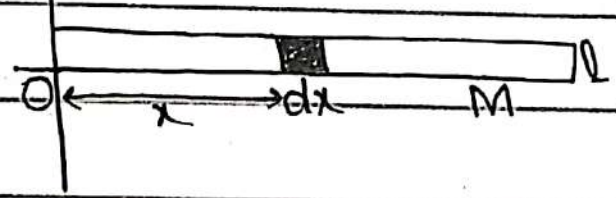
Mechanics

Chapter #6

Moment of Inertia:

Question #1:

Calculate the moment of inertia of uniform rigid rod of length about an axis perpendicular to the rod and passing through an endpoint.



Let

Total mass of rod = M

Length of rod = l

$\rho = \frac{M}{V}$ $\quad$ $\because$ For rod $V = l$

$\rho = \frac{M}{l}$

let

mass of small strip = m

width of small strip = dx

mass element of small strip $dm = \rho\, dx$

Now,

For small particle

$\because I = mr^2$

$I_L = mx^2$

$dI_L = (dm)x^2$

$\because dm = \rho\, dx$

$\int dI_L = \int \rho x^2 dx$

$I_L = \rho \int x^2 dx = \rho \left| \frac{x^3}{3} \right|_0^l$

$I_L = \rho \frac{l^3}{3}$

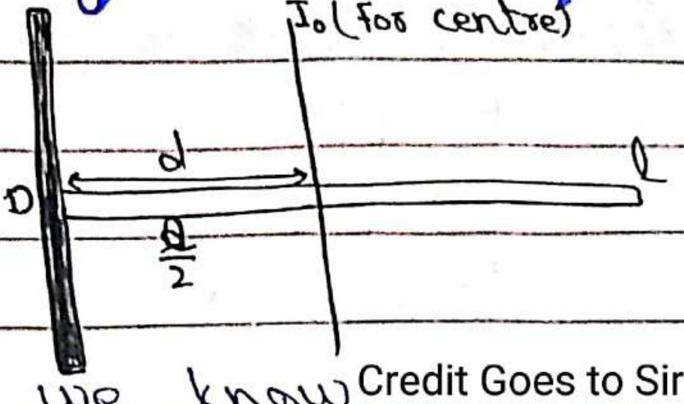
$\rho = \frac{m}{l}$

$I_L = \frac{m}{l} \frac{l^3}{3}$

$$\boxed{I_L = \frac{1}{3} \underset{\text{mass of rod}}{M} \underset{\text{length of rod}}{l^2}}$$

Now,

Find moment of inertia through centre of Rod?



As we know

$$I_1 = \frac{1}{3} ml^2$$

Through parallel axis theorem

$$I_1 = I_0 + md^2$$

$$\frac{1}{3} ml^2 = I_0 + m\left(\frac{l}{2}\right)^2$$

$$\frac{1}{3} ml^2 = I_0 + \frac{ml^2}{4}$$

$$\frac{ml^2}{3} - \frac{ml^2}{4} = I_0$$

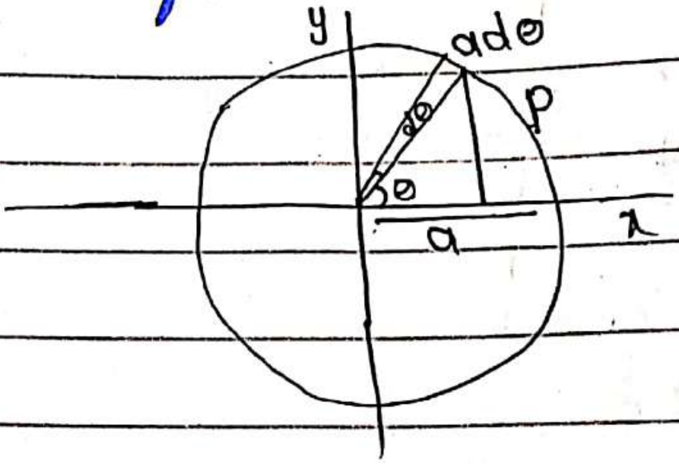
$$ml^2\left[\frac{1}{3} - \frac{1}{4}\right] = I_0$$

$$I_0 = \left[\frac{4-3}{12}\right] ml^2$$

$$\boxed{I_0 = \frac{1}{12} ml^2}$$

Question#2

Find moment of inertia of uniform circular wire?



$x = a\cos\theta \quad ; \quad y = a\sin\theta$

Total of circular wire $= m$

Total length of wire $= 2\pi a$

length of small arc $\overline{PQ} = ad\theta$

mass of small arc $\overline{PQ} = m$

mass element of small arc $dm = \rho a d\theta$

$I_{xx} = my^2$

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$dI_{xx} = dm\,y^2$

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$dI_{xx} = y^2(\rho a d\theta)$

$dI_{xx} = (a\sin\theta)^2 \rho a d\theta$

$dI_{xx} = a^2\sin^2\theta \rho a d\theta$

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$$\int dI_{xx} = \rho a^3 \int_0^{2\pi} \sin^2\theta \, d\theta$$

$$I_{xx} = \rho a^3 \int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta$$

$$I_{xx} = \frac{\rho a^3}{2} \int_0^{2\pi} (1-\cos 2\theta) \, d\theta$$

$$I_{xx} = \frac{\rho a^3}{2} \left| \theta - \frac{\sin 2\theta}{2} \right|_0^{2\pi}$$

$\because \rho = \frac{m}{V}$

$\rho = \frac{m}{2\pi a}$

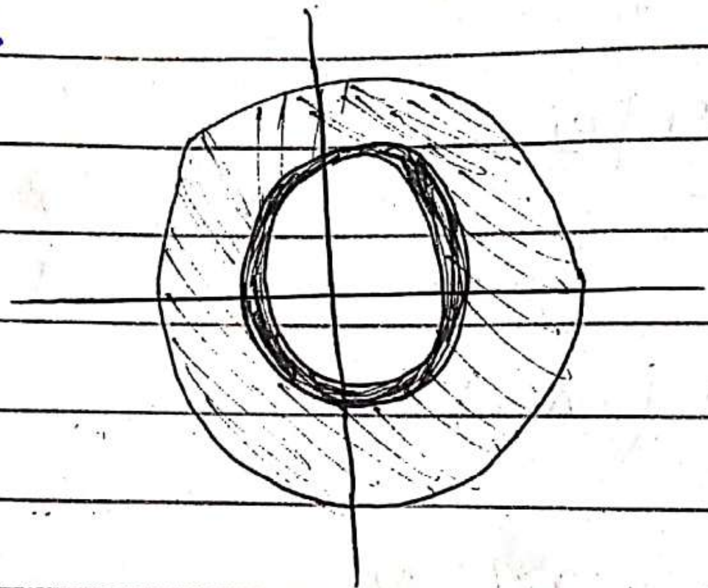
$$I_{xx} = \frac{\rho a^3}{2} (2\pi - 0)$$

$$I_{xx} = \frac{m}{\cancel{2\pi a}} \cdot \frac{a^{\cancel{3}}}{2} (\cancel{2\pi})$$

$$\boxed{I_{xx} = \frac{ma^2}{2}}$$

Question#3

Find moment of inertia of disc of radius 'a' from centre axis is perpendicular to disc and parallel to disc.



let

mass of disc $= m$

Area of disc $= \pi a^2$

$$\rho = \frac{m}{\pi a^2}$$

mass of small strip $= m$

Area of small strip $= 2\pi r\, dr$

mass element of small strip

$dm = \rho(2\pi r)\, dr$

We know

$$I_{zz} = \frac{mr^2}{2}$$

$$dI_{zz} = \frac{(dm) r^2}{2}$$

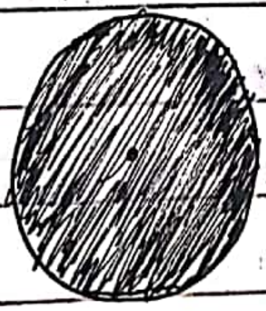
$$dI_{zz} = \frac{1}{\cancel{2}} r^2 (\rho \cancel{2}\pi r)\, dr$$

$$\int dI_{zz} = \rho\pi \int_0^a r^3\, dr$$

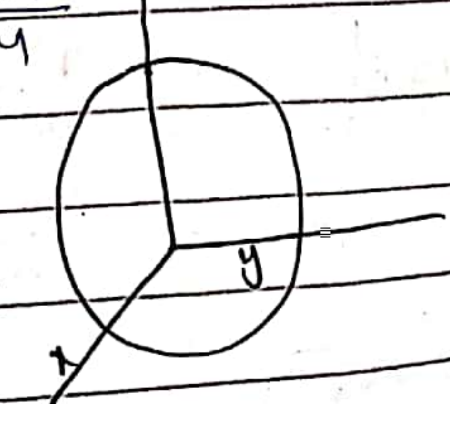
$$I_{zz} = \pi\rho \left|\frac{r^4}{4}\right|_0^a$$

$$I_{zz} = \cancel{\pi} \cdot \frac{m}{\cancel{\pi} a^2} \left|\frac{a^4}{4}\right|$$

$$I_{zz} = \frac{ma^2}{4}$$



$$I_{zz} = \frac{ma^2}{4}$$



We know by $\perp$ axis then

$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = I_{diameter} + I_{diameter}$$

$$I_{zz} = 2 I_{diameter}$$

$$\frac{ma^2}{4} = 2 I_{diameter}$$

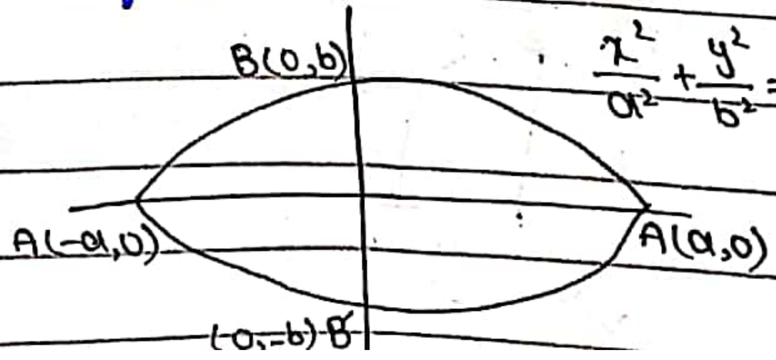
$$I_{diameter} = \frac{ma^2}{8}$$

$$\boxed{I_{diameter} = I_{xx} = I_{yy} = \frac{ma^2}{8}}$$

Question#4

Find moment of inertia of uniform elliptical plate with semi-major and semi-minor axes a, b respectively about (i) a major axis (ii) a minor axis (iii) about an axis through centre and $\perp$ to plate.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



let,

Total mass of elliptic plate $= m$

Area of elliptic plate $= \pi ab$

$$\rho = \frac{m}{\pi ab}$$

mass of small strip $= m$

Area of small strip $= 2y\,dx$

mass element of small strip

$$dm = \rho 2y\,dx$$

We know

$$I_{xx} = \frac{1}{3} m y^2$$

$$dI_{xx} = \frac{1}{3} y^2 (dm)$$

$$dI_{xx} = \frac{1}{3} y^2 (\rho 2y)\,dx$$

$$\int_{-a}^{a} dI_{xx} = \int_{-a}^{a} \frac{2}{3} y^3 \rho\,dx$$

Given

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\sqrt{y^2} = \sqrt{\frac{b^2}{a^2}(a^2 - x^2)}$$

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

$$I_{xx} = \frac{4}{3}\rho\int_0^a y^3\,dx$$

$$I_{xx} = \frac{4}{3}\rho\int_0^a \left(\pm\frac{b}{a}\sqrt{a^2-x^2}\right)^3$$

$$I_{xx} = \frac{4}{3}\cdot\frac{b^3}{a^3}\rho\int_0^a (a^2-x^2)^{\frac{3}{2}}$$

$$x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$$

x	0	a
θ	0	$\frac{\pi}{2}$

$$I_{xx} = \frac{4}{3}\rho\cdot\frac{b^3}{a^3}\int_0^{\frac{\pi}{2}} (a^2-a^2\sin^2\theta)^{\frac{3}{2}}\,(a\cos\theta\,d\theta)$$

$$I_{xx} = \frac{4}{3}\rho\,\frac{b^3}{a^3}\int_0^{\pi/2} \left(a^2(1-\sin^2\theta)\right)^{\frac{3}{2}}\,(a\cos\theta)\,d\theta$$

$$I_{xx} = \frac{4}{3}\rho\,\frac{b^3}{a^2}\int_0^{\pi/2} (a^2\cos^2\theta)^{\frac{3}{2}}\,(a\cos\theta)\,d\theta$$

$$I_{xx} = \frac{4}{3}\rho\,\frac{b^3}{a^3}\int_0^{\frac{\pi}{2}} (a^3\cos^3\theta)(a\cos\theta)\,d\theta$$

$$I_{xx} = \frac{\cancel{4}}{\cancel{3}}\rho\,\frac{b^3}{\cancel{a^3}}\,a^4\cdot\frac{\cancel{3}}{\cancel{4}}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$I_{xx} = \frac{b^3}{4}\pi a\rho$$

$$I_{xx} = \frac{b^3}{4} \cancel{\pi a} \cdot \frac{m}{\cancel{\pi a} b}$$

$$I_{xx} = \frac{mb^2}{4}$$

Similarly,

$$I_{yy} = \frac{ma^2}{4}$$

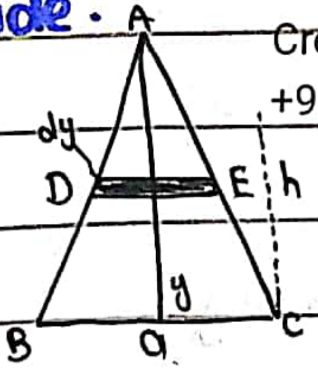
$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = \frac{mb^2}{4} + \frac{ma^2}{4}$$

$$\boxed{I_{zz} = \frac{m}{4}(a^2+b^2)}$$

Question#5

Find moment of inertia of triangular plate through any side.



let

Total mass of plate = m

Area of triangle $= \frac{1}{2}ah$

$$\rho = \frac{m}{\frac{1}{2}ah}$$

mass of small strip $= m$

$\because \Delta ABC$ and ΔADF are similar

$$\frac{DE}{a} = \frac{h-y}{h}$$

$$DE = \left(\frac{h-y}{h}\right)a$$

Area of small strip $= \frac{a(h-y)}{h}dy$

Mass element of strip

$$dm = \rho\left(\frac{a(h-y)}{h}\right)dy$$

Now,

$$I_{BC} = my^2$$

$$dI_{BC} = (dm)y^2$$

$$\int_0^h dI_{BC} = \int_0^h y^2 \rho a\left(\frac{h-y}{h}\right)dy$$

$$I_{BC} = \frac{\rho a}{h}\int_0^h (h-y)y^2\,dy$$

$$I_{BC} = \frac{\rho a}{h}\int_0^h (hy^2 - y^3)\,dy$$

$$I_{BC} = \frac{\rho\alpha}{h}\left|\frac{hy^3}{3} - \frac{y^4}{4}\right|_0^h$$

$$I_{BC} = \frac{\rho\alpha}{h}\left|\frac{h^4}{3} - \frac{h^4}{4}\right|$$

$$I_{BC} = \frac{h^4\rho\alpha}{\cancel{h}}\left|\frac{1}{3} - \frac{1}{4}\right|$$

$$I_{BC} = \rho h^3\alpha\left|\frac{4-3}{12}\right|$$

$$I_{BC} = \rho\alpha\left|\frac{h^3}{12}\right|$$

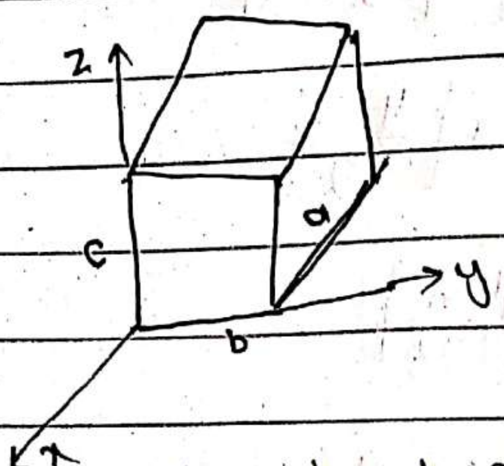
$$I_{BC} = \frac{M\cancel{\alpha}}{\frac{1}{2}\cancel{\alpha h}}\left|\frac{h^3}{12}\right|$$

$$I_{BC} = 2M\left(\frac{h^2}{\cancel{12}\,6}\right)$$

$$\boxed{I_{BC} = \frac{Mh^2}{6}}$$

Question#6

Find moment of inertia of cuboid throwing one edge.



let

Total mass of cuboid = m

Volume = abc

$$\rho = \frac{m}{abc}$$

$$I_{xx} = \iiint_R \rho\,(y^2+z^2)\,dV$$

$$I_{xx} = \int_0^c\int_0^b\int_0^a \rho\,(y^2+z^2)\,dx\,dy\,dz$$

$$I_{xx} = \rho\int_0^c\int_0^b\int_0^a (y^2+z^2)\,dx\,dy\,dz$$

$$\int_0^c\int_0^b\int_0^a (y^2+z^2)\,dx\,dy\,dz$$

$$a\int_0^c\int_0^b y^2\,dy\,dz+\int_0^b\int_0^c z^2\,dz\,dy$$

$$a\left[c\cdot\frac{b^3}{3}+b\,\frac{c^3}{3}\right]$$

$$\frac{abc}{3}\left(b^2+c^2\right)$$

$$\frac{m}{\cancel{abc}}\cdot\frac{\cancel{abc}}{3}\left(b^2+c^2\right)$$

$$\frac{m}{3}\left(b^2+c^2\right)$$

Symmetry

$$\frac{m}{3}\left(a^2+c^2\right)$$

$$\frac{m}{3}\left(a^2+b^2\right)$$

$$I_{xx} = \{\rho a \int_0^c\int_0^b y^2\,dy\,dz + \rho a \int_0^b\int_0^c z^2\,dz\,dy$$

$$I_{xx} = \rho a\left[a\cdot\frac{a^3}{3} + a\cdot\frac{a^3}{3}\right]$$

$$I_{xx} = \frac{m}{a^2}\cdot\cancel{a}\left[\frac{a^4}{3} + \frac{a^4}{3}\right]$$

$$I_{xx} = \frac{m}{\cancel{a^2}}\left[\frac{2a^{\cancel{4}}}{3}\right]$$

$$\boxed{I_{xx} = \frac{2}{3}ma^2}$$

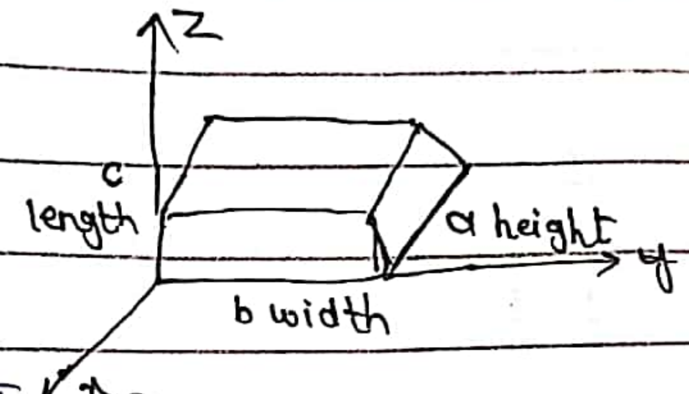
By Symmetry

$$\boxed{I_{yy} = \frac{2}{3}ma^2}$$

$$\boxed{I_{zz} = \frac{2}{3}ma^2}$$

Question#7

Find product of inertia of cuboid. or moment of inertia.



$I_{xy} = I_{12} = ?$

let

Total mass of cuboid $= M$

volume of cuboid $= abc$

density $= \rho = \frac{M}{abc}$

Now,

Product of inertia

$$I_{xy} = -\rho \iiint_R xy\, dV$$

$$I_{xy} = -\rho \int_0^a \int_0^b \int_0^c xy\, dz\, dy\, dx$$

$$I_{xy} = -\rho\, c \cdot \frac{a^2}{2} \cdot \frac{b^2}{2}$$

$$I_{xy} = -\rho c \cdot \frac{a^2b^2}{4}$$

$$I_{xy} = -\frac{M}{\cancel{abc}} \frac{a^2b^2\cancel{c}}{4}$$

$$\boxed{I_{xy} = \frac{-Mab}{4}}$$

By symmetry

$$\boxed{I_{yz} = \frac{-Mbc}{4}}$$

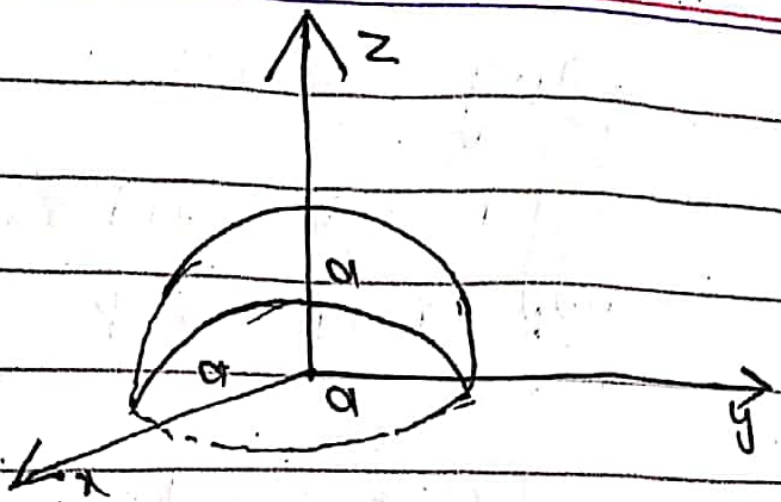
$$\boxed{I_{zx} = \frac{-Mac}{4}}$$

Question#8

Find moment of Inertia of uniform hemi-sphere and radius about

(i) axis of symmetry

(ii) An axis perpendicular to the axis of symmetry and passing through centre of base.



let Total mass $= M$

Volume $= \frac{2}{3}\pi a^3 \rightarrow$ For half sphere

$$\rho = \frac{M}{V} = \frac{3M}{2\pi a^3}$$

As we know that moment Inertia along z-axis

$$I_{zz} = \iiint_R \rho(x^2+y^2)\,dV$$

In Spherical co-ordinate

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$0 \le r \le a$ Credit Goes to Sir Fazal Abbas

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$0 \le \theta \le \frac{\pi}{2}$

$0 < \phi < 2\pi$

Using values

$$I_{zz} = \rho \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$I_{zz} = \rho \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} r^4 (\sin^3\theta \cos^2\phi + \sin^3\theta \sin^2\phi) \, dr \, d\theta \, d\phi$$

$$I_{zz} = \rho \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} r^4 (\sin^3\theta \cos^2\phi + \sin^3\theta \sin^2\phi) \, dr \, d\theta \, d\phi$$

$$I_{zz} = \rho \int_0^{2\pi}\int_0^{\pi/2} \left|\frac{r^5}{5}\right|_0^a \sin^3\theta (\cos^2\phi + \sin^2\phi) \, d\theta \, d\phi$$

$$I_{zz} = \rho \frac{a^5}{5} \int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta \, d\theta \, d\phi$$

$$I_{zz} = \rho \frac{a^5}{5} \int_0^{2\pi} \left(\frac{2}{3}\right) d\phi$$

$$I_{zz} = \rho \frac{2a^5}{15} \left|\phi\right|_0^{2\pi}$$

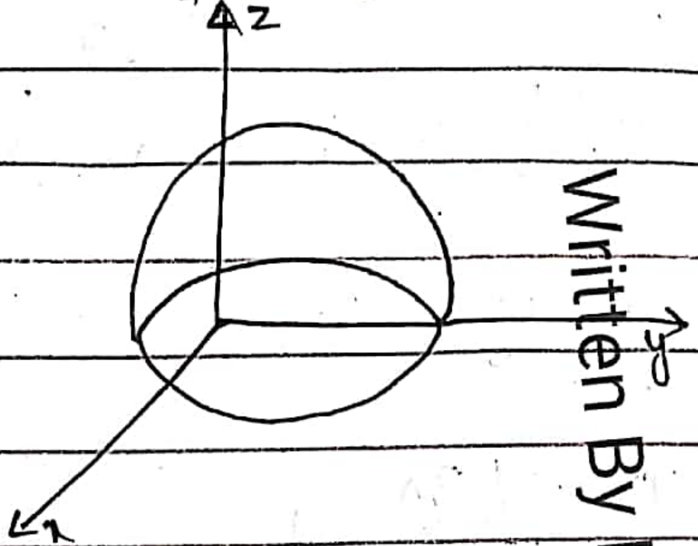
$$I_{zz} = \frac{4\pi a^5}{15} \rho$$

$$I_{zz} = \frac{\cancel{4}^{3}}{15} a^5 \cancel{\pi} \left(\frac{3\pi}{\cancel{2\pi a^3}}\right)$$

$$I_{zz} = \frac{2}{5} m a^2$$

diff سے along < y اور x M.I along < z

change violation ہے کیونکہ half sphere ہے اس لیے ہے۔



same rotation ہے یہاں کیونکہ M.I ہوگا same میں y اور x ہے۔

Now,

moment Inertia along x-axis

$$I_{xx} = \iiint_R \rho (y^2 + z^2)\, dV$$

using spherical co-ordinate

$$I_{xx} = \rho \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} r^2 \sin^2\theta \sin^2\phi$$

$$I_{xx} = \rho \frac{a^5}{5} \int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta \sin^2\phi \, d\theta \, d\phi +$$

$$\int_0^{2\pi}\int_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta \, d\phi$$

$$I_{xx} = \frac{\rho a^5}{5}\left(\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\phi - \frac{\sin 2\phi}{2}\right)_0^{2\pi}\right.$$

$$\left.(2\pi - 0)\left(\frac{\cos^3\theta}{-3}\right)_0^{\pi/2}\right)$$

$$I_{xx} = \frac{\rho a^5}{5}\left(\frac{1}{3}(2\pi) + 2\pi\left(0 + \frac{1}{3}\right)\right)$$

$$I_{xx} = \frac{\rho a^5}{5}\left(\frac{4\pi}{3}\right)$$

$$I_{xx} = \left(\frac{4\pi a^5}{15}\right)\left(\frac{3m}{4\pi a^3}\right)$$

$$\boxed{I_{xx} = \frac{2}{5} m a^2}$$

By Symmetry

Similarly,

$$\boxed{I_{yy} = \frac{2}{5} m a^2}$$

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