Question 1: **Prove that** $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta d\theta}{5-4\cos\theta} = \frac{\pi}{12}$

Solution: Let, $\displaystyle I = \int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta} = \text{Re}\int_0^{2\pi} \frac{e^{3i\theta}\, d\theta}{5-4\cos\theta},$

Put , $\displaystyle e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$

$\displaystyle I = \text{Re}\int_C \frac{z^3}{5-2(z+\frac{1}{z})}\cdot\frac{dz}{iz} = \text{Re}\left[-\frac{1}{i}\int_C \frac{z^3 dz}{2z^2-5z+2}\right],$ where, C: $|z| = 1$ (unit circle with centre at origin)

Poles of $\displaystyle f(z) = \frac{z^3}{2z^2-5z+2}$ are given by $\displaystyle 2z^2-5z+2 = 0 \Rightarrow z = \frac{5\pm\sqrt{25-16}}{4} = \frac{5\pm 3}{4} = 2, \frac{1}{2}.$

Only the pole $z = \frac{1}{2}$ lies inside C: $|z| = 1$ and it has order one.

$$R_1\left(f,\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \frac{\left(z-\frac{1}{2}\right)z^3}{2(z-2)\left(z-\frac{1}{2}\right)} = \frac{\frac{1}{8}}{2\left(\frac{1}{2}-2\right)} = \frac{\frac{1}{8}}{2\left(-\frac{3}{2}\right)} = -\frac{1}{24}$$

By Cauchy's residue theorem

$$I = \text{Re}\left[-\frac{1}{i}\left(2\pi i\sum_i R_i\right)\right] = \text{Re}\left[-\frac{1}{i}\left(2\pi i\left(-\frac{1}{24}\right)\right)\right] = \frac{\pi}{12}$$ **Hence proved**.

Question 2: **Prove that** $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$

Solution: Let, $\displaystyle I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Put , $\displaystyle e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$

$\displaystyle I = \int_C \frac{\frac{dz}{iz}}{2+\frac{1}{2}\left(\frac{z^2+1}{z}\right)} = \int_C \frac{2dz}{i(4z+z^2+1)} = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1},$ where, C: $|z| = 1$ (unit circle with centre at origin)

Poles of $\displaystyle f(z) = \frac{1}{z^2+4z+1}$ are given by $\displaystyle z^2+4z+1 = 0 \Rightarrow z = \frac{-4\pm\sqrt{16-4}}{2} = -2\pm\sqrt{3}.$

As, $|-2+\sqrt{3}| < 1$ and $|-2-\sqrt{3}| > 1$. So, only the pole $z = -2+\sqrt{3}$ lies inside C: $|z| = 1$ and it has order one.

$$R_1(f,-2+\sqrt{3}) = \lim_{z\to-2+\sqrt{3}} \frac{z-(-2+\sqrt{3})}{\left(z-(-2+\sqrt{3})\right)\left(z-(-2-\sqrt{3})\right)} = \frac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})} = \frac{1}{2\sqrt{3}}$$

By Cauchy's residue theorem

$\mathrm{I} = \frac{2}{i}(2\pi i)\sum_i R_i = \frac{2}{i}(2\pi i)\left(\frac{1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$ **Hence proved.**

Question 3: **Prove that** $\displaystyle\int_0^{2\pi} \frac{\mathbf{d}\boldsymbol{\theta}}{\mathbf{1} + \boldsymbol{a}\cos\boldsymbol{\theta}} = \frac{\mathbf{2}\boldsymbol{\pi}}{\sqrt{\mathbf{1} - \boldsymbol{a}^{\mathbf{2}}}}$, $\quad (\boldsymbol{a} > \mathbf{0},\ \boldsymbol{a}^{\mathbf{2}} < \mathbf{1})$

Solution: Let, $I = \displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + a\cos\theta}$

Put, $\mathrm{e}^{i\theta} = z \ \Rightarrow\ \mathrm{d}\theta = \frac{\mathrm{d}z}{i\mathrm{e}^{i\theta}} = \frac{\mathrm{d}z}{iz}$, $\quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$I = \displaystyle\int_C \frac{\frac{\mathrm{d}z}{iz}}{1 + a\left(\frac{z^2+1}{2z}\right)} = \frac{2}{i}\int_C \frac{\mathrm{d}z}{az^2 + 2z + a}$, where, C: $|z| = 1$ (unit circle with centre at origin)

Poles of $f(z) = \frac{1}{az^2 + 2z + a}$ are given by $az^2 + 2z + a = 0 \Rightarrow z = \frac{-2 \pm \sqrt{4 - 4a^2}}{2a} = \frac{-1 \pm \sqrt{1 - a^2}}{a}$

Let, $\alpha = \frac{-1 + \sqrt{1 - a^2}}{a}$ and $\beta = \frac{-1 - \sqrt{1 - a^2}}{a} \quad \Rightarrow \quad \alpha\beta = \frac{1 - (1 - a^2)}{a^2} = 1 \ \Rightarrow\ |\alpha\beta| = 1$

Now, $1 - a^2 > 0 \ \Rightarrow\ a^2 < 1 \Rightarrow |a| < 1 \Rightarrow \ |\beta| = \left|\frac{-1 - \sqrt{1 - a^2}}{a}\right| = \frac{\left|1 + \sqrt{1 - a^2}\right|}{|a|} > 1 \Rightarrow \ |\alpha| = \frac{1}{|\beta|} < 1$

$\Rightarrow$ Only the pole $z = \alpha = \frac{-1 + \sqrt{1 - a^2}}{a}$ lies inside $C\colon |z| = 1$ and it has order one.

$R_1(f, \alpha) = \lim_{z \to \alpha} \frac{z - \alpha}{a(z - \alpha)(z - \beta)} = \frac{1}{a(\alpha - \beta)} = \frac{1}{a\left[\frac{-1 + \sqrt{1 - a^2}}{a} - \left(\frac{-1 - \sqrt{1 - a^2}}{a}\right)\right]} = \frac{1}{2\sqrt{1 - a^2}}$

By Cauchy's residue theorem

$\mathrm{I} = \frac{2}{i}(2\pi i)\sum_i R_i = \frac{2}{i}(2\pi i)\frac{1}{2\sqrt{1 - a^2}} = \frac{2\pi}{\sqrt{1 - a^2}}$ **Hence proved.**

Question 4: **Prove that** $\displaystyle\int_0^{2\pi} \frac{\mathbf{d}\boldsymbol{\theta}}{\mathbf{5} + \mathbf{3}\cos\boldsymbol{\theta}} = \frac{\boldsymbol{\pi}}{\mathbf{2}}$

Solution: Let, $I = \displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 3\cos\theta}$

Put, $\mathrm{e}^{i\theta} = z \ \Rightarrow\ \mathrm{d}\theta = \frac{\mathrm{d}z}{i\mathrm{e}^{i\theta}} = \frac{\mathrm{d}z}{iz}$, $\quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I=\int_C \frac{\frac{dz}{iz}}{5+3\left(\frac{z^2+1}{2z}\right)}=\frac{2}{i}\int_C \frac{dz}{3z^2+10z+3},$$ where, C: $|z|=1$ (unit circle with centre at origin)

Poels of $f(z)=\dfrac{1}{3z^2+10z+3}$ are given by $3z^2+10z+3=0 \Rightarrow z=\dfrac{-10\pm\sqrt{64}}{6}=\dfrac{-10\pm 8}{6}=-\dfrac{1}{3},-3$

Only the pole $z=-\dfrac{1}{3}$ lie inside $C:|z|=1$ and it has order one.

$$R_1\left(f,-\frac{1}{3}\right)=\lim_{z\to-\frac{1}{3}}\frac{z+\frac{1}{3}}{3\left(z+\frac{1}{3}\right)(z+3)}=\frac{1}{3\left(-\frac{1}{3}+3\right)}=\frac{1}{8}$$

By Cauchy's residue theorem

$$I=\frac{2}{i}\left(2\pi i\sum_i R_i\right)=\frac{2}{i}(2\pi i)\left(\frac{1}{8}\right)=\frac{\pi}{2}$$ **Hence Proved.**

Question 5: **Prove that** $\displaystyle\int_0^{2\pi}\frac{\cos^2 3\theta\, d\theta}{5-4\cos\theta}=\frac{3\pi}{8}$

Solution: Let, $$I=\int_0^{2\pi}\frac{\cos^2 3\theta\ d\theta}{5-4\cos\theta}=\frac{1}{2}\int_0^{2\pi}\frac{2\cos^2 3\theta\ d\theta}{5-4\cos\theta}=\frac{1}{2}\int_0^{2\pi}\frac{(1+\cos 6\theta)d\theta}{5-4\cos\theta}=\text{Re}\left[\frac{1}{2}\int_0^{2\pi}\frac{(1+e^{6i\theta})d\theta}{5-4\cos\theta}\right]$$

Put , $e^{i\theta}=z \Rightarrow d\theta=\dfrac{dz}{ie^{i\theta}}=\dfrac{dz}{iz}$, $\cos\theta=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)=\dfrac{z^2+1}{2z}$

$$I=\text{Re}\left[\frac{1}{2}\int_C\frac{(1+z^6)\frac{dz}{iz}}{5-4\left(\frac{z^2+1}{2z}\right)}\right]=\text{Re}\left[-\frac{1}{2i}\int_C\frac{(1+z^6)dz}{2z^2-5z+2}\right],$$ where, C: $|z|=1$ (unit circle with centre at origin)

Poles of $f(z)=\dfrac{1+z^6}{2z^2-5z+2}$ are given by $2z^2-5z+2=0 \Rightarrow z=\dfrac{5\pm\sqrt{25-16}}{4}=\dfrac{5\pm 3}{4}=2,\dfrac{1}{2}$

Only the pole $z=\dfrac{1}{2}$ lies inside $C:|z|=1$ and it has order one.

$$R_1\left(f,\frac{1}{2}\right)=\lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)(1+z^6)}{2\left(z-\frac{1}{2}\right)(z-2)}=\lim_{z\to\frac{1}{2}}\frac{(1+z^6)}{2(z-2)}=\frac{1+\frac{1}{64}}{2\left(\frac{1}{2}-2\right)}=\frac{\frac{65}{64}}{2\left(\frac{-3}{2}\right)}=\frac{-65}{192}$$

By Cauchy's residue theorem

$$\text{I}=\text{Re}\left[\frac{1}{-2i}\left(2\pi i\sum_i R_i\right)\right]=\text{Re}\left[-\pi\left(\frac{-65}{192}\right)\right]=\frac{65\pi}{192}$$ **Hence Proved**.

Question 6: **If $a^2>b^2+c^2$, then proved that** $\displaystyle\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta+c\sin\theta}=\frac{2\pi}{\sqrt{a^2-b^2-c^2}}$

Solution: Let, $I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta}$

Put , $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2-1}{2iz}$

$$I = \int_C \frac{\frac{dz}{iz}}{a + b\left(\frac{z^2+1}{2z}\right) + c\left(\frac{z^2-1}{2iz}\right)},$$ where, $C: |z| = 1$ (unit circle with centre at origin)

$$= \int_C \frac{\frac{dz}{iz}}{\frac{2aiz + z^2bi + bi + cz^2 - c}{2iz}} = 2\int_C \frac{dz}{2aiz + z^2bi + bi + cz^2 - c}$$

$$= 2\int_C \frac{dz}{(c+bi)z^2 + 2iaz - (c-bi)}$$

Poles of $f(z) = \frac{1}{(c+bi)z^2 + 2iaz - (c-bi)}$ are given by $(c+bi)z^2 + 2iaz - (c-bi) = 0$

$$z = \frac{-2ia \pm \sqrt{(2ia)^2 + 4(c+bi)(c-bi)}}{2(c+bi)} = \frac{-2ai \pm \sqrt{-4a^2 + 4c^2 + 4b^2}}{2(c+bi)} = \frac{-ai \pm \sqrt{-a^2 + c^2 + b^2}}{c+bi}$$
$$= \frac{\left(-a \pm \sqrt{a^2 - c^2 - b^2}\right)i}{c+bi}$$

Let, $\alpha = \frac{\left(-a + \sqrt{a^2 - c^2 - b^2}\right)i}{c+bi}$ and $\beta = \frac{\left(-a - \sqrt{a^2 - c^2 - b^2}\right)i}{c+bi}$

$$\Rightarrow \alpha\beta = \frac{-\left[a^2 - (a^2 - c^2 - b^2)\right]}{(c+bi)^2} = \frac{-(c^2+b^2)}{(c+bi)^2} \Rightarrow |\alpha\beta| = \frac{|-(c^2+b^2)|}{|(c+bi)^2|} = \frac{c^2+b^2}{|c+bi|^2} = \frac{c^2+b^2}{c^2+b^2} = 1$$

Now, $a^2 > b^2 + c^2 \Rightarrow a > \sqrt{c^2+b^2} \Rightarrow a + \sqrt{a^2 - c^2 - b^2} > \sqrt{c^2+b^2} \Rightarrow |\beta| = \left|\frac{a + \sqrt{a^2 - c^2 - b^2}}{\sqrt{c^2+b^2}}\right| > 1$

$\Rightarrow |\alpha| = \frac{1}{|\beta|} < 1$. Only the pole $z = \alpha = \frac{\left(-a + \sqrt{a^2 - c^2 - b^2}\right)i}{c+bi}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f,\alpha) = \lim_{z\to\alpha} \frac{z-\alpha}{(c+bi)(z-\alpha)(z-\beta)} = \frac{1}{(c+bi)(\alpha-\beta)}$$
$$= \frac{1}{(c+bi)\left[\frac{\left(-a + \sqrt{a^2 - c^2 - b^2}\right)i}{(c+bi)} - \left(\frac{\left(-a - \sqrt{a^2 - c^2 - b^2}\right)i}{(c+bi)}\right)\right]} = \frac{1}{2i\sqrt{a^2 - b^2 - c^2}}$$

By Cauchy's residue theorem

$I = 2(2\pi i)\sum_i R_i = 2(2\pi i)\left[\frac{1}{2i\sqrt{a^2 - b^2 - c^2}}\right] = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}$ **Hence Proved**.

Question 7: **Prove that** $\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{(5 - 3\cos\theta)^4} = \frac{135\,\pi}{16384}$

Solution: Let, $I = \int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{(5 - 3\cos\theta)^4} = \text{Re} \int_0^{2\pi} \frac{e^{3i\theta} \, d\theta}{(5 - 3\cos\theta)^4}$

Put, $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}, \qquad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I = \text{Re} \int_C \frac{z^3 \frac{dz}{iz}}{\left(5 - 3\frac{z^2+1}{2z}\right)^4} \quad \text{where, } C: |z| = 1 \text{ (unit circle with centre at origin)}$$

$$= \text{Re}\left[\frac{1}{i}\int_C \frac{z^2 dz}{\left(\frac{-3z^2+10z-3}{2z}\right)^4}\right] = \text{Re}\left[\frac{16}{i}\int_C \frac{z^6 \, dz}{(3z^2-10z+3)^4}\right] = \text{Re}\left[\frac{16}{i}\int_C \frac{z^6 \, dz}{(3z^2-9z-z+3)^4}\right]$$

$$= \text{Re}\left[\frac{16}{i}\int_C \frac{z^6 \, dz}{[3z(z-3)-(z-3)]^4}\right] = \text{Re}\left[\frac{16}{i}\int_C \frac{z^6 \, dz}{(z-3)^4(3z-1)^4}\right]$$

Poles of $f(z) = \frac{z^6}{(z-3)^4(1-3z)^4}$ are $z = 3, \frac{1}{3}$. Only the pole $z = \frac{1}{3}$ lies inside C: $|z| = 1$ and it has order four.

$$R_1\left(f, \frac{1}{3}\right) = \lim_{z\to\frac{1}{3}} \frac{1}{3!}\frac{d^3}{dz^3}\left[\frac{z^6\left(z-\frac{1}{3}\right)^4}{(3)^4\left(z-\frac{1}{3}\right)^4(z-3)^4}\right] = \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\frac{d^3}{dz^3}\left[\frac{z^6}{(z-3)^4}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\frac{d^2}{dz^2}\left[\frac{(z-3)^4(6z^5) - z^6\cdot 4\cdot(z-3)^3}{(z-3)^8}\right] = \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\frac{d^2}{dz^2}\left[\frac{(z-3)(6z^5)-4z^6}{(z-3)^5}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\frac{d^2}{dz^2}\left[\frac{2z^6-18z^5}{(z-3)^5}\right] = \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\frac{d}{dz}\left[\frac{(z-3)^5(12z^5-90z^4)-5(z-3)^4(2z^6-18z^5)}{(z-3)^{10}}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\frac{d}{dz}\left[\frac{(z-3)(12z^5-90z^4)-5(2z^6-18z^5)}{(z-3)^6}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\frac{d}{dz}\left[\frac{12z^6-90z^5-36z^5+270z^4-10z^6+90z^5}{(z-3)^6}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\frac{d}{dz}\left[\frac{2z^6-36z^5+270z^4}{(z-3)^6}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\left[\frac{(z-3)^6(12z^5-180z^4+1080z^3)-6(z-3)^5(2z^6-36z^5+270z^4)}{(z-3)^{12}}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\left[\frac{(z-3)(12z^5-180z^4+1080z^3)-6(2z^6-36z^5+270z^4)}{(z-3)^7}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\left[\frac{12z^6-180z^5+1080z^4-36z^5+540z^4-3240z^3-12z^6+216z^5-1620z^4}{(z-3)^7}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3!(3)^4}\left[\frac{-3240z^3}{(z-3)^7}\right] = \frac{1}{3!(3)^4}\left[\frac{-3240\left(\frac{1}{27}\right)}{\left(\frac{1}{3}-3\right)^7}\right] = \frac{1}{6(3)^4}\left[\frac{-120}{\left(-\frac{8}{3}\right)^7}\right] = \frac{120}{6(3)^4}\left(\frac{3}{8}\right)^7 = \frac{20(3)^3}{(8)^7}$$

By Cauchy's residue theorem

$$I=\operatorname{Re}\left[\frac{16}{i}(2\pi i)\sum_i R_i\right]=\operatorname{Re}\left[\frac{16}{i}(2\pi i)\left(\frac{20(3)^3}{(8)^7}\right)\right]=\frac{135\pi}{16384}$$ **Hence proved**.

Alternate solution: Let, $I=\int_0^{2\pi}\frac{\cos 3\theta\, d\theta}{(5-3\cos\theta)^4}=\operatorname{Re}\int_0^{2\pi}\frac{e^{-3i\theta}\, d\theta}{(5-3\cos\theta)^4}$

Put , $e^{i\theta}=z \Rightarrow d\theta=\frac{dz}{ie^{i\theta}}=\frac{dz}{iz}$, $\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$

$$I=\operatorname{Re}\int_C\frac{z^{-3}\frac{dz}{iz}}{\left(5-3\frac{z^2+1}{2z}\right)^4}$$ where, C: $|z|=1$ (unit circle with centre at origin)

$$=\operatorname{Re}\left[\frac{1}{i}\int_C\frac{z^{-4}dz}{\left(\frac{-3z^2+10z-3}{2z}\right)^4}\right]=\operatorname{Re}\left[\frac{16}{i}\int_C\frac{dz}{(3z^2-10z+3)^4}\right]=\operatorname{Re}\left[\frac{16}{i}\int_C\frac{dz}{(3z^2-9z-z+3)^4}\right]$$

$$=\operatorname{Re}\left[\frac{16}{i}\int_C\frac{dz}{[3z(z-3)-(z-3)]^4}\right]=\operatorname{Re}\left[\frac{16}{i}\int_C\frac{dz}{(z-3)^4(3z-1)^4}\right]$$

Poles of $f(z)=\frac{1}{(z-3)^4(1-3z)^4}$ are $z=3,\frac{1}{3}$. Only the pole $z=\frac{1}{3}$ lies inside C: $|z|=1$ and it has order four.

$$R_1\left(f,\frac{1}{3}\right)=\lim_{z\to\frac{1}{3}}\frac{1}{3!}\frac{d^3}{dz^3}\left[\frac{\left(z-\frac{1}{3}\right)^4}{(3)^4\left(z-\frac{1}{3}\right)^4(z-3)^4}\right]=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d^3}{dz^3}\left[\frac{1}{(z-3)^4}\right]=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d^2}{dz^2}\left[-4(z-3)^{-5}\right]$$

$$=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\frac{d}{dz}\left[20(z-3)^{-6}\right]=\lim_{z\to\frac{1}{3}}\frac{1}{3!\,(3)^4}\left[-120(z-3)^{-7}\right]=\frac{1}{3!\,(3)^4}\left[-120\left(\frac{1}{3}-3\right)^{-7}\right]$$

$$=\frac{1}{6(3)^4}\left[-120\left(-\frac{8}{3}\right)^{-7}\right]=\frac{1}{6(3)^4}\left[-120\left(-\frac{3}{8}\right)^7\right]=\frac{20(3)^3}{(8)^7}\quad\frac{120}{6(3)^4}\left(\frac{3}{8}\right)^7=\frac{20(3)^3}{(8)^7}$$

By Cauchy's residue theorem

$$I=\operatorname{Re}\left[\frac{16}{i}(2\pi i)\sum_i R_i\right]=\operatorname{Re}\left[\frac{16}{i}(2\pi i)\left(\frac{20(3)^3}{(8)^7}\right)\right]=\frac{135\pi}{16384}$$ **Hence proved**.

Question 8: **Prove that** $\displaystyle\int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos(n\theta)\, d\theta}{5-4\cos\theta}=\frac{2\pi}{3}\left(\frac{7}{4}\right)^n$, $\quad n=0,1,2,3,\cdots$

Solution: Let, $I=\int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos(n\theta)\, d\theta}{5-4\cos\theta}=\operatorname{Re}\int_0^{2\pi}\frac{(1+2\cos\theta)^n e^{ni\theta}d\theta}{5-4\cos\theta}$

Put , $e^{i\theta}=z \Rightarrow d\theta=\frac{dz}{ie^{i\theta}}=\frac{dz}{iz}$, $\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$

$$I = \text{Re}\left[\int_C \frac{\left[1+2\left(\frac{z^2+1}{2z}\right)\right]^n z^n \frac{dz}{iz}}{5-4\left(\frac{z^2+1}{2z}\right)}\right], \quad \text{where, } C: |z| = 1 \text{ (unit circle with centre at origin)}$$

$$= \text{Re}\left[\frac{2}{i}\int_C \frac{(z^2+z+1)^n dz}{-4z^2+10z-4}\right] = \text{Re}\left[\frac{-1}{i}\int_C \frac{(z^2+z+1)^n dz}{2z^2-5z+2}\right] = \text{Re}\left[\frac{-1}{i}\int_C \frac{(z^2+z+1)^n dz}{(z-2)(2z-1)}\right]$$

Poles of $f(z) = \frac{(z^2+z+1)^n}{(z-2)(2z-1)}$ are $z = \frac{1}{2}, 2$. Only the pole $z = \frac{1}{2}$ lies inside $C: |z| = 1$and it has order one.

$$R_1\left(f, \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \frac{\left(z-\frac{1}{2}\right)(z^2+z+1)^n}{2(z-2)\left(z-\frac{1}{2}\right)} = \lim_{z\to\frac{1}{2}} \frac{(z^2+z+1)^n}{2(z-2)} = \frac{\left(\frac{1}{4}+\frac{1}{2}+1\right)^n}{2\left(\frac{1}{2}-2\right)} = \frac{1}{-3}\left(\frac{7}{4}\right)^n$$

By Cauchy's residue theorem

$$\text{I} = \text{Re}\left[\left(-\frac{1}{i}\right)\left(2\pi i\sum_i R_i\right)\right] = \text{Re}\left[\left(-\frac{1}{i}\right)2\pi i\left(\frac{1}{-3}\left(\frac{7}{4}\right)^n\right)\right] = \frac{2\pi}{3}\left(\frac{7}{4}\right)^n$$ **Hence proved**.

Question 9: **Prove that** $\int_0^{2\pi} \cot\left(\frac{\theta-\alpha}{2}\right) d\theta = \begin{cases} 2\pi i, & \text{If } \text{Im}(\alpha) > 0 \\ -2\pi i, & \text{If } \text{Im}(\alpha) < 0 \\ \text{diverges}, & \text{If } \text{Im}(\alpha) = 0 \end{cases}$

Solution: Let, $I = \int_0^{2\pi} \cot\left(\frac{\theta-\alpha}{2}\right) d\theta = i\int_0^{2\pi} \left[\frac{e^{i\left(\frac{\theta-\alpha}{2}\right)} + e^{-i\left(\frac{\theta-\alpha}{2}\right)}}{e^{i\left(\frac{\theta-\alpha}{2}\right)} - e^{-i\left(\frac{\theta-\alpha}{2}\right)}}\right] d\theta = i\int_0^{2\pi} \left[\frac{e^{i(\theta-\alpha)}+1}{e^{i(\theta-\alpha)}-1}\right] d\theta = i\int_0^{2\pi} \left[\frac{e^{i\theta}+e^{\alpha i}}{e^{i\theta}-e^{\alpha i}}\right] d\theta$

Put , $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$

$$I = i\int_C \left[\frac{z+e^{\alpha i}}{z-e^{\alpha i}}\right]\frac{dz}{iz} = \int_C \frac{z+e^{\alpha i}}{z(z-e^{\alpha i})} dz, \quad \text{where, } C: |z| = 1 \text{ (unit circle with centre at origin)}$$

Poles of $f(z) = \frac{z+e^{\alpha i}}{z(z-e^{\alpha i})}$ are $z = 0, \ e^{i\alpha}$.

Let, $\alpha = a + bi \Rightarrow |e^{i\alpha}| = |e^{i(a+bi)}| = |e^{-b+ai}| = |e^{-b}e^{ai}| = e^{-b}|e^{ai}| = e^{-b} \quad \because |e^{ai}| = 1$

There are three cases:

Case I: If $b = \text{Im}(\alpha) > 0 \Rightarrow |e^{i\alpha}| = e^{-b} < 1 \Rightarrow$ the pole $z = e^{i\alpha}$ lies inside $C: |z| = 1$ and it has order one.

Case II: If $b = \text{Im}(\alpha) < 0 \Rightarrow |e^{i\alpha}| = e^{-b} > 1 \Rightarrow$ the pole $z = e^{i\alpha}$ lies outside $C: |z| = 1$.

Case III: If $b = \text{Im}(\alpha) = 0 \Rightarrow |e^{i\alpha}| = e^{-b} = 1 \Rightarrow$ the pole $z = e^{i\alpha}$ lies on $C: |z| = 1$.

Furthermore, the pole $z = 0$ lies inside $C: |z| = 1$ in any case and it has order 1.

$$R(f, 0) = \lim_{z\to 0} \frac{z(z+e^{\alpha i})}{z(z-e^{\alpha i})} = \lim_{z\to 0} \frac{z+e^{\alpha i}}{z-e^{\alpha i}} = \frac{e^{\alpha i}}{-e^{\alpha i}} = -1$$

$$R_2\left(f, \mathrm{e}^{i\alpha}\right) = \lim_{z \to \mathrm{e}^{i\alpha}} \frac{(z - \mathrm{e}^{\alpha i})(z + \mathrm{e}^{\alpha i})}{z(z - \mathrm{e}^{\alpha i})} = \lim_{z \to \mathrm{e}^{i\alpha}} \frac{z + \mathrm{e}^{\alpha i}}{z} = \frac{2\mathrm{e}^{2i\alpha}}{\mathrm{e}^{i\alpha}} = 2$$

By Cauchy's residue theorem

$$I = 2\pi i \sum_i R_i = \begin{cases} 2\pi i(-1 + 2) = 2\pi i, & \text{If } \operatorname{Im}(\alpha) > 0 \\ 2\pi i(-1) = -2\pi i, & \text{If } \operatorname{Im}(\alpha) < 0 \\ \text{diverges}, & \text{If } \operatorname{Im}(\alpha) = 0 \end{cases}$$ **Hence Proved**.

Note: $\int_C f(z)\mathrm{d}z$ diverges, if any of the pole (or poles) of $f(z)$ lies (or lie) on C.

Question 10: **Prove that** $\displaystyle\int_0^{2\pi} \mathbf{e^{\cos\theta} \cos(\sin\theta) \cos\theta \, d\theta = \pi}$

Solution: Let, $\displaystyle I = \int_0^{2\pi} \mathrm{e}^{\cos\theta} \cos(\sin\theta) \cos\theta \, \mathrm{d}\theta = \operatorname{Re}\left[\int_0^{2\pi} \mathrm{e}^{\cos\theta} \mathrm{e}^{i\sin\theta} \cos\theta \, \mathrm{d}\theta\right]$

$\because \mathrm{e}^{i\sin\theta} = \cos(\sin\theta) + i\sin(\sin\theta) \quad \Rightarrow \quad \operatorname{Re}\left[\mathrm{e}^{i\sin\theta}\right] = \cos(\sin\theta)$

Put, $\mathrm{e}^{i\theta} = z \ \Rightarrow \ \mathrm{d}\theta = \dfrac{\mathrm{d}z}{i\mathrm{e}^{i\theta}} = \dfrac{\mathrm{d}z}{iz}, \qquad \cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = \dfrac{z^2 + 1}{2z}$

$$I = \operatorname{Re}\left[\int_C \mathrm{e}^z \left(\frac{z^2 + 1}{2z}\right) \frac{\mathrm{d}z}{iz}\right] = \operatorname{Re}\left[\frac{1}{2i} \int_C \mathrm{e}^z \left(\frac{z^2 + 1}{z^2}\right) \mathrm{d}z\right],$$ where, C: $|z| = 1$ (unit circle with centre at origin)

Pole of $f(z) = \mathrm{e}^z \left(\dfrac{z^2 + 1}{z^2}\right)$ is $z = 0$, which lie inside C: $|z| = 1$ and it has order two.

$$R_1(f, 0) = \lim_{z \to 0} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^2 \mathrm{e}^z (z^2 + 1)}{z^2}\right] = \lim_{z \to 0} \frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{e}^z (z^2 + 1)\right] = \lim_{z \to 0}\left[\mathrm{e}^z (z^2 + 1) + \mathrm{e}^z (2z)\right] = [1 + 0] = 1$$

By Cauchy's residue theorem

$$I = \operatorname{Re}\left[\left(\frac{1}{2i}\right)(2\pi i) \sum_i R_i\right] = \operatorname{Re}\left[\left(\frac{1}{2i}\right) 2\pi i \times 1\right] = \pi$$ **Hence proved**.

Question 11: **If** $|\boldsymbol{a}| < \mathbf{1}$, **then prove that** $\displaystyle\int_0^{2\pi} \frac{\mathbf{d}\boldsymbol{\theta}}{\mathbf{1} + \boldsymbol{a}^2 - \mathbf{2}\boldsymbol{a}\cos\boldsymbol{\theta}} = \frac{\mathbf{2}\boldsymbol{\pi}}{\mathbf{1} - \boldsymbol{a}^2}$

Solution: Let, $\displaystyle I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + a^2 - 2a\cos\theta}$

Put, $\mathrm{e}^{i\theta} = z \ \Rightarrow \ \mathrm{d}\theta = \dfrac{\mathrm{d}z}{i\mathrm{e}^{i\theta}} = \dfrac{\mathrm{d}z}{iz}, \qquad \cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = \dfrac{z^2 + 1}{2z}$

$$I = \int_C \frac{\frac{dz}{iz}}{1 + a^2 - 2a\left(\frac{z^2+1}{2z}\right)} \qquad C\colon\ |z| = 1 \text{ (unit circle with centre at origin)}$$

$$= \frac{1}{i}\int_C \frac{dz}{za^2 - az^2 + z - a} = \frac{1}{i}\int_C \frac{dz}{-az(z-a) + (z-a)} = \frac{1}{i}\int_C \frac{dz}{(z-a)(1-az)}$$

Poles of $f(z) = \frac{z}{(z-a)(1-az)}$ are $z = \frac{1}{a}, a$.

As, $|a| < 1$ (given), therefore, only the pole $z = a$ lies inside $C\colon |z| = 1$ and it has order one.

$$R_1(f, a) = \lim_{z \to a} \frac{z-a}{(z-a)(1-az)} = \lim_{z \to a} \frac{1}{1-az} = \frac{1}{1-a^2}$$

By Cauchy's residue theorem

$$I = \frac{1}{i}(2\pi i)\sum_i R_i = \frac{1}{i}\left(2\pi i\left(\frac{1}{1-a^2}\right)\right) = \frac{2\pi}{1-a^2}$$ **Hence proved**.

Question 12: Prove that $\displaystyle\int_{-\pi}^{\pi} \frac{\boldsymbol{a\cos\theta\, d\theta}}{\boldsymbol{a+\cos\theta}} = \boldsymbol{2\pi a}\left(\boldsymbol{1} - \frac{\boldsymbol{a}}{\sqrt{\boldsymbol{a^2-1}}}\right), \qquad \boldsymbol{a > 1}$

Solution: Let, $\displaystyle I = \int_{-\pi}^{\pi} \frac{a\cos\theta\, d\theta}{a+\cos\theta} = 2\int_0^{\pi} \frac{a\cos\theta\, d\theta}{a+\cos\theta}$ $\qquad \because$ if f is even then $\displaystyle\int_{-a}^{a} f(\theta)d\theta = 2\int_0^a f(\theta)d\theta$

$$\Rightarrow I = 2\int_0^{\pi} \frac{a\cos\theta\, d\theta}{a+\cos\theta} = 2\cdot\frac{1}{2}\int_0^{2\pi} \frac{a\cos\theta\, d\theta}{a+\cos\theta} \qquad \because \text{ if } f(2b-\theta) = f(\theta) \text{ then } \int_0^{2b} f(\theta)d\theta = 2\int_0^b f(\theta)d\theta$$

$$I = a\int_0^{2\pi} \frac{(a+\cos\theta - a)d\theta}{a+\cos\theta} = a\int_0^{2\pi} d\theta - a^2\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = 2\pi a - a^2\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} \qquad \rightarrow (A)$$

Let $\displaystyle I_1 = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$

Put, $\quad e^{i\theta} = z \ \Rightarrow\ d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I_1 = \int_C \frac{\frac{dz}{iz}}{a + \left(\frac{z^2+1}{2z}\right)} = \frac{2}{i}\int_C \frac{dz}{z^2+2az+1}, \qquad \text{where,} \qquad C\colon |z| = 1 \text{ (unit circle with centre at origin)}$$

Poles of $f(z) = \frac{1}{z^2+2az+1}$ are given by $z^2 + 2az + 1 = 0 \Rightarrow z = \frac{-2a \pm \sqrt{4a^2-4}}{2} = -a \pm \sqrt{a^2-1}$.

Let, $\alpha = -a + \sqrt{a^2-1}$ and $\beta = -a - \sqrt{a^2-1} \ \Rightarrow\ \alpha\beta = a^2 - (a^2-1) = 1 \ \Rightarrow\ |\alpha\beta| = 1$

Now, $\ a > 1 \ \Rightarrow\ a^2 > 1 \ \Rightarrow\ |\beta| = \left|-a - \sqrt{a^2-1}\right| = \left|a + \sqrt{a^2-1}\right| > 1 \ \Rightarrow\ |\alpha| = \frac{1}{|\beta|} < 1$

$\Rightarrow$ Only the pole $z = \alpha = -a + \sqrt{a^2 - 1}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f, \alpha) = \lim_{z \to \alpha} \frac{z - \alpha}{(z - \alpha)(z - \beta)} = \frac{1}{\alpha - \beta} = \frac{1}{-a + \sqrt{a^2 - 1} - \left(-a - \sqrt{a^2 - 1}\right)} = \frac{1}{2\sqrt{a^2 - 1}}$$

By Cauchy's residue theorem

$$I_1 = \frac{2}{i}\left[2\pi i \sum_i R_i\right] = \frac{2}{i}\left[2\pi i\left(\frac{1}{2\sqrt{a^2 - 1}}\right)\right] = \frac{2\pi}{\sqrt{a^2 - 1}}$$

Put the value of I_1 in equation (A)

$$I = 2\pi a - \frac{2\pi a^2}{\sqrt{a^2 - 1}} = 2\pi a\left(1 - \frac{a}{\sqrt{a^2 - 1}}\right)$$ **Hence proved.**

Question 13: **prove that** $\displaystyle\int_0^{2\pi} \mathrm{e}^{-\cos\theta} \cos(n\theta + \sin\theta)\, \mathrm{d}\theta = (-1)^n \frac{2\pi}{n!}$ **(*n* being positive integer)**

Solution: Let,

$$I = \int_0^{2\pi} \mathrm{e}^{-\cos\theta} \cos(n\theta + \sin\theta)\, \mathrm{d}\theta = \mathrm{Re}\left[\int_0^{2\pi} \mathrm{e}^{-\cos\theta} e^{-i(n\theta + \sin\theta)}\, \mathrm{d}\theta\right]$$

$$= \mathrm{Re}\left[\int_0^{2\pi} \mathrm{e}^{-(\cos\theta + i\sin\theta)} \mathrm{e}^{-in\theta}\, \mathrm{d}\theta\right] = \mathrm{Re}\left[\int_0^{2\pi} \mathrm{e}^{-\mathrm{e}^{i\theta}} \mathrm{e}^{-in\theta}\, \mathrm{d}\theta\right]$$

Put, $\mathrm{e}^{i\theta} = z \Rightarrow \mathrm{d}\theta = \dfrac{\mathrm{d}z}{i\mathrm{e}^{i\theta}} = \dfrac{\mathrm{d}z}{iz}$

$$I = \mathrm{Re}\left[\int_C \mathrm{e}^{-z} z^{-n} \frac{\mathrm{d}z}{iz}\right] = \mathrm{Re}\left[\frac{1}{i}\int_C \frac{\mathrm{e}^{-z}}{z^{n+1}}\, \mathrm{d}z\right],$$ where, $C: |z| = 1$ (unit circle with centre at origin)

Pole of $f(z) = \dfrac{\mathrm{e}^{-z}}{z^{n+1}}$ is $z = 0$, which lies inside $C: |z| = 1$ and it has order $n + 1$.

$$R_1(f, 0) = \lim_{z \to 0} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[\frac{z^{n+1}\mathrm{e}^{-z}}{z^{n+1}}\right] = \lim_{z \to 0} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n}(\mathrm{e}^{-z}) = \frac{1}{n!}(-1)^n$$

By Cauchy's residue theorem

$$I = \mathrm{Re}\left[\frac{1}{i}\left(2\pi i \sum_i R_i\right)\right] = \mathrm{Re}\left[\frac{1}{i}\left(2\pi i\left(\frac{1}{n!}(-1)^n\right)\right)\right] = (-1)^n \frac{2\pi}{n!}$$ **Hence proved.**

Question 14: **Prove that** $\displaystyle\int_0^{2\pi} (\cos\theta)^{2n} \mathrm{d}\theta = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}$, **where *n* is a positive integer.**

Solution: Let,

$$I = \int_0^{2\pi} (\cos\theta)^{2n} \mathrm{d}\theta$$

Put , $e^{i\theta} = z \Rightarrow d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = \dfrac{z^2+1}{2z}$

$$I = \int_C \left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^{2n} \frac{dz}{iz} = \frac{1}{(2^{2n})i}\int_C \frac{(z^2+1)^{2n}}{z^{2n+1}}dz, \qquad \text{where,} \qquad C\text{: } |z| = 1 \text{ (unit circle with centre at origin)}$$

Pole of $f(z) = \dfrac{(z^2+1)^{2n}}{z^{2n+1}}$ is $z = 0$, which lies inside C: $|z| = 1$ and it has order $2n+1$.

$$f(z) = \frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} = \frac{1}{z}\left[\binom{2n}{0}\cdot z^{2n} + \binom{2n}{1}\cdot z^{2n-1}\cdot\frac{1}{z} + \cdots + \binom{2n}{n}\cdot z^{2n-n}\cdot\frac{1}{z^n} + \cdots + \binom{2n}{2n}\cdot z^{2n-2n}\cdot\frac{1}{z^{2n}}\right]$$

Residue of $f(z)$ at $z = 0$ is the coefficient of $\dfrac{1}{z}$.

$$R_1(f,0) = \binom{2n}{n} = \frac{(2n)!}{n!\,(2n-n)!} = \frac{(2n)!}{n!\,n!} = \frac{(2n)!}{(n!)^2}$$

By Cauchy's residue theorem

$$I = \frac{1}{(2^{2n})i}\left(2\pi i\sum_i R_i\right) = \frac{1}{(2^{2n})i}\left[2\pi i\left(\frac{(2n)!}{(n!)^2}\right)\right] = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}$$ **Hence proved.**

Question 15: **Prove that** $\displaystyle\int_0^{\pi}\frac{\sin^2\theta\, d\theta}{a+b\cos\theta} = \frac{\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$, **where,** $a > b > 0$.

Solution: Let, $\displaystyle I = \int_0^{\pi}\frac{\sin^2\theta d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{\pi}\frac{(1-\cos 2\theta)d\theta}{a+b\cos\theta}$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{2\pi}\frac{(1-\cos 2\theta)d\theta}{a+b\cos\theta} \qquad \because \text{ if } f(2B-\theta) = f(\theta) \text{ then } \int_0^{2B} f(\theta)d\theta = 2\int_0^{B} f(\theta)d\theta$$

$$I = \frac{1}{4}\text{Re}\left[\int_0^{2\pi}\frac{(1-e^{2i\theta})d\theta}{a+b\cos\theta}\right],$$

Put , $e^{i\theta} = z \Rightarrow d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right) = \dfrac{z^2+1}{2z}$

$$I = \text{Re}\left[\frac{1}{4}\int_C \frac{(1-z^2)\frac{dz}{iz}}{a+b\left(\frac{z^2+1}{2z}\right)}\right] = \text{Re}\left[\frac{1}{2i}\int_C\frac{(1-z^2)dz}{bz^2+2az+b}\right], \qquad \text{where,} \quad C\text{: } |z| = 1 \text{ (unit circle with centre at origin)}$$

Poles of $f(z) = \dfrac{1-z^2}{bz^2+2az+b}$ are given by $bz^2+2az+b = 0 \Rightarrow z = \dfrac{-2a\pm\sqrt{4a^2-4b^2}}{2b} = \dfrac{-a\pm\sqrt{a^2-b^2}}{b}$.

Let, $\alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \dfrac{-a-\sqrt{a^2-b^2}}{b} \Rightarrow \alpha\beta = \dfrac{a^2-(a^2-b^2)}{b^2} = 1 \Rightarrow |\alpha\beta| = 1$

Now, $a > b > 0 \Rightarrow a^2-b^2 > 0 \Rightarrow |\beta| = \left|\dfrac{-a-\sqrt{a^2-b^2}}{b}\right| = \left|\dfrac{a+\sqrt{a^2-b^2}}{b}\right| > 1 \Rightarrow |\alpha| = \dfrac{1}{|\beta|} < 1$

$\Rightarrow$ Only the pole $z = \alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f,\alpha) = \lim_{z\to\alpha} \frac{(z-\alpha)(1-z^2)}{b(z-\alpha)(z-\beta)} = \frac{1-\alpha^2}{b(\alpha-\beta)} = \frac{1-\left(\dfrac{-a+\sqrt{a^2-b^2}}{b}\right)^2}{b\left[\dfrac{-a+\sqrt{a^2-b^2}}{b}-\left(\dfrac{-a-\sqrt{a^2-b^2}}{b}\right)\right]}$$

$$= \frac{b^2-a^2-(a^2-b^2)+2a\sqrt{a^2-b^2}}{2b^2\sqrt{a^2-b^2}} = \frac{2b^2-2a^2+2a\sqrt{a^2-b^2}}{2b^2\sqrt{a^2-b^2}} = \frac{-(a^2-b^2)+a\sqrt{a^2-b^2}}{b^2\sqrt{a^2-b^2}}$$

$$= \frac{\sqrt{a^2-b^2}\left(-\sqrt{a^2-b^2}+a\right)}{b^2\sqrt{a^2-b^2}} = \frac{a-\sqrt{a^2-b^2}}{b^2}$$

By Cauchy's residue theorem

$$I = \operatorname{Re}\left[\frac{1}{2i}\left(2\pi i \sum_i R_i\right)\right] = \operatorname{Re}\left[\frac{1}{2i}.2\pi i\left(\frac{a-\sqrt{a^2-b^2}}{b^2}\right)\right] = \frac{\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$ **Hence proved**.

Note: In statement of the question 15 (in Iqbal's book), it is given that $|a| > |b| > 0$, which is wrong. As if $a = -5$ and $b = 3$, then $|\beta| = \left|\frac{-a-\sqrt{a^2-b^2}}{b}\right| = \left|\frac{a+\sqrt{a^2-b^2}}{b}\right| = \left|\frac{-5+\sqrt{25-9}}{3}\right| = \frac{1}{3} < 1$ and we cannot reach the required result in this case. To get the required result, we must have $|\beta| > 1$, which can be attained if we take $a > b > 0$.

Question 16: **Prove that** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\alpha+\sin^2\theta} = \frac{\pi}{2\sqrt{\alpha^2+\alpha}}$, **where $\alpha \ge 0$**

Solution: Let, $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\alpha+\sin^2\theta} = \frac{1}{2}\int_0^{\pi} \frac{\mathrm{d}\theta}{\alpha+\sin^2\theta}$ $\quad\because$ if $f(2a-\theta) = f(\theta)$ then $\displaystyle\int_0^{2a} f(\theta)\mathrm{d}\theta = 2\int_0^{a} f(\theta)\mathrm{d}\theta$

$$= \frac{1}{2}\int_0^{\pi} \frac{2\mathrm{d}\theta}{2\alpha+2\sin^2\theta} = \frac{1}{2}\int_0^{\pi} \frac{2\mathrm{d}\theta}{2\alpha+1-\cos 2\theta}$$

Put, $2\theta = \varphi$, $\theta = \dfrac{\varphi}{2} \Rightarrow \mathrm{d}\theta = \dfrac{\mathrm{d}\varphi}{2}$, $\theta = 0 \Rightarrow \varphi = 0$ and $\theta = \pi \Rightarrow \varphi = 2\pi$

$$I = \frac{1}{2}\int_0^{2\pi} \frac{2\left(\dfrac{\mathrm{d}\varphi}{2}\right)}{2\alpha+1-\cos\varphi} = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\alpha+1-\cos\varphi}$$

Put, $\mathrm{e}^{i\varphi} = z \Rightarrow \mathrm{d}\varphi = \dfrac{\mathrm{d}z}{i\mathrm{e}^{i\varphi}} = \dfrac{\mathrm{d}z}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right) = \dfrac{z^2+1}{2z}$

$$I = \frac{1}{2}\int_C \frac{\frac{dz}{iz}}{2\alpha + 1 - \frac{z^2+1}{2z}}, \qquad \text{where,} \quad C:\ |z| = 1 \text{ (unit circle with centre at origin)}$$

$$= \frac{1}{2}\int_C \frac{\frac{dz}{iz}}{\frac{2(2\alpha+1)z - z^2 - 1}{2z}} = -\frac{1}{i}\int_C \frac{dz}{z^2 - (2+4\alpha)z + 1}$$

Poles of $f(z) = \frac{1}{z^2 - (2+4\alpha)z + 1}$ are given by $z^2 - (2+4\alpha)z + 1 = 0$

$$\Rightarrow \quad z = \frac{2 + 4\alpha \pm \sqrt{(2+4\alpha)^2 - 4}}{2} \quad \Rightarrow \quad z = \frac{2 + 4\alpha \pm \sqrt{4 + 16\alpha^2 + 16\alpha - 4}}{2} = 1 + 2\alpha \pm 2\sqrt{\alpha^2 + \alpha}$$

Let, $\alpha' = 1 + 2\alpha + 2\sqrt{\alpha^2+\alpha}$ and $\beta' = 1 + 2\alpha - 2\sqrt{\alpha^2+\alpha}$ $\Rightarrow \alpha'\beta' = (1+2\alpha)^2 - 4(\alpha^2+\alpha) = 1 \Rightarrow |\alpha'\beta'| = 1$

Now, $\alpha \geq 0$ (given) $\Rightarrow$ $|\alpha'| = \left|1 + 2\alpha + 2\sqrt{\alpha^2+\alpha}\right| > 1 \Rightarrow |\beta'| = \frac{1}{|\alpha'|} < 1$

$\Rightarrow$ Only the pole $z = \beta' = 1 + 2\alpha - 2\sqrt{\alpha^2+\alpha}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f, \beta') = \lim_{z\to\beta'} \frac{z - \beta'}{(z-\alpha')(z-\beta')} = \frac{1}{\beta' - \alpha'} = \frac{1}{1 + 2\alpha - 2\sqrt{\alpha^2+\alpha} - \left(1 + 2\alpha + 2\sqrt{\alpha^2+\alpha}\right)} = \frac{1}{-4\sqrt{\alpha^2+\alpha}}$$

By Cauchy's residue theorem

$$I = -\frac{1}{i}\left(2\pi i \sum_i R_i\right) = -\frac{1}{i}\left[2\pi i\left(\frac{1}{-4\sqrt{\alpha^2+\alpha}}\right)\right] = \frac{\pi}{2\sqrt{\alpha^2+\alpha}}$$

Hence proved.

Question 17: **Prove that** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\alpha^2\cos^2\theta + \beta^2\sin^2\theta} = \frac{\pi}{2\alpha\beta}$, **where** $\beta \geq \alpha > 0$

Solution: **Case I**: **When** $\beta > \alpha > 0$: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\alpha^2\cos^2\theta + \beta^2\sin^2\theta} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\alpha^2\left(\frac{1+\cos 2\theta}{2}\right) + \beta^2\left(\frac{1-\cos 2\theta}{2}\right)}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{d\theta}{\alpha^2(1+\cos 2\theta) + \beta^2(1-\cos 2\theta)}$$

Put, $2\theta = \varphi, \theta = \frac{\varphi}{2} \Rightarrow d\theta = \frac{d\varphi}{2}$, $\quad \theta = 0 \Rightarrow \varphi = 0, \quad \theta = \frac{\pi}{2} \Rightarrow \varphi = \pi$

$$I = 2\int_0^{\pi} \frac{\frac{d\varphi}{2}}{\alpha^2(1+\cos\varphi) + \beta^2(1-\cos\varphi)} = \int_0^{\pi} \frac{d\varphi}{\alpha^2(1+\cos\varphi) + \beta^2(1-\cos\varphi)}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{d\varphi}{\alpha^2(1+\cos\varphi) + \beta^2(1-\cos\varphi)} \quad \because \text{if } f(2b-\varphi) = f(\varphi) \text{ then } \int_0^{2b} f(\varphi)d\varphi = 2\int_0^{b} f(\varphi)d\varphi$$

Put , $e^{i\varphi} = z \Rightarrow d\varphi = \frac{dz}{ie^{i\varphi}} = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

$$I = \frac{1}{2}\int_C \frac{\frac{dz}{iz}}{\alpha^2\left(1 + \frac{z^2+1}{2z}\right) + \beta^2\left[1 - \left(\frac{z^2+1}{2z}\right)\right]}, \quad \text{where,} \quad C:\ |z| = 1 \text{ (unit circle with centre at origin)}$$

$$= \frac{1}{i}\int_C \frac{dz}{2\alpha^2 z + \alpha^2 z^2 + \alpha^2 + 2\beta^2 z - \beta^2 z^2 - \beta^2} = \frac{1}{i}\int_C \frac{dz}{(\alpha^2-\beta^2)z^2 + 2(\alpha^2+\beta^2)z + \alpha^2 - \beta^2}$$

Poles of $f(z) = \frac{1}{(\alpha^2-\beta^2)z^2 + 2(\alpha^2+\beta^2)z + \alpha^2 - \beta^2}$ are given by $(\alpha^2-\beta^2)z^2 + 2(\alpha^2+\beta^2)z + \alpha^2 - \beta^2 = 0$

$$\Rightarrow z = \frac{-2\alpha^2 - 2\beta^2 \pm \sqrt{4(\alpha^2+\beta^2)^2 - 4(\alpha^2-\beta^2)(\alpha^2-\beta^2)}}{2(\alpha^2-\beta^2)}$$

$$= \frac{-2\alpha^2 - 2\beta^2 \pm \sqrt{4\alpha^4 + 4\beta^4 + 8\alpha^2\beta^2 - 4\alpha^4 - 4\beta^4 + 8\alpha^2\beta^2}}{2(\alpha^2-\beta^2)} = \frac{-2\alpha^2 - 2\beta^2 \pm 4\alpha\beta}{2(\alpha^2-\beta^2)}$$

$$= \frac{-\alpha^2 - \beta^2 \pm 2\alpha\beta}{\alpha^2-\beta^2}.$$

Let, $\alpha' = \frac{-\alpha^2 - \beta^2 + 2\alpha\beta}{\alpha^2-\beta^2} = \frac{-(\alpha-\beta)^2}{(\alpha+\beta)(\alpha-\beta)} = \frac{-(\alpha-\beta)}{\alpha+\beta}$

and $\beta' = \frac{-\alpha^2 - \beta^2 - 2\alpha\beta}{\alpha^2-\beta^2} = \frac{-(\alpha+\beta)^2}{(\alpha+\beta)(\alpha-\beta)} = \frac{-(\alpha+\beta)}{\alpha-\beta}$ $\Rightarrow \alpha'\beta' = 1 \Rightarrow |\alpha'\beta'| = 1$

Now, $\beta > \alpha > 0$ (given) $\Rightarrow |\alpha+\beta| > |\alpha-\beta| \Rightarrow |\beta'| = \left|\frac{-(\alpha+\beta)}{\alpha-\beta}\right| = \frac{|\alpha+\beta|}{|\alpha-\beta|} > 1 \Rightarrow |\alpha'| = \frac{1}{|\beta'|} < 1$

$\Rightarrow$ Only the pole $z = \alpha' = \frac{-(\alpha-\beta)}{\alpha+\beta}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f, \alpha') = \lim_{z\to\alpha'} \frac{z-\alpha'}{(\alpha^2-\beta^2)(z-\alpha')(z-\beta')} = \frac{1}{(\alpha^2-\beta^2)(\alpha'-\beta')} = \frac{1}{(\alpha^2-\beta^2)\left[\frac{-(\alpha-\beta)}{\alpha+\beta} - \left(\frac{-(\alpha+\beta)}{\alpha-\beta}\right)\right]}$$

$$= \frac{1}{(\alpha^2-\beta^2)\left[\frac{-(\alpha-\beta)^2 + (\alpha+\beta)^2}{\alpha^2-\beta^2}\right]} = \frac{1}{-\alpha^2 - \beta^2 + 2\alpha\beta + \alpha^2 + \beta^2 + 2\alpha\beta} = \frac{1}{4\alpha\beta}$$

By Cauchy's residue theorem

$$I = \frac{1}{i}\left(2\pi i \sum_i R_i\right) = \frac{1}{i}\left(2\pi i\left(\frac{1}{4\alpha\beta}\right)\right) = \frac{\pi}{2\alpha\beta}$$

Case II: When $\boldsymbol{\beta = \alpha > 0}$: In this case, given integral becomes $I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\alpha^2(\cos^2\theta + \sin^2\theta)} = \frac{1}{\alpha^2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2\alpha^2}$

Combining both cases, we have, $I = \frac{\pi}{2\alpha\beta}$

Question 18: **Prove that** $\displaystyle\int_0^{2\pi} \frac{e^{ni\theta}d\theta}{1+2r\cos\theta+r^2} = \frac{(-r)^n 2\pi}{1-r^2}$, **where** $-1<r<1$, $n=0,1,2,3,\cdots$

Solution: Let, $\displaystyle I = \int_0^{2\pi} \frac{e^{ni\theta}d\theta}{1+2r\cos\theta+r^2}$

Put , $e^{i\theta} = z \;\Rightarrow\; d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right) = \dfrac{z^2+1}{2z}$

$$I = \int_C \frac{z^n}{1+2r\left(\frac{z^2+1}{2z}\right)+r^2}\cdot\frac{dz}{iz}, \qquad C:\ |z|=1 \text{ (unit circle with centre at origin)}$$

$$= \frac{1}{i}\int_C \frac{z^n dz}{rz^2+zr^2+z+r} = \frac{1}{i}\int_C \frac{z^n dz}{rz(z+r)+(z+r)} = \int_C \frac{z^n dz}{(rz+1)(z+r)}$$

Poles of $f(z) = \dfrac{z^n}{(rz+1)(z+r)}$ are given by $z=-r, -\dfrac{1}{r}$.

Now, $-1<r<1$ (given) $\Rightarrow$ Only the pole $z=-r$ lies inside C: $|z|=1$ and it has order one.

$$R_1(f,-r) = \lim_{z\to -r} \frac{(z+r)z^n}{(rz+1)(z+r)} = \frac{(-r)^n}{1-r^2}$$

By Cauchy's residue theorem

$$I = \frac{1}{i}\left(2\pi i \sum_i R_i\right) = \frac{2\pi i}{i}\left(\frac{(-r)^n}{1-r^2}\right) = \frac{(-r)^n 2\pi}{1-r^2}$$ **Hence proved.**

Question 19: **Prove that** $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sin\theta-2\cos\theta+3} = \pi$

Solution: Let, $\displaystyle I = \int_0^{2\pi} \frac{d\theta}{\sin\theta-2\cos\theta+3}$

Put , $e^{i\theta} = z \;\Rightarrow\; d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right) = \dfrac{z^2+1}{2z}$, $\sin\theta = \dfrac{1}{2i}\left(z+\dfrac{1}{z}\right) = \dfrac{z^2-1}{2zi}$,

$$I = \int_C \frac{\frac{dz}{iz}}{\frac{z^2-1}{2zi}-2\left(\frac{z^2+1}{2z}\right)+3}, \qquad C:\ |z|=1 \text{ (unit circle with centre at origin}$$

$$= 2\int_C \frac{dz}{z^2-1-2z^2i-2i+6iz} = 2\int_C \frac{dz}{(1-2i)z^2+6iz-(1+2i)}$$

Poles of $f(z) = \dfrac{1}{(1-2i)z^2+6iz-(1+2i)}$ are given by $(1-2i)z^2+6iz-(1+2i)=0$

$$\Rightarrow z = \frac{-6i\pm\sqrt{-36+4(1-4i^2)}}{2(1-2i)} = \frac{-6i\pm\sqrt{-36+4+16}}{2(1-2i)} = \frac{-6i\pm\sqrt{16}i}{2(1-2i)} = \frac{-3i\pm 2i}{1-2i}.$$

Let, $\alpha = \dfrac{-3i+2i}{1-2i} = \dfrac{-i}{1-2i}$ and $\beta = \dfrac{-3i-2i}{1-2i} = \dfrac{-5i}{1-2i}$ $\Rightarrow$ $|\alpha| = \dfrac{1}{\sqrt{5}} < 1$ and $|\beta| = \dfrac{5}{\sqrt{5}} = \sqrt{5} > 1$

$\Rightarrow$ Only the pole $z = \alpha = \dfrac{-i}{1-2i}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1(f,\alpha) = \lim_{z\to\alpha} \frac{z-\alpha}{(1-2i)(z-\alpha)(z-\beta)} = \frac{1}{(1-2i)(\alpha-\beta)} = \frac{1}{(1-2i)\left[\frac{-i}{1-2i} - \left(\frac{-5i}{1-2i}\right)\right]} = \frac{1}{(1-2i)\left(\frac{4i}{1-2i}\right)} = \frac{1}{4i}$$

By Cauchy's residue theorem

$$I = 2\left(2\pi i \sum_i R_i\right) = 4\pi i\left(\frac{1}{4i}\right) = \pi$$ **Hence proved**.

Question 20: **Prove that** $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{10 - 8\cos\theta} = \frac{\pi}{24}$

Solution: Let, $I = \displaystyle\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{10 - 8\cos\theta} = \mathrm{Re}\left[\int_0^{2\pi} \frac{e^{3i\theta} \, d\theta}{10 - 8\cos\theta}\right]$

Put, $e^{i\theta} = z \Rightarrow d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = \dfrac{z^2+1}{2z}$

$$I = \mathrm{Re}\int_C \frac{z^3}{10 - 4\left(\frac{z^2+1}{z}\right)} \cdot \frac{dz}{iz},$$ where, $C: |z| = 1$ (unit circle with centre at origin)

$$= \mathrm{Re}\left[\int_C \frac{z^3}{\frac{10z - 4z^2 - 4}{z}} \cdot \frac{dz}{iz}\right] = \mathrm{Re}\left[-\frac{1}{2i}\int_C \frac{z^3 dz}{2z^2 - 5z + 2}\right] = \mathrm{Re}\left[-\frac{1}{2i}\int_C \frac{z^3 dz}{2z^2 - 4z - z + 2}\right]$$

$$= \mathrm{Re}\left[-\frac{1}{2i}\int_C \frac{z^3 dz}{2z(z-2) - (z-2)}\right] = \mathrm{Re}\left[-\frac{1}{2i}\int_C \frac{z^3 dz}{(z-2)(2z-1)}\right]$$

Poles of $f(z) = \dfrac{z^3}{(z-2)(2z-1)}$ are $z = 2, \dfrac{1}{2}$. Only the pole $z = \dfrac{1}{2}$ lies inside $C: |z| = 1$ and it has order one.

$$R_1\left(f, \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \frac{\left(z - \frac{1}{2}\right) z^3}{2(z-2)\left(z - \frac{1}{2}\right)} = \frac{\frac{1}{8}}{2\left(\frac{1}{2} - 2\right)} = \frac{\frac{1}{8}}{2\left(-\frac{3}{2}\right)} = -\frac{1}{24}$$

By Cauchy's residue theorem

$$I = \mathrm{Re}\left[-\frac{1}{2i}\left(2\pi i \sum_i R_i\right)\right] = \mathrm{Re}\left[-\frac{1}{2i}\left(2\pi i\left(-\frac{1}{24}\right)\right)\right] = \frac{\pi}{24}$$ **Hence proved**.

Question 21: **Prove that** $\displaystyle\int_0^{\pi} \cot(\theta - \alpha) d\theta = \begin{cases} \pi i, & \text{If } \mathrm{Im}(\alpha) > 0 \\ -\pi i, & \text{If } \mathrm{Im}(\alpha) < 0 \\ \text{diverges}, & \text{If } \mathrm{Im}(\alpha) = 0 \end{cases}$

Solution: Let, $I = \int_0^{\pi} \cot(\theta - \alpha)\mathrm{d}\theta$

$$= \int_0^{2\pi} \cot(\theta - \alpha)\mathrm{d}\theta \qquad \because \text{ if } f(\theta+\beta) = f(\theta) \text{ then } \int_0^{n\beta} f(\theta)\mathrm{d}\theta = n\int_0^{\beta} f(\theta)\mathrm{d}\theta, \ \forall\ n \in \{1,2,3,\cdots\}$$

$$= i\int_0^{\pi}\left[\frac{\mathrm{e}^{i(\theta-\alpha)} + \mathrm{e}^{-i(\theta-\alpha)}}{\mathrm{e}^{i(\theta-\alpha)} - \mathrm{e}^{-i(\theta-\alpha)}}\right]\mathrm{d}\theta = i\int_0^{\pi}\left[\frac{\mathrm{e}^{2i(\theta-\alpha)} + 1}{\mathrm{e}^{2i(\theta-\alpha)} - 1}\right]\mathrm{d}\theta = i\int_0^{\pi}\left[\frac{\mathrm{e}^{2i\theta} + \mathrm{e}^{2i\alpha}}{\mathrm{e}^{2i\theta} - \mathrm{e}^{2i\alpha}}\right]\mathrm{d}\theta = \frac{i}{2}\int_0^{2\pi}\left[\frac{\mathrm{e}^{2i\theta} + \mathrm{e}^{2i\alpha}}{\mathrm{e}^{2i\theta} - \mathrm{e}^{2i\alpha}}\right]\mathrm{d}\theta$$

Put , $\mathrm{e}^{i\theta} = z \ \Rightarrow\ \mathrm{d}\theta = \frac{\mathrm{d}z}{i\mathrm{e}^{i\theta}} = \frac{\mathrm{d}z}{iz}$

$$I = \frac{i}{2}\int_C \left[\frac{z^2 + \mathrm{e}^{2i\alpha}}{z^2 - \mathrm{e}^{2i\alpha}}\right]\cdot\frac{\mathrm{d}z}{iz} = \frac{1}{2}\int_C \frac{z^2 + \mathrm{e}^{2i\alpha}}{z(z^2 - \mathrm{e}^{2i\alpha})}\mathrm{d}z, \qquad \text{where,} \qquad C\colon |z| = 1 \text{ (unit circle with centre at origin)}$$

Poles of $f(z) = \frac{z^2 + \mathrm{e}^{2i\alpha}}{z(z^2 - \mathrm{e}^{2i\alpha})}$ are $z = 0,\ \ \pm\mathrm{e}^{i\alpha}$.

Let, $\alpha = a + bi \quad \Rightarrow \quad |\mathrm{e}^{i\alpha}| = |\mathrm{e}^{i(a+bi)}| = |\mathrm{e}^{-b+ai}| = |\mathrm{e}^{-b}\mathrm{e}^{ai}| = \mathrm{e}^{-b}|\mathrm{e}^{ai}| = \mathrm{e}^{-b} \qquad \because |\mathrm{e}^{ai}| = 1$

There are three cases:

Case I: If $b = \mathrm{Im}(\alpha) > 0 \ \Rightarrow\ |\pm\mathrm{e}^{i\alpha}| = \mathrm{e}^{-b} < 1 \Rightarrow$ the poles $z = \pm\mathrm{e}^{i\alpha}$ lie inside $C\colon |z| = 1$ and they have order one.

Case II: If $b = \mathrm{Im}(\alpha) < 0 \ \Rightarrow\ |\pm\mathrm{e}^{i\alpha}| = \mathrm{e}^{-b} > 1 \Rightarrow$ the poles $z = \pm\mathrm{e}^{i\alpha}$ lie outside $C\colon |z| = 1$.

Case III: If $b = \mathrm{Im}(\alpha) = 0 \Rightarrow |\pm\mathrm{e}^{i\alpha}| = \mathrm{e}^{-b} = 1 \Rightarrow$ the poles $z = \pm\mathrm{e}^{i\alpha}$ lie on $C\colon |z| = 1$.

Furthermore, the pole $z = 0$ lies inside $C\colon |z| = 1$ in any case and it has order 1.

$$R_1(f,0) = \lim_{z\to 0}\frac{z(z^2 + \mathrm{e}^{2i\alpha})}{z(z^2 - \mathrm{e}^{2i\alpha})} = \lim_{z\to 0}\frac{z^2 + \mathrm{e}^{2i\alpha}}{z^2 - \mathrm{e}^{2i\alpha}} = \frac{\mathrm{e}^{2i\alpha}}{-\mathrm{e}^{2i\alpha}} = -1$$

$$R_2(f,\mathrm{e}^{i\alpha}) = \lim_{z\to\mathrm{e}^{i\alpha}}\frac{(z - \mathrm{e}^{i\alpha})(z^2 + \mathrm{e}^{2i\alpha})}{z(z + \mathrm{e}^{i\alpha})(z - \mathrm{e}^{i\alpha})} = \lim_{z\to\mathrm{e}^{i\alpha}}\frac{z^2 + \mathrm{e}^{2i\alpha}}{z(z + \mathrm{e}^{i\alpha})} = \frac{2\mathrm{e}^{2i\alpha}}{\mathrm{e}^{i\alpha}(2\mathrm{e}^{i\alpha})} = \frac{2\mathrm{e}^{2i\alpha}}{2\mathrm{e}^{2i\alpha}} = 1$$

$$R_3(f,-\mathrm{e}^{i\alpha}) = \lim_{z\to-\mathrm{e}^{i\alpha}}\frac{(z + \mathrm{e}^{i\alpha})(z^2 + \mathrm{e}^{2i\alpha})}{z(z + \mathrm{e}^{i\alpha})(z - \mathrm{e}^{i\alpha})} = \lim_{z\to-\mathrm{e}^{i\alpha}}\frac{z^2 + \mathrm{e}^{2i\alpha}}{z(z - \mathrm{e}^{i\alpha})} = \frac{2\mathrm{e}^{2i\alpha}}{-\mathrm{e}^{i\alpha}(-2\mathrm{e}^{i\alpha})} = \frac{2\mathrm{e}^{2i\alpha}}{2\mathrm{e}^{2i\alpha}} = 1$$

By Cauchy's residue theorem

$$I = \frac{1}{2}\left(2\pi i\sum_i R_i\right) = \pi i\sum_i R_i = \begin{cases} \pi i(-1+1+1) = \pi i, & \text{If } \mathrm{Im}(\alpha) > 0 \\ \pi i(-1) = -\pi i, & \text{If } \mathrm{Im}(\alpha) < 0 \\ \text{diverges}, & \text{If } \mathrm{Im}(\alpha) = 0 \end{cases}$$

Hence Proved.

Note: $\int_C f(z)\mathrm{d}z$ diverges, if any of the pole (or poles) of $f(z)$ lies (or lie) on C.

Question 1: **Prove that** $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx = \frac{\pi}{4}$

Solution: Let, $\displaystyle I = \int_0^\infty \frac{x^2}{(x^2+1)^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,\mathrm{d}x = \frac{1}{2}\int_C \frac{z^2}{(z^2+1)^2}\,\mathrm{d}z$

Here, $\displaystyle f(z) = \frac{z^2}{(z^2+1)^2} \quad\Rightarrow\quad zf(z) = \frac{z^3}{(z^2+1)^2} \to 0 \quad \text{as} \quad |z| \to \infty$

Poles of $f(z)$ are $z = \pm i$. Only the pole $z = i$ lies in upper half of the complex plane and it has order two.

$$R_1^+(f,i) = \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^2(z-i)^2}{(z+i)^2(z-i)^2}\right] = \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^2}{(z+i)^2}\right] = \lim_{z\to i}\frac{(z+i)^2(2z)-2(z+i)z^2}{(z+i)^4} = \lim_{z\to i}\frac{(z+i)(2z)-2z^2}{(z+i)^3}$$
$$= \frac{(2i)(2i)+2}{(2i)^3} = \frac{-4+2}{-8i} = \frac{-2}{-8i} = \frac{1}{4i}$$

$$I = \frac{1}{2}\left[2\pi i \sum_i R_i^+\right] = \pi i \sum_i R_i^+ = \pi i\left(\frac{1}{4i}\right) = \frac{\pi}{4}$$ **Hence proved**.

Question 2: **Prove that** $\displaystyle\int_0^\infty \frac{1}{x^4+6x^2+25}\,dx = \frac{\pi}{20}$

Solution: Let, $\displaystyle I = \int_0^\infty \frac{1}{x^4-6x^2+25}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^4-6x^2+25}\,\mathrm{d}x = \frac{1}{2}\int_C \frac{1}{z^4-6z^2+25}\,\mathrm{d}z$

Here, $\displaystyle f(z) = \frac{1}{z^4-6z^2+25} \quad\Rightarrow\quad zf(z) = \frac{z}{z^4-6z^2+25} \to 0 \quad \text{as} \quad |z| \to \infty$

Poles of $f(z)$ are given by $z^4-6z^2+25=0 \quad\Rightarrow\quad z^2 = \frac{6\pm\sqrt{36-100}}{2} = \frac{6\pm\sqrt{-64}}{2} = \frac{6\pm 8i}{2} = 3\pm 4i$

If $z^2 = 3+4i = (2+i)^2 \Rightarrow z = \pm(2+i)$ and if $z^2 = 3-4i = (2-i)^2 \Rightarrow z = \pm(2-i)$

Let, $z_1 = 2+i,\quad z_2 = -2-i,\quad z_3 = 2-i,\quad z_4 = -2+i,$

Only the poles z_1 and z_4 lie in upper half of the complex plane and they have order one.

$$R_1^+(f,z_1) = \lim_{z\to z_1}\frac{(z-z_1)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = \lim_{z\to z_1}\frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \frac{1}{(4+2i)(2i)(4)} = \frac{1}{16i(2+i)}$$

$$R_2^+(f,z_3) = \lim_{z\to z_4}\frac{(z-z_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = \lim_{z\to z_4}\frac{1}{(z_4-z_1)(z_4-z_2)(z_4-z_3)} = \frac{1}{(-4)(2i)(-4+2i)} = \frac{1}{16i(2-i)}$$

$$\sum_i R_i^+ = \frac{1}{16i}\left[\frac{1}{2+i}+\frac{1}{2-i}\right] = \frac{1}{16i}\left[\frac{2-i+2+i}{(2+i)(2-i)}\right] = \frac{1}{16i}\times\frac{4}{5} = \frac{1}{20i}$$

$$I = \frac{1}{2}\left[2\pi i\sum_i R_i^+\right] = \pi i\sum_i R_i^+ = \frac{1}{2}\left[2\pi i\left(\frac{1}{20i}\right)\right] = \frac{\pi}{20}$$ **Hence proved**.

Alternate solution: Let, $I=\int\limits_{0}^{\infty}\dfrac{1}{x^4-6x^2+25}\mathrm{d}x=\dfrac{1}{2}\int\limits_{-\infty}^{\infty}\dfrac{1}{x^4-6x^2+25}\mathrm{d}x=\dfrac{1}{2}\int\limits_{C}\dfrac{1}{z^4-6z^2+25}\mathrm{d}z$

Here, $f(z)=\dfrac{1}{z^4-6z^2+25}\quad\Rightarrow\quad zf(z)=\dfrac{z}{z^4-6z^2+25}\to 0\quad$ as $\quad|z|\to\infty$

Poles of $f(z)$ are given by $z^4-6z^2+25=0\quad\Rightarrow\quad z^2=\frac{6\pm\sqrt{36-100}}{2}=\frac{6\pm\sqrt{-64}}{2}=\frac{6\pm 8i}{2}=3\pm 4i$

If $z^2=3+4i=(2+i)^2\Rightarrow z=\pm(2+i)$ and if $z^2=3-4i=(2-i)^2\Rightarrow z=\pm(2-i)$

Only the poles $z=\pm 2+i$ lie in upper half of the complex plane and they have order one.

Let α be one of these poles of $f(z)$ (which lie in upper half of the complex plane), then

$$R^+_\alpha(f,\alpha)=\lim_{z\to\alpha}\frac{z-\alpha}{z^4-6z^2+25}\qquad\left(\frac{0}{0}\right)\text{ form}$$

$$=\lim_{z\to\alpha}\frac{1}{4z^3-12z}=\frac{1}{4\alpha^3-12\alpha}=\frac{1}{4(\alpha^3-3\alpha)}=\frac{\alpha}{4(\alpha^4-3\alpha^2)}=\frac{\alpha}{4(3\alpha^2-25)}\qquad\because\alpha^4-6\alpha^2+25=0$$

$$\sum_\alpha R^+_\alpha=\frac{2+i}{4[3(2+i)^2-25]}+\frac{-2+i}{4[3(-2+i)^2-25]}=\frac{2+i}{4[3(3+4i)-25]}+\frac{-2+i}{4[3(3-4i)-25]}$$

$$=\frac{2+i}{4(-16+12i)}+\frac{-2+i}{4(-16-12i)}=\frac{2+i}{-16(4-3i)}+\frac{-2+i}{-16(4+3i)}$$

$$=-\frac{(2+i)(4+3i)+(-2+i)(4-3i)}{16(4-3i)(4+3i)}=-\frac{5+10i-5+10i}{16(25)}=-\frac{20i}{400}=-\frac{i}{20}$$

$$I=\frac{1}{2}\left[2\pi i\sum_i R^+_i\right]=\pi i\sum_i R^+_i=\frac{1}{2}\left[2\pi i\left(-\frac{i}{20}\right)\right]=\frac{\pi}{20}$$ **Hence proved**.

Question 3: **Prove that** $\displaystyle\int\limits_{-\infty}^{\infty}\frac{\mathbf{1}}{(\boldsymbol{x^2+a^2})(\boldsymbol{x^2+b^2})^2}\mathbf{dx}=\frac{\boldsymbol{\pi(a+2b)}}{\boldsymbol{2ab^3(a+b)^2}}$, $\boldsymbol{a>0}$, $\boldsymbol{b>0}$

Solution: Let, $I=\int\limits_{-\infty}^{\infty}\dfrac{1}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x=\int\limits_{-\infty}^{\infty}\dfrac{1}{(z^2+a^2)(x^2+b^2)}\mathrm{d}z$

Here, $f(z)=\dfrac{1}{(z^2+a^2)(x^2+b^2)}\quad\Rightarrow\quad zf(z)=\dfrac{z}{(z^2+a^2)(x^2+b^2)}\to 0\quad$ as $\quad|z|\to\infty$

Poles of $f(z)$ are $z=\pm ai$, $\pm bi$. Only the poles ai and bi lie in upper half of the complex plane due to the fact that $a>0$, $b>0$ (given) and they have order one.

$$R^+_1(f,ai)=\lim_{z\to ai}\frac{(z-ai)}{(z+ai)(z-ai)(z^2+b^2)}=\lim_{z\to ai}\frac{1}{(z+ai)(z^2+b^2)}=\frac{1}{2ai(-a^2+b^2)}=\frac{-1}{2ai(a^2-b^2)}$$

$$R^+_2(f,bi)=\lim_{z\to bi}\frac{(z-bi)}{(z^2+a^2)(z+bi)(z-bi)}=\lim_{z\to bi}\frac{1}{(z^2+a^2)(z+bi)}=\frac{1}{2bi(a^2-b^2)}$$

$$I=\int\limits_{-\infty}^{\infty}\frac{1}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x=2\pi i\sum_i R^+_i=\frac{2\pi i}{2(a^2-b^2)}\left[\frac{-1}{ai}+\frac{1}{bi}\right]=\frac{\pi}{(a^2-b^2)}\left[\frac{-1}{a}+\frac{1}{b}\right]=\frac{-\pi}{(a+b)(a-b)}\left[\frac{a-b}{ab}\right]$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{-\pi}{ab(a+b)}$$

Differentiating both sides with respect to b, we get

$$\int_{-\infty}^{\infty} \frac{-2b}{(x^2+a^2)(x^2+b^2)^2}\,dx = -\frac{\pi}{a}\left(\frac{a+2b}{b^2(a+b)^2}\right)$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)^2}\,dx = \frac{\pi}{2ab}\left(\frac{a+2b}{b^2(a+b)^2}\right) = \frac{\pi(a+2b)}{2ab^3(a+b)^2}$$ **Hence proved**.

Question 4: **Prove that** $\displaystyle\int_0^{\infty} \frac{x^6}{(x^4+a^4)^2}\,dx = \frac{3\sqrt{2}\,\pi}{16a}\ ,\ a>0$

Solution: Let, $\displaystyle I = \int_0^{\infty} \frac{x^6}{x^4+a^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^6}{x^4+a^4}\,dx = \frac{1}{2}\int_C \frac{z^6}{z^4+a^4}\,dz$

Here, $f(z) = \dfrac{z^6}{z^4+a^4} \Rightarrow zf(z) = \dfrac{z^7}{z^4+a^4} \to 0$ as $|z| \to \infty$

Poles of $f(z)$ are given by $z^4+a^4=0 \Rightarrow z^4 = -a^4 = a^4e^{i\pi} \Rightarrow z_k = ae^{\frac{\pi i(1+2k)}{4}}$, $k = 0,1,2,3.$

$z_0 = ae^{\frac{i\pi}{4}} = \frac{a}{\sqrt{2}}(1+i)$, $z_1 = ae^{\frac{3i\pi}{4}} = \frac{a}{\sqrt{2}}(-1+i)$, $z_2 = ae^{\frac{5i\pi}{4}} = -\frac{a}{\sqrt{2}}(1+i)$, $z_3 = ae^{\frac{7i\pi}{4}} = \frac{a}{\sqrt{2}}(1-i)$. Only the poles z_0 and z_1 lie in upper half of the complex plane due to the fact that $a>0$ (given) and they all have order one.

$$R_1^+(f,z_0) = \lim_{z\to z_0} \frac{(z-z_0)z^6}{z^4+a^4}, \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$R_1^+(f,z_0) = \lim_{z\to z_0} \frac{z^6+6(z-z_0)z^5}{4z^3} = \frac{z_0^6}{4z_0^3} = \frac{z_0^3}{4} = \frac{z_0^4}{4z_0} = \frac{-a^4}{4z_0}, \qquad \because\ z_0^4 = -a^4$$

Similarly, $R_2^+(f,z_1) = \dfrac{-a^4}{4z_1}$

$$I = \frac{1}{2}\left[2\pi i\sum_i R_i^+\right] = \pi i\sum_i R_i^+ = \frac{-\pi i a^4}{4}\left[\frac{1}{z_0}+\frac{1}{z_1}\right] = \frac{-\pi i a^4}{4}\left[\frac{z_0+z_1}{z_0z_1}\right] = \frac{-\pi i a^4}{4}\left[\frac{\frac{2ai}{\sqrt{2}}}{\frac{a^2}{2}(-2)}\right] = -\frac{\pi a^3}{2\sqrt{2}}$$

$$\int_0^{\infty} \frac{x^6}{x^4+a^4}\,dx = -\frac{\pi a^3}{2\sqrt{2}}$$

Differentiating both sides with respect to a, we get

$$\int_0^{\infty} \frac{-4a^3x^6}{(x^4+a^4)^2}\,dx = -\frac{3\pi a^2}{2\sqrt{2}}$$

$$\int_0^\infty \frac{x^6}{(x^4+a^4)^2}\,\mathrm{d}x = \frac{3\pi}{8\sqrt{2}a} = \left(\frac{3\pi}{8\sqrt{2}a}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{3\sqrt{2}\ \pi}{16a}$$ **Hence proved.**

Question 5: **Prove that** $\displaystyle\int_0^\infty \frac{x^{2m}}{1+\ x^{2n}}\,\mathrm{d}x = \frac{\pi}{2n\sin\left[\frac{(2m+1)\pi}{2n}\right]}$, $m, n \in \mathbb{Z}$, $0 \le m < n$.

Solution: Let, $\displaystyle I = \int_0^\infty \frac{x^{2m}}{1+\ x^{2n}}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^{2m}}{1+\ x^{2n}}\,\mathrm{d}x = \frac{1}{2}\int_C \frac{z^{2m}}{1+\ z^{2n}}\,\mathrm{d}z$

Here, $\displaystyle f(z) = \frac{z^{2m}}{1+\ z^{2n}} \Rightarrow zf(z) = \frac{z^{2m+1}}{1+\ z^{2n}} \to 0$ as $|z| \to \infty$

Poles of $f(z)$ are given by $1+\ z^{2n} = 0 \Rightarrow z^{2n} = -1 = e^{i\pi} \Rightarrow z_k = e^{\frac{(1+2k)}{2n}\pi i}$, $k = 0, 1, 2, 3, \cdots, 2n-1.$

Only the poles $z_0 = e^{\frac{1}{2n}\pi i}$, $z_1 = e^{\frac{3}{2n}\pi i}$, $z_2 = e^{\frac{5}{2n}\pi i}, \cdots$, $z_{n-1} = e^{\frac{(2n-1)}{2n}\pi i}$ lie in upper half of the complex plane and they all have order one.

Let α be one of these poles of $f(z)$ (which lie in upper half of the complex plane), then

$$R_\alpha^+(f,\alpha) = \lim_{z\to\alpha}\frac{(z-\ \alpha)z^{2m}}{1+\ z^{2n}} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to\alpha}\frac{(z-\ \alpha)2mz^{2m-1}+\ z^{2m}}{2nz^{2n-1}} = \frac{\alpha^{2m}}{2n\alpha^{2n-1}} = \frac{\alpha^{2m+1}}{2n\alpha^{2n}} = -\frac{1}{2n}\alpha^{2m+1} \qquad \because \alpha^{2n} = -1$$

$$\sum_\alpha R_\alpha^+ = -\frac{1}{2n}\left[e^{\left(\frac{2m+1}{2n}\right)\pi i} +\ e^{3\left(\frac{2m+1}{2n}\right)\pi i} + \cdots + e^{(2n-1)\left(\frac{2m+1}{2n}\right)\pi i}\right]$$

This is finite geometric series with a = first term = $\frac{1}{2n}e^{\left(\frac{2m+1}{2n}\right)\pi i}$, r = ratio = $e^{\left(\frac{2m+1}{n}\right)\pi i}$ and N = number of terms = n

$$\Rightarrow \sum_\alpha R_\alpha^+ = \frac{a(1-\ r^N)}{1-r} = -\frac{\frac{1}{2n}e^{\left(\frac{2m+1}{2n}\right)\pi i}\left[1-\ \left(e^{\left(\frac{2m+1}{n}\right)\pi i}\right)^n\right]}{1-\ e^{\left(\frac{2m-2n+1}{n}\right)\pi i}} = -\frac{1}{2n}\left[\frac{e^{\left(\frac{2m+1}{2n}\right)\pi i}\left[1-\ e^{(2m+1)\pi i}\right]}{e^{\left(\frac{2m+1}{2n}\right)\pi i}\left[e^{-\left(\frac{2m+1}{2n}\right)\pi i}-\ e^{\left(\frac{2m+1}{2n}\right)\pi i}\right]}\right]$$

$$= -\frac{1}{2n}\left[\frac{1-\ e^{(2m+1)\pi i}}{e^{-\left(\frac{2m+1}{2n}\right)\pi i}-\ e^{\left(\frac{2m+1}{2n}\right)\pi i}}\right] = -\frac{1}{2n}\left[\frac{1-\ \cos[(2m+1)\pi]\ -\ \sin[(2m+1)\pi]}{-2i\sin\left[\frac{(2m+1)\pi}{2n}\right]}\right]$$

$$= -\frac{1}{4ni}\left[\frac{1-\ (-1)\ -\ 0}{\sin\left(\frac{(2m+1)\pi}{2n}\right)}\right] = \frac{1}{2ni}\left[\frac{1}{\sin\left[\frac{(2m+1)\pi}{2n}\right]}\right] \qquad \left(\begin{array}{c}\because\ m\in\mathbb{Z} \Rightarrow 2m+1 \text{ is odd integer} \\ \therefore \quad \cos[(2m+1)\pi] = -1 \\ \text{and} \quad \sin[(2m+1)\pi] = 0\end{array}\right)$$

$$I = \frac{1}{2}\left[2\pi i\sum_\alpha R_\alpha^+\right] = \pi i\sum_\alpha R_\alpha^+ = \frac{\pi i}{2ni}\left[\frac{1}{\sin\left[\frac{(2m+1)\pi}{2n}\right]}\right] = \frac{\pi}{2n\sin\left[\frac{(2m+1)\pi}{2n}\right]}$$ **Hence proved.**

Question 6: **Prove that** $\displaystyle\int_0^\infty \frac{1}{1+x^6}\,\mathrm{d}x = \frac{\pi}{3}$

Solution: Let, $I = \int_0^{\infty} \frac{1}{1+x^6} \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+x^6} \mathrm{d}x = \frac{1}{2} \int_C \frac{1}{1+z^6} \mathrm{d}z$

Here, $f(z) = \frac{1}{1+z^6} \quad \Rightarrow \quad zf(z) = \frac{z}{1+z^6} \to 0 \quad \text{as} \quad |z| \to \infty$

Poles of $f(z)$ are given by $1 + z^6 = 0 \Rightarrow z^6 = -1 = e^{i\pi} \Rightarrow z_k = e^{\frac{\pi i(1+2k)}{6}}, \quad k = 0, 1, 2, 3, 4, 5.$

$z_0 = e^{\frac{i\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ z_1 = e^{\frac{i\pi}{2}} = i,\ z_2 = e^{\frac{5i\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i,\ z_3 = e^{\frac{7i\pi}{6}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i,\ z_4 = e^{\frac{3i\pi}{2}} - i,\ z_5 = e^{\frac{11i\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i.$

Only the poles z_0, z_1 and z_2 lie in upper half of the complex plane and they all have order one.

$R_1^+(f, z_0) = \lim_{z \to z_0} \frac{z - z_0}{1+z^6}, \quad \left(\frac{0}{0}\right) \text{ form}$

$R_1^+(f, z_0) = \lim_{z \to z_0} \frac{1}{6z^5} = \frac{1}{6z_0^5} = \frac{z_0}{6z_0^6} = -\frac{z_0}{6}, \quad \because z_0^6 = -1$

Similarly, $R_2^+(f, z_1) = -\frac{z_1}{6}$ and $R_3^+(f, z_2) = -\frac{z_2}{6}$

$I = \frac{1}{2}\left[2\pi i \sum_i R_i^+\right] = \pi i \sum_i R_i^+ = \frac{-\pi i}{6}[z_0 + z_1 + z_2] = \frac{-\pi i}{6}\left[\frac{\sqrt{3}}{2} + \frac{1}{2}i + i - \frac{\sqrt{3}}{2} + \frac{1}{2}i\right] = \frac{-\pi i}{6}[2i] = \frac{\pi}{3}$ **Hence proved.**

Question 7: **Prove that** $\int_0^{\infty} \frac{1}{1+x^4} dx = \frac{\pi\sqrt{2}}{4}$

Solution: Let, $I = \int_0^{\infty} \frac{1}{1+x^4} \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+x^4} \mathrm{d}x = \frac{1}{2} \int_C \frac{1}{1+z^4} \mathrm{d}z$

Here, $f(z) = \frac{1}{1+z^4} \quad \Rightarrow \quad zf(z) = \frac{z}{1+z^4} \to 0 \quad \text{as} \quad |z| \to \infty$

Poles of $f(z)$ are given by $z^4 + 1 = 0 \Rightarrow z^4 = -1 = e^{i\pi} \Rightarrow z_k = e^{\frac{\pi i(1+2k)}{4}}, \quad k = 0, 1, 2, 3.$

$z_0 = e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}}(1+i),\quad z_1 = e^{\frac{3i\pi}{4}} = \frac{1}{\sqrt{2}}(-1+i),\quad z_2 = e^{\frac{5i\pi}{4}} = -\frac{1}{\sqrt{2}}(1+i),\quad z_3 = e^{\frac{7i\pi}{4}} = \frac{1}{\sqrt{2}}(1-i).$ Only the poles z_0 and z_1 lie in upper half of the complex plane due to the fact that $a > 0$ (given) and they have order one.

$R_1^+(f, z_0) = \lim_{z \to z_0} \frac{z - z_0}{z^4 + 1}, \quad \left(\frac{0}{0}\right) \text{ form}$

$R_1^+(f, z_0) = \lim_{z \to z_0} \frac{1}{4z^3} = \frac{1}{4z_0^3} = \frac{z_0}{4z_0^4} = -\frac{z_0}{4}, \quad \because z_0^4 = -1$

Similarly, $R_2^+(f, z_1) = -\frac{z_1}{4}$

$I = \frac{1}{2}\left[2\pi i \sum_i R_i^+\right] = \pi i \sum_i R_i^+ = \frac{-\pi i}{4}[z_0 + z_1] = \frac{-\pi i}{4}\left[\frac{1}{\sqrt{2}}(1+i) + \frac{1}{\sqrt{2}}(-1+i)\right] = \frac{-\pi i}{4}\left[\frac{2i}{\sqrt{2}}\right] = \frac{\pi\sqrt{2}}{4}$ **Hence proved.**

Question 8: **Prove that** $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx = -\frac{\pi}{5}$

Solution: Let, $\displaystyle I = \int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx = \int_C \frac{z}{(z^2+1)(z^2+2z+2)}\,dz$

Here, $\displaystyle f(z) = \frac{z}{(z^2+1)(z^2+2z+2)} \quad \Rightarrow \quad zf(z) = \frac{z^2}{(z^2+1)(z^2+2z+2)} \to 0 \quad \text{as} \quad |z| \to \infty$

Poles of $f(z)$ are given by $(z^2+1)(z^2+2z+2) = 0 \Rightarrow z^2+1 = 0$ or $z^2+2z+2 = 0$

$z^2+1=0 \Rightarrow z=\pm i$	$z^2+2z+2=0 \Rightarrow z=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm\sqrt{-4}}{2}=-1\pm i$

Only the poles i and $-1+i$ lie in upper half of the complex plane and they have order one.

$$R_1^+(f,i) = \lim_{z\to i}\frac{z(z-i)}{(z^2+1)(z^2+2z+2)} = \lim_{z\to i}\frac{z(z-i)}{(z+i)(z-i)(z^2+2z+2)} = \lim_{z\to i}\frac{z}{(z+i)(z^2+2z+2)} = \frac{1}{2(2i+1)}$$

$$R_2^+(f,-1+i) = \lim_{z\to -1+i}\frac{z\big(z-(-1+i)\big)}{(z^2+1)(z^2+2z+2)} = \lim_{z\to -1+i}\frac{z\big(z-(-1+i)\big)}{(z^2+1)\big(z-(-1+i)\big)\big(z-(-1-i)\big)}$$

$$= \lim_{z\to -1+i}\frac{z}{(z^2+1)\big(z-(-1-i)\big)} = \frac{-1+i}{((-1+i)^2+1)(-1+i+1+i)} = \frac{-1+i}{(1-2i)(2i)} = \frac{-1+i}{2i(1-2i)}$$

$$I = 2\pi i\sum_i R_i^+ = \frac{2\pi i}{2}\left[\frac{1}{1+2i}+\frac{i-1}{i(1-2i)}\right] = \pi i\left[\frac{i(1-2i)+(1+2i)(i-1)}{i(1+2i)(1-2i)}\right] = \pi i\left[\frac{i+2+i-1-2-2i}{5i}\right] = \pi i\left[\frac{-1}{5i}\right]$$

$= -\dfrac{\pi}{5}$ **Hence proved**.

Question 9: **Statement of the question given in the Iqbal's book is wrong. So it cannot be solved.**

Question 10: **Prove that** $\displaystyle\int_{-\infty}^{\infty} \frac{\log(1-x^2)}{1+x^2}\,dx = \pi\log(2)$

Solution: Let, $\displaystyle I = \int_{-\infty}^{\infty} \frac{\log(1-x^2)}{1+x^2}\,dx = \int_C \frac{\log(1-z^2)}{1+z^2}\,dz$

Here, $\displaystyle f(z) = \frac{\log(1-z^2)}{1+z^2} \quad \Rightarrow \quad zf(z) = \frac{z\log(1-z^2)}{1+z^2} \to 0 \quad \text{as} \quad |z|\to\infty$

Poles of $f(z)$ are given by $1+z^2 = 0 \Rightarrow z = \pm i$. Only the pole i lies in upper half of the complex plane and it has order one.

$$R_1^+(f,i) = \lim_{z\to i}\frac{(z-i)\log(1-z^2)}{(z+i)(z-i)} = \lim_{z\to i}\frac{\log(1-z^2)}{z+i} = \frac{\log(2)}{2i}$$

$I = 2\pi i\displaystyle\sum_i R_i^+ = 2\pi i\left[\frac{\log(2)}{2i}\right] = \pi\log(2)$ **Hence proved**.

Question 11: **Prove that** $\displaystyle\int_0^{\infty}\frac{x^2}{x^8+1}dx=\frac{\pi}{8\sin\left(\frac{3\pi}{8}\right)}=\frac{\pi}{8}\operatorname{cosec}\left(\frac{3\pi}{8}\right)$

Solution: Let, $\displaystyle I=\int_0^{\infty}\frac{x^2}{x^8+1}\mathrm{d}x=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{x^8+1}\mathrm{d}x=\frac{1}{2}\int_C\frac{z^2}{z^8+1}\mathrm{d}z$

Here, $f(z)=\dfrac{z^2}{z^8+1}$ $\Rightarrow$ $zf(z)=\dfrac{z^3}{z^8+1}\to 0$ as $|z|\to\infty$

Poles of $f(z)$ are given by $z^6+1=0\Rightarrow z^8=-1=e^{i\pi}\Rightarrow z_k=e^{\frac{\pi i(1+2k)}{8}}$, $k=0,1,2,3,\cdots,7.$

Only the poles $z_0=e^{\frac{\pi i}{8}}$, $z_1=e^{\frac{3\pi i}{8}}$, $z_2=e^{\frac{5\pi i}{8}}$ and $z_3=e^{\frac{7\pi i}{8}}$ lie in the upper half plane and they all have order one.

Let α be one of these poles of $f(z)$ (which lie in upper half of the complex plane), then

$$R_{\alpha}^{+}(f,\alpha)=\lim_{z\to\alpha}\frac{(z-\alpha)z^2}{1+z^8}\qquad\left(\frac{0}{0}\right)\text{ form}$$
$$=\lim_{z\to\alpha}\frac{(z-\alpha)2z+z^2}{8z^7}=\frac{\alpha^2}{8\alpha^7}=\frac{\alpha^3}{8\alpha^8}=-\frac{\alpha^3}{8}\qquad\because\alpha^8=-1$$

$$\sum_{\alpha}R_{\alpha}^{+}=-\frac{1}{8}\left[\left(e^{\frac{\pi i}{8}}\right)^3+\left(e^{\frac{3\pi i}{8}}\right)^3+\left(e^{\frac{5\pi i}{8}}\right)^3+\left(e^{\frac{7\pi i}{8}}\right)^3\right]=-\frac{1}{8}\left[e^{\frac{3\pi i}{8}}+e^{\frac{9\pi i}{8}}+e^{\frac{15\pi i}{8}}+e^{\frac{21\pi i}{8}}\right]$$

This is finite geometric series with $a=$ first term $=-\frac{1}{8}e^{\frac{3\pi i}{8}}$, $r=$ ratio $=e^{\frac{6\pi i}{8}}=e^{\frac{3\pi i}{4}}$ and $n=$ number of terms $=4$

$$\Rightarrow\sum_{\alpha}R_{\alpha}^{+}=\frac{a(1-r^n)}{1-r}=\frac{-\frac{1}{8}e^{\frac{3\pi i}{8}}\left[1-\left(e^{\frac{3\pi i}{4}}\right)^4\right]}{1-e^{\frac{3\pi i}{4}}}=-\frac{1}{8}\frac{e^{\frac{3\pi i}{8}}\left[1-e^{3\pi i}\right]}{e^{\frac{3\pi i}{8}}\left[e^{-\frac{3\pi i}{8}}-e^{\frac{3\pi i}{8}}\right]}=-\frac{1}{8}\left(\frac{1+1}{e^{-\frac{3\pi i}{8}}-e^{\frac{3\pi i}{8}}}\right)$$
$$=-\frac{1}{8}\left(\frac{2}{-2i\sin\left(\frac{3\pi}{8}\right)}\right)=\frac{1}{8i\sin\left(\frac{3\pi}{8}\right)}$$

$$I=\frac{1}{2}\left[2\pi i\sum_{\alpha}R_{\alpha}^{+}\right]=\pi i\sum_{\alpha}R_{\alpha}^{+}=\pi i\left[\frac{1}{8i\sin\left(\frac{3\pi}{8}\right)}\right]=\frac{\pi}{8\sin\left(\frac{3\pi}{8}\right)}=\frac{\pi}{8}\operatorname{cosec}\left(\frac{3\pi}{8}\right)\qquad\textbf{Hence proved.}$$

Note: The result of question 11 can be achieved by substituting $m=1$ and $n=4$ in question 5.

Question 1: **Prove that** $\displaystyle\int_0^\infty \frac{\cos^2 x}{(1+x^2)^2}\,dx = \frac{\pi}{8}(1+3e^{-2})$

Solution: Let, $\displaystyle I = \int_0^\infty \frac{\cos^2 x}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos^2 x}{(1+x^2)^2}\,dx = \frac{1}{4}\int_{-\infty}^\infty \frac{1+\cos(2x)}{(1+x^2)^2}\,dx = \frac{1}{4}\mathrm{Re}\int_{-\infty}^\infty \frac{1+e^{2ix}}{(1+x^2)^2}\,dx$

$$= \frac{1}{4}\mathrm{Re}\int_C \frac{1+e^{2iz}}{(1+z^2)^2}\,dz$$

Poles of $f(z) = \frac{1+e^{2iz}}{(1+z^2)^2}$ are $z = \pm i$. Only the pole $z = i$ lies in upper half of the complex plane and it has order two.

$$R_1(f,i) = \lim_{z\to i}\frac{d}{dz}\left[\frac{(z-i)^2(1+e^{2iz})}{(z+i)^2(z-i)^2}\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{1+e^{2iz}}{(z+i)^2}\right] = \lim_{z\to i}\frac{(z+i)^2(2ie^{2iz}) - 2(z+i)(1+e^{2iz})}{(z+i)^4}$$

$$= \lim_{z\to i}\frac{(z+i)(2ie^{2iz}) - 2(1+e^{2iz})}{(z+i)^3} = \frac{(2i)(2ie^{-2}) - 2(1+e^{-2})}{(2i)^3} = \frac{-4e^{-2} - 2(1+e^{-2})}{-8i}$$

$$= \frac{-2e^{-2} - (1+e^{-2})}{-4i} = \frac{1+3e^{-2}}{4i}$$

$$I = \frac{1}{4}\mathrm{Re}\left[2\pi i\sum_n R_n\right] = \frac{1}{4}\mathrm{Re}\left[2\pi i\left(\frac{1+3e^{-2}}{4i}\right)\right] = \frac{\pi}{8}(1+3e^{-2})$$ Hence proved.

Question 2: **Prove that** $\displaystyle\int_0^\infty \frac{x\sin x\,dx}{(x^2+a^2)^2} = \frac{\pi e^{-a}}{4a}$, **where** $a > 0$

Solution: Let, $\displaystyle I = \int_0^\infty \frac{x\sin x}{(x^2+a^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x}{(x^2+a^2)^2}\,dx = \frac{1}{2}\mathrm{Im}\int_{-\infty}^\infty \frac{x\,e^{ix}}{(x^2+a^2)^2}\,dx = \frac{1}{2}\mathrm{Im}\int_C \frac{z\,e^{iz}}{(z^2+a^2)^2}\,dz$

Poles of $f(z) = \frac{z\,e^{iz}}{(z^2+a^2)^2}$ are $z = \pm ai$. Only the pole $z = i$ lies in upper half plane due to the fact that $a > 0$ (given) and it has order two.

$$R_1(f,ai) = \lim_{z\to ai}\frac{d}{dz}\left[\frac{(z-ai)^2(z\,e^{iz})}{(z+ai)^2(z-ai)^2}\right] = \lim_{z\to ai}\frac{d}{dz}\left[\frac{z\,e^{iz}}{(z+ai)^2}\right] = \lim_{z\to ai}\frac{(z+ai)^2(ize^{iz}+e^{iz}) - 2(z+ai)(ze^{iz})}{(z+ai)^4} =$$

$$= \lim_{z\to ai}\frac{(z+ai)(ize^{iz}+e^{iz}) - 2ze^{iz}}{(z+ai)^3} = \frac{(2ai)(-ae^{-a}+e^{-a}) - 2aie^{-a}}{(2ai)^3} = \frac{(-2a^2i + 2ai - 2ai)e^{-a}}{-8a^3i}$$

$$= \frac{-2a^2i\,e^{-a}}{-8a^3i} = \frac{e^{-a}}{4a}$$

$$I = \frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_n\right] = \frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{e^{-a}}{4a}\right)\right] = \frac{\pi e^{-a}}{4a}$$ Hence proved.

Question 3: **Prove that** $\displaystyle\int_{-\infty}^\infty \frac{\sin x\;dx}{x^2-2x+5} = \frac{\pi}{2e^2}\sin 1$

Solution: Let, $\displaystyle I = \int_{-\infty}^\infty \frac{\sin x\;dx}{x^2-2x+5} = \mathrm{Im}\int_{-\infty}^\infty \frac{e^{ix}\,dx}{x^2-2x+5} = \mathrm{Im}\int_C \frac{e^{iz}\,dz}{z^2-2z+5}$

Poles of $f(z)=\frac{e^{iz}}{z^2-2z+5}$ are $z=\frac{2\pm\sqrt{4-20}}{2}=\frac{2\pm\sqrt{-16}}{2}=1\pm 2i$. Only the pole $z=1+2i$ lies in upper half plane and it has order one.

$$R_1(f,1+2i)=\lim_{z\to 1+2i}\frac{(z-1-2i)e^{iz}}{(z-1-2i)(z-1+2i)}=\lim_{z\to 1+2i}\frac{e^{iz}}{z-1+2i}=\frac{e^{i(1+2i)}}{1+2i-1+2i}=\frac{e^{-2+i}}{4i}=\frac{e^{-2}e^{i}}{4i}$$
$$=\frac{e^{-2}}{4i}(\cos 1+i\sin 1)$$

$$I=\operatorname{Im}\left[2\pi i\sum_n R_n\right]=\operatorname{Im}\left[2\pi i\left(\frac{e^{-2}}{4i}(\cos 1+i\sin 1)\right)\right]=\operatorname{Im}\left[\frac{\pi}{2e^2}(\cos 1+i\sin 1)\right]=\frac{\pi}{2e^2}\sin 1$$ Hence proved.

Question 4: **Prove that** $\displaystyle\int_0^\infty \frac{\cos(ax)}{x^4+4}\,dx=\frac{\pi}{8}[\cos a+\sin a]e^{-a},\qquad a>0$

Solution: Let, $\displaystyle I=\int_0^\infty\frac{\cos(ax)}{x^4+4}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos(ax)}{x^4+4}dx=\frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty}\frac{e^{iax}}{x^4+4}dx=\frac{1}{2}\operatorname{Re}\int_C\frac{e^{iaz}}{z^4+4}dz$

Poles of $f(z)=\frac{e^{iaz}}{z^4+4}$ are given by $z^4=-4=4e^{(2k\pi+\pi)i}\ \Rightarrow\ z_k=\sqrt{2}e^{\frac{(2k+1)\pi i}{4}},\ k=0,1,2,3.$

Here, $z_0=\sqrt{2}e^{\frac{\pi i}{4}}=\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=1+i,\quad z_1=\sqrt{2}e^{\frac{3\pi i}{4}}=-1+i,\quad z_2=\sqrt{2}e^{\frac{5\pi i}{4}}=-1-i,\quad z_1=\sqrt{2}e^{\frac{3\pi i}{4}}=1-i$

Only the poles z_0 and z_1 lie in upper half plane and they have order one.

$$R_1(f,z_0)=\lim_{z\to z_0}\frac{(z-z_0)e^{iaz}}{z^4+4}\qquad\left(\frac{0}{0}\right)\text{ form}$$
$$=\lim_{z\to z_0}\frac{iae^{iaz}(z-z_0)+e^{iaz}}{4z^3}=\frac{e^{iaz_0}}{4z_0^{\,3}}=\frac{z_0e^{iaz_0}}{4z_0^{\,4}}=\frac{z_0e^{iaz_0}}{4(-4)}=\frac{z_0e^{iaz_0}}{-16}\qquad\because\ z_0^{\,4}=-4$$

Similarly, $R_2(f,z_1)=\frac{z_1e^{iaz_1}}{-16}$

$$\text{Sum of residues}=\sum_n R_n=-\frac{1}{16}\left(z_0e^{iaz_0}+z_1e^{iaz_1}\right)=-\frac{1}{16}\left[(1+i)e^{ia(1+i)}+(-1+i)e^{ia(-1+i)}\right]$$
$$=-\frac{1}{16}\left[(1+i)e^{ia}+(-1+i)e^{-ia}\right]e^{-a}=-\frac{1}{16}\left[e^{ia}+ie^{ia}-e^{-ia}+ie^{-ia}\right]e^{-a}$$
$$=-\frac{1}{16}\left[(e^{ia}-e^{-ia})+i(e^{ia}+e^{-ia})\right]e^{-a}=-\frac{1}{16}[2i\sin a+2i\cos a]e^{-a}=-\frac{i}{8}[\sin a+\cos a]e^{-a}$$

$$I=\frac{1}{2}\operatorname{Re}\left[2\pi i\sum_n R_n\right]=\operatorname{Re}\left[2\pi i\left(-\frac{i}{8}[\sin a+\cos a]e^{-a}\right)\right]=\frac{\pi}{8}[\cos a+\sin a]e^{-a}$$ Hence proved.

Question 5: **Prove that** $\displaystyle\int_0^\infty\frac{x^3\sin x\,dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{2(a^2-b^2)}\left[a^2e^{-a}-b^2e^{-b}\right],\qquad a,b>0$

Solution: Let, $I=\int_0^{\infty}\frac{x^3\sin x\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^3\sin x\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}=\frac{1}{2}\mathrm{Im}\int_{-\infty}^{\infty}\frac{x^3\mathrm{e}^{ix}\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}$

$$=\frac{1}{2}\mathrm{Im}\int_C\frac{z^3\mathrm{e}^{iz}\mathrm{d}z}{(z^2+a^2)(z^2+b^2)}$$

Poles of $f(z)=\frac{z^3\mathrm{e}^{iz}}{(z^2+a^2)(z^2+b^2)}$ are $z=\pm ai,\pm bi$. Only the poles $z=ai,bi$ lie in upper half plane due to the fact that $a,b>0$ (given) and they have order one.

$$R_1(f,ia)=\lim_{z\to ia}\frac{(z-ia)z^3\,\mathrm{e}^{iz}}{(z+ia)(z-ia)(z+b^2)}=\lim_{z\to ia}\frac{z^3\,\mathrm{e}^{iz}}{(z+ia)(z^2+b^2)}=\frac{-ia^3\,\mathrm{e}^{-a}}{2ia(b^2-a^2)}=\frac{a^2\,\mathrm{e}^{-a}}{2(a^2-b^2)}$$

$$R_2(f,ib)=\lim_{z\to ib}\frac{(z-ib)z^3\,\mathrm{e}^{iz}}{(z+a^2)(z+ib)(z-ib)}=\lim_{z\to ib}\frac{z^3\,\mathrm{e}^{iz}}{(z+a^2)(z+ib)}=\frac{-ib^3\,\mathrm{e}^{-b}}{(-b^2+a^2)(2ib)}=\frac{-b^2\,\mathrm{e}^{-b}}{2(a^2-b^2)}$$

$$I=\frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_n\right]=\frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{a^2\,\mathrm{e}^{-a}}{2(a^2-b^2)}-\frac{b^2\,\mathrm{e}^{-b}}{2(a^2-b^2)}\right)\right]=\frac{\pi}{2(a^2-b^2)}[a^2\mathrm{e}^{-a}-b^2\mathrm{e}^{-b}]$$ Hence proved.

Question 6: **Prove that** $\int_0^{\infty}\frac{x^3\sin(mx)}{x^4+a^4}\mathrm{d}x=\frac{\pi}{2}\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\cos\left(\frac{ma}{\sqrt{2}}\right),\quad m,a>0$

Solution: Let, $I=\int_0^{\infty}\frac{x^3\sin(mx)}{x^4+a^4}\mathrm{d}x=\frac{1}{2}\int_{-\infty}^{-\infty}\frac{x^3\sin(mx)}{x^4+a^4}\mathrm{d}x=\frac{1}{2}\mathrm{Im}\int_C\frac{x^3\mathrm{e}^{imx}}{x^4+a^4}\mathrm{d}x=\frac{1}{2}\mathrm{Im}\int_C\frac{z^3\mathrm{e}^{imz}}{z^4+a^4}\mathrm{d}z$

Poles of $f(z)=\frac{z^3\mathrm{e}^{imz}}{z^4+a^4}$ are given by $z^4=-a^4=a^4\mathrm{e}^{(2k\pi+\pi)i}\quad\Rightarrow\quad z_k=a\mathrm{e}^{\frac{(2k+1)\pi i}{4}},\quad k=0,1,2,3.$

Here, $z_0=a\mathrm{e}^{\frac{\pi i}{4}}=\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i,\quad z_1=a\mathrm{e}^{\frac{3\pi i}{4}}=-\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i,\quad z_2=a\mathrm{e}^{\frac{\pi i}{4}}=-\frac{a}{\sqrt{2}}-\frac{a}{\sqrt{2}}i,\quad z_1=a\mathrm{e}^{\frac{\pi i}{4}}=\frac{a}{\sqrt{2}}-\frac{a}{\sqrt{2}}i$

Only the poles z_0 and z_1 lie in upper half plane due to the fact that $a>0$ (given) and they have order one.

$$R_1(f,z_0)=\lim_{z\to z_0}\frac{(z-z_0)z^3\mathrm{e}^{imz}}{z^4+a^4}\qquad\left(\frac{0}{0}\right)\text{ form}$$

$$=\lim_{z\to z_0}\frac{z^3\mathrm{e}^{imz}+(z-z_0)(3z^2)\mathrm{e}^{imz}+(z-z_0)z^3\left(im\mathrm{e}^{imz}\right)}{4z^3}=\frac{z_0^3\mathrm{e}^{imz_0}}{4z_0^3}=\frac{\mathrm{e}^{imz_0}}{4}$$

Similarly, $R_2(f,z_1)=\frac{\mathrm{e}^{imz_1}}{4}$

$$\text{Sum of residues}=\sum_n R_n=\frac{1}{4}\left[\mathrm{e}^{imz_0}+\mathrm{e}^{imz_1}\right]=\frac{1}{4}\left[\mathrm{e}^{ima\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}+\mathrm{e}^{ima\left(\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}\right]=\frac{1}{4}\left[\mathrm{e}^{\frac{ima}{\sqrt{2}}}\cdot\mathrm{e}^{\frac{-ma}{\sqrt{2}}}+\mathrm{e}^{\frac{-ima}{\sqrt{2}}}\cdot\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\right]$$

$$=\frac{1}{4}\left[\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\left(\mathrm{e}^{\frac{ima}{\sqrt{2}}}+\mathrm{e}^{\frac{-ima}{\sqrt{2}}}\right)\right]=\frac{1}{4}\left[2\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\cos\left(\frac{ma}{\sqrt{2}}\right)\right]=\frac{1}{2}\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\cos\left(\frac{ma}{\sqrt{2}}\right)$$

$$I=\frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_n\right]=\frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{1}{2}\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\cos\left(\frac{ma}{\sqrt{2}}\right)\right)\right]=\frac{\pi}{2}\mathrm{e}^{\frac{-ma}{\sqrt{2}}}\cos\left(\frac{ma}{\sqrt{2}}\right)$$ Hence proved.

Question 7: **Prove that** $\displaystyle\int_0^\infty \frac{x\sin(mx)}{x^4+a^4}\,dx = \frac{\pi}{4b^2}e^{-mb}\sin(mb)$, **where**, $b=\frac{a}{\sqrt{2}}$, $m, a>0$

Solution: Let, $\displaystyle I=\int_0^\infty \frac{x\sin(mx)}{x^4+a^4}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin(mx)}{x^4+a^4}dx = \frac{1}{2}\mathrm{Im}\int_C \frac{xe^{imx}}{x^4+a^4} = \frac{1}{2}\mathrm{Im}\int_C \frac{ze^{imz}}{z^4+a^4}dz$

Poles of $f(z)=\frac{ze^{imz}}{z^4+a^4}$ are given by $z^4=-a^4=a^4e^{(2k\pi+\pi)i} \Rightarrow z_k = ae^{\frac{(2k+1)\pi i}{4}}$, $k=0,1,2,3.$

Here, $z_0 = ae^{\frac{\pi i}{4}} = \frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i$, $z_1 = ae^{\frac{3\pi i}{4}} = -\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i$, $z_2 = ae^{\frac{\pi i}{4}} = -\frac{a}{\sqrt{2}}-\frac{a}{\sqrt{2}}i$, $z_1 = ae^{\frac{\pi i}{4}} = \frac{a}{\sqrt{2}}-\frac{a}{\sqrt{2}}i$

Only the poles z_0 and z_1 lie in upper half plane due to the fact that $a>0$ (given) and they have order one.

$$R_1(f,z_0)=\lim_{z\to z_0}\frac{(z-z_0)ze^{imz}}{z^4+a^4} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$=\lim_{z\to z_0}\frac{ze^{imz}+(z-z_0)e^{imz}+(z-z_0)z\big(ime^{imz}\big)}{4z^3} = \frac{z_0e^{imz_0}}{4z_0^3} = \frac{z_0^2e^{imz_0}}{4z_0^4} = \frac{z_0^2e^{imz_0}}{-4a^4} \quad \because z_0^4=-a^4$$

Similarly, $R_2(f,z_1) = \dfrac{z_1^2e^{imz_1}}{-4a^4}$

$$\text{Sum of residues} = \sum_n R_n = \frac{-1}{4a^4}\left[{z_0}^2e^{imz_0}+{z_1}^2e^{imz_1}\right]$$

$$=\frac{-1}{4a^4}\left[\left(\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i\right)^2 ie^{ima\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)} + \left(-\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i\right)^2 e^{ima\left(\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}\right]$$

$$=\frac{-1}{4a^4}\left[a^2ie^{\frac{ima}{\sqrt{2}}}\cdot e^{\frac{-ma}{\sqrt{2}}} - a^2ie^{\frac{-ima}{\sqrt{2}}}\cdot e^{\frac{-ma}{\sqrt{2}}}\right] = \frac{-i}{4a^2}e^{\frac{-ma}{\sqrt{2}}}\left[e^{\frac{ima}{\sqrt{2}}}-e^{\frac{-ima}{\sqrt{2}}}\right] = \frac{-i}{4a^2}e^{\frac{-ma}{\sqrt{2}}}\left[2i\sin\left(\frac{ma}{\sqrt{2}}\right)\right]$$

$$=\frac{1}{2a^2}e^{\frac{-ma}{\sqrt{2}}}\sin\left(\frac{ma}{\sqrt{2}}\right)$$

$$I=\frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_n\right] = \frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{1}{2a^2}e^{\frac{-ma}{\sqrt{2}}}\sin\left(\frac{ma}{\sqrt{2}}\right)\right)\right] = \frac{\pi}{2a^2}e^{\frac{-ma}{\sqrt{2}}}\sin\left(\frac{ma}{\sqrt{2}}\right)$$

$$=\frac{\pi}{4b^2}e^{-mb}\sin(mb), \quad \text{where, } b=\frac{a}{\sqrt{2}}$$ Hence proved.

Question 8: **Prove that** $\displaystyle\int_0^\infty \frac{\cos^2 x}{(1+x^2)^2}dx = \frac{\pi}{8}\left(1+3e^{-2}\right)$

Solution : Question 1 is repeated here in Iqbal's book. See solution of question 1.

Question 9: **Prove that** $\displaystyle\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}dx = \frac{\pi(1+ab)e^{-ab}}{4b^3}$, $b>0$

Solution: Let, $\displaystyle I=\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(ax)}{(x^2+b^2)^2}dx = \frac{1}{2}\mathrm{Re}\int_C \frac{e^{iax}}{(x^2+b^2)^2}dx = \frac{1}{2}\mathrm{Re}\int_C \frac{e^{iaz}}{(z^2+b^2)^2}dz$

Poles of $f(z)=\frac{e^{iaz}}{(z^2+b^2)^2}$ are $z=\pm bi$. Only the pole $z=bi$ lies in upper half plane due to the fact that $b>0$ (given) and it has order two.

$$R_2(f,bi)=\lim_{z\to ib}\frac{d}{dz}\left[\frac{(z-ib)^2 e^{iaz}}{(z+ib)^2(z-ib)^2}\right]=\lim_{z\to ib}\frac{d}{dz}\left[\frac{e^{iaz}}{(z+ib)^2}\right]=\lim_{z\to ib}\frac{(z+ib)^2 iae^{iaz}-2(z+ib)e^{iaz}}{(z+ib)^4}$$

$$=\lim_{z\to ib}\frac{[(z+ib)ia-2]e^{iaz}}{(z+ib)^3}=\frac{[(2ib)ia-2]e^{-ab}}{(2ib)^3}=\frac{[-2ab-2]e^{-ab}}{-8ib^3}=\frac{[ab+1]e^{-ab}}{4ib^3}$$

$$I=\frac{1}{2}\mathrm{Re}\left[2\pi i\sum_n R_n\right]=\frac{1}{2}\mathrm{Re}\left[2\pi i\left(\frac{[ab+1]e^{-ab}}{4ib^3}\right)\right]=\frac{\pi(1+ab)e^{-ab}}{4b^3}$$ Hence proved.

Question 10: **Prove that** $\displaystyle\int_0^\infty \frac{x(x^2+1)\sin x}{x^4+x^2+1}dx=\frac{\pi e^{-\sqrt{3}/2}}{2\sqrt{3}}\left[\sin\frac{1}{2}+\sqrt{3}\cos\frac{1}{2}\right]$

Solution: Let, $\displaystyle I=\int_0^\infty\frac{x(x^2+1)\sin x}{x^4+x^2+1}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x(x^2+1)\sin x}{x^4+x^2+1}dx=\frac{1}{2}\mathrm{Im}\int_C\frac{x(x^2+1)e^{ix}}{x^4+x^2+1}dx$

$$=\frac{1}{2}\mathrm{Im}\int_C\frac{z(z^2+1)e^{iz}}{z^4+z^2+1}dz$$

Poles of $f(z)=\frac{z(z^2+1)e^{iz}}{z^4+z^2+1}$ are given by $z^4+z^2+1=0$. But $(z^2-1)(z^4+z^2+1)=z^6-1$. Thus the roots of the polynomial z^4+z^2+1 are the roots of polynomial z^6-1 other than $z=\pm 1$

Now $z^6-1=0\Rightarrow z^6=1=e^{2k\pi i}\Rightarrow z_k=e^{\frac{k\pi i}{3}},\quad k=0,1,2,3,4,5$

Here, $z_0=1,\quad z_1=\frac{1}{2}+\frac{\sqrt{3}i}{2},\quad z_2=-\frac{1}{2}+\frac{\sqrt{3}i}{2},\quad z_3=-1,\quad z_4=-\frac{1}{2}-\frac{\sqrt{3}i}{2},\quad z_5=\frac{1}{2}-\frac{\sqrt{3}i}{2}$

Poles of $f(z)$, which lie in upper half plane, are z_1 and z_2 which are simple poles. Let α is one of the pole of $f(z)$

$$R(f,\alpha)=\lim_{z\to\alpha}(z-\alpha)f(z)=\lim_{z\to\alpha}\frac{(z-\alpha)z(z^2+1)e^{iz}}{z^4+z^2+1}\qquad\left(\frac{0}{0}\right)\text{ form}$$

$$=\lim_{z\to\alpha}\left[\frac{(z-\alpha)\frac{d}{dz}\left[z(z^2+1)e^{iz}\right]+z(z^2+1)e^{iz}}{4z^3+2z}\right]=\frac{\alpha(\alpha^2+1)e^{i\alpha}}{4\alpha^3+2\alpha}=\frac{(\alpha^2+1)e^{\alpha i}}{4\alpha^2+2}$$

Sum of residues at $z=z_1$ and $z=z_2$ is given by

$$R_1+R_2=\frac{e^{i\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}\left(\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2+1\right)}{4\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2+2}+\frac{e^{i\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}\left(\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2+1\right)}{4\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2+2}$$

$$=\frac{e^{i\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}\left(\frac{1}{4}-\frac{3}{4}+\frac{\sqrt{3}i}{2}+1\right)}{4\left(\frac{1}{4}-\frac{3}{4}+\frac{\sqrt{3}i}{2}\right)+2}+\frac{e^{i\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}\left(\frac{1}{4}-\frac{3}{4}-\frac{\sqrt{3}i}{2}+1\right)}{4\left(\frac{1}{4}-\frac{3}{4}-\frac{\sqrt{3}i}{2}\right)+2}$$

$$R_{1+}R_2 = \frac{e^{\frac{i}{2}}e^{\frac{-\sqrt{3}}{2}}\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}{-2+2\sqrt{3}i+2} + \frac{e^{-\frac{i}{2}}e^{\frac{-\sqrt{3}}{2}}\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)}{-2-2\sqrt{3}i+2} = \frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}i}\left[e^{\frac{i}{2}}\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)-e^{\frac{-i}{2}}\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right]$$

$$= \frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}i}\left[\left(\frac{e^{\frac{i}{2}}-e^{\frac{-i}{2}}}{2}\right)+\sqrt{3}i\left(\frac{e^{\frac{i}{2}}+e^{\frac{-i}{2}}}{2}\right)\right] = \frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}i}\left[i\sin\frac{1}{2}+\sqrt{3}i\cos\frac{1}{2}\right] = \frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}}\left[\sin\frac{1}{2}+\sqrt{3}\cos\frac{1}{2}\right]$$

$$I = \frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_n\right] = \frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{e^{\frac{-\sqrt{3}}{2}}}{2\sqrt{3}}\left[\sin\frac{1}{2}+\sqrt{3}\cos\frac{1}{2}\right]\right)\right] = \frac{\pi e^{-\sqrt{3}/2}}{2\sqrt{3}}\left[\sin\frac{1}{2}+\sqrt{3}\cos\frac{1}{2}\right]$$ **Hence proved**.

Question 11: **Prove that** $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(nx)}{1+x^2}dx = \begin{cases}\pi e^{-n} & \text{for } n>0\\ \pi & \text{for } n=0\end{cases}$

Solution: Let, $\displaystyle I = \int_{-\infty}^{\infty}\frac{\cos(nx)}{1+x^2}dx = \mathrm{Re}\int_C \frac{e^{inx}}{1+x^2}dx = \mathrm{Re}\int_C \frac{e^{inz}}{1+z^2}dz$

Poles of $f(z) = \frac{e^{inz}}{1+z^2}$ are $z = \pm i$. Only the pole $z = i$ lies in upper half plane and it has order one.

$$R_1(f,bi) = \lim_{z\to i}\frac{(z-i)e^{inz}}{(z+i)(z-i)} = \lim_{z\to i}\frac{e^{inz}}{z+i} = \frac{e^{-n}}{2i}$$

$$I = \mathrm{Re}\left[2\pi i\sum_n R_n\right] = \mathrm{Re}\left[2\pi i\left(\frac{e^{-n}}{2i}\right)\right] = \pi e^{-n} = \begin{cases}\pi e^{-n} & \text{for } n>0\\ \pi & \text{for } n=0\end{cases}$$ **Hence proved**.

Question 1: **Prove that** $\displaystyle\int_0^\infty \frac{\sin mx\,dx}{x(x^2+a^2)^2} = \frac{\pi}{2a^4}\left[1-\frac{e^{-ma}(ma+2)}{2}\right]$, $a>0$, $m>0$

Solution: Let, $\displaystyle I=\int_0^\infty \frac{\sin mx\ dx}{x(x^2+a^2)} = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin mx\ dx}{x(x^2+a^2)} = \frac{1}{2}\mathrm{Im}\int_C \frac{e^{imz}\,dz}{z(z^2+a^2)}$

Poles of $f(z)=\frac{e^{imz}}{z(z^2+a^2)}$ are $z=0, \pm ai$. The pole $z=0$ lies on real axis, having order one and the pole $z=ai$ lies in upper half of the complex plane and it has order one.

$$R_{1,x}(f,z)=\lim_{z\to 0}\frac{ze^{imz}}{z(z^2+a^2)}=\lim_{z\to 0}\frac{e^{imz}}{(z^2+a^2)}=\frac{1}{a^2}$$

$$R_{1,p}(f,ia)=\lim_{z\to ia}\frac{(z-ia)e^{imz}}{z(z+ia)(z-ia)}=\frac{e^{m(ia)i}}{(ai)(2ai)}=\frac{e^{mai^2}}{-2a^2}=\frac{e^{-ma}}{-2a^2}$$

$$I=\int_0^\infty \frac{\sin mx\ dx}{x(x^2+a^2)}=\frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{e^{-ma}}{-2a^2}\right)+\pi i\left(\frac{1}{a^2}\right)\right]=\frac{1}{2}\left[\pi\left(\frac{e^{-ma}}{-a^2}\right)+\pi\left(\frac{1}{a^2}\right)\right]$$

$$=\frac{1}{2}\left[\pi\left(\frac{1}{a^2}\right)-\pi\left(\frac{e^{-ma}}{a^2}\right)\right]=\frac{\pi}{2a^2}[1-e^{-ma}]$$

Now, differentiate both sides w.r.t "a"

$$\int_0^\infty \frac{-2a\sin mx\,dx}{x(x^2+a^2)^2}=\frac{\pi}{2a^2}(me^{-ma})-\frac{2\pi}{2a^3}(1-e^{-ma})=\frac{\pi}{a^3}\left[\frac{ma}{2}e^{-ma}-1+e^{-ma}\right]=\frac{\pi}{a^3}\left[\frac{e^{-ma}(ma+2)}{2}-1\right]$$

$$\int_0^\infty \frac{\sin mx\,dx}{x(x^2+a^2)^2}=\frac{\pi}{2a^4}\left[1-\frac{e^{-ma}(ma+2)}{2}\right]$$ **Hence proved.**

Question 2: **Prove that** $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,dx}{a^2-x^2}=\frac{\pi}{a}\sin a$

Solution: Let, $\displaystyle I=\int_{-\infty}^\infty \frac{\cos x\ dx}{a^2-x^2}=\mathrm{Re}\int_C \frac{e^{iz}dz}{a^2-z^2}$

Poles of $f(z)=\frac{e^{iz}}{a^2-z^2}$ are $z=\pm a$. The only pole which lie in upper half of the complex plane is $z=a$, having order 1.

$$R_{1,p}(f,a)=\lim_{z\to a}\frac{-e^{iz}(z-a)}{(z-a)(z+a)}=\frac{-e^{ia}}{2a}$$

$$I=\mathrm{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\mathrm{Re}\left[2i\pi\left(\frac{-e^{ia}}{2a}\right)+\pi i(0)\right]=-\mathrm{Re}\left[(i\pi)\left(\frac{e^{ia}}{a}\right)\right]$$

$$=-\mathrm{Re}\left[\left(\frac{\pi i}{a}\right)(\cos a+i\sin a)\right]=\mathrm{Re}\left[\frac{\pi}{a}\sin a-\frac{i\pi}{a}\cos a\right]=\frac{\pi}{a}\sin a$$ **Hence proved.**

Question 3: **Prove that** $\displaystyle\int_0^\infty \frac{\sin mx\,dx}{x^3+4x}=\frac{\pi}{8}(1-e^{-2m})$

Solution: Let, $I=\int_0^{\infty}\frac{\sin mx\,\mathrm{d}x}{x^3+4x}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin mx\,\mathrm{d}x}{x(x^2+4)}=\frac{1}{2}\int_C\frac{\mathrm{e}^{imz}\,\mathrm{d}z}{z(z^2+4)}$

Poles of $f(z)=\frac{\mathrm{e}^{imz}}{z(z^2+4)}$ are $z=0,\ \pm 2i$. The pole $z=0$ lies on real axis, having order one and the pole $z=2i$ lies in upper half of the complex plane and it has order one.

$$R_{1,x}(f,0)=\frac{z\mathrm{e}^{imz}}{z(z^2+4)}=\frac{\mathrm{e}^{imz}}{z^2+4}=\frac{1}{4}$$

$$R_{1,p}(f,2i)=\frac{(z-2i)\mathrm{e}^{imz}}{z(z+2i)(z-2i)}=\lim_{z\to 2i}\frac{\mathrm{e}^{imz}}{z(z+2i)}=\frac{\mathrm{e}^{-2m}}{2i(4i)}=-\frac{\mathrm{e}^{-2m}}{8}$$

$$I=\frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\mathrm{Im}\left[2\pi i\left(-\frac{\mathrm{e}^{-2m}}{8}\right)+\pi i\left(\frac{1}{4}\right)\right]=\pi\left(-\frac{\mathrm{e}^{-2m}}{8}\right)+\pi\left(\frac{1}{8}\right)=\frac{\pi}{8}-\frac{\pi\mathrm{e}^{-2m}}{8}$$

$$=\frac{\pi}{8}(1-\mathrm{e}^{-2m})$$

Hence proved.

Question 4: **Proved that** $\int_0^{\infty}\frac{\sin mx\,\mathrm{d}x}{x(x^2+a^2)}=\frac{\pi}{2a^2}[1-\mathrm{e}^{-ma}]\quad a>0, m>0$

Solution: Let, $I=\int_0^{\infty}\frac{\sin mx\,\mathrm{d}x}{x(x^2+a^2)}=\frac{1}{2}\int_0^{\infty}\frac{\sin mx\,\mathrm{d}x}{x(x^2+a^2)}=\frac{1}{2}\mathrm{Im}\int_C\frac{\mathrm{e}^{imz}\,\mathrm{d}z}{z(z^2+a^2)}$

The poles of $f(z)=\frac{\mathrm{e}^{imz}}{z(z^2+a^2)}$ are $z=0,\pm ai$. Only the pole $z=0$ lies on real axis, having order 1 and the pole $z=ai$ lies in upper half plane and it has order 1.

$$R_{1,x}(f,z)=\lim_{z\to 0}\left[\frac{z\mathrm{e}^{imz}}{z(z^2+a^2)}\right]=\lim_{z\to 0}\left[\frac{\mathrm{e}^{imz}}{(z^2+a^2)}\right]=\frac{1}{a^2}$$

$$R_{1,p}(f,ia)=\lim_{z\to ai}\frac{(z-ia)\mathrm{e}^{imz}}{z(z+ia)(z-ia)}=\frac{\mathrm{e}^{m(ai)i}}{(ai)(2ai)}=\frac{\mathrm{e}^{mai^2}}{-2a^2}=\frac{\mathrm{e}^{-ma}}{-2a^2}$$

$$I=\frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{\mathrm{e}^{-ma}}{-2a^2}\right)+\pi i\left(\frac{1}{a^2}\right)\right]=\frac{1}{2}\left[\pi\left(\frac{1}{a^2}\right)-\pi\left(\frac{\mathrm{e}^{-ma}}{a^2}\right)\right]$$

$$=\frac{\pi}{2a^2}[1-\mathrm{e}^{-ma}]$$

Hence proved.

Questio 5: **Prove that** $\int_{-\infty}^{\infty}\frac{\cos mx\,\mathrm{d}x}{x^2-a^2}=-\frac{\pi}{a}\sin ma$

Solution: Let, $I=\int_{-\infty}^{\infty}\frac{\cos mx\,\mathrm{d}x}{x^2-a^2}=\mathrm{Re}\int_C\frac{\mathrm{e}^{imz}\,\mathrm{d}z}{z^2-a^2}$

Poles of $f(z)=\frac{\mathrm{e}^{imz}}{z^2-a^2}$ are $z=\pm a$. Only the pole which lies in upper half of the complex plane is $z=a$, having order 1.

$$R_{1,p}(f,a)=\frac{\mathrm{e}^{imz}(z-a)}{(z-a)(z+a)}=\frac{\mathrm{e}^{ima}}{2a}$$

$$I = \text{Re}\left[2\pi i \sum_n R_{n,p} + \pi i \sum_n R_{n,x}\right] = \text{Re}\left[2i\pi\left(\frac{e^{ima}}{2a}\right) + \pi i(0)\right] = \text{Re}\left[(i\pi)\left(\frac{e^{ima}}{a}\right)\right]$$

$$= \text{Re}\left[\left(\frac{\pi i}{a}\right)(\cos ma + i\sin ma)\right] = \text{Re}\left[\left(-\frac{\pi}{a}\sin ma + \frac{i\pi}{a}\cos ma\right)\right]$$

$$= -\frac{\pi}{a}\sin ma$$ **Hence proved.**

Question 6: **Prove that** $\int_0^\infty \frac{\sin mx\,dx}{x(x^2-a^2)} = \frac{\pi}{2a^2}(\cos ma - 1)$ $a > 0$, $m > 0$

Solution: Let, $I = \int_0^\infty \frac{\sin mx\,dx}{x(x^2-a^2)} = \frac{1}{2}\int_0^\infty \frac{\sin mx\,dx}{x(x^2-a^2)} = \frac{1}{2}\text{Im}\int_C \frac{e^{imz}\,dz}{z(z^2-a^2)}$

Poles of $f(z) = \frac{e^{imz}}{z(z^2-a^2)}$ are $z = 0, \pm a$. All poles have order 1 and lie on real axis.

$$R_{1,x}(f,0) = \lim_{z\to 0}\frac{ze^{imz}}{z(z^2-a^2)} = \lim_{z\to 0}\frac{e^{imz}}{(z^2-a^2)} = \frac{1}{-a^2}$$

$$R_{2,x}(f,a) = \lim_{z\to a}\frac{(z-a)e^{imz}}{z(z^2-a^2)} = \lim_{z\to a}\frac{(z-a)e^{imz}}{z(z-a)(z+a)} = \frac{e^{imz}}{z(z+a)} = \frac{e^{ima}}{2a^2}$$

$$R_{3,x}(f,-a) = \lim_{z\to -a}\frac{(z+a)e^{imz}}{z(z^2-a^2)} = \lim_{z\to -a}\frac{(z+a)e^{imz}}{z(z-a)(z+a)} = \lim_{z\to -a}\frac{e^{imz}}{z(z+a)} = \frac{e^{-ima}}{2a^2}$$

$$I = \frac{1}{2}\text{Im}\left[\pi i \sum_n R_{n,x} + 2\pi i \sum_n R_{n,p}\right] = \frac{1}{2}\text{Im}\left[\pi i\left(\frac{1}{-a^2} + \frac{e^{ima}}{2a^2} + \frac{e^{-ima}}{2a^2}\right) + 2\pi i(0)\right]$$

$$= \frac{1}{2}\text{Im}\left[\pi i\left(\frac{1}{-a^2} + \frac{\cos ma + i\sin ma}{2a^2} + \frac{\cos ma - i\sin ma}{2a^2}\right)\right] = \frac{1}{2}\text{Im}\left[\pi i\left(\frac{2\cos ma - 2}{2a^2}\right)\right]$$

$$= \frac{\pi}{a^2}[\cos ma - 1]$$ **Hence proved.**

Question 7: **Prove that** $\int_0^\infty \frac{\sin^2 mx\,dx}{x^2(a^2+x^2)^2} = \frac{\pi}{8a^5}\left[e^{-2ma}(2ma+3) + 4ma - 3\right], m > 0, a > 0$

Solution: Let, $I = \int_0^\infty \frac{\sin^2 mx\,dx}{x^2(a^2+x^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\frac{1}{2}(1-\cos 2mx)\,dx}{x^2(a^2+x^2)^2} = \frac{1}{4}\text{Re}\int_C \frac{(1-e^{2imz})dz}{z^2(z^2+a^2)^2}$

Poles of $f(z) = \frac{(1-e^{2imz})}{z^2(z^2+a^2)^2}$ are $z = 0, \pm ai$. Only the pole $z = 0$ lies on real axis, having order 2 and $z = ai$ has order 2 and lies in upper half plane.

$$R_{1,x}(f,0) = \lim_{z\to 0}\frac{d}{dz}\left[\frac{z^2(1-e^{2imz})}{z^2(z^2+a^2)^2}\right] = \lim_{z\to 0}\frac{(z^2+a^2)^2(-2mie^{2miz}) - (1-e^{2miz})2(z^2+a^2)2z}{(z^2+a^2)^4}$$

$$= \lim_{z\to 0}\frac{(z^2+a^2)(-2mie^{2miz}) - 4z(1-e^{2miz})}{(z^2+a^2)^3} = \frac{-2ima^2}{a^6} = \frac{-2im}{a^4}$$

$$R_{1,p}(f,ai) = \lim_{z\to ai}\frac{d}{dz}\left[\frac{(z-ai)^2(1-e^{2imz})}{z^2(z+ai)^2(z-ai)^2}\right] = \lim_{z\to ai}\frac{d}{dz}\left[\frac{(1-e^{2imz})}{z^2(z+ai)^2}\right]$$

$$R_{1,p}(f,ai)=\lim_{z\to ai}\frac{z^2(z+ai)^2\left(-2mie^{2imz}\right)-\left(1-e^{2imz}\right)[2z\,(z+ai)^2+2z^2(z+ai)\,]}{z^4(z+ai)^4}$$
$$=\frac{(ai)^2(2ai)^2(-2mie^{-2ma})-(1-e^{-2ma})[2ai\,(2ai)^2+2(ai)^2(2ai)\,]}{(ai)^4(2ai)^4}$$
$$=\frac{-8mia^4e^{-2ma}+(1-e^{-2ma})(12a^3i)}{16a^8}$$

$$I=\frac{1}{4}\mathrm{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{4}\mathrm{Re}\left[2\pi i\left(\frac{-8mia^4e^{-2ma}+(1-e^{-2ma})(12a^3i)}{16a^8}\right)+\pi i\left(\frac{-2im}{a^4}\right)\right]$$
$$=\frac{1}{4}\mathrm{Re}\left[\pi i\left(\frac{-8mia^4e^{-2ma}+(1-e^{-2ma})(12a^3i)}{8a^8}\right)+\left(\frac{2\pi m}{a^4}\right)\right]$$
$$=\frac{\pi a^3}{4}\mathrm{Re}\left[\frac{8ma\,e^{-2ma}-(1-e^{-2ma})(12)+16\pi ma}{8a^8}\right]$$
$$=\frac{\pi}{8a^5}[e^{-2ma}(2ma+3)+4ma-3]$$ **Hence proved**.

Alternate solution: Let $I=\int_0^\infty\frac{\sin^2mx\,dx}{x^2(a^2+x^2)}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\frac{1}{2}(1-\cos 2mx)\,dx}{x^2(a^2+x^2)}=\frac{1}{4}\mathrm{Re}\int_C\frac{\left(1-e^{2imz}\right)dz}{z^2(z^2+a^2)}$

Poles of $f(z)=\frac{(1-e^{2imz})}{z^2(z^2+a^2)}$ are $z=0,\pm ai$. Only the pole $z=0$ lies on real axis, having order 2 while $z=ai$ has order 1 and lies in upper half plane.

$$R_{1,x}(f,0)=\lim_{z\to 0}\frac{d}{dz}\left[\frac{z^2\left(1-e^{2imz}\right)}{z^2(z^2+a^2)}\right]=\lim_{z\to 0}\frac{(z^2+a^2)\left(-2mie^{2miz}\right)-\left(1-e^{2miz}\right)2z}{(z^2+a^2)^2}=\frac{-2ima^2}{a^4}=\frac{-2im}{a^2}$$

$$R_{1,p}(f,ai)=\lim_{z\to ai}\frac{(z-ai)\left(1-e^{2imz}\right)}{z^2(z+ai)(z-ai)}=\lim_{z\to ai}\frac{\left(1-e^{2imz}\right)}{z^2(z+ai)}=\frac{\left(1-e^{-2ma}\right)}{(ai)^2(ai+ai)}=\frac{1-e^{-2ma}}{-2a^3i}=\frac{e^{-2ma}-1}{2a^3i}$$

$$I=\int_0^\infty\frac{\sin^2mx\,dx}{x^2(a^2+x^2)}=\frac{1}{4}\mathrm{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{4}\mathrm{Re}\left[2\pi i\left(\frac{e^{-2ma}-1}{2a^3i}\right)+\pi i\left(\frac{-2im}{a^2}\right)\right]$$
$$=\frac{\pi}{4}\mathrm{Re}\left[\frac{e^{-2ma}-1}{a^3}+\frac{2m}{a^2}\right]=\frac{\pi}{4a^3}[e^{-2ma}-1+2ma]$$

Differentiate w.r.t "a", on both sides, we get

$$\int_0^\infty\frac{-2a\sin^2mx\,dx}{x^2(a^2+x^2)^2}=\frac{\pi}{4a^6}[(-2me^{-2ma}+2m)a^3-(e^{-2ma}-1+2ma)3a^2]$$
$$=\frac{\pi}{4a^4}[-2mae^{-2ma}-3e^{-2ma}+2ma-6ma+3]$$

$$\int_0^\infty\frac{\sin^2mx\,dx}{x^2(a^2+x^2)^2}=\frac{\pi}{8a^5}[e^{-2ma}(2ma+3)+4ma-3]$$ **Hence proved**.

Question 8: prove that if $a<0$ **(*i*)** $\int_0^\infty\frac{b\cos ax\,dx}{x^2-b^2}=\frac{-\pi}{2}\sin ab$ **(*ii*)** $\int_0^\infty\frac{x\sin ax\,dx}{x^2-b^2}=\frac{\pi}{2}\cos ab$

Solution (*i*): Let $I=\int_0^\infty \frac{b\cos ax\ \mathrm{d}x}{x^2-b^2}=\frac{1}{2}\int_{-\infty}^\infty \frac{b\cos ax\ \mathrm{d}x}{x^2-b^2}=\frac{1}{2}\mathrm{Re}\int_C \frac{b\ \mathrm{e}^{iaz}\ \mathrm{d}z}{z^2-b^2}$

Poles of $f(z)=\frac{b\mathrm{e}^{iaz}}{z^2-b^2}$ are $z=\pm b$. Only the pole which lies in upper half of the complex plane is $z=b$, having order 1.

$$R_{1,p}(f,b)=\lim_{z\to b}\left[\frac{\mathrm{b}\mathrm{e}^{iaz}(z-b)}{(z-b)(z+b)}\right]=\frac{\mathrm{e}^{iab}}{2}$$

$$I=\frac{1}{2}\mathrm{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\mathrm{Re}\left[2i\pi\left(\frac{\mathrm{e}^{iab}}{2}\right)+\pi i(0)\right]=Re\left[i\pi\left(\frac{\mathrm{e}^{iab}}{2}\right)\right]=\mathrm{Re}\left[\frac{\pi i}{2}(\cos ab+i\sin ab)\right]$$

$$=\mathrm{Re}\left[\left(-\frac{\pi}{2}\sin ab+\frac{i\pi}{2}\cos ab\right)\right]=-\frac{\pi}{2}\sin ab$$ **Hence proved.**

Solution (*ii*): We have just proved in (i) that $\int_0^\infty \frac{b\cos ax\ \mathrm{d}x}{x^2-b^2}=-\frac{\pi}{2}\sin ab$

Differentiate both side w.r.t "a"

$$\int_0^\infty \frac{bx\sin ax\ \mathrm{d}x}{x^2-b^2}=\frac{b\pi}{2}\cos ab$$

$$\int_0^\infty \frac{x\sin ax\ \mathrm{d}x}{x^2-b^2}=\frac{\pi}{2}\cos ab$$ **Hence proved.**

Alternate solution (*ii*): Let $I=\int_0^\infty \frac{x\sin ax\ \mathrm{d}x}{x^2-b^2}=\frac{1}{2}\int_{-\infty}^\infty \frac{b\sin ax\ \mathrm{d}x}{x^2-b^2}=\frac{1}{2}\mathrm{Im}\int_C \frac{z\ \mathrm{e}^{iaz}\ \mathrm{d}z}{z^2-b^2}$

Poles of $f(z)=\frac{z\ \mathrm{e}^{iaz}}{z^2-b^2}$ are $z=\pm b$. Only the pole which lies in upper half of the complex plane is $z=b$, having order 1.

$$R_{1,p}(f,b)=\lim_{z\to b}\frac{z\mathrm{e}^{iaz}(z-b)}{(z-b)(z+b)}=\frac{b\mathrm{e}^{iab}}{2b}=\frac{b\mathrm{e}^{iab}}{2}$$

$$I=\frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\mathrm{Im}\left[2i\pi\left(\frac{\mathrm{e}^{iab}}{2}\right)+\pi i(0)\right]=\mathrm{Im}\left[i\pi\left(\frac{\mathrm{e}^{iab}}{2}\right)\right]=\mathrm{Im}\left[\frac{\pi i}{2}(\cos ab+i\sin ab)\right]$$

$$=\mathrm{Im}\left[\left(-\frac{\pi}{2}\sin ab+\frac{i\pi}{2}\cos ab\right)\right]=\frac{\pi}{2}\cos ab$$ **Proved.**

Question* 9**: ***Prove that $\int_0^\infty \frac{\sin^2 mx\,dx}{x^2(a^2+x^2)}=\frac{\pi}{4a^3}(e^{-2ma}-1+2ma),\quad m>0,\quad a>0$

Solution: Let $I=\int_0^\infty \frac{\sin^2 mx\ \mathrm{d}x}{x^2(a^2+x^2)}=\frac{1}{2}\int_{-\infty}^\infty \frac{\frac{1}{2}(1-\cos 2mx)\ \mathrm{d}x}{x^2(a^2+x^2)}=\frac{1}{4}\mathrm{Re}\int_C \frac{(1-\mathrm{e}^{2imz})\mathrm{d}z}{z^2(z^2+a^2)}$

Poles of $f(z)=\frac{(1-\mathrm{e}^{2imz})}{z^2(z^2+a^2)}$ are $z=0,\pm ai$. Only the pole $z=0$ lies on real axis, having order 2 while $z=ai$ has order 1 and lies in upper half plane.

$$R_{1,x}(f,0)=\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^2(1-\mathrm{e}^{2imz})}{z^2(z^2+a^2)}\right]=\lim_{z\to 0}\frac{(z^2+a^2)(-2mi\mathrm{e}^{2miz})-(1-\mathrm{e}^{2miz})2z}{(z^2+a^2)^2}=\frac{-2ima^2}{a^4}=\frac{-2im}{a^2}$$

$$R_{1,p}(f,ai)=\lim_{z\to ai}\left[\frac{(z-ai)\left(1-e^{2imz}\right)}{z^2(z+ai)(z-ai)}\right]=\lim_{z\to ai}\frac{\left(1-e^{2imz}\right)}{z^2(z+ai)}=\frac{\left(1-e^{-2ma}\right)}{(ai)^2(ai+ai)}=\frac{1-e^{-2ma}}{-2a^3i}=\frac{e^{-2ma}-1}{2a^3i}$$

$$I=\frac{1}{4}\mathrm{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{4}\mathrm{Re}\left[2\pi i\left(\frac{e^{-2ma}-1}{2a^3i}\right)+\pi i\left(\frac{-2im}{a^2}\right)\right]=\frac{\pi}{4}\mathrm{Re}\left[\frac{e^{-2ma}-1}{a^3}+\frac{2m}{a^2}\right]$$
$$=\frac{\pi}{4a^3}\left[e^{-2ma}-1+2ma\right]$$

Question 10: **prove that** $\displaystyle\int_0^\infty\frac{(x-\sin x)dx}{x^3(x^2+a^2)}=\frac{\pi}{2a^4}\left(\frac{a^2}{2}+1-e^{-a}\right)$ **where** $a>0$

Solution: $\displaystyle\int_0^\infty\frac{(x-\sin x)dx}{x^3(x^2+a^2)}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{(x-\sin x)dx}{x^3(x^2+a^2)}=\frac{1}{2}\mathrm{Im}\int_C\frac{\left(z-e^{iz}\right)dz}{z^3(z^2+a^2)}$

Poles of $f(z)=\frac{z-e^{iz}}{z^3(z^2+a^2)}$ are $z=0,\pm ai$. The pole $z=0$ lies on real axis, having order 3 and the pole $z=ai$ lies in upper half of the complex plane with order 1.

$$R_{1,x}(f,0)=\lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{z^3\left(z-e^{iz}\right)}{z^3(z^2+a^2)}\right]=\lim_{z\to 0}\frac{1}{2}\frac{d^2}{dz^2}\left[\frac{z-e^{iz}}{z^2+a^2}\right]=\lim_{z\to 0}\frac{1}{2}\frac{d}{dz}\left[\frac{(z^2+a^2)\left(1-ie^{iz}\right)-\left(z-e^{iz}\right)(2z)}{(z^2+a^2)^2}\right]$$

$$=\lim_{z\to 0}\frac{1}{2}\left[\frac{\begin{array}{c}(z^2+a^2)^2\left[(2z)\left(1-ie^{iz}\right)+(z^2+a^2)\left(e^{iz}\right)-\left(z-e^{iz}\right)(2)-\left(1-ie^{iz}\right)(2z)\right]\\-\left[(z^2+a^2)\left(1-ie^{iz}\right)-\left(z-e^{iz}\right)(2z)\right]\left[4z(z^2+a^2)\right]\end{array}}{(z^2+a^2)^4}\right]$$

$$=\frac{1}{2}\left[\frac{a^4(a^2+2)}{a^8}\right]=\frac{1}{2a^4}(a^2+2)$$

$$R_{1,p}(f,ai)=\lim_{z\to ai}\frac{(z-ai)\left(z-e^{iz}\right)}{z^3(z+ai)(z-ai)}=\lim_{z\to ai}\left[\frac{z-e^{iz}}{z^3(z+ai)}\right]=\frac{ai-e^{-a}}{(-a^3i)(2ai)}=\frac{ai-e^{-a}}{2a^4}$$

$$I=\frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{ai-e^{-a}}{2a^4}\right)+\pi i\left(\frac{1}{2a^4}(a^2+2)\right)\right]$$
$$=\frac{\pi}{2}\mathrm{Im}\left[\frac{-2a-2ie^{-a}+ia^2+2i}{2a^4}\right]=\frac{\pi}{2a^4}\left[\frac{a^2}{2}-e^{-a}+1\right]$$ **Hence proved**.

Question 11: **Prove that** $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x\,dx}{(x^2+4)(x-1)}=\frac{\pi}{5}\left(\cos 1-\frac{1}{e^2}\right)$

Solution: Let $\displaystyle I=\int_{-\infty}^{\infty}\frac{\sin x\,dx}{(x^2+4)(x-1)}=\mathrm{Im}\int_C\frac{e^{iz}dz}{(z^2+4)(z-1)}$

Poles of $f(z)=\frac{e^{iz}}{(z^2+4)(z-1)}$ are $z=1,\pm 2i$. The pole $z=1$ lies on real axis, having order 1 and the $z=2i$ lies in upper half of the complex plane with order 1.

$$R_{1,x}(f,1)=\frac{(z-1)e^{iz}}{(z^2+4)(z-1)}=\lim_{z\to 1}\frac{e^{iz}}{(z^2+4)}=\frac{e^{i}}{5}$$

$$R_{1,p}(f,2i)=\lim_{z\to 2i}\frac{(z-2i)e^{iz}}{(z+2i)(z-2i)(z-1)}=\frac{e^{iz}}{(z+2i)(z-1)}=\frac{e^{-2}}{(4i)(2i-1)}=\frac{-e^{-2}}{(4)(2+i)}\cdot\frac{2-i}{2-i}=\frac{e^{-2}(i-2)}{20}$$

$$I=\text{Im}\left[2\pi i\left(\frac{e^{-2}(i-2)}{20}\right)+\pi i\left(\frac{e^{i}}{5}\right)\right]=\text{Im}\left[\frac{-\pi e^{-2}-2\pi i e^{-2}+2\pi i e^{i}}{10}\right]$$

$$=\text{Im}\left[\frac{-\pi e^{-2}-2\pi i e^{-2}+2\pi i(\cos 1+i\sin 1)}{10}\right]=\frac{\pi}{5}(\cos 1-e^{-2})$$ **Hence proved**.

Form: $\int_0^\infty x^{\alpha-1} f(x)dx$

Working Rule:

Case I: For the poles of $f(z)$, which do not lie on positive part of the real axis.

Step I: Replace x by z and $x^{\alpha-1}$ by $\mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]}$ and consider $\varphi(z) = \mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]} f(z)$

Step II: Calculate all poles of $f(z)$ and compute residues of $\varphi(z)$ at all these poles. (Note that, each pole of $f(z)$ is also a pole of $\varphi(z)$, but the converse is not true, as $\varphi(z)$ may have singularities other than the poles of $f(z)$.)

StepIII: $\int_0^\infty x^{\alpha-1} f(x)dx = \dfrac{-\pi}{\sin(\pi\alpha)} \mathrm{e}^{-\alpha\pi i} \sum_i R_i$,

where $\sum_i R_i =$ sum of residues of all the poles which do not lie on positive part of real axis.

Case II: For the poles of $f(z)$, which do lie on positive part of the real axis.

Step I: Replace $x$by z and $x^{\alpha-1}$ by $\mathrm{e}^{(\alpha-1)\log(z)}$ and consider $\varphi(z) = \mathrm{e}^{[(\alpha-1)\log(z)]} f(z)$

Step II: Calculate all poles of $f(z)$ and compute residues of $\varphi(z)$ at all these poles. (Note that, each pole of $f(z)$ is also a pole of $\varphi(z)$, but the converse is not true, as $\varphi(z)$ may have singularities other than the poles of $f(z)$.)

StepIII: $\int_0^\infty x^{\alpha-1} f(x)dx = -\pi\cot(\pi\alpha) \sum_i R_i'$,

where $\sum_i R_i' =$ sum of residues of all the poles which do lie on positive part of real axis.

Question 1: **Prove that** $\displaystyle \int_0^\infty \frac{x^{\alpha-1}\mathrm{d}x}{x+\mathrm{e}^{i\beta}} = \pi\frac{\mathrm{e}^{(\alpha-1)\beta i}}{\sin(\alpha\pi)}, \quad 0 < \alpha < 1, \quad -\pi < \beta < \pi$

Solution: Pole of $f(z) = \frac{1}{z+\mathrm{e}^{i\beta}}$ is $z = -\mathrm{e}^{i\beta}$ (having order one), which do not lie on positive part of the real axis due to the fact that $-\pi < \beta < \pi$

Let $\displaystyle I = \int_0^\infty \frac{x^{\alpha-1}\mathrm{d}x}{x+\mathrm{e}^{i\beta}} = \int_C \frac{\mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]}\mathrm{d}z}{z+\mathrm{e}^{i\beta}}$ and $\displaystyle \varphi(z) = \frac{\mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]}}{z+\mathrm{e}^{i\beta}}$

$$R(\varphi, -\mathrm{e}^{i\beta}) = \frac{\mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]}}{z+\mathrm{e}^{i\beta}}(z+\mathrm{e}^{i\beta}) = \mathrm{e}^{[(\alpha-1)(\log \mathrm{e}^{i\beta}+i\pi)]} = \mathrm{e}^{[(\alpha-1)(i\beta+i\pi)]} = \mathrm{e}^{i(\alpha-1)\beta}.\mathrm{e}^{i(\alpha-1)\pi}$$

$$= \mathrm{e}^{i(\alpha-1)\beta}\mathrm{e}^{i\alpha\pi}\mathrm{e}^{-i\pi} = -\,\mathrm{e}^{i(\alpha-1)\beta}\mathrm{e}^{i\pi\alpha} \qquad \because \mathrm{e}^{-i\pi} = -1$$

$$I = \frac{-\pi}{\sin(\pi\alpha)}\mathrm{e}^{i\alpha\pi}\sum_i R_i = \frac{-\pi}{\sin(\pi\alpha)}\mathrm{e}^{-i\alpha\pi}[-\mathrm{e}^{i(\alpha-1)\beta}\mathrm{e}^{i\pi\alpha}] = \pi\frac{\mathrm{e}^{(\alpha-1)\beta i}}{\sin(\alpha\pi)}$$ **Hence proved**.

Question 2: **Prove that** $\displaystyle \int_0^\infty \frac{x^\alpha\,\mathrm{d}x}{(x+a)(x+b)} = \frac{\pi}{\sin(\alpha\pi)}\left[\frac{a^\alpha - b^\alpha}{a-b}\right], \quad -1 < \alpha < 1, \quad a > 0.$

Solution: Poles of $f(z)=\frac{1}{(z+a)(z+b)}$ are $z=-a,-b$ (both of them have order one), which do not lie on positive part of the real axis due to the fact that $a>0,\ b>0$

Let, $$I=\int_0^\infty \frac{x^\alpha\,dx}{(x+a)(x+b)}=\int_C \frac{e^{[(\alpha)(\log(-z)+i\pi)]}}{(z+a)(z+b)}dz \quad \text{and} \quad \varphi(z)=\frac{e^{[(\alpha)(\log(-z)+i\pi)]}}{(z+a)(z+b)}$$

The poles of $\varphi(z)$ (or equivalently that of $f(z)$) are $z=-a$ and $z=-b$ and both of them have order 1.

$$R_1(\varphi,-a)=\lim_{z\to-a}\frac{e^{[(\alpha)(\log(-z)+i\pi)]}(z+a)}{(z+a)(z+b)}=\frac{e^{\alpha(\log(a)+i\pi)}}{b-a}=\frac{e^{\alpha\log a}\,e^{i\pi\alpha}}{b-a}=\frac{e^{\log a^\alpha}e^{i\pi\alpha}}{-(a-b)}=\frac{-a^\alpha e^{i\pi\alpha}}{a-b}$$

$$R_2(\varphi,-b)=\lim_{z\to-b}\frac{e^{[(\alpha)(\log(-z)+i\pi)]}(z+b)}{(z+a)(z+b)}=\frac{e^{[(\alpha)(\log(b)+i\pi)]}}{a-b}=\frac{b^\alpha e^{i\pi\alpha}}{a-b}$$

$$I=\frac{-\pi}{\sin((\alpha+1)\pi)}e^{-(\alpha+1)\pi i}\sum_{i=1}^{2}R_i$$

$$\because \sin((\alpha+1)\pi)=\sin(\alpha\pi+\pi)=-\sin(\alpha\pi) \quad \text{and} \quad e^{-(\alpha+1)\pi i}=e^{-\alpha\pi i}e^{-\pi i}=e^{-\alpha\pi i}(-1)=-e^{-\alpha\pi i}$$

$$\therefore I=\frac{-\pi}{\sin(\alpha\pi)}e^{-\alpha\pi i}\left[\frac{b^\alpha e^{i\pi\alpha}}{a-b}-\frac{a^\alpha e^{i\pi a}}{a-b}\right]=\frac{-\pi}{\sin(\alpha\pi)}e^{-\alpha\pi i}\left[\frac{e^{\alpha\pi i}(b^\alpha-a^\alpha)}{a-b}\right]$$
$$=\frac{\pi}{\sin(\alpha\pi)}\left[\frac{(a^\alpha-b^\alpha)}{a-b}\right] \quad \textbf{Hence proved.}$$

Question 3: Prove that $\displaystyle\int_0^\infty \frac{x^\alpha}{x^4+1}dx=\frac{\pi}{4\sin\left(\frac{\pi(\alpha+1)}{4}\right)}$, **where** $-1<\alpha<3$

Solution: Poles of $f(z)=\frac{1}{z^4+1}$ are $z_k=e^{(\pi i+2k\pi i)/4},\ \ k=0,1,2,3$ (all of them have order one), which do not lie on positive part of the real axis.

Let, $$I=\int_0^\infty \frac{x^\alpha}{x^4+1}dx=\int_C \frac{e^{[(\alpha)(\log(-z)+i\pi)]}}{z^4+1}dz \quad \text{and} \quad \varphi(z)=\frac{e^{[(\alpha)(\log(-z)+i\pi)]}}{z^4+1}$$

Let β be a root of $z^4=-1$, then

$$R(\varphi,\beta)=\lim_{z\to\beta}\frac{(z-\beta)e^{[(\alpha)(\log(-z)+i\pi)]}}{z^4+1}, \quad \left(\frac{0}{0}\right)\text{form}$$

Apply L, Hospital's rule

$$\boldsymbol{R(\varphi,\beta)=\lim_{z\to\beta}\frac{e^{[(\alpha)(\log(-z)+i\pi)]}+(z-\beta)\frac{d}{dz}\left(e^{[(\alpha)(\log(-z)+i\pi)]}\right)}{4z^3}=\frac{e^{[(\alpha)(\log(-\beta)+i\pi)]}}{4\beta^3}=\frac{\beta e^{[(\alpha)(\log(-\beta)+i\pi)]}}{-4}}$$
$$\boldsymbol{\because \beta^4=-1}$$

Sum of residues at all poles is

$$\boldsymbol{\sum_\beta R_\beta=\frac{-1}{4}\Big[e^{\pi i/4}e^{\alpha[\log(-e^{\pi i/4})+\pi i]}+e^{3\pi i/4}e^{\alpha[\log(-e^{3\pi i/4})+\pi i]}+e^{5\pi i/4}e^{\alpha[\log(-e^{5\pi i/4})+\pi i]}}$$
$$\boldsymbol{+e^{7\pi i/4}e^{\alpha[\log(-e^{7\pi i/4})+\pi i]}\Big]}$$

Now, polar forms of $-e^{\frac{\pi i}{4}}, -e^{\frac{3\pi i}{4}}, -e^{\frac{5\pi i}{4}}$ and $-e^{\frac{7\pi i}{4}}$, respectively, are given by

$$-e^{\frac{\pi i}{4}} = \frac{-1}{\sqrt{2}}(1+i) = e^{\frac{-3\pi i}{4}}, \qquad -e^{\frac{3\pi i}{4}} = \frac{-1}{\sqrt{2}}(-1+i) = e^{\frac{-\pi i}{4}},$$

$$-e^{\frac{5\pi i}{4}} = \frac{-1}{\sqrt{2}}(-1-i) = e^{\frac{\pi i}{4}}, \qquad -e^{\frac{7\pi i}{4}} = \frac{-1}{\sqrt{2}}(1-i) = e^{\frac{3\pi i}{4}}$$

$$\Rightarrow \sum_{\beta} R_\beta = \frac{-1}{4}\Big[e^{\pi i/4}e^{\alpha[\log(e^{-3\pi i/4})+\pi i]} + e^{3\pi i/4}e^{\alpha[\log(e^{-\pi i/4})+\pi i]} + e^{5\pi i/4}e^{\alpha[\log(e^{\pi i/4})+\pi i]} + e^{7\pi i/4}e^{\alpha[\log(e^{3\pi i/4})+\pi i]}\Big]$$

$$= \frac{-1}{4}\left[e^{\frac{\pi i}{4}}e^{\alpha\left(\frac{-3\pi i}{4}+\pi i\right)} + e^{\frac{3\pi i}{4}}e^{\alpha\left(\frac{-\pi i}{4}+\pi i\right)} + e^{\frac{5\pi i}{4}}e^{\alpha\left(\frac{\pi i}{4}+\pi i\right)} + e^{\frac{7\pi i}{4}}e^{\alpha\left(\frac{3\pi i}{4}+\pi i\right)}\right]$$

$$= \frac{-1}{4}\left[\frac{1}{\sqrt{2}}(1+i)e^{\alpha\left(\frac{\pi i}{4}\right)} + \frac{1}{\sqrt{2}}(i-1)e^{\alpha\left(\frac{3\pi i}{4}\right)} + \frac{1}{\sqrt{2}}(-1-i)e^{\alpha\left(\frac{5\pi i}{4}\right)} + \frac{1}{\sqrt{2}}(1-i)e^{\alpha\left(\frac{7\pi i}{4}\right)}\right]$$

$$= \frac{-1}{4\sqrt{2}}\left[(1+i)e^{\frac{\alpha\pi i}{4}}\left(1-e^{\alpha\pi i}\right) + (-1+i)e^{\frac{3\alpha\pi i}{4}}\left(1-e^{\alpha\pi i}\right)\right]$$

$$= \frac{-1}{4\sqrt{2}}\left(1-e^{\alpha\pi i}\right)\left((1+i)e^{\frac{\alpha\pi i}{4}} + (-1+i)e^{\frac{3\alpha\pi i}{4}}\right)$$

$$= \frac{-1}{4\sqrt{2}}e^{\frac{\alpha\pi i}{2}}\left(e^{-\frac{\alpha\pi i}{2}} - e^{\frac{\alpha\pi i}{2}}\right)\left[\left(e^{\frac{\alpha\pi i}{4}} - e^{\frac{3\alpha\pi i}{4}}\right) + i\left(e^{\frac{\alpha\pi i}{4}} + e^{\frac{3\alpha\pi i}{4}}\right)\right]$$

$$= \frac{-1}{4\sqrt{2}}e^{\frac{\alpha\pi i}{2}}\left(-2i\sin\left(\frac{\alpha\pi}{2}\right)\right)e^{\frac{\alpha\pi i}{4}}e^{\frac{\alpha\pi i}{4}}\left[\left(e^{-\frac{\alpha\pi i}{4}} - e^{\frac{\alpha\pi i}{4}}\right) + i\left(e^{-\frac{\alpha\pi i}{4}} + e^{\frac{\alpha\pi i}{4}}\right)\right]$$

$$= \frac{1}{2\sqrt{2}}\left(i\sin\left(\frac{\alpha\pi}{2}\right)\right)e^{\alpha\pi i}\left[2i\sin\left(\frac{\alpha\pi}{4}\right) - 2i\cos\left(\frac{\alpha\pi}{4}\right)\right]$$

$$= i^2 e^{\alpha\pi i}\sin\left(\frac{\alpha\pi}{2}\right)\left[\frac{1}{\sqrt{2}}\cos\left(\frac{\alpha\pi}{4}\right) - \frac{1}{\sqrt{2}}\sin\left(\frac{\alpha\pi}{4}\right)\right]$$

$$= -e^{\alpha\pi i}\sin\left(\frac{\alpha\pi}{2}\right)\left[\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\alpha\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\alpha\pi}{4}\right)\right] = -e^{\alpha\pi i}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)$$

$$I = \frac{-\pi}{\sin((\alpha+1)\pi)}e^{-(\alpha+1)\pi i}\sum_{\beta} R(f,\beta)$$

$\because \sin((\alpha+1)\pi) = \sin(\alpha\pi+\pi) = -\sin(\alpha\pi)$ and $e^{-(\alpha+1)\pi i} = e^{-\alpha\pi i}e^{-\pi i} = e^{-\alpha\pi i}(-1) = -e^{-\alpha\pi i}$

$$I = \frac{-\pi(-e^{-\alpha\pi i})}{-\sin(\alpha\pi)}\left[-e^{\alpha\pi i}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)\right] = \frac{\pi}{\sin(\alpha\pi)}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{2\cos\left(\frac{\alpha\pi}{2}\right)}\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)$$

$$I = \frac{\pi}{2\sin\left(\frac{\alpha\pi}{2}+\frac{\pi}{2}\right)}\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right) \qquad \because \sin\left(\frac{\alpha\pi}{2}+\frac{\pi}{2}\right) = \cos\left(\frac{\alpha\pi}{2}\right)$$

$$= \frac{\pi}{4\sin\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)}\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{4\sin\left(\frac{\pi(\alpha+1)}{4}\right)}$$

Hence proved.

Question 4: Prove that $\displaystyle\int_0^\infty \frac{x^b dx}{1+x^2} = \frac{\pi}{2}\sec\left(\frac{b\pi}{2}\right), \quad -1 < b < 1$

Solution: Poles of $f(z)=\frac{1}{1+z^2}$ are $z=\pm i$ (both of them have order one), which do not lie on positive part of the real axis.

Let, $$I=\int_0^{\infty}\frac{x^b\,dx}{1+x^2}=\int_C\frac{e^{b[\log(-z)+\pi i]}dz}{1+z^2} \quad \text{and} \quad \varphi(z)=\frac{e^{b[\log(-z)+\pi i]}}{1+z^2}$$

$$R_1(f,i)=\lim_{z\to i}\left[\frac{(z-i)e^{b[\log(-z)+\pi i]}}{(z-i)(z+i)}\right]=\lim_{z\to i}\left[\frac{e^{b[\log(-z)+\pi i]}}{(z+i)}\right]=\frac{e^{b[\log(-i)+\pi i]}}{2i}=\frac{e^{b\left[\log\left(e^{\frac{-\pi i}{2}}\right)+\pi i\right]}}{2i}=\frac{e^{b\left[\frac{-\pi i}{2}+\pi i\right]}}{2i}$$
$$=\frac{e^{\frac{b\pi i}{2}}}{2i}$$

$$R_1(f,-i)=\lim_{z\to -i}\left[\frac{(z+i)e^{b[\log(-z)+\pi i]}}{(z-i)(z+i)}\right]=\lim_{z\to -i}\left[\frac{e^{b[\log(-z)+\pi i]}}{(z-i)}\right]=\frac{e^{b[\log(i)+\pi i]}}{2i}=\frac{e^{b\left[\log\left(e^{\frac{\pi i}{2}}\right)+\pi i\right]}}{2i}=\frac{e^{b\left[\frac{\pi i}{2}+\pi i\right]}}{2i}$$
$$=\frac{e^{\frac{3b\pi i}{2}}}{2i}$$

$$\sum_{i=1}^{2}R_i=\left[\frac{e^{\frac{b\pi i}{2}}}{2i}-\frac{e^{\frac{3b\pi i}{2}}}{2i}\right]=\frac{1}{i}e^{b\pi i}\left[\frac{e^{\frac{-b\pi i}{2}}-e^{\frac{b\pi i}{2}}}{2}\right]=\frac{1}{i}e^{b\pi i}\left(-i\sin\left(\frac{b\pi}{2}\right)\right)=-e^{b\pi i}\sin\left(\frac{b\pi}{2}\right)$$

$$I=\frac{-\pi}{\sin((b+1)\pi)}e^{-(b+1)\pi i}\sum_{i=1}^{2}R_i$$

$$\because \sin((b+1)\pi)=\sin(b\pi+\pi)=-\sin(b\pi) \quad \text{and} \quad e^{-(b+1)\pi i}=e^{-b\pi i}e^{-\pi i}=e^{-b\pi i}(-1)=-e^{-\alpha\pi i}$$

$$\Rightarrow I=\frac{-\pi}{-\sin(b\pi)}\left(-e^{-b\pi i}\right)\left[-e^{b\pi i}\sin\left(\frac{b\pi}{2}\right)\right]=\frac{\pi}{\sin(b\pi)}\sin\left(\frac{b\pi}{2}\right)=\frac{\pi}{2\sin\left(\frac{b\pi}{2}\right)\cos\left(\frac{b\pi}{2}\right)}\sin\left(\frac{b\pi}{2}\right)=\frac{\pi}{2\cos\left(\frac{b\pi}{2}\right)}$$
$$=\frac{\pi}{2}\sec\left(\frac{b\pi}{2}\right)$$ **Hence proved.**

Question 5: prove that $\int_0^{\infty}\frac{\sqrt{x}}{x^2+x+1}dx=\frac{\pi}{\sqrt{3}}$

Solution: Poles of $f(z)=\frac{1}{z^2+z+1}$ are $z=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}\,i}{2}$ (both of them have order one), which do not lie on positive part of the real axis.

Let $$I=\int_0^{\infty}\frac{\sqrt{x}}{x^2+x+1}dx=\int_C\frac{e^{\frac{1}{2}[\log(-z)+\pi i]}\,dz}{z^2+z+1} \quad \text{and} \quad \varphi(z)=\frac{e^{\frac{1}{2}[\log(-z)+\pi i]}}{z^2+z+1}$$

$$R_1\left(\varphi,\frac{-1+\sqrt{3}i}{2}\right)=\lim_{z\to\frac{-1+\sqrt{3}i}{2}}\frac{e^{\frac{1}{2}[\log(-z)+\pi i]}\left(z+\frac{1-\sqrt{3}i}{2}\right)}{\left(z+\frac{1-\sqrt{3}i}{2}\right)\left(z+\frac{1+\sqrt{3}i}{2}\right)}=\frac{e^{\frac{1}{2}\left[\log\left(\frac{1-\sqrt{3}i}{2}\right)+\pi i\right]}}{2\left(\frac{\sqrt{3}i}{2}\right)}=\frac{e^{\frac{1}{2}\left[\log\left(e^{\frac{-\pi i}{3}}\right)+\pi i\right]}}{\sqrt{3}i}$$
$$=\frac{e^{\frac{1}{2}\left[\frac{-\pi i}{3}+\pi i\right]}}{\sqrt{3}i}=\frac{e^{\frac{1}{2}\left[\frac{2\pi i}{3}\right]}}{\sqrt{3}i}=\frac{e^{\frac{\pi i}{3}}}{\sqrt{3}i} \qquad \because \frac{1-\sqrt{3}i}{2}=e^{\frac{-\pi i}{3}} \text{ (in polar form)}$$

$$R_2\left(\varphi,\frac{-1-\sqrt{3}i}{2}\right)=\lim_{z\to\frac{-1-\sqrt{3}i}{2}}\frac{\mathrm{e}^{\frac{1}{2}[\log(-z)+\pi i]}\left(z+\frac{1+\sqrt{3}i}{2}\right)}{\left(z+\frac{1-\sqrt{3}i}{2}\right)\left(z+\frac{1+\sqrt{3}i}{2}\right)}=\frac{\mathrm{e}^{\frac{1}{2}\left[\log\left(\frac{1+\sqrt{3}i}{2}\right)+\pi i\right]}}{-2\left(\frac{\sqrt{3}i}{2}\right)}=\frac{\mathrm{e}^{\frac{1}{2}\left[\log\left(\mathrm{e}^{\frac{\pi i}{3}}\right)+\pi i\right]}}{-\sqrt{3}i}=\frac{\mathrm{e}^{\frac{1}{2}\left[\frac{\pi i}{3}+\pi i\right]}}{-\sqrt{3}i}$$

$$=\frac{\mathrm{e}^{\frac{1}{2}\left[\frac{4\pi i}{3}\right]}}{-\sqrt{3}i}=\frac{\mathrm{e}^{\frac{2\pi i}{3}}}{-\sqrt{3}i}\qquad\because\frac{1+\sqrt{3}i}{2}=\mathrm{e}^{\frac{\pi i}{3}}\text{ (in polar form)}$$

$$I=\frac{-\pi}{\sin\left(\frac{3\pi}{2}\right)}\mathrm{e}^{-\frac{3\pi i}{2}}\sum_{i=1}^{2}R_i=\frac{-\pi}{-1}\mathrm{e}^{-\frac{3\pi i}{2}}\left[\frac{\mathrm{e}^{\frac{\pi i}{3}}}{\sqrt{3}i}-\frac{\mathrm{e}^{\frac{2\pi i}{3}}}{\sqrt{3}i}\right]$$

$$=\pi\left[\cos\left(-\frac{3\pi}{2}\right)+i\sin\left(-\frac{3\pi}{2}\right)\right]\left[\frac{\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)-\cos\left(\frac{2\pi}{3}\right)-i\sin\left(\frac{2\pi}{3}\right)}{\sqrt{3}i}\right]$$

$$=\pi\frac{i}{\sqrt{3}i}\left[\frac{1}{2}+i\frac{\sqrt{3}}{2}+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right]=\frac{\pi}{\sqrt{3}}$$

Hence proved.

Question 6: **Prove that** $\displaystyle\int_0^\infty\frac{x^{\alpha-1}}{x^2+x+1}\,dx=\frac{2\pi}{3}\frac{\cos\left[\frac{2\alpha\pi+\pi}{6}\right]}{\sin(\pi\alpha)}$, **where** $0<\alpha<2$

Solution: Poles of $f(z)=\frac{1}{z^2+z+1}$ are $z=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}\,i}{2}$ (both of them have order one), which do not lie on positive part of the real axis.

Let, $\displaystyle I=\int_0^\infty\frac{x^{\alpha-1}}{x^2+x+1}\mathrm{d}x=\int_C\frac{\mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi])]}}{z^2+z+1}dz$ and $\displaystyle\varphi(z)=\frac{\mathrm{e}^{[(\alpha-1)[\log(-z)+i\pi]]}}{z^2+z+1}$

$$R_1\left(\varphi,\frac{-1+\sqrt{3}i}{2}\right)=\lim_{z\to\frac{-1+\sqrt{3}i}{2}}\frac{\left(\mathrm{e}^{(\alpha-1)(\log(-z)+i\pi)}\right)\left(z+\frac{1-\sqrt{3}i}{2}\right)}{\left(z+\frac{1-\sqrt{3}i}{2}\right)\left(z-\left(\frac{-1-\sqrt{3}i}{2}\right)\right)}=\frac{\mathrm{e}^{(\alpha-1)\left[\log\left(\frac{1-\sqrt{3}i}{2}\right)+i\pi\right]}}{\frac{-1+\sqrt{3}i}{2}-\left(\frac{-1-\sqrt{3}i}{2}\right)}$$

$$R_1\left(\varphi,\frac{-1+\sqrt{3}i}{2}\right)=\frac{\mathrm{e}^{(\alpha-1)\left[\log\left(\mathrm{e}^{\frac{-\pi i}{3}}\right)+i\pi\right]}}{\sqrt{3}i}=\frac{\mathrm{e}^{(\alpha-1)\left[\frac{-\pi i}{3}+\pi i\right]}}{\sqrt{3}i}=\frac{\mathrm{e}^{\frac{2\pi(\alpha-1)i}{3}}}{\sqrt{3}i}\qquad\because\frac{1-\sqrt{3}i}{2}=\mathrm{e}^{\frac{-\pi i}{3}}\text{ (in polar form)}$$

$$R_2\left(\varphi,\frac{-1-\sqrt{3}i}{2}\right)=\lim_{z\to\frac{-1-\sqrt{3}i}{2}}\frac{\mathrm{e}^{(\alpha-1)[\log(-z)+i\pi]}\left(z+\frac{1+\sqrt{3}i}{2}\right)}{\left(z+\frac{1+\sqrt{3}i}{2}\right)\left(z-\left(\frac{-1+\sqrt{3}i}{2}\right)\right)}=\frac{\mathrm{e}^{(\alpha-1)\left[\log\left(\frac{1+\sqrt{3}i}{2}\right)+i\pi\right]}}{\left(\frac{-1-\sqrt{3}i}{2}-\left(\frac{-1+\sqrt{3}i}{2}\right)\right)}$$

$$=\frac{\mathrm{e}^{(\alpha-1)\left[\log\left(\mathrm{e}^{\frac{\pi i}{3}}\right)+i\pi\right]}}{-\sqrt{3}i}=\frac{\mathrm{e}^{(\alpha-1)\left[\frac{\pi i}{3}+i\pi\right]}}{-\sqrt{3}i}=\frac{\mathrm{e}^{\frac{4\pi(\alpha-1)i}{3}}}{-\sqrt{3}i}\qquad\because\frac{1+\sqrt{3}i}{2}=\mathrm{e}^{\frac{\pi i}{3}}\text{ (in polar form)}$$

$$I=\frac{-\pi}{\sin(\alpha\pi)}\mathrm{e}^{-\alpha\pi i}\cdot\sum_{i=1}^{2}R_i=\frac{-\pi}{\sin(\alpha\pi)}\mathrm{e}^{-\alpha\pi i}\left[\frac{\mathrm{e}^{\frac{2\pi(\alpha-1)i}{3}}}{\sqrt{3}i}-\frac{\mathrm{e}^{\frac{4\pi(\alpha-1)i}{3}}}{\sqrt{3}i}\right]$$

$$=\frac{-\pi}{\sqrt{3}i\sin(\alpha\pi)}\mathrm{e}^{-\alpha\pi i}\mathrm{e}^{\frac{2\pi(\alpha-1)i}{3}}\left[1-\mathrm{e}^{\frac{2\pi(\alpha-1)i}{3}}\right]$$

$$I = \frac{-\pi}{\sqrt{3}i\,\sin(\alpha\pi)}\,\mathrm{e}^{-\alpha\pi i+\frac{2\pi(\alpha-1)i}{3}}\,\mathrm{e}^{\frac{2\pi(\alpha-1)i}{6}}\left[\mathrm{e}^{-\frac{\pi(\alpha-1)i}{3}} - \mathrm{e}^{\frac{2\pi(\alpha-1)i}{3}}\right]$$

$$= \frac{-\pi}{\sqrt{3}i\,\sin(\alpha\pi)}\,\mathrm{e}^{\frac{-\alpha\pi i-2\pi i}{3}}\,\mathrm{e}^{\frac{\pi(\alpha-1)i}{3}}\left[-2i\sin\left(\frac{\pi(\alpha-1)}{3}\right)\right]$$

$$= \frac{2\pi}{\sqrt{3}\sin(\alpha\pi)}\,\mathrm{e}^{\frac{-\alpha\pi i-2\pi i}{3}+\frac{\pi(\alpha-1)i}{3}}\sin\left(\frac{\pi(\alpha-1)}{3}\right) = \frac{2\pi}{\sqrt{3}\sin(\alpha\pi)}\,\mathrm{e}^{-\pi i}\sin\left(\frac{\pi(\alpha-1)}{3}\right)$$

$$= \frac{2\pi}{\sqrt{3}\sin(\alpha\pi)}\cos\left(\frac{\pi(\alpha-1)}{3}+\frac{\pi}{2}\right) \qquad \because \sin\theta = \cos\left(\theta+\frac{\pi}{2}\right)$$

$$= \frac{2\pi}{\sqrt{3}\,\sin(\alpha\pi)}\cos\left(\frac{2\pi\alpha-2\pi+3\pi}{6}\right) = \frac{2\pi}{3}\,\frac{\cos\left[\frac{2\alpha\pi+\pi}{6}\right]}{\sin(\pi\alpha)}$$ **Hence proved**

Question 7: **Prove that** $\displaystyle\int_0^\infty \frac{x^\alpha dx}{(1+x^2)^2} = \frac{\pi(1-\alpha)}{4\cos\left(\frac{\pi\alpha}{2}\right)}, \qquad -1<\alpha<3$

Solution: Poles of $f(z) = \frac{1}{(1+z^2)^2}$ are $z = \pm i$ (both of them have order two), which do not lie on positive part of the real axis.

Let, $\displaystyle I = \int_0^\infty \frac{x^\alpha dx}{(1+x^2)^2} = \int_C \frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}dz}{(1+z^2)^2}$ and $\displaystyle\varphi(z) = \frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{(1+z^2)^2}$

$$R_1(\varphi,i) = \lim_{z\to i}\frac{d}{dz}\left[\frac{(z-i)^2\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{(z-i)^2(z+i)^2}\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{(z+i)^2}\right]$$

$$R_1(\varphi,i) = \lim_{z\to i}\frac{(z+i)^2\mathrm{e}^{\alpha[\log(-z)+\pi i]}\frac{\alpha}{z} - \mathrm{e}^{\alpha[\log(-z)+\pi i]}\cdot 2(z+i)}{(z+i)^4} = \frac{(2i)^2\mathrm{e}^{\alpha[\log(-i)+\pi i]}\frac{\alpha}{i} - \mathrm{e}^{\alpha[\log(-i)+\pi i]}\cdot 2(2i)}{(2i)^4}$$

$$= \frac{4\alpha i\mathrm{e}^{\alpha\left[\log\left(\mathrm{e}^{\frac{-\pi i}{2}}\right)+\pi i\right]} - 4i\mathrm{e}^{\alpha\left[\log\left(\mathrm{e}^{\frac{-\pi i}{2}}\right)+\pi i\right]}}{16} = \frac{4\alpha i\mathrm{e}^{\alpha\left[\frac{-\pi i}{2}+\pi i\right]} - 4i\mathrm{e}^{\alpha\left[\frac{-\pi i}{2}+\pi i\right]}}{16} = \frac{4\alpha i\mathrm{e}^{\frac{\alpha\pi i}{2}} - 4i\mathrm{e}^{\frac{\alpha\pi i}{2}}}{16}$$

$$R_1(\varphi,i) = \frac{4i\mathrm{e}^{\frac{\alpha\pi i}{2}}(\alpha-1)}{16} = \frac{i\mathrm{e}^{\frac{\alpha\pi i}{2}}(\alpha-1)}{4}$$

$$R_2(\varphi,-i) = \lim_{z\to -i}\frac{d}{dz}\left[\frac{(z+i)^2\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{(z-i)^2(z+i)^2}\right] = \lim_{z\to -i}\frac{d}{dz}\left[\frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{(z-i)^2}\right]$$

$$= \lim_{z\to -i}\left[\frac{(z-i)^2\mathrm{e}^{\alpha[\log(-z)+\pi i]}\frac{\alpha}{z} - \mathrm{e}^{\alpha[\log(-z)+\pi i]}2(z-i)}{(z-i)^4}\right]$$

$$= \frac{(-2i)^2\mathrm{e}^{\alpha[\log(i)+\pi i]}\frac{\alpha}{-i} - \mathrm{e}^{\alpha[\log(i)+\pi i]}2(-2i)}{(-2i)^4} = \frac{-4\alpha i\mathrm{e}^{\alpha\left[\log\left(\mathrm{e}^{\frac{\pi i}{2}}\right)+\pi i\right]} + 4i\mathrm{e}^{\alpha\left[\log\left(\mathrm{e}^{\frac{\pi i}{2}}\right)+\pi i\right]}}{16}$$

$$= \frac{-4\alpha i\mathrm{e}^{\alpha\left[\frac{\pi i}{2}+\pi i\right]} + 4i\mathrm{e}^{\alpha\left[\frac{\pi i}{2}+\pi i\right]}}{16} = \frac{-4\alpha i\mathrm{e}^{\frac{3\alpha\pi i}{2}} + 4i\mathrm{e}^{\frac{3\alpha\pi i}{2}}}{16} = \frac{-4i\mathrm{e}^{\frac{\alpha\pi i}{2}}(\alpha-1)}{16} = \frac{-i\mathrm{e}^{\frac{3\alpha\pi i}{2}}(\alpha-1)}{4}$$

$$I = \frac{-\pi}{\sin(\alpha+1)\pi}\mathrm{e}^{-(\alpha+1)\pi i}\cdot\sum_{i=1}^{2}R_i$$

$\because \sin\big((\alpha+1)\pi\big) = \sin(\alpha\pi+\pi) = -\sin(\alpha\pi)$ and $\mathrm{e}^{-(\alpha+1)\pi i} = \mathrm{e}^{-\alpha\pi i}\mathrm{e}^{-\pi i} = \mathrm{e}^{-\alpha\pi i}(-1) = -\mathrm{e}^{-\alpha\pi i}$

$$\Rightarrow I=\frac{-\pi}{-\sin(\alpha\pi)}\left(-e^{-\alpha\pi i}\right)\left[\frac{ie^{\frac{\alpha\pi i}{2}}(\alpha-1)}{4}-\frac{ie^{\frac{3\alpha\pi i}{2}}(\alpha-1)}{4}\right]=\frac{-\pi}{\sin(\alpha\pi)}e^{-\alpha\pi i}e^{\alpha\pi i}i(\alpha-1)\left[\frac{e^{\frac{-\alpha\pi i}{2}}-e^{\frac{\alpha\pi i}{2}}}{4}\right]$$

$$=\frac{-\pi}{\sin(\alpha\pi)}i(\alpha-1)\left[\frac{-2i\sin\left(\frac{\alpha\pi}{2}\right)}{4}\right]=\frac{-\pi(\alpha-1)}{\sin(\alpha\pi)}\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{2}=\frac{\pi(1-\alpha)}{2\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)}\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{2}$$

$$=\frac{\pi(1-\alpha)}{4\cos\left(\frac{\alpha\pi}{2}\right)}$$ **Hence proved.**

Question 8: **Question 1 of book is repeated here** (**see the book**). **No need to repeat same question.**

Questio 9: **Prove that** $\int_0^\infty \frac{x^\alpha dx}{1+2x\cos\theta+x^2}=\frac{\pi}{\sin(\alpha\pi)}\frac{\sin(\alpha\theta)}{\sin\theta}$ $\quad 0<\alpha<1, -\pi<\theta<\pi$

Solution: Poles of $f(z)=\frac{1}{1+2z\cos\theta+z^2}$ are $z=\frac{-2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}=-\cos\theta\pm i\sin\theta=-e^{-i\theta},-e^{i\theta}$ (both of them have order one), which do not lie on positive part of the real axis.

Let, $$I=\int_0^\infty \frac{x^\alpha dx}{1+2x\,\mathrm{cso}\theta+x^2}=\int_C \frac{e^{\alpha[\log(-z)+\pi i]}dz}{1+2z\cos\theta+z^2}$$ and $$\varphi(z)=\frac{e^{\alpha[\log(-z)+\pi i]}}{1+2z\cos\theta+z^2}$$

$$R_1(f,-e^{-i\theta})=\lim_{z\to-e^{-i\theta}}\frac{e^{\alpha[\log(-z)+\pi i]}}{(z+e^{i\theta})(z+e^{-i\theta})}=\lim_{z\to-e^{-i\theta}}\frac{e^{\alpha[\log(-z)+\pi i]}}{z+e^{i\theta}}=\frac{e^{\alpha[\log(e^{-i\theta})+\pi i]}}{-e^{-i\theta}+e^{i\theta}}=\frac{e^{\alpha[-i\theta+\pi i]}}{2i\sin\theta}=\frac{e^{\alpha i[\pi-\theta]}}{2i\sin\theta}$$

$$R_2(f,-e^{i\theta})=\lim_{z\to-e^{i\theta}}\frac{(z+e^{i\theta})e^{\alpha[\log(-z)+\pi i]}}{(z+e^{i\theta})(z+e^{-i\theta})}=\lim_{z\to-e^{i\theta}}\frac{e^{\alpha[\log(-z)+\pi i]}}{z+e^{-i\theta}}=\frac{e^{\alpha[\log(e^{i\theta})+\pi i]}}{-e^{i\theta}+e^{-i\theta}}=\frac{e^{\alpha[i\theta+\pi i]}}{-2i\sin\theta}=\frac{-e^{\alpha i[\pi+\theta]}}{2i\sin\theta}$$

$$I=\frac{-\pi}{\sin(\alpha+1)\pi}e^{-(\alpha+1)\pi i}\sum_{i=1}^{2}R_i$$

$\because \sin((\alpha+1)\pi)=\sin(\alpha\pi+\pi)=-\sin(\alpha\pi)$ and $e^{-(\alpha+1)\pi i}=e^{-\alpha\pi i}e^{-\pi i}=e^{-\alpha\pi i}(-1)=-e^{-\alpha\pi i}$

$$\Rightarrow I=\frac{-\pi}{-\sin\alpha\pi}\left(-e^{-\alpha\pi i}\right)\left[\frac{e^{\alpha i[\pi-\theta]}}{2i\sin\theta}-\frac{e^{\alpha i[\pi+\theta]}}{2i\sin\theta}\right]=\frac{-\pi}{-2i\sin(\alpha\pi)\sin\theta}\left(-e^{-\alpha\pi i}\right)e^{\alpha\pi i}\left[e^{-\alpha i\theta}-e^{\alpha i\theta}\right]$$

$$=\frac{-\pi}{2i\sin(\alpha\pi)\sin\theta}[-2i\sin(\alpha\theta)]=\frac{\pi}{\sin(\alpha\pi)}\frac{\sin(\alpha\theta)}{\sin\theta}$$ **Hence proved.**

Question 10: **Prove that** $\int_0^\infty \frac{x^\alpha}{x^4+1}dx=\frac{\pi}{4}\,\mathrm{coses}\left(\frac{\pi(\alpha+1)}{4}\right)$, **where** $-1<\alpha<3$

Solution: From result of question 3, the result of question 10 is obvious.

Question 11: **prove that** $\int_0^\infty \frac{x^{1/3}dx}{(x+a)(x+b)}=\frac{2\pi}{\sqrt{3}}\left[\frac{a^{1/3}-b^{1/3}}{a-b}\right]$, $a>0$, $b>0$, $a\neq b$

Solution: Poles of $f(z)=\frac{1}{(z+a)(z+b)}$ are $z=-a,-b$ (both of them have order one), which do not lie on positive part of the real axis due to the fact that $a>0,\ b>0$.

Let, $$I=\int_0^\infty \frac{x^{1/3}dx}{(x+a)(x+b)}=\int_C \frac{e^{\left[\left(\frac{1}{3}\right)(\log(-z)+i\pi)\right]}}{(z+a)(z+b)}dz$$ and $$\varphi(z)=\frac{e^{\left[\left(\frac{1}{3}\right)(\log(-z)+i\pi)\right]}}{(z+a)(z+b)}$$

$$R_1(\varphi,-a)=\lim_{z\to -a}\frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(-z)+i\pi)\right]}(z+a)}{(z+a)(z+b)}=\frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(a)+i\pi)\right]}}{-a+b}=\frac{\mathrm{e}^{\frac{1}{3}(\log(a)+i\pi)}}{b-a}=\frac{\mathrm{e}^{\frac{1}{3}\log a}\mathrm{e}^{\frac{i\pi}{3}}}{b-a}=\frac{-a^{\frac{1}{3}}\mathrm{e}^{\frac{i\pi}{3}}}{a-b}$$

$$R_2(\varphi,-b)=\lim_{z\to -b}\frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(-z)+i\pi)\right]}(z+b)}{(z+a)(z+b)}=\frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(b)+i\pi)\right]}}{a-b}=\frac{b^{\frac{1}{3}}\mathrm{e}^{\frac{i\pi}{3}}}{a-b}$$

$$I=\frac{-\pi}{\sin\left(\left(\frac{1}{3}+1\right)\pi\right)}\mathrm{e}^{-\left(\frac{1}{3}+1\right)\pi i}\sum_{i=1}^{2}R_i=\frac{-\pi}{\sin\left(\frac{1}{3}\pi+\pi\right)}\mathrm{e}^{-\frac{\pi i}{3}}\mathrm{e}^{-\pi i}\left[\frac{b^{\frac{1}{3}}\mathrm{e}^{\frac{i\pi}{3}}}{a-b}-\frac{a^{\frac{1}{3}}\mathrm{e}^{\frac{i\pi}{3}}}{a-b}\right]$$

$$=\frac{-\pi}{\sin\left(\frac{\pi}{3}+\pi\right)}\mathrm{e}^{-\frac{\pi i}{3}}\mathrm{e}^{-\pi i}\left[\frac{\mathrm{e}^{\frac{i\pi}{3}}\left(b^{\frac{1}{3}}-a^{\frac{1}{3}}\right)}{a-b}\right]=\frac{-\pi}{-\sin\left(\frac{\pi}{3}\right)}\mathrm{e}^{-\frac{\pi i}{3}}(-1)\left[\frac{\mathrm{e}^{\frac{i\pi}{3}}\left(b^{\frac{1}{3}}-a^{\frac{1}{3}}\right)}{a-b}\right]$$

$$=\frac{-\pi}{\frac{\sqrt{3}}{2}}\mathrm{e}^{-\frac{\pi i}{3}}\left[\frac{\mathrm{e}^{\frac{i\pi}{3}}\left(b^{\frac{1}{3}}-a^{\frac{1}{3}}\right)}{a-b}\right]=\frac{2\pi}{\sqrt{3}}\left[\frac{a^{1/3}-b^{1/3}}{a-b}\right]$$ **Hence proved**.

Question 12: **Prove that** $\displaystyle\int_0^\infty \frac{x^\alpha \mathrm{d}x}{(x^2-b^2)(x^2-c^2)}=\frac{\pi}{2}\left[\frac{b^{\alpha-1}-c^{\alpha-1}}{b^2-c^2}\right][\operatorname{cosec}(\pi\alpha)-\cot(\pi\alpha)],\quad a>0,\ b>0$

Note: This question is not in the book

Solution: Poles of $f(z)=\frac{1}{(z^2-b^2)(z^2-c^2)}$ are $z=\pm b,\pm c$ (all of them have order one). $z=b$ and $z=c$ do lie on positive part of the real axis, while $z=-b$ and $z=-c$ do not lie on the positive part of the real axis due to the fact that $a>0,\ b>0$.

For the poles $z=b,c$, we let, $I=\displaystyle\int_0^\infty \frac{x^\alpha \mathrm{d}x}{(x^2-b^2)(x^2-c^2)}=\int_C \frac{e^{\alpha\log z}}{(z^2-b^2)(z^2-c^2)}\mathrm{d}z,\ \varphi_1(z)$

$$=\frac{e^{\alpha\log z}}{(z^2-b^2)(z^2-c^2)}$$

For the poles $z=-b,-c$, we let,

$$I=\int_0^\infty \frac{x^\alpha \mathrm{d}x}{(x^2-b^2)(x^2-c^2)}=\int_C \frac{e^{\alpha[\log(-z)+\pi i]}}{(z^2-b^2)(z^2-c^2)}\mathrm{d}z,\ \varphi_2(z)=\frac{e^{\alpha[\log(-z)+\pi i]}}{(z^2-b^2)(z^2-c^2)}$$

$$R_1(\varphi_1,b)=\lim_{z\to b}\frac{(z-b)\mathrm{e}^{\alpha\log z}}{(z-b)(z+b)(z^2-c^2)}=\lim_{z\to b}\frac{\mathrm{e}^{\alpha\log z}}{(z+b)(z^2-c^2)}=\frac{\mathrm{e}^{\alpha\log b}}{(2b)(b^2-c^2)}=\frac{\mathrm{e}^{\log b^\alpha}}{(2b)(b^2-c^2)}$$
$$=\frac{b^\alpha}{2b(b^2-c^2)}=\frac{b^{\alpha-1}}{2(b^2-c^2)}$$

$$R_2(\varphi_1,c)=\lim_{z\to c}\frac{(z-c)\mathrm{e}^{\alpha\log z}}{(z-c)(z+c)(z^2-b^2)}=\lim_{z\to c}\frac{\mathrm{e}^{\alpha\log z}}{(z+c)(z^2-b^2)}=\frac{\mathrm{e}^{\alpha\log c}}{(2c)(c^2-b^2)}=\frac{-\mathrm{e}^{\log c^\alpha}}{(2b)(b^2-c^2)}$$
$$=\frac{-c^\alpha}{2c(b^2-c^2)}=\frac{-c^{\alpha-1}}{2(b^2-c^2)}$$

$$R_3(\varphi_2,-b)=\lim_{z\to -b}\frac{(z+b)e^{\alpha[\log(-z)+\pi i]}}{(z-b)(z+b)(z^2-c^2)}=\lim_{z\to -b}\frac{e^{\alpha[\log(-z)+\pi i]}}{(z-b)(z^2-c^2)}=\frac{e^{\alpha[\log b+\pi i]}}{(-2b)(b^2-c^2)}=\frac{e^{\alpha\log b}e^{\alpha\pi i}}{(-2b)(b^2-c^2)}$$

$$=\frac{e^{\log b^{\alpha}}e^{\alpha\pi i}}{(-2b)(b^2-c^2)}=\frac{-b^{\alpha}e^{\alpha\pi i}}{2b(b^2-c^2)}=\frac{-b^{\alpha-1}e^{\alpha\pi i}}{2(b^2-c^2)}$$

$$R_4(\varphi_2,-c)=\lim_{z\to -c}\frac{(z+c)e^{\alpha[\log(-z)+\pi i]}}{(z-c)(z+c)(z^2-b^2)}=\lim_{z\to -c}\frac{e^{\alpha[\log(-z)+\pi i]}}{(z-c)(z^2-b^2)}=\frac{e^{\alpha[\log c+\pi i]}}{(-2c)(c^2-b^2)}=\frac{e^{\alpha\log c}e^{\alpha\pi i}}{(2c)(b^2-c^2)}$$

$$=\frac{e^{\log c^{\alpha}}e^{\alpha\pi i}}{(2c)(b^2-c^2)}=\frac{c^{\alpha}e^{\alpha\pi i}}{2c(b^2-c^2)}=\frac{c^{\alpha-1}e^{\alpha\pi i}}{2(b^2-c^2)}$$

$$I=-\pi\cot\big((\alpha+1)\pi\big)[R_1+R_2]-\frac{\pi}{\sin\big((\alpha+1)\pi\big)}e^{-(\alpha+1)\pi i}[R_3+R_4]$$

$\because \cot\big((\alpha+1)\pi\big)=\cot(\alpha\pi+\pi)=\cot(\alpha\pi)$, $\quad e^{-(\alpha+1)\pi i}=e^{-\alpha\pi i}e^{-\pi i}=-e^{-\alpha\pi i}$, $\quad \sin\big((\alpha+1)\pi\big)=-\sin(\alpha\pi)$

$$\int_0^\infty\frac{x^{\alpha}dx}{(x^2-b^2)(x^2-c^2)}=-\pi\cot(\alpha\pi)\left[\frac{b^{\alpha-1}-c^{\alpha-1}}{2(b^2-c^2)}\right]-\frac{\pi}{\sin(\alpha\pi)}e^{-\alpha\pi i}e^{\alpha\pi i}\left[\frac{c^{\alpha-1}-b^{\alpha-1}}{2(b^2-c^2)}\right]$$

$$=\frac{\pi}{2}\left[\frac{b^{\alpha-1}-c^{\alpha-1}}{b^2-c^2}\right][\operatorname{cosec}(\pi\alpha)-\cot(\alpha\pi)]$$ **Hence proved**.

Question 13: **Proved that** $\displaystyle\int_0^\infty\frac{x^{\alpha-1}dx}{x^2+b^2}=\frac{\pi}{2}\frac{b^{\alpha-2}}{\sin\left(\frac{\alpha\pi}{2}\right)},\quad b>0$

Note: This question is not in the book

Solution: Poles of $f(z)=\frac{1}{z^2+b^2}$ are $z=\pm bi$ (both of them have order one), which do not lie on positive part of the real axis.

Let, $\displaystyle I=\int_0^\infty\frac{x^{\alpha-1}dx}{x^2+b^2}=\int_C\frac{e^{(\alpha-1)[\log(-z)+\pi i]}dz}{z^2+b^2}$ and $\displaystyle\varphi(z)=\frac{e^{(\alpha-1)[\log(-z)+\pi i]}}{z^2+b^2}$

$$R_i(f,b)=\lim_{z\to bi}\frac{(z-bi)e^{(\alpha-1)[\log(-z)+\pi i]}}{(z-bi)(z+bi)}=\lim_{z\to bi}\frac{e^{(\alpha-1)[\log(-z)+\pi i]}}{z+bi}=\frac{e^{(\alpha-1)[\log(-bi)+\pi i]}}{2bi}=\frac{e^{(\alpha-1)\left[\log b-\frac{\pi i}{2}+\pi i\right]}}{2bi}$$

$$=\frac{e^{(\alpha-1)\left[\log b+\frac{\pi i}{2}\right]}}{2bi}=\frac{e^{(\alpha-1)\log b}e^{(\alpha-1)\frac{\pi i}{2}}}{2bi}=\frac{e^{\log b^{(\alpha-1)}}e^{(\alpha-1)\frac{\pi i}{2}}}{2bi}=\frac{b^{(\alpha-1)}e^{(\alpha-1)\frac{\pi i}{2}}}{2bi}$$

$$\because\ e^{[\log(-bi)]}=\log b+\log(-i)=\log b+\log\left(e^{\frac{-\pi i}{2}}\right)=\log b+\frac{-\pi i}{2}$$

$$R_2(f,-bi)=\lim_{z\to -bi}\frac{(z+bi)e^{(\alpha-1)[\log(-z)+\pi i]}}{(z-bi)(z+bi)}=\lim_{z\to -bi}\frac{e^{(\alpha-1)[\log(-z)+\pi i]}}{z-bi}=\frac{e^{(\alpha-1)[\log(bi)+\pi i]}}{-2bi}=\frac{e^{(\alpha-1)\left[\log b+\frac{\pi i}{2}+\pi i\right]}}{-2bi}$$

$$=\frac{e^{(\alpha-1)\left[\log b+\frac{3\pi i}{2}\right]}}{-2bi}=\frac{e^{(\alpha-1)\log b}e^{(\alpha-1)\frac{3\pi i}{2}}}{-2bi}=\frac{e^{\log b^{(\alpha-1)}}e^{(\alpha-1)\frac{3\pi i}{2}}}{-2bi}=\frac{b^{(\alpha-1)}e^{(\alpha-1)\frac{3\pi i}{2}}}{-2bi}$$

$$\because\ e^{[\log(bi)]}=\log b+\log(i)=\log b+\log\left(e^{\frac{\pi i}{2}}\right)=\log b+\frac{\pi i}{2}$$

$$I=\frac{-\pi}{\sin\alpha\pi}e^{-\alpha\pi i}\sum_i R_i=\frac{-\pi}{\sin(\alpha\pi)}e^{-\alpha\pi i}\left[\frac{b^{(\alpha-1)}e^{(\alpha-1)\frac{\pi i}{2}}}{2bi}-\frac{b^{(\alpha-1)}e^{(\alpha-1)\frac{3\pi i}{2}}}{2bi}\right]$$

$$=\frac{-\pi}{2i\sin(\alpha\pi)}e^{-\alpha\pi i}b^{(\alpha-2)}\left[e^{\frac{\alpha\pi i}{2}}e^{\frac{-\pi i}{2}}-e^{\frac{3\alpha\pi i}{2}}e^{\frac{-3\pi i}{2}}\right]=\frac{-\pi}{2i\sin(\alpha\pi)}e^{-\alpha\pi i}b^{(\alpha-2)}\left[-ie^{\frac{\alpha\pi i}{2}}-ie^{\frac{3\alpha\pi i}{2}}\right]$$

$$=\frac{-\pi}{2i\sin(\alpha\pi)}e^{-\alpha\pi i}e^{\alpha\pi i}b^{(\alpha-2)}(-i)\left[e^{\frac{-\alpha\pi i}{2}}+e^{\frac{\alpha\pi i}{2}}\right]=\frac{\pi}{2\sin(\alpha\pi)}b^{(\alpha-2)}\left[2\cos\left(\frac{\alpha\pi}{2}\right)\right]$$

$$=\frac{\pi}{\sin(\alpha\pi)}b^{(\alpha-2)}\cos\left(\frac{\alpha\pi}{2}\right)=\frac{\pi}{2\,\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)}b^{(\alpha-2)}\cos\left(\frac{\alpha\pi}{2}\right)$$

$$=\frac{\pi}{2}\frac{b^{\alpha-2}}{\sin\left(\frac{\alpha\pi}{2}\right)}$$ **Hence proved**.