Question 1: **Prove that** $\displaystyle\int_0^\infty \frac{\sin mx\,dx}{x(x^2+a^2)^2} = \frac{\pi}{2a^4}\left[1 - \frac{e^{-ma}}{2}(ma+2)\right], \quad a > 0, \quad m > 0$

Solution: Let $\displaystyle I = \int_0^\infty \frac{\sin mx\ dx}{x(x^2+a^2)} = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin mx\ dx}{x(x^2+a^2)} = \frac{1}{2}\text{Im}\int_C \frac{e^{imz}\,dz}{z(z^2+a^2)}$

Poles of $f(z) = \frac{e^{imz}}{z(z^2+a^2)}$ are $z = 0, \pm ai$. The pole $z = 0$ lies on real axis, having order one and the pole $z = ai$ lies in upper half of the complex plane and it has order one.

$$R_{1,x}(f,z) = \lim_{z\to 0}\left[\frac{z e^{imz}}{z(z^2+a^2)}\right] = \lim_{z\to 0}\left[\frac{e^{imz}}{(z^2+a^2)}\right] = \frac{1}{a^2}$$

$$R_{1,p}(f,ia) = \lim_{z\to ia}\left[\frac{(z-ia)e^{imz}}{z(z+ia)(z-ia)}\right] = \frac{e^{m(ia)i}}{(ai)(2ai)} = \frac{e^{mai^2}}{-2a^2} = \frac{e^{-ma}}{-2a^2}$$

$$\int_0^\infty \frac{\sin mx\ dx}{x(x^2+a^2)} = \frac{1}{2}\text{Im}\left[2\pi i\sum_n R_{n,p} + \pi i \sum_n R_{n,x}\right] = \frac{1}{2}\text{Im}\left[2\pi i\left(\frac{e^{-ma}}{-2a^2}\right) + \pi i\left(\frac{1}{a^2}\right)\right]$$
$$= \frac{1}{2}\left[\pi\left(\frac{e^{-ma}}{-a^2}\right) + \pi\left(\frac{1}{a^2}\right)\right] = \frac{1}{2}\left[\pi\left(\frac{1}{a^2}\right) - \pi\left(\frac{e^{-ma}}{a^2}\right)\right] = \frac{\pi}{2a^2}[1 - e^{-ma}]$$

Now, differentiate both sides w.r.t "a"

$$\int_0^\infty \frac{-2a\sin mx\,dx}{x(x^2+a^2)^2} = \frac{\pi}{2a^2}(me^{-ma}) - \frac{2\pi}{2a^3}(1-e^{-ma}) = \frac{\pi}{a^3}\left[\frac{ma}{2}e^{-ma} - 1 + e^{-ma}\right]$$
$$= \frac{\pi}{a^3}\left[\left(\frac{ma+2}{2}\right)e^{-ma} - 1\right]$$ **Hence proved**.

Question 2: **Prove that** $\displaystyle\int_{-\infty}^\infty \frac{\cos x\ dx}{a^2 - x^2} = \frac{\pi}{a}\sin a$

Solution: Let $\displaystyle I = \int_{-\infty}^\infty \frac{\cos x\ dx}{a^2-x^2} = \text{Re}\int_C \frac{e^{iz}dz}{a^2-z^2}$

Poles of $f(z) = \frac{e^{iz}}{a^2-z^2}$ are $z = \pm a$. The only pole which lie in upper half of the complex plane is $z = a$, having order 1.

$$R_{1,p}(f,a) = \lim_{z\to a}\left[\frac{-e^{iz}(z-a)}{(z-a)(z+a)}\right] = \frac{-e^{ia}}{2a}$$

$$\int_{-\infty}^\infty \frac{\cos x\ dx}{a^2-x^2} = \text{Re}\left[2\pi i\sum_n R_{n,p} + \pi i\sum_n R_{n,x}\right] = \text{Re}\left[2i\pi\left(\frac{-e^{ia}}{2a}\right) + \pi i(0)\right] = -\text{Re}\left[(i\pi)\left(\frac{e^{ia}}{a}\right)\right]$$

$$\int_{-\infty}^{\infty}\frac{\cos x\ dx}{a^2-x^2}=-\text{Re}\left[\left(\frac{\pi i}{a}\right)(\cos a+i\sin a)\right]=\text{Re}\left[\frac{\pi}{a}\sin a-\frac{i\pi}{a}\cos a\right]=\frac{\pi}{a}\sin a$$ **Hence proved**.

Question 3: **Prove that** $\displaystyle\int_{0}^{\infty}\frac{\sin mx\ \mathrm{d}x}{x^3+4x}=\frac{\pi}{8}(1-\mathrm{e}^{-2m})$

Solution: Let $\displaystyle I=\int_{0}^{\infty}\frac{\sin mx\ \mathrm{d}x}{x^3+4x}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin mx\ \mathrm{d}x}{x(x^2+4)}=\frac{1}{2}\int_{C}\frac{\mathrm{e}^{imz}\ \mathrm{d}z}{z(z^2+4)}$

Poles of $f(z)=\frac{\mathrm{e}^{imz}}{z(z^2+4)}$ are $z=0,\ \pm 2i$. The pole $z=0$ lies on real axis, having order one and the pole $z=2i$ lies in upper half of the complex plane and it has order one.

$$R_{1,x}(f,0)=\lim_{z\to 0}\left[\frac{z\mathrm{e}^{imz}}{z(z^2+4)}\right]=\lim_{z\to 0}\left[\frac{\mathrm{e}^{imz}}{z^2+4}\right]=\frac{1}{4}$$

$$R_{1,p}(f,2i)=\lim_{z\to 2i}\left[\frac{(z-2i)\mathrm{e}^{imz}}{z(z+2i)(z-2i)}\right]=\lim_{z\to 2i}\left[\frac{\mathrm{e}^{imz}}{z(z+2i)}\right]=\frac{\mathrm{e}^{-2m}}{2i(4i)}=-\frac{\mathrm{e}^{-2m}}{8}$$

$$\int_{0}^{\infty}\frac{\sin mx\ \mathrm{d}x}{x^2+4x}=\frac{1}{2}\text{Im}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\text{Im}\left[2\pi i\left(-\frac{\mathrm{e}^{-2m}}{8}\right)+\pi i\left(\frac{1}{4}\right)\right]$$
$$=\pi\left(-\frac{\mathrm{e}^{-2m}}{8}\right)+\pi\left(\frac{1}{8}\right)=\frac{\pi}{8}-\frac{\pi\mathrm{e}^{-2m}}{8}=\frac{\pi}{8}(1-\mathrm{e}^{-2m})$$ **Hence proved**.

Question 4: **Proved that** $\displaystyle\int_{0}^{\infty}\frac{\sin mx\ \mathrm{d}x}{x(x^2+a^2)}=\frac{\pi}{2a^2}[1-\mathrm{e}^{-ma}]\quad a>0, m>0$

Solution: Let $\displaystyle I=\int_{0}^{\infty}\frac{\sin mx\ \mathrm{d}x}{x(x^2+a^2)}=\frac{1}{2}\int_{0}^{\infty}\frac{\sin mx\ \mathrm{d}x}{x(x^2+a^2)}=\frac{1}{2}\text{Im}\int_{C}\frac{\mathrm{e}^{imz}\ \mathrm{d}z}{z(z^2+a^2)}$

The poles of $f(z)=\frac{\mathrm{e}^{imz}}{z(z^2+a^2)}$ are $z=0,\pm ai$. Only the pole $z=0$ lies on real axis, having order 1 and the pole $z=ai$ lies in upper half plane and it has order 1.

$$R_{1,x}(f,z)=\lim_{z\to 0}\left[\frac{z\mathrm{e}^{imz}}{z(z^2+a^2)}\right]=\lim_{z\to 0}\left[\frac{\mathrm{e}^{imz}}{(z^2+a^2)}\right]=\frac{1}{a^2}$$

$$R_{1,p}(f,ia)=\lim_{z\to ai}\left[\frac{(z-ia)\mathrm{e}^{imz}}{z(z+ia)(z-ia)}\right]=\frac{\mathrm{e}^{m(ai)i}}{(ai)(2ai)}=\frac{\mathrm{e}^{mai^2}}{-2a^2}=\frac{\mathrm{e}^{-ma}}{-2a^2}$$

$$\int_{0}^{\infty}\frac{\sin mx\ \mathrm{d}x}{x(x^2+a^2)}=\frac{1}{2}\text{Im}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\text{Im}\left[2\pi i\left(\frac{\mathrm{e}^{-ma}}{-2a^2}\right)+\pi i\left(\frac{1}{a^2}\right)\right]$$
$$=\frac{1}{2}\left[\pi\left(\frac{1}{a^2}\right)-\pi\left(\frac{\mathrm{e}^{-ma}}{a^2}\right)\right]=\frac{\pi}{2a^2}[1-\mathrm{e}^{-ma}]$$ **Hence proved**.

Questio 5: **Prove that** $\displaystyle\int_{-\infty}^{\infty}\frac{\cos mx\ dx}{x^2-a^2}=-\frac{\pi}{a}\sin ma$

Solution: Let $\displaystyle I=\int_{-\infty}^{\infty}\frac{\cos mx\ dx}{x^2-a^2}=\text{Re}\int_C\frac{e^{imz}\,dz}{z^2-a^2}$

Poles of $f(z)=\frac{e^{imz}}{z^2-a^2}$ are $z=\pm a$. Only the pole which lies in upper half of the complex plane is $z=a$, having order 1.

$$R_{1,p}(f,a)=\lim_{z\to a}\left[\frac{e^{imz}(z-a)}{(z-a)(z+a)}\right]=\frac{e^{ima}}{2a}$$

$$\int_{-\infty}^{\infty}\frac{\cos mx\ dx}{a^2-x^2}=\text{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\text{Re}\left[2i\pi\left(\frac{e^{ima}}{2a}\right)+\pi i(0)\right]=\text{Re}\left[(i\pi)\left(\frac{e^{ima}}{a}\right)\right]$$
$$=\text{Re}\left[\left(\frac{\pi i}{a}\right)(\cos ma+i\sin ma)\right]=\text{Re}\left[\left(-\frac{\pi}{a}\sin ma+\frac{i\pi}{a}\cos ma\right)\right]$$
$$=-\frac{\pi}{a}\sin ma$$ **Hence proved**.

Question 6: **Prove that** $\displaystyle\int_0^{\infty}\frac{\sin mx\,dx}{x(x^2-a^2)}=\frac{\pi}{2a^2}(\cos ma-1)\quad a>0,\quad m>0$

Solution: Let $\displaystyle I=\int_0^{\infty}\frac{\sin mx\ dx}{x(x^2-a^2)}=\frac{1}{2}\int_0^{\infty}\frac{\sin mx\ dx}{x(x^2-a^2)}=\frac{1}{2}\text{Im}\int_C\frac{e^{imz}\,dz}{z(z^2-a^2)}$

Poles of $f(z)=\frac{e^{imz}}{z(z^2-a^2)}$ are $z=0,\pm a$. All poles have order 1 and lie on real axis.

$$R_{1,x}(f,0)=\lim_{z\to 0}\left[\frac{ze^{imz}}{z(z^2-a^2)}\right]=\lim_{z\to 0}\left[\frac{e^{imz}}{(z^2-a^2)}\right]=\frac{1}{-a^2}$$

$$R_{2,x}(f,a)=\lim_{z\to a}\left[\frac{(z-a)e^{imz}}{z(z^2-a^2)}\right]=\lim_{z\to a}\left[\frac{(z-a)e^{imz}}{z(z-a)(z+a)}\right]=\lim_{z\to a}\left[\frac{e^{imz}}{z(z+a)}\right]=\frac{e^{ima}}{2a^2}$$

$$R_{3,x}(f,-a)=\lim_{z\to -a}\left[\frac{(z+a)e^{imz}}{z(z^2-a^2)}\right]=\lim_{z\to -a}\left[\frac{(z+a)e^{imz}}{z(z-a)(z+a)}\right]=\lim_{z\to -a}\left[\frac{e^{imz}}{z(z+a)}\right]=\frac{e^{-ima}}{2a^2}$$

$$\int_0^{\infty}\frac{\sin mx\ dx}{x(x^2-a^2)}=\frac{1}{2}\text{Im}\left[\pi i\sum_n R_{n,x}+2\pi i\sum_n R_{n,p}\right]=\frac{1}{2}\text{Im}\left[\pi i\left(\frac{1}{-a^2}+\frac{e^{ima}}{2a^2}+\frac{e^{-ima}}{2a^2}\right)+2\pi i(0)\right]$$
$$=\frac{1}{2}\text{Im}\left[\pi i\left(\frac{1}{-a^2}+\frac{\cos ma+i\sin ma}{2a^2}+\frac{\cos ma-i\sin ma}{2a^2}\right)\right]$$
$$=\frac{1}{2}\text{Im}\left[\pi i\left(\frac{2\cos ma-2}{2a^2}\right)\right]=\frac{\pi}{a^2}\,[\cos ma-1]$$ **Hence proved**.

Question 7: **Prove that** $\displaystyle\int_0^\infty \frac{\sin^2 mx\,dx}{x^2(a^2+x^2)^2} = \frac{\pi}{8a^5}\left[e^{-2ma}(2ma+3)+4ma-3\right], m>0, a>0$

Solution: Let $\displaystyle I=\int_0^\infty \frac{\sin^2 mx\,dx}{x^2(a^2+x^2)^2}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\frac{1}{2}(1-\cos 2mx)\,dx}{x^2(a^2+x^2)^2}=\frac{1}{4}\mathrm{Re}\int_C \frac{(1-e^{2imz})dz}{z^2(z^2+a^2)^2}$

Poles of $f(z)=\frac{(1-e^{2imz})}{z^2(z^2+a^2)^2}$ are $z=0,\pm ai$. Only the pole $z=0$ lies on real axis, having order 2 and $z=ai$ has order 2 and lies in upper half plane.

$$R_{1,x}(f,0)=\lim_{z\to 0}\frac{d}{dz}\left[\frac{z^2(1-e^{2imz})}{z^2(z^2+a^2)^2}\right]=\lim_{z\to 0}\left[\frac{(z^2+a^2)^2(-2mie^{2miz})-(1-e^{2miz})2(z^2+a^2)2z}{(z^2+a^2)^4}\right]$$
$$=\lim_{z\to 0}\left[\frac{(z^2+a^2)\left(-2mie^{2miz}\right)-4z(1-e^{2miz})}{(z^2+a^2)^3}\right]=\frac{-2ima^2}{a^6}=\frac{-2im}{a^4}$$

$$R_{1,p}(f,ai)=\lim_{z\to ai}\frac{d}{dz}\left[\frac{(z-ai)^2(1-e^{2imz})}{z^2(z+ai)^2(z-ai)^2}\right]=\lim_{z\to ai}\frac{d}{dz}\left[\frac{(1-e^{2imz})}{z^2(z+ai)^2}\right]$$
$$=\lim_{z\to ai}\left[\frac{z^2(z+ai)^2(-2mie^{2imz})-(1-e^{2imz})[2z\,(z+ai)^2+2z^2(z+ai)\,]}{z^4(z+ai)^4}\right]$$
$$=\frac{(ai)^2(2ai)^2(-2mie^{-2ma})-(1-e^{-2ma})[2ai\,(2ai)^2+2(ai)^2(2ai)\,]}{(ai)^4(2ai)^4}$$
$$=\frac{-8mia^4e^{-2ma}+(1-e^{-2ma})(12a^3i)}{16a^8}$$

$$\int_0^\infty \frac{\sin^2 mx\,dx}{x^2(a^2+x^2)^2}=\frac{1}{4}\mathrm{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]$$
$$=\frac{1}{4}\mathrm{Re}\left[2\pi i\left(\frac{-8mia^4e^{-2ma}+(1-e^{-2ma})(12a^3i)}{16a^8}\right)+\pi i\left(\frac{-2im}{a^4}\right)\right]$$
$$=\frac{1}{4}\mathrm{Re}\left[\pi i\left(\frac{-8mia^4e^{-2ma}+(1-e^{-2ma})(12a^3i)}{8a^8}\right)+\left(\frac{2\pi m}{a^4}\right)\right]$$
$$=\frac{\pi a^3}{4}\mathrm{Re}\left[\frac{8ma\,e^{-2ma}-(1-e^{-2ma})(12)+16\pi ma}{8a^8}\right]$$
$$=\frac{\pi}{8a^5}\left[e^{-2ma}(2ma+3)+4ma-3\right]$$ **Hence proved**.

Alternate solution: Let $\displaystyle I=\int_0^\infty \frac{\sin^2 mx\,dx}{x^2(a^2+x^2)}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\frac{1}{2}(1-\cos 2mx)\,dx}{x^2(a^2+x^2)}=\frac{1}{4}\mathrm{Re}\int_C \frac{(1-e^{2imz})dz}{z^2(z^2+a^2)}$

Poles of $f(z)=\frac{(1-e^{2imz})}{z^2(z^2+a^2)}$ are $z=0,\pm ai$. Only the pole $z=0$ lies on real axis, having order 2 while $z=ai$ has order 1 and lies in upper half plane.

$$R_{1,x}(f,0)=\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{dz}}\left[\frac{z^2\left(1-\mathrm{e}^{2imz}\right)}{z^2(z^2+a^2)}\right]=\lim_{z\to 0}\left[\frac{(z^2+a^2)\left(-2mi\mathrm{e}^{2miz}\right)-\left(1-\mathrm{e}^{2miz}\right)2z}{(z^2+a^2)^2}\right]=\frac{-2ima^2}{a^4}$$
$$=\frac{-2im}{a^2}$$

$$R_{1,p}(f,ai)=\lim_{z\to ai}\left[\frac{(z-ai)\left(1-\mathrm{e}^{2imz}\right)}{z^2(z+ai)(z-ai)}\right]=\lim_{z\to ai}\left[\frac{\left(1-\mathrm{e}^{2imz}\right)}{z^2(z+ai)}\right]=\left[\frac{(1-\mathrm{e}^{-2ma})}{(ai)^2(ai+ai)}\right]=\frac{1-\mathrm{e}^{-2ma}}{-(2a^3i)}$$
$$=\frac{\mathrm{e}^{-2ma}-1}{(2a^3i)}$$

$$\int_0^\infty\frac{\sin^2 mx\,dx}{x^2(a^2+x^2)}=\frac{1}{4}\mathrm{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{4}\mathrm{Re}\left[2\pi i\left(\frac{\mathrm{e}^{-2ma}-1}{2a^3i}\right)+\pi i\left(\frac{-2im}{a^2}\right)\right]$$
$$=\frac{\pi}{4}\mathrm{Re}\left[\frac{\mathrm{e}^{-2ma}-1}{a^3}+\frac{2m}{a^2}\right]=\frac{\pi}{4a^3}\left[\mathrm{e}^{-2ma}-1+2ma\right]$$

Differentiate w.r.t "a", on both sides, we get

$$\int_0^\infty\frac{-2\mathrm{a}\sin^2 mx\,dx}{x^2(a^2+x^2)^2}=\frac{\pi}{4a^6}\left[(-2m\mathrm{e}^{-2ma}+2m)a^3-(\mathrm{e}^{-2ma}-1+2ma)3a^2\right]$$
$$=\frac{\pi}{4a^4}\left[-2ma\mathrm{e}^{-2ma}-3\mathrm{e}^{-2ma}+2ma-6ma+3\right]$$

$$\int_0^\infty\frac{\sin^2 mx\,dx}{x^2(a^2+x^2)^2}=\frac{\pi}{8a^5}\left[\mathrm{e}^{-2ma}(2ma+3)+4ma-3\right]$$ **Hence proved**.

Question 8: **prove that if $a<0$** (i) $\displaystyle\int_0^\infty\frac{b\cos ax\,dx}{x^2-b^2}=\frac{-\pi}{2}\sin ab$

(ii) $\displaystyle\int_0^\infty\frac{x\sin ax\,dx}{x^2-b^2}=\frac{\pi}{2}\cos ab$

Solution (i): Let $\displaystyle I=\int_0^\infty\frac{b\cos ax\ \mathrm{d}x}{x^2-b^2}=\frac{1}{2}\int_{-\infty}^\infty\frac{b\cos ax\ \mathrm{d}x}{x^2-b^2}=\frac{1}{2}\mathrm{Re}\int_C\frac{b\,\mathrm{e}^{iaz}\,\mathrm{d}z}{z^2-b^2}$

Poles of $f(z)=\frac{b\mathrm{e}^{iaz}}{z^2-b^2}$ are $z=\pm b$. Only the pole which lies in upper half of the complex plane is $z=b$, having order 1.

$$R_{1,p}(f,b)=\lim_{z\to b}\left[\frac{\mathrm{b}\mathrm{e}^{iaz}(z-b)}{(z-b)(z+b)}\right]=\frac{\mathrm{e}^{iab}}{2}$$

$$\int_0^\infty\frac{b\cos ax\ \mathrm{d}x}{x^2-b^2}=\frac{1}{2}\mathrm{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\mathrm{Re}\left[2i\pi\left(\frac{\mathrm{e}^{iab}}{2}\right)+\pi i(0)\right]=Re\left[i\pi\left(\frac{\mathrm{e}^{iab}}{2}\right)\right]$$

$$\int_0^\infty \frac{b\cos ax\ \mathrm{d}x}{x^2-b^2} = \mathrm{Re}\left[\frac{\pi i}{2}(\cos ab + i\sin ab)\right] = \mathrm{Re}\left[\left(-\frac{\pi}{2}\sin ab + \frac{i\pi}{2}\cos ab\right)\right]$$
$$= -\frac{\pi}{2}\sin ab \qquad \textbf{Hence proved.}$$

Solution $(\boldsymbol{ii})$: We have just proved in (i) that $\int_0^\infty \frac{b\cos ax\ \mathrm{d}x}{x^2-b^2} = -\frac{\pi}{2}\sin ab$

Differentiate both side w.r.t "a"

$$\int_0^\infty \frac{bx\sin ax\ \mathrm{d}x}{x^2-b^2} = \frac{b\pi}{2}\cos ab$$

$$\int_0^\infty \frac{x\sin ax\ \mathrm{d}x}{x^2-b^2} = \frac{\pi}{2}\cos ab \qquad \textbf{Hence proved.}$$

Alternate solution $(\boldsymbol{ii})$: Let $I = \int_0^\infty \frac{x\sin ax\ \mathrm{d}x}{x^2-b^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{b\sin ax\ \mathrm{d}x}{x^2-b^2} = \frac{1}{2}\mathrm{Im}\int_C \frac{z\,\mathrm{e}^{iaz}\,\mathrm{d}z}{z^2-b^2}$

Poles of $f(z) = \frac{z\,\mathrm{e}^{iaz}}{z^2-b^2}$ are $z = \pm b$. Only the pole which lies in upper half of the complex plane is $z = b$, having order 1.

$$R_{1,p}(f,b) = \lim_{z\to b}\left[\frac{z\mathrm{e}^{iaz}(z-b)}{(z-b)(z+b)}\right] = \frac{b\mathrm{e}^{iab}}{2b} = \frac{b\mathrm{e}^{iab}}{2}$$

$$\int_0^\infty \frac{x\sin ax\ \mathrm{d}x}{x^2-b^2} = \frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_{n,p} + \pi i\sum_n R_{n,x}\right] = \frac{1}{2}\mathrm{Im}\left[2i\pi\left(\frac{\mathrm{e}^{iab}}{2}\right) + \pi i(0)\right] = \mathrm{Im}\left[i\pi\left(\frac{\mathrm{e}^{iab}}{2}\right)\right]$$
$$= \mathrm{Im}\left[\frac{\pi i}{2}(\cos ab + i\sin ab)\right] = \mathrm{Im}\left[\left(-\frac{\pi}{2}\sin ab + \frac{i\pi}{2}\cos ab\right)\right] = \frac{\pi}{2}\cos ab \qquad \textbf{Proved.}$$

Question 9: **Prove that** $\boldsymbol{\int_0^\infty \frac{\sin^2 mx\,\mathrm{d}x}{x^2(a^2+x^2)} = \frac{\pi}{4a^3}(e^{-2ma} - 1 + 2ma)\,, m > 0, a > 0}$

Solution: Let $I = \int_0^\infty \frac{\sin^2 mx\ \mathrm{d}x}{x^2(a^2+x^2)} = \frac{1}{2}\int_{-\infty}^\infty \frac{\frac{1}{2}(1-\cos 2mx)\ \mathrm{d}x}{x^2(a^2+x^2)} = \frac{1}{4}\mathrm{Re}\int_C \frac{(1-\mathrm{e}^{2imz})\mathrm{d}z}{z^2(z^2+a^2)}$

Poles of $f(z) = \frac{(1-\mathrm{e}^{2imz})}{z^2(z^2+a^2)}$ are $z = 0, \pm ai$. Only the pole $z = 0$ lies on real axis, having order 2 while $z = ai$ has order 1 and lies in upper half plane.

$$R_{1,x}(f,0) = \lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^2(1-\mathrm{e}^{2imz})}{z^2(z^2+a^2)}\right] = \lim_{z\to 0}\left[\frac{(z^2+a^2)(-2mi\mathrm{e}^{2miz}) - (1-\mathrm{e}^{2miz})2z}{(z^2+a^2)^2}\right] = \frac{-2ima^2}{a^4}$$
$$= \frac{-2im}{a^2}$$

$$R_{1,p}(f,ai)=\lim_{z\to ai}\left[\frac{(z-ai)\left(1-\mathrm{e}^{2imz}\right)}{z^2(z+ai)(z-ai)}\right]=\lim_{z\to ai}\left[\frac{\left(1-\mathrm{e}^{2imz}\right)}{z^2(z+ai)}\right]=\left[\frac{(1-\mathrm{e}^{-2ma})}{(ai)^2(ai+ai)}\right]=\frac{1-\mathrm{e}^{-2ma}}{-(2a^3i)}$$
$$=\frac{\mathrm{e}^{-2ma}-1}{(2a^3i)}$$

$$\int_0^\infty \frac{\sin^2 mx\,dx}{x^2(a^2+x^2)}=\frac{1}{4}\mathrm{Re}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{4}\mathrm{Re}\left[2\pi i\left(\frac{\mathrm{e}^{-2ma}-1}{2a^3i}\right)+\pi i\left(\frac{-2im}{a^2}\right)\right]$$
$$=\frac{\pi}{4}\mathrm{Re}\left[\frac{\mathrm{e}^{-2ma}-1}{a^3}+\frac{2m}{a^2}\right]=\frac{\pi}{4a^3}\left[\mathrm{e}^{-2ma}-1+2ma\right]$$

Question: 10 prove that $\displaystyle\int_0^\infty \frac{(x-\sin x)dx}{x^3(x^2+a^2)}=\frac{\pi}{2a^4}\left(\frac{a^2}{2}+1-\mathrm{e}^{-a}\right)$ **where $a>0$**

Solution: $\displaystyle\int_0^\infty \frac{(x-\sin x)dx}{x^3(x^2+a^2)}=\frac{1}{2}\int_{-\infty}^\infty \frac{(x-\sin x)dx}{x^3(x^2+a^2)}=\frac{1}{2}\mathrm{Im}\int_C \frac{\left(z-\mathrm{e}^{iz}\right)dz}{z^3(z^2+a^2)}$

Poles of $f(z)=\frac{z-\mathrm{e}^{iz}}{z^3(z^2+a^2)}$ are $z=0,\pm ai$. The pole $z=0$ lies on real axis, having order 3 and the pole $z=ai$ lies in upper half of the complex plane with order 1.

$$R_{1,x}(f,0)=\lim_{z\to 0}\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{z^3\left(z-\mathrm{e}^{iz}\right)}{z^3(z^2+a^2)}\right]=\lim_{z\to 0}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{z-\mathrm{e}^{iz}}{z^2+a^2}\right]$$
$$=\lim_{z\to 0}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z^2+a^2)\left(1-ie^{iz}\right)-\left(z-e^{iz}\right)(2z)}{(z^2+a^2)^2}\right]$$
$$=\lim_{z\to 0}\frac{1}{2}\left[\frac{\begin{array}{c}(z^2+a^2)^2\left[(2z)\left(1-i\mathrm{e}^{iz}\right)+(z^2+a^2)\left(\mathrm{e}^{iz}\right)-\left(z-\mathrm{e}^{iz}\right)(2)-\left(1-i\mathrm{e}^{iz}\right)(2z)\right]\\ -\left[(z^2+a^2)\left(1-i\mathrm{e}^{iz}\right)-\left(z-\mathrm{e}^{iz}\right)(2z)\right]\left[4z(z^2+a^2)\right]\end{array}}{(z^2+a^2)^4}\right]$$
$$=\frac{1}{2}\left[\frac{a^4(a^2+2)}{a^8}\right]=\frac{1}{2a^4}\left[a^2+2\right]$$

$$R_{1,p}(f,ai)=\lim_{z\to ai}\left[\frac{(z-ai)\left(z-\mathrm{e}^{iz}\right)}{z^3(z+ai)(z-ai)}\right]=\lim_{z\to ai}\left[\frac{z-\mathrm{e}^{iz}}{z^3(z+ai)}\right]=\frac{ai-\mathrm{e}^{-a}}{(-a)^3i(2ai)}=\frac{ai-\mathrm{e}^{-a}}{2a^4}$$

$$\int_0^\infty \frac{(x-\sin x)dx}{x^3(x^2+a^2)}=\frac{1}{2}\mathrm{Im}\left[2\pi i\sum_n R_{n,p}+\pi i\sum_n R_{n,x}\right]=\frac{1}{2}\mathrm{Im}\left[2\pi i\left(\frac{ai-\mathrm{e}^{-a}}{2a^4}\right)+\pi i\left(\frac{1}{2a^4}(a^2+2)\right)\right]$$
$$=\frac{\pi}{2}\mathrm{Im}\left[\frac{-2a-2i\mathrm{e}^{-a}+ia^2+2i}{2a^4}\right]=\frac{\pi}{2a^4}\left[\frac{a^2}{2}-\mathrm{e}^{-a}+1\right]$$ **Hence proved.**

Question 11: Prove that $\displaystyle\int_{-\infty}^\infty \frac{\sin x\,dx}{(x^2+4)(x-1)}=\frac{\pi}{5}\left(\cos 1-\frac{1}{\mathrm{e}^2}\right)$

Solution: Let $I = \int_{-\infty}^{\infty} \frac{\sin x \, dx}{(x^2+4)(x-1)} = \text{Im} \int_{C} \frac{e^{iz} dz}{(z^2+4)(z-1)}$

Poles of $f(z) = \frac{e^{iz}}{(z^2+4)(z-1)}$ are $z = 1, \pm 2i$. The pole $z = 1$ lies on real axis, having order 1 and the $z = 2i$ lies in upper half of the complex plane with order 1.

$$R_{1,x}(f,1) = \lim_{z\to 1}\left[\frac{(z-1)e^{iz}}{(z^2+4)(z-1)}\right] = \lim_{z\to 1}\left[\frac{e^{iz}}{(z^2+4)}\right] = \frac{e^{i}}{5}$$

$$R_{1,p}(f,2i) = \lim_{z\to 2i}\left[\frac{(z-2i)e^{iz}}{(z+2i)(z-2i)(z-1)}\right] = \lim_{z\to 2i}\left[\frac{e^{iz}}{(z+2i)(z-1)}\right] = \left[\frac{e^{-2}}{(4i)(2i-1)}\right]$$

$$= \frac{-e^{-2}}{(4)(2+i)} \cdot \frac{2-i}{2-i} = \frac{e^{-2}(i-2)}{20}$$

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{(x^2+4)(x-1)} = \text{Im}\left[2\pi i\left(\frac{e^{-2}(i-2)}{20}\right) + \pi i\left(\frac{e^{i}}{5}\right)\right] = \text{Im}\left[\frac{-\pi e^{-2} - 2\pi i e^{-2} + 2\pi i e^{i}}{10}\right]$$

$$= \text{Im}\left[\frac{-\pi e^{-2} - 2\pi i e^{-2} + 2\pi i(\cos 1 + i \sin 1)}{10}\right] = \frac{\pi}{5}(\cos 1 - e^{-2})$$ **Hence proved**.