}$ $\quad\Rightarrow \alpha'\beta' = 1 \Rightarrow |\alpha'\beta'| = 1$

Now, $\beta > \alpha > 0$ (given) $\Rightarrow |\alpha+\beta| > |\alpha-\beta| \Rightarrow |\beta'| = \left|\frac{-(\alpha+\beta)}{\alpha-\beta}\right| = \frac{|\alpha+\beta|}{|\alpha-\beta|} > 1 \Rightarrow |\alpha'| = \frac{1}{|\beta'|} < 1$

$\Rightarrow$ Only the pole $z = \alpha' = \frac{-(\alpha-\beta)}{\alpha+\beta}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f,\alpha') = \lim_{z\to\alpha'} \frac{z-\alpha'}{(\alpha^2-\beta^2)(z-\alpha')(z-\beta')} = \frac{1}{(\alpha^2-\beta^2)(\alpha'-\beta')} = \frac{1}{(\alpha^2-\beta^2)\left[\frac{-(\alpha-\beta)}{\alpha+\beta} - \left(\frac{-(\alpha+\beta)}{\alpha-\beta}\right)\right]}$$

$$= \frac{1}{(\alpha^2-\beta^2)\left[\frac{-(\alpha-\beta)^2 + (\alpha+\beta)^2}{\alpha^2-\beta^2}\right]} = \frac{1}{-\alpha^2-\beta^2+2\alpha\beta+\alpha^2+\beta^2+2\alpha\beta} = \frac{1}{4\alpha\beta}$$

By Cauchy's residue theorem

$$I = \frac{1}{i}\left(2\pi i\sum_i R_i\right) = \frac{1}{i}\left(2\pi i\left(\frac{1}{4\alpha\beta}\right)\right) = \frac{\pi}{2\alpha\beta}$$

Case II: When $\beta = \alpha > 0$: In this case, given integral becomes $I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\alpha^2(\cos^2\theta + \sin^2\theta)} = \frac{1}{\alpha^2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2\alpha^2}$

Combining both cases, we have, $I = \frac{\pi}{2\alpha\beta}$

Question 18: **Prove that** $\displaystyle\int_0^{2\pi} \frac{e^{ni\theta}d\theta}{1+2r\cos\theta+r^2} = \frac{(-r)^n 2\pi}{1-r^2}$, **where** $-1<r<1$, $n=0,1,2,3,\cdots$

Solution: Let, $$I = \int_0^{2\pi} \frac{e^{ni\theta}d\theta}{1+2r\cos\theta+r^2}$$

Put, $e^{i\theta}=z \Rightarrow d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right) = \dfrac{z^2+1}{2z}$

$$I = \int_C \frac{z^n}{1+2r\left(\frac{z^2+1}{2z}\right)+r^2}\cdot\frac{dz}{iz}, \qquad C:\ |z|=1 \text{ (unit circle with centre at origin)}$$

$$= \frac{1}{i}\int_C \frac{z^n dz}{rz^2+zr^2+z+r} = \frac{1}{i}\int_C \frac{z^n dz}{rz(z+r)+(z+r)} = \int_C \frac{z^n dz}{(rz+1)(z+r)}$$

Poles of $f(z) = \dfrac{z^n}{(rz+1)(z+r)}$ are given by $z = -r, -\dfrac{1}{r}$.

Now, $-1<r<1$ (given) $\Rightarrow$ Only the pole $z=-r$ lies inside C: $|z|=1$ and it has order one.

$$R_1(f,-r) = \lim_{z\to -r}\frac{(z+r)z^n}{(rz+1)(z+r)} = \frac{(-r)^n}{1-r^2}$$

By Cauchy's residue theorem

$$I = \frac{1}{i}\left(2\pi i\sum_i R_i\right) = \frac{2\pi i}{i}\left(\frac{(-r)^n}{1-r^2}\right) = \frac{(-r)^n 2\pi}{1-r^2}$$ **Hence proved.**

Question 19: **Prove that** $\displaystyle\int_0^{2\pi}\frac{d\theta}{\sin\theta-2\cos\theta+3} = \pi$

Solution: Let, $$I = \int_0^{2\pi}\frac{d\theta}{\sin\theta-2\cos\theta+3}$$

Put, $e^{i\theta}=z \Rightarrow d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right) = \dfrac{z^2+1}{2z}$, $\sin\theta = \dfrac{1}{2i}\left(z+\dfrac{1}{z}\right) = \dfrac{z^2-1}{2zi}$,

$$I = \int_C \frac{\frac{dz}{iz}}{\frac{z^2-1}{2zi}-2\left(\frac{z^2+1}{2z}\right)+3}, \qquad C:\ |z|=1 \text{ (unit circle with centre at origin}$$

$$= 2\int_C \frac{dz}{z^2-1-2z^2i-2i+6iz} = 2\int_C\frac{dz}{(1-2i)z^2+6iz-(1+2i)}$$

Poles of $f(z) = \dfrac{1}{(1-2i)z^2+6iz-(1+2i)}$ are given by $(1-2i)z^2+6iz-(1+2i)=0$

$$\Rightarrow z = \frac{-6i\pm\sqrt{-36+4(1-4i^2)}}{2(1-2i)} = \frac{-6i\pm\sqrt{-36+4+16}}{2(1-2i)} = \frac{-6i\pm\sqrt{16}i}{2(1-2i)} = \frac{-3i\pm 2i}{1-2i}.$$

Let, $\alpha = \dfrac{-3i+2i}{1-2i} = \dfrac{-i}{1-2i}$ and $\beta = \dfrac{-3i-2i}{1-2i} = \dfrac{-5i}{1-2i}$ $\Rightarrow$ $|\alpha| = \dfrac{1}{\sqrt{5}} < 1$ and $|\beta| = \dfrac{5}{\sqrt{5}} = \sqrt{5} > 1$

$\Rightarrow$ Only the pole $z = \alpha = \dfrac{-i}{1-2i}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f,\alpha) = \lim_{z\to\alpha} \frac{z-\alpha}{(1-2i)(z-\alpha)(z-\beta)} = \frac{1}{(1-2i)(\alpha-\beta)} = \frac{1}{(1-2i)\left[\frac{-i}{1-2i} - \left(\frac{-5i}{1-2i}\right)\right]} = \frac{1}{(1-2i)\left(\frac{4i}{1-2i}\right)} = \frac{1}{4i}$$

By Cauchy's residue theorem

$$I = 2\left(2\pi i \sum_i R_i\right) = 4\pi i\left(\frac{1}{4i}\right) = \pi \qquad \textbf{Hence proved.}$$

Question 20: **Prove that** $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{10 - 8\cos\theta} = \frac{\pi}{24}$

Solution: Let, $\displaystyle I = \int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{10 - 8\cos\theta} = \mathrm{Re}\left[\int_0^{2\pi} \frac{e^{3i\theta}\, d\theta}{10 - 8\cos\theta}\right]$

Put , $e^{i\theta} = z \Rightarrow d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = \dfrac{z^2+1}{2z}$

$$I = \mathrm{Re}\int_C \frac{z^3}{10 - 4\left(\frac{z^2+1}{z}\right)} \cdot \frac{dz}{iz}, \quad \text{where,} \quad C: |z| = 1 \text{ (unit circle with centre at origin)}$$

$$= \mathrm{Re}\left[\int_C \frac{z^3}{\frac{10z - 4z^2 - 4}{z}} \cdot \frac{dz}{iz}\right] = \mathrm{Re}\left[-\frac{1}{2i}\int_C \frac{z^3 dz}{2z^2 - 5z + 2}\right] = \mathrm{Re}\left[-\frac{1}{2i}\int_C \frac{z^3 dz}{2z^2 - 4z - z + 2}\right]$$

$$= \mathrm{Re}\left[-\frac{1}{2i}\int_C \frac{z^3 dz}{2z(z-2) - (z-2)}\right] = \mathrm{Re}\left[-\frac{1}{2i}\int_C \frac{z^3 dz}{(z-2)(2z-1)}\right]$$

Poles of $f(z) = \dfrac{z^3}{(z-2)(2z-1)}$ are $z = 2, \dfrac{1}{2}$. Only the pole $z = \dfrac{1}{2}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1\left(f, \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \frac{\left(z - \frac{1}{2}\right)z^3}{2(z-2)\left(z - \frac{1}{2}\right)} = \frac{\frac{1}{8}}{2\left(\frac{1}{2} - 2\right)} = \frac{\frac{1}{8}}{2\left(-\frac{3}{2}\right)} = -\frac{1}{24}$$

By Cauchy's residue theorem

$$I = \mathrm{Re}\left[-\frac{1}{2i}\left(2\pi i \sum_i R_i\right)\right] = \mathrm{Re}\left[-\frac{1}{2i}\left(2\pi i\left(-\frac{1}{24}\right)\right)\right] = \frac{\pi}{24} \qquad \textbf{Hence proved.}$$

Question 21: **Prove that** $\displaystyle\int_0^{\pi} \cot(\theta - \alpha)d\theta = \begin{cases} \pi i, & \text{If } \mathrm{Im}(\alpha) > 0 \\ -\pi i, & \text{If } \mathrm{Im}(\alpha) < 0 \\ \text{diverges}, & \text{If } \mathrm{Im}(\alpha) = 0 \end{cases}$

Solution: Let, $I = \int_0^{\pi} \cot(\theta - \alpha)\mathrm{d}\theta$

$$= \int_0^{2\pi} \cot(\theta-\alpha)\mathrm{d}\theta \qquad \because \text{ if } f(\theta+\beta) = f(\theta) \text{ then } \int_0^{n\beta} f(\theta)\mathrm{d}\theta = n\int_0^{\beta} f(\theta)\mathrm{d}\theta, \ \forall\ n \in \{1,2,3,\cdots\}$$

$$= i\int_0^{\pi}\left[\frac{\mathrm{e}^{i(\theta-\alpha)} + \mathrm{e}^{-i(\theta-\alpha)}}{\mathrm{e}^{i(\theta-\alpha)} - \mathrm{e}^{-i(\theta-\alpha)}}\right]\mathrm{d}\theta = i\int_0^{\pi}\left[\frac{\mathrm{e}^{2i(\theta-\alpha)}+1}{\mathrm{e}^{2i(\theta-\alpha)}-1}\right]\mathrm{d}\theta = i\int_0^{\pi}\left[\frac{\mathrm{e}^{2i\theta}+\mathrm{e}^{2i\alpha}}{\mathrm{e}^{2i\theta}-\mathrm{e}^{2i\alpha}}\right]\mathrm{d}\theta = \frac{i}{2}\int_0^{2\pi}\left[\frac{\mathrm{e}^{2i\theta}+\mathrm{e}^{2i\alpha}}{\mathrm{e}^{2i\theta}-\mathrm{e}^{2i\alpha}}\right]\mathrm{d}\theta$$

Put , $\mathrm{e}^{i\theta} = z \Rightarrow \mathrm{d}\theta = \frac{\mathrm{d}z}{i\mathrm{e}^{i\theta}} = \frac{\mathrm{d}z}{iz}$

$$I = \frac{i}{2}\int_C \left[\frac{z^2+\mathrm{e}^{2i\alpha}}{z^2-\mathrm{e}^{2i\alpha}}\right]\cdot\frac{\mathrm{d}z}{iz} = \frac{1}{2}\int_C \frac{z^2+\mathrm{e}^{2i\alpha}}{z(z^2-\mathrm{e}^{2i\alpha})}\mathrm{d}z, \qquad \text{where,} \qquad C: |z| = 1 \text{ (unit circle with centre at origin)}$$

Poles of $f(z) = \frac{z^2+\mathrm{e}^{2i\alpha}}{z(z^2-\mathrm{e}^{2i\alpha})}$ are $z = 0, \ \pm\mathrm{e}^{i\alpha}$.

Let, $\alpha = a + bi \Rightarrow \left|\mathrm{e}^{i\alpha}\right| = \left|\mathrm{e}^{i(a+bi)}\right| = \left|\mathrm{e}^{-b+ai}\right| = \left|\mathrm{e}^{-b}\mathrm{e}^{ai}\right| = \mathrm{e}^{-b}\left|\mathrm{e}^{ai}\right| = \mathrm{e}^{-b} \qquad \because \left|\mathrm{e}^{ai}\right| = 1$

There are three cases:

Case I: If $b = \mathrm{Im}(\alpha) > 0 \Rightarrow \left|\pm\mathrm{e}^{i\alpha}\right| = \mathrm{e}^{-b} < 1 \Rightarrow$ the poles $z = \pm\mathrm{e}^{i\alpha}$ lie inside $C: |z| = 1$ and they have order one.

Case II: If $b = \mathrm{Im}(\alpha) < 0 \Rightarrow \left|\pm\mathrm{e}^{i\alpha}\right| = \mathrm{e}^{-b} > 1 \Rightarrow$ the poles $z = \pm\mathrm{e}^{i\alpha}$ lie outside $C: |z| = 1$.

Case III: If $b = \mathrm{Im}(\alpha) = 0 \Rightarrow \left|\pm\mathrm{e}^{i\alpha}\right| = \mathrm{e}^{-b} = 1 \Rightarrow$ the poles $z = \pm\mathrm{e}^{i\alpha}$ lie on $C: |z| = 1$.

Furthermore, the pole $z = 0$ lies inside $C: |z| = 1$ in any case and it has order 1.

$$R_1(f,0) = \lim_{z\to 0}\frac{z\left(z^2+\mathrm{e}^{2i\alpha}\right)}{z(z^2-\mathrm{e}^{2i\alpha})} = \lim_{z\to 0}\frac{z^2+\mathrm{e}^{2i\alpha}}{z^2-\mathrm{e}^{2i\alpha}} = \frac{\mathrm{e}^{2i\alpha}}{-\mathrm{e}^{2i\alpha}} = -1$$

$$R_2\left(f,\mathrm{e}^{i\alpha}\right) = \lim_{z\to\mathrm{e}^{i\alpha}}\frac{(z-\mathrm{e}^{i\alpha})(z^2+\mathrm{e}^{2i\alpha})}{z(z+\mathrm{e}^{i\alpha})(z-\mathrm{e}^{i\alpha})} = \lim_{z\to\mathrm{e}^{i\alpha}}\frac{z^2+\mathrm{e}^{2i\alpha}}{z(z+\mathrm{e}^{i\alpha})} = \frac{2\mathrm{e}^{2i\alpha}}{\mathrm{e}^{i\alpha}(2\mathrm{e}^{i\alpha})} = \frac{2\mathrm{e}^{2i\alpha}}{2\mathrm{e}^{2i\alpha}} = 1$$

$$R_3\left(f,-\mathrm{e}^{i\alpha}\right) = \lim_{z\to -\mathrm{e}^{i\alpha}}\frac{(z+\mathrm{e}^{i\alpha})(z^2+\mathrm{e}^{2i\alpha})}{z(z+\mathrm{e}^{i\alpha})(z-\mathrm{e}^{i\alpha})} = \lim_{z\to -\mathrm{e}^{i\alpha}}\frac{z^2+\mathrm{e}^{2i\alpha}}{z(z-\mathrm{e}^{i\alpha})} = \frac{2\mathrm{e}^{2i\alpha}}{-\mathrm{e}^{i\alpha}(-2\mathrm{e}^{i\alpha})} = \frac{2\mathrm{e}^{2i\alpha}}{2\mathrm{e}^{2i\alpha}} = 1$$

By Cauchy's residue theorem

$$I = \frac{1}{2}\left(2\pi i\sum_i R_i\right) = \pi i\sum_i R_i = \begin{cases}\pi i(-1+1+1) = \pi i, & \text{If } \mathrm{Im}(\alpha) > 0\\ \pi i(-1) = -\pi i, & \text{If } \mathrm{Im}(\alpha) < 0\\ \text{diverges}, & \text{If } \mathrm{Im}(\alpha) = 0\end{cases}$$

Hence Proved.

Note: $\int_C f(z)\mathrm{d}z$ diverges, if any of the pole (or poles) of $f(z)$ lies (or lie) on C.