Date:

# Notes:

## Mathmatics

Credit goes to:

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0301-4695644

Written by:

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Class:

MSC-Mathmatics

Date:

# Chapter 05

# "Line Surface Volume Integrals And Related Integrals"

## Theorems

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# "Exercise" Chap # 5

## Tangential Line Integrals:

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**Q#1** If $\vec{A} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$. evaluate $\int_c \vec{A} \cdot d\vec{r}$ along the curve in the xy-plane $y = x^3$ from the point $(1,1)$ to $(2,8)$

**Solution:-** Since the Integration performed in xy-plane So $(z=0)$

$\Rightarrow$ We can take

$$\vec{r} = x\hat{i} + y\hat{j}$$

then

$$\int_c \vec{A} \cdot d\vec{r} = \int_c \left[(5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}\right] \cdot \left[dx\hat{i} + dy\hat{j}\right]$$

$$= \int_c (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

Substitute $\quad y = x^3$

$$dy = 3x^2\,dx$$

Where $x$ goes from

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1 to 2 then

**Note**
چونکہ اب سارے values
x میں ہیں تو
limits بھی x کی
ہی لیں گے
1 to 2

$$\int_C \vec{A}\cdot d\vec{r} = \int_{x=1}^{2} (5x.x^3 - 6x^2)\,dx + (2.x^3 - 4x)\,3x^2\,dx$$

$$= \int_1^2 (5x^4 - 6x^2)\,dx + (6x^5 - 12x^3)\,dx$$

$$= \int_1^2 (5x^4 - 6x^2)\,dx + \int_1^2 (6x^5 - 12x^3)\,dx$$

$$= \left| \frac{5x^5}{5} - \frac{6x^3}{3} \right|_1^2 + \left| \frac{6x^6}{6} - \frac{12x^4}{4} \right|_1^2$$

$$= \left|x^5\right|_1^2 - \left|2x^3\right|_1^2 + \left|x^6\right|_1^2 - \left|3x^4\right|_1^2$$

$$= 31 - 14 + 63 - 45$$

$$= 35$$

Hence

$$\int_C \vec{A}\cdot d\vec{r} = 35 \text{ Ans}$$

**Q#2** If $\vec{A} = y^2\hat{i} - x^2\hat{j}$, evaluate $\int_C \vec{A}\cdot d\vec{r}$ along the following paths.

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i) the straight line $y=4x$ from $(0,0)$ to $(1,4)$

ii) the curve $y=4x^2$ from $(0,0)$ to $(1,4)$

**Solution:-**

i) $y=4x$ point $(0,0)$ to $(1,4)$

Now

Since the Integration performed in $xy$-plane So $(z=0)$

Take $\vec{r} = x\hat{i} + y\hat{j}$

Now

$$\int_C \vec{A}\cdot d\vec{r} = \int_C (y^2\hat{i} - x^2\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_C y^2 dx - x^2 dy$$

Substitute $y = 4x$

$dy = 4dx$

Where $x$ goes to $0 \to 1$

$$\int_C \vec{A}\cdot d\vec{r} = \int_{x=0}^{1} (4x)^2 dx - x^2(4dx)$$

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$$\int_C \vec{A}\cdot d\vec{r} = \int_0^1 (16x^2\,dx - 4x^2\,dx)$$

$$= 16\int_0^1 x^2\,dx - 4\int_0^1 x^2\,dx$$

$$= 16\left|\frac{x^3}{3}\right|_0^1 - 4\left|\frac{x^3}{3}\right|_0^1$$

$$= \frac{16}{3} - \frac{4}{3} = \frac{16-4}{3} = \frac{\cancel{12}^{4}}{\cancel{3}} = 4$$

Hence

$$\int_C \vec{A}\cdot d\vec{r} = 4 \quad \underline{\text{Ans}}$$

Now ii) $y = 4x^2$ point $(0,0)$ to $(1,4)$

Take

$$\int_C \vec{A}\cdot d\vec{r} = \int_C y^2\,dx - x^2\,dy$$

put

$$y = 4x^2$$

$$dy = 8x\,dx$$

where $x$ goes from 0 to 1

$$= \int_{x=0}^{1} (4x^2)^2\,dx - x^2(8x\,dx)$$

$$= \int_0^1 16x^4\,dx - \int_0^1 8x^3\,dx$$

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$$\int_C \vec{A}\circ d\vec{r} = \left|\frac{16x^5}{5}\right|_0^1 - \left|\frac{8x^4}{4}\right|_0^1$$

$$= \frac{16}{5} - \frac{8}{4} \Rightarrow \frac{16}{5} - 2$$

$$= \frac{16-10}{5} = \frac{6}{5}$$

Hence

$$\int_C \vec{A}\circ d\vec{r} = \frac{6}{5} \quad \text{Ans}$$

**Q#3** If $\vec{A} = xy\hat{i} + x^2y^2\hat{j}$, evaluate $\int_C \vec{A}\circ d\vec{r}$ along the following paths.

i) the straight line $y = 2 - x$ from $(2,0)$ to $(0,2)$

limit 0 to $\pi/2$

ii) the quarter circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$.

**Solution:** i) $y = 2 - x$

Point $(2,0)$ to $(0,2)$

Now

$$\int_C \vec{A}\circ d\vec{r} = \int_C (xy\hat{i} + x^2y^2\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

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$$\int_C \vec{A}\cdot d\vec{r} = \int_C xy\,dx + x^2y^2\,dy$$

put $y = 2 - x$

$dy = -dx$ where $x$ goes from 2 to 0

$$\int_C \vec{A}\cdot d\vec{r} = \int_{x=2}^{0} x(2-x)\,dx + x^2(2-x)^2(-dx)$$

$$= \int_2^0 (2x - x^2)\,dx - x^2(4 + x^2 - 4x)\,dx$$

$$= \int_2^0 (2x - x^2)\,dx - \int_2^0 (4x^2 + x^4 - 4x^3)\,dx$$

$$\int_C \vec{A}\cdot d\vec{r} = \left|\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right|_2^0 - \left|\frac{4x^3}{3} + \frac{x^5}{5} - \frac{\cancel{4}x^4}{\cancel{4}}\right|_2^0$$

$$= -4 + \frac{8}{3} + \frac{32}{3} + \frac{32}{5} - 16$$

$$= \frac{40}{3} + \frac{32}{5} - 20$$

$$= \frac{200 + 96 - 300}{15} = \frac{-4}{15}$$

Hence

$$\int_C \vec{A}\cdot d\vec{r} = \frac{-4}{15} \quad \text{Ans}$$

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ii) $x^2+y^2=4$ point $(2,0),(0,2)$

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Solution: $\vec{A} = xy\hat{i} + x^2y^2\hat{j}$

Since Integration performed in xy-plane So $(z=0)$

We can take

$\vec{r} = x\hat{i} + y\hat{j}$

Now

$$\int_C \vec{A}\cdot d\vec{r} = \int_C (xy\hat{i} + x^2y^2\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_C xy\,dx + x^2y^2\,dy \quad \text{---(I)}$$

$\Rightarrow$ By parametric equation of circle

$x = r\cos\theta \quad , \quad y = r\sin\theta$

$x^2+y^2=4$

$x^2+y^2=2^2$ Here $r=2$

So

$x = 2\cos\theta \quad , \quad y = 2\sin\theta$

$dx = -2\sin\theta\, d\theta \quad , \quad dy = 2\cos\theta\, d\theta$

$\Rightarrow$ By Substituting Values in I)

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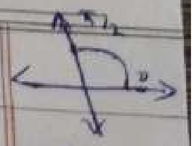

$$\int_C \vec{A}\cdot d\vec{r} = \int_0^{\pi/2} (2\cos\theta \cdot 2\sin\theta)(-2\sin\theta\, d\theta) + (2\cos\theta)^2 \cdot (2\sin\theta)^2 (2\cos\theta)\, d\theta$$ for quarter circle

$$= \int_0^{\pi/2} -8\sin^2\theta \cos\theta\, d\theta + \int_0^{\pi/2} 4\cos^2\theta \cdot 4\sin^2\theta (2\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} -8\sin^2\theta \cos\theta\, d\theta + 32\int_0^{\pi/2} \cos^3\theta \sin^2\theta\, d\theta$$

$$= \int_0^{\pi/2} -8\sin^2\theta \cos\theta\, d\theta + 32\int_0^{\pi/2} \cos^3\theta (1-\cos^2\theta)\, d\theta$$

$$= \int_0^{\pi/2} -8\sin^2\theta \cos\theta\, d\theta + 32\int_0^{\pi/2} \cos^3\theta\, d\theta - 32\int_0^{\pi/2} \cos^5\theta\, d\theta$$

$$= -8\int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta + 32\int_0^{\pi/2} \cos^3\theta\, d\theta - 32\int_0^{\pi/2} \cos^5\theta\, d\theta$$

$\Rightarrow$ By value the

$$= -8\left|\frac{\sin^3\theta}{3}\right|_0^{\pi/2} + 32\left(\frac{2}{3}\right) - 32\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)$$

$$= \frac{-8}{3} + \frac{64}{3} - \frac{256}{15}$$

$$= \frac{-40+320-256}{15} = \frac{24}{15}$$

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$$\int_C \vec{A}\cdot d\vec{r} = \frac{-40+320-256}{15}$$

$$= \frac{24}{15} = \frac{8}{5}$$

Hence

$$\int_C \vec{A}\cdot d\vec{r} = \frac{8}{5} \quad \text{Ans}$$

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**Q#4** Find the work done in moving a particle in the force field $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy)\hat{j}$ along parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.

Note: اس Q کو y میں convert کرنا زیادہ easy ہے چونکہ $x = y^2$

**Solution:** $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy)\hat{j}$

$y^2 = x$ from $(0,0)$ to $(1,1)$

Now

Consider $\vec{r} = x\hat{i} + y\hat{j}$

So $d\vec{r} = dx\hat{i} + dy\hat{j}$

Take

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left((x^2 - y^2 + x)\hat{i} - (2xy)\hat{j}\right)\cdot(dx\hat{i} + dy\hat{j})$$

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$$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2 - y^2 + x)\,dx - (2xy)\,dy \qquad \text{--- (I)}$$

Since $y^2 = x$

$2y\,dy = dx$

and $y$ varies from 0 to 1

$\Rightarrow$ By using in (I)

$$\int_C \vec{F}\cdot d\vec{r} = \int_{y=0}^{1} (y^4 - y^2 + \cancel{y^2})\,2y\,dy - (2y^2 y)\,dy$$

$$= \int_0^1 2y^5\,dy - 2y^3\,dy$$

$$= \int_0^1 (2y^5 - 2y^3)\,dy$$

$$= \left| \frac{2y^6}{6} - \frac{2y^4}{4} \right|_0^1 \Rightarrow \frac{1}{3} - \frac{1}{2}$$

$$= \frac{2-3}{6} = -\frac{1}{6}$$

Now

$$\int_C \vec{F}\cdot d\vec{r} = -\frac{1}{6} \qquad \underline{\text{Ans}}$$

"الٰہی تو قادرِ مطلق ہے مولا میری مشکل آساں کر دے
کشتی ساحل کی طرف نہ سہی ساحل کشتی کی طرف کر دے"

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**Q#5** If $\vec{A} = (2x + y^2)\hat{i} + (3y - 4x)\hat{j}$, evaluate $\oint_C \vec{A} \cdot d\vec{r}$ around the triangle C consisting the line segment.

$C_1$ from $(0,0)$ to $(2,0)$

the line segment $C_2$ from $(2,0)$ to $(2,1)$

the line segment $C_3$ from $(2,1)$ to $(0,0)$

**Solution:** $\vec{A} = (2x + y^2)\hat{i} + (3y - 4x)\hat{j}$

Consider

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

Now

$$\oint_C \vec{A} \cdot d\vec{r} = \int_C (2x + y^2)dx + (3y - 4x)dy \longrightarrow \text{(I)}$$

Since the triangle consisting the line segment $C_1$, $C_2$ and $C_3$ therefore

$$\oint_C \vec{A} \cdot d\vec{r} = \int_{C_1} \vec{A} \cdot d\vec{r} + \int_{C_2} \vec{A} \cdot d\vec{r} + \int_{C_3} \vec{A} \cdot d\vec{r} \longrightarrow \text{II)}$$

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Along the line Segment $C_1$.
from $(0,0)$ to $(2,0)$

Here $y = 0$ | and $x$ varies
$dy = 0$ | from 0 to 2

$\Rightarrow$ By Substituting in I)

We have

$$\int_{C_1} \vec{A} \cdot d\vec{r} = \int_{x=0}^{2} (2x + 0)\,dx + (0 - 4x)\,0$$

$$= \int_0^2 2x\,dx = \left| \frac{2x^2}{2} \right|_0^2 = 4$$

So

$$\int_{C_1} \vec{A} \cdot d\vec{r} = 4 \quad \_\_\_\_ \text{A)}$$

Along the line Segment $C_2$.
from $(2,0)$ to $(2,1)$

Here $x = 2$ | and $y$ varies
$dx = 0$ | from 0 to 1

$\Rightarrow$ By substituting in I)

We have

$$\int_{C_2} \vec{A} \cdot d\vec{r} = \int_{y=0}^{1} (2(2) + y^2)\,0 + (3y - 4(2))\,dy$$

$$= \int_0^1 (3y - 8)\,dy$$

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$$\int_C \vec{A}\cdot d\vec{r} = \left. \frac{3y^2}{2} - 8y \right|_0$$

$$= \frac{3}{2} - 8 \Rightarrow \frac{3-16}{2} = -\frac{13}{2} \quad —8)$$

## Along the line Segment $C_3$

from $(2,1)$ to $(0,0)$

Now

The caluation of the line Segment $C_3$

⇒ By

$$y - y_1 = m(x - x_1)$$

$$\because m = \frac{y_2 - y_1}{x_2 - x_1}$$

So

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 1 = \frac{0-1}{0-2}(x - 2)$$

$$y - 1 = \frac{1}{2}(x - 2)$$

$$y = \frac{x-2}{2} + 1$$

$$y = \frac{x-2+2}{2} \Rightarrow \frac{x}{2}$$

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$$y = \frac{x}{2}$$

Here $x = 2y$

and $dx = 2dy$

and $y$ varies from 1 to 0

Take

$$\int_{C_3} \vec{A} \circ d\vec{r} = \int_{C_3} (2x + y^2)\,dx + (3y - 4x)\,dy$$

$\Rightarrow$ By using above values

$$\int_{C_3} \vec{A} \circ d\vec{r} = \int_{y=1}^{0} (2(2y) + y^2).2dy + (3y - 4(2y))dy$$

$$= \int_{1}^{0} (\cancel{8y} + 2y^2 + 3y - \cancel{8y})\,dy$$

$$= \int_{1}^{0} (2y^2 + 3y)\,dy$$

$$= \left| \frac{2y^3}{3} + \frac{3y^2}{2} \right|_{1}^{0} \Rightarrow 0 - \left(\frac{2}{3} + \frac{3}{2}\right)$$

$$= -\frac{2}{3} - \frac{3}{2} \Rightarrow \frac{-4-9}{6} = \frac{-13}{6}$$

Hence

$$\int_{C_3} \vec{A} \circ d\vec{r} = -\frac{13}{6} \quad \text{———} \quad c)$$

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$\Rightarrow$ By using A), B) and C) in II) we have

$$\oint_C \vec{A}\cdot d\vec{r} = \int_{C_1} \vec{A}\cdot d\vec{r} + \int_{C_2} \vec{A}\cdot d\vec{r} + \int_{C_3} \vec{A}\cdot d\vec{r}$$

$$= 4 - \frac{13}{2} - \frac{13}{6} = \frac{24-39-13}{6}$$

$$= \frac{24-52}{6} \Rightarrow -\frac{28}{6} \Rightarrow -\frac{14}{3}$$

Hence

$$\oint_C \vec{A}\cdot dr = -\frac{14}{3} \quad \text{Ans}$$

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**Q#6** If $\vec{A} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$ evaluate $\oint_C \vec{A}\cdot d\vec{r}$ where C is the rectangle with vertices $(0,0), (1,0), (1,\frac{\pi}{2}), (0,\frac{\pi}{2})$.

**Solution:**

As $\vec{A} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$

&

$(0,0), (1,0), (1,\frac{\pi}{2}), (0,\frac{\pi}{2})$

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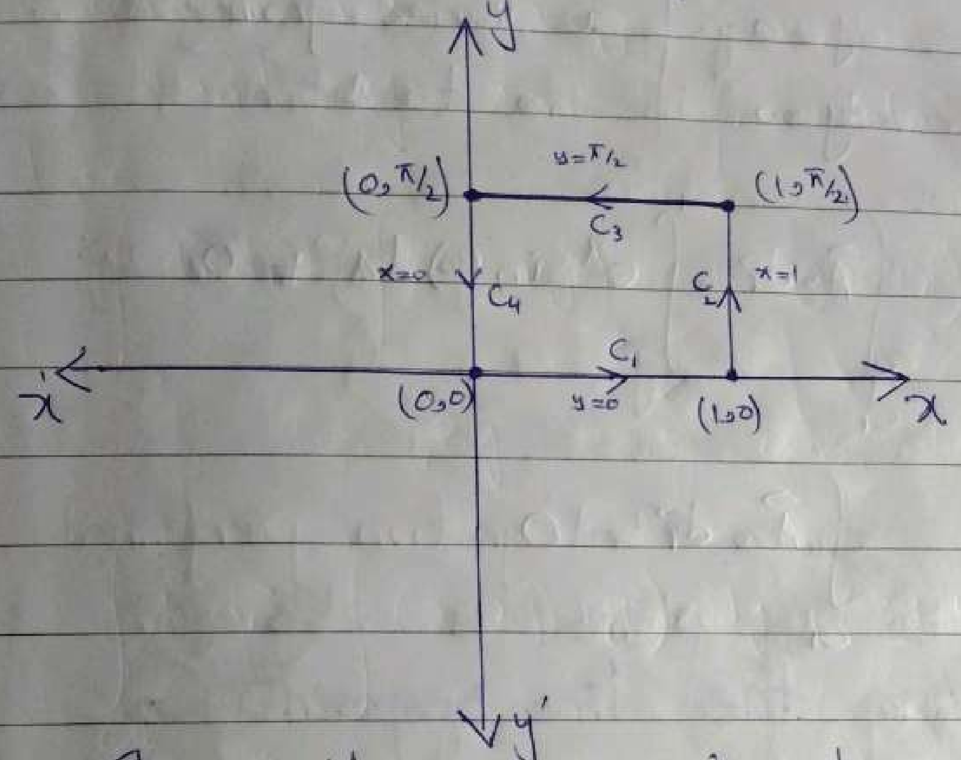
The rectangle by the given vertices is given as




⇒ Since the rectangle has four vertices therefore

$$\oint_C \vec{A}\cdot d\vec{r} = \int_{C_1}^{\rightarrow} \vec{A}\cdot d\vec{r} + \int_{C_2}^{\uparrow} \vec{A}\cdot d\vec{r} + \int_{C_3}^{\leftarrow} \vec{A}\cdot d\vec{r} + \int_{C_4}^{\downarrow} \vec{A}\cdot d\vec{r}$$

Now

$$\int_C \vec{A}\cdot d\vec{r} = \int_C \left(e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}\right)\cdot\left(dx\,\hat{i} + d\right.$$

$$= \int_C (e^x \sin y)\,dx + (e^x \cos y)\,dy$$

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## Along the line Segment $C_1$

Since

$y = 0$ | $x$ varies from
$dy = 0$ | 0 to 1

So

$$\int_{C_1} \vec{A} \cdot d\vec{r} = \int_0^1 e^x \sin(0)\, dx + 0$$

$$\int_{C_1} \vec{A} \cdot d\vec{r} = 0$$

## Along the line Segment $C_2$

Since

$x = 1$ | $y$ varies from
$dx = 0$ | 0 to $\pi/2$

So

$$\int_{C_2} \vec{A} \cdot d\vec{r} = \int_0^{\pi/2} 0 + e \cos y\, dy$$

$$= e \left| \sin y \right|_0^{\pi/2} = e$$

## Along the line Segment $C_3$

Since $y = \pi/2$ | $x$ varies from
$dy = 0$ | 1 to 0

So

$$\int_{C_3} \vec{A}\cdot d\vec{r} = \int_1^0 e^x \sin\left(\frac{\pi}{2}\right) dx$$

$$= \int_1^0 e^x dx = \left|e^x\right|_1^0 = e^0 - e^1$$

$$\int \vec{A}\cdot d\vec{r} = 1 - e$$

**Along the line segment $C_4$.**

Since $x = 0$ | $y$ varies from

$dx = 0$ | $\pi/2$ to $0$

So

$$\int_{C_4} \vec{A}\cdot d\vec{r} = \int_{\pi/2}^0 \cos y\, dy = \left|\sin y\right|_{\pi/2}^0 = -1$$

$\Rightarrow$ By using values in I)

We have

$$\oint_C \vec{A}\cdot d\vec{r} = 0 + \cancel{e} + \cancel{1} - \cancel{e} - \cancel{1}$$

Hence

$$\oint_C \vec{A}\cdot d\vec{r} = 0 \quad \underline{\text{Ans}}$$

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**Q#7** If $\vec{A} = (y-2x)\hat{i} + (3x+2y)\hat{j}$. evaluate $\oint_C \vec{A}\cdot d\vec{r}$ where C is a circle $x^2+y^2=4$ in the xy-plane traversed in the counter clock-wise direction.

**Solution:** $\vec{A} = (y-2x)\hat{i} + (3x+2y)\hat{j}$

$\Leftrightarrow \quad x^2+y^2=4$

Since Integration performed in xy-plane So $z=0$

Take

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

Now

$$\oint_C \vec{A}\cdot d\vec{r} = \oint_C (y-2x)dx + (3x+2y)dy \qquad \text{——— (I)}$$

$\Rightarrow$ By parametric equation of circle

$$x = r\cos\theta \;,\; y = r\sin\theta$$

$$\because r=2$$

$$x = 2\cos\theta \;,\; y = 2\sin\theta$$

$$dx = -2\sin\theta\, d\theta \;,\; dy = 2\cos\theta\, d\theta$$

and $\qquad 0 \le \theta \le 2\pi$

By substituting values in I)

$$\oint \vec{A}\cdot d\vec{r} = \int_0^{2\pi} (2\sin\theta - 4\cos\theta)(-2\sin\theta)\,d\theta + (6\cos\theta + 4\sin\theta)(2\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} (-4\sin^2\theta + 8\sin\theta\cos\theta + 12\cos^2\theta + 8\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} (-4(1-\cos^2\theta) + 16\sin\theta\cos\theta + 12\cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi} (-4 + 4\cos^2\theta + 16\cos\theta\sin\theta + 12\cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi} (16\cos^2\theta + 16\cos\theta\sin\theta - 4)\,d\theta$$

$$= \int_0^{2\pi} 16\left(\frac{1+\cos 2\theta}{2}\right)d\theta + 16\int_0^{2\pi}(\sin\theta\cos\theta)\,d\theta - \int_0^{2\pi} 4\,d\theta$$

$$= \left|8\left(\theta + \frac{\sin 2\theta}{2}\right)\right|_0^{2\pi} + \left|\frac{\sin^2\theta}{2}\right|_0^{2\pi} - 4\left|\theta\right|_0^{2\pi}$$

$$= 16\pi + 0 - 8\pi \Rightarrow 8\pi$$

Hence

$$\oint \vec{A}\cdot d\vec{r} = 8\pi \quad \text{Ans}$$

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**Q#8** Find the work done by the force field $\vec{F} = y^2 x\hat{i} + 2x\hat{j}$ in moving a particle once around the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the counter clockwise direction.

**Solution:**

$$\vec{F} = -y^2 x\hat{i} + 2x\hat{j}$$

& $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow$ The work done by given force field is given as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left(-y^2 x\hat{i} + 2x\hat{j}\right) \cdot \left(dx\hat{i} + dy\hat{j}\right)$$

$$= \int_C (-y^2 x)\,dx + 2x\,dy \quad \text{--- (I)}$$

$\Rightarrow$ The parametric equations of ellipse are

$$x = a\cos\theta \quad , \quad y = b\sin\theta$$

$$dx = -a\sin\theta\,d\theta \quad , \quad dy = b\cos\theta\,d\theta$$

and

$$0 \leq \theta \leq 2\pi$$

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⇒ By substituting values in I)

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (-b^2\sin^2\theta \cdot a\cos\theta)(-a\sin\theta)d\theta + (2a\cos\theta)(b\cos\theta)d\theta$$

$$= \int_0^{2\pi} (a^2b^2\sin^3\theta\cos\theta)d\theta + (2ab\cos^2\theta)d\theta$$

$$= \int_0^{2\pi} (a^2b^2\sin^3\theta\cos\theta)d\theta + \int_0^{2\pi} 2ab\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \int_0^{2\pi} (a^2b^2\sin^3\theta\cos\theta)d\theta + ab\int_0^{2\pi}(1+\cos 2\theta)d\theta$$

$$= a^2b^2\left|\frac{\sin^4\theta}{4}\right|_0^{2\pi} + ab\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= 0 + 2\pi ab$$

Hence

$$\int_C \vec{F}\cdot d\vec{r} = 2\pi ab \quad \underline{\text{Ans}}$$

**Q#9** If $\vec{A} = yz\hat{i} + zx\hat{j} + xy\hat{k}$, evaluate $\int_C \vec{A}\cdot d\vec{r}$ where C is the straight line from (0, 0, 0) to (1, 1, 1).

**Solution:** $\vec{A} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

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points $(0,0,0)$ to $(1,1,1)$

Now

$$\int_C \vec{A}\cdot d\vec{r} = \int_C yz\,dx + zx\,dy + xy\,dz \quad \text{—— (I)}$$

Now $\Rightarrow x = x_1 + (x_2 - x_1)t$

$$x = 0 + (1-0)t$$
$$x = t$$
$$dx = dt$$

Also $\Rightarrow y = y_1 + (y_2 - y_1)t$

$$y = 0 + (1-0)t$$
$$y = t$$
$$dy = dt$$

Also $\Rightarrow z = z_1 + (z_2 - z_1)t$

$$z = 0 + (1-0)t = t$$
$$dz = dt \quad \text{and} \quad 0 \le t \le 1$$

$\Rightarrow$ By using values in I)

$$\int_C \vec{A}\cdot d\vec{r} = \int_0^1 t\cdot t\,dt + t\cdot t\,dt + t\,t\,dt$$

$$= \int_0^1 (t^2 + t^2 + t^2)\,dt \Rightarrow \int_0^1 3t^2\,dt$$

$$\int_C \vec{A}\cdot d\vec{r} = \left|\frac{3t^3}{3}\right|_0^1 = 1$$

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Hence $\int_C \vec{A}\cdot d\vec{r} = 1$ Ans

Q#10 If $\vec{A} = z\hat{i} + x\hat{j} + y\hat{k}$, evaluate $\int_C \vec{A}\cdot d\vec{r}$ ⇒ where C is the curve $x = \cos t$, $y = \sin t$ $z = 3t$ and $0 \le t \le 2\pi$.

**Solution:** $\vec{A} = z\hat{i} + x\hat{j} + y\hat{k}$

$$x = \cos t \,,\; y = \sin t \,,\; z = 3t$$

$$0 \le t \le 2\pi$$

Now $\int_C \vec{A}\cdot d\vec{r} = \int_C z\,dx + x\,dy + y\,dz \longrightarrow \text{I})$

⇒ $dx = -\sin t\,dt$, $dy = \cos t\,dt$, $dz = 3dt$

using values in I)

$$\int_C \vec{A}\cdot d\vec{r} = \int_0^{2\pi} 3t(-\sin t)dt + \cos t(\cos t)dt + \sin t.(3dt)$$

$$= \int_0^{2\pi} -3t\sin t.dt + \cos^2 t\,dt + 3\sin t\,dt$$

$$= -3\int_0^{2\pi} \underbrace{t.\sin t\,dt}_{\text{by parts}} + \int_0^{2\pi} \frac{1+\cos 2t}{2}dt + 3\int \sin t\,d$$

Now we find $-3\int t.\sin t\,dt$

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$$-3\int_0^{2\pi} t.\sin t\,dt = -3\left[t\int \sin t\,dt - \int\left(\frac{d}{dt}t\int \sin t\right)dt\right]_0^{2\pi}$$

$$= -3\left[t(-\cos t) - \int 1(-\cos t)\,dt\right]_0^{2\pi}$$

$$= -3\left[-t\cos t + \sin t\right]_0^{2\pi}$$

$$= -3(-2\pi) = 6\pi$$

using in II)

$$\int_C \vec{A}\cdot d\vec{r} = 6\pi + \frac{1}{2}\left|t + \frac{\sin 2t}{2}\right|_0^{2\pi} + 3\left|-\cos t\right|_0^{2\pi}$$

$$= 6\pi + \frac{1}{\cancel{2}}(\cancel{2}\pi) + 3(-1+1)$$

$$= 6\pi + \pi = 7\pi$$

Hence

$$\int_C \vec{A}\cdot d\vec{r} = 7\pi \quad \text{Ans}$$

**Q#11** If $\vec{A} = x^2\hat{i} + y\hat{j} + xyz\hat{k}$, evaluate $\int_c \vec{A}\cdot d\vec{r}$ along the following paths.

i) $x=t$, $y=t^2$, $z=t^3$ from $t=0$, $t=1$

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## Solution:

$$\vec{A} = x^2\hat{i} + y\hat{j} + xyz\hat{k}$$

Now

$$\int_C \vec{A}\cdot d\vec{r} = \int_C x^2 dx + y dy + xyz dz \quad \text{— (I)}$$

As

$$x = t \Rightarrow dx = dt$$

$$y = t^2 \Rightarrow dy = 2t\,dt$$

$$z = t^3 \Rightarrow dz = 3t^2 dt$$

& $t=0, t=1$ using in I)

$$\int_C \vec{A}\cdot d\vec{r} = \int_0^1 t^2 dt + t^2 . 2t dt + t . t^2 . t^3 . 3t^2 dt$$

$$= \int_0^1 (t^2 + 2t^3 + 3t^8)\, dt$$

$$= \left| \frac{t^3}{3} + \frac{2t^4}{4} + \frac{3t^9}{9} \right|_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{3}{9} \Rightarrow \frac{1}{3} + \frac{1}{2} + \frac{1}{3}$$

$$= \frac{2+3+2}{6} = \frac{7}{6}$$

Hence

i) $\int_C \vec{A}\cdot d\vec{r} = \frac{7}{6}$ Ans

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$$\int_{C_1} \vec{A}\cdot d\vec{r} = \int_0^1 x^2\,dx = \left|\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

**Along the line Segment $C_2$.**

$(1,0,0)$ to $(1,1,0)$

Here $x=1$, $z=0$ | $y$ varies
$dx=0$ $dz=0$ | from 0 to 1

So

$$\int_{C_2} \vec{A}\cdot d\vec{r} = \int_0^1 0 + y\,dy + 0$$

$$= \left|y^2/2\right|_0^1 \Rightarrow \frac{1}{2}$$

**Along the line Segment $C_3$.**

$(1,1,0)$ to $(1,1,1)$

Here $x=1$, $y=1$ | $z$ varies from
$dx=0$, $dy=0$ | 0 to 1

So

$$\int_{C_3} \vec{A}\cdot d\vec{r} = \int_0^1 0 + 0 + z\,dz$$

$$= \left|\frac{z^2}{2}\right|_0^1 = \frac{1}{2}$$

Hence

$$\int_C \vec{A}\cdot d\vec{r} = \frac{1}{3} + \frac{1}{2} + \frac{1}{2} = \frac{2+3+3}{6}$$

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Hence

$$\therefore \int_C \vec{A}\cdot d\vec{r} = \frac{4}{3} \quad \text{Ans}$$

**Q#12** If $\vec{A} = (x^2+y^2)y\hat{i} - (x^2+y^2)x\hat{j} + (a^3+z^3)\hat{k}$, evaluate $\oint_C \vec{A}\cdot d\vec{r}$ where C is the circle $x^2+y^2=a^2$, $z=0$.

**Solution:** $\vec{A} = (x^2+y^2)y\hat{i} - (x^2+y^2)x\hat{j} + (a^3+z^3)\hat{k}$

and $x^2+y^2 = a^2$

Now

$$\oint_C \vec{A}\cdot d\vec{r} = \int_C (x^2+y^2)y\,dx - (x^2+y^2)x\,dy + (a^3+z^3)\,dz \quad \text{--- (I)}$$

$\Rightarrow$ By parametric equation of circle are

$x = r\cos\theta$ , $y = r\sin\theta$

$x = a\cos\theta$ , $y = a\sin\theta$ $\quad \because r=a$

$dx = -a\sin\theta\, d\theta$ , $dy = a\cos\theta\, d\theta$

$\because z=0 \Rightarrow dz = 0$

and $0 \le \theta \le 2\pi$

$\Rightarrow$ By substituting values in I) we have

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$z = 4t^2 - 1$ from $t = 0$ to $t = 1$.

**Solution:** $\vec{F} = 3x^2 + (2xz - y)\hat{j} + z\hat{k}$

Let $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C 3x^2 dx + (2xz - y)\,dy + z\,dz \quad \text{---} \; \text{(I)}$$

Since

$$x = 2t^2 \quad \Big| \quad y = t \quad \Big| \quad z = 4t^2 - 1$$

$$dx = 4t\,dt \quad \Big| \quad dy = dt \quad \Big| \quad dz = 8t\,dt$$

$$0 \leq t \leq 1$$

$\Rightarrow$ using values in I)

$$\int_C \vec{A} \cdot d\vec{r} = \int_0^1 3(2t^2)^2 \overset{\overset{dx}{\uparrow}}{4t\,dt} + (2(2t^2)(4t^2 - 1) - t)\,dt + (4t^2 - 1)(8t)\,dt$$

$$= \int_0^1 48t^5 dt + (16t^4 - 4t^2 - t)\,dt + (32t^3 - 8t)\,dt$$

$$= \int_0^1 (48t^5 + 16t^4 + 32t^3 - 4t^2 - 9t)\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (48t^5 + 16t^4 + 32t^3 - 4t^2 - 9t)\,dt$$

$$= \left| \frac{48t^6}{6} + \frac{16t^5}{5} + \frac{32t^4}{4} - \frac{4t^3}{3} - \frac{9t^2}{2} \right|_0^1$$

$$= 8 + \frac{16}{5} + 8 - \frac{4}{3} - \frac{9}{2}$$

$$= 16 + \frac{16}{5} - \frac{4}{3} - \frac{9}{2}$$

$$= \frac{80 + 16}{5} - \frac{8 - 27}{6} \Rightarrow \frac{96}{5} + \frac{19}{6}$$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{576 + 96}{30} \Rightarrow \frac{671}{30} \rightarrow 22.3 \text{ Ans}$$

**Q#14** Show that $\vec{A} = (4xy - 3x^2z^2)\hat{i} + 2x^2\hat{j} - 2x^3z\hat{k}$, is a conservative vector field. Find the Scalar potential function $\phi$ Such that $\vec{A} = \nabla\phi$.

**Solution:**

$$\vec{A} = (4xy - 3x^2z^2)\hat{i} + 2x^2\hat{j} - 2x^3\hat{k}$$

As $\vec{A} = \nabla\phi$

$\Rightarrow$ The condition for

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Conservative force field is that

$\nabla \times$ (any vector)

$\nabla \times \vec{A} = 0$

$\nabla \times \vec{F} = 0$

$$\nabla \times \vec{A} = 0$$

Now

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (4xy - 3x^2z^2) & 2x^2 & -2x^3z \end{vmatrix}$$

$$\nabla \times \vec{A} = (0-0)\hat{i} + \hat{j}(\cancel{-6x^2z} + \cancel{6x^2z}) + \hat{k}(\cancel{4x} - \cancel{4x})$$

$$\nabla \times \vec{A} = 0$$

Which show that $\vec{A}$ is conservative vector field. Ans

Now

We find Scalar function $\phi$

As

$$\nabla \phi = \vec{A}$$

$$\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = (4xy - 3x^2z^2)\hat{i} + 2x^2\hat{j} - 2x^3z\hat{k}$$

$\Rightarrow$ By comparing we get these values

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$$\frac{\partial \phi}{\partial x} = 4xy - 3x^2z^2 \quad \text{--- I)}$$

$$\frac{\partial \phi}{\partial y} = 2x^2 \quad \text{--- II)}$$

$$\frac{\partial \phi}{\partial z} = -2x^3z \quad \text{--- III)}$$

Integrating I, II, III)

We get

$$\phi = 2x^2y - x^3z^2 + f(y,z)$$

$$\phi = 2x^2y + g(x,z)$$

$$\phi = -x^3z + h(x,y)$$

these agree if we choose

Note

$$f(y,z) = 0 \ , \quad g(x,z) = -x^3z$$

$$h(x,y) = 2x^2y$$

Hence we get

$$\phi = 2x^2y - x^3z^2 + \text{constant} \quad \text{Ans}$$

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**Q#15** Prove that $\vec{A} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$ is a conservative force field. Find the Scalar Potential function for the $\vec{A}$.

**Solution:** $\vec{A} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$

$\Rightarrow$ The condition for conservative force field is that

$$\nabla \times \vec{A} = 0$$

Now

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$\nabla \times \vec{A} = \hat{i}(0-0) + \hat{j}(\cancel{3z^2} - \cancel{3z^2}) + \hat{k}(\cancel{2y\cos x} - \cancel{2y\cos x})$$

$$\nabla \times \vec{A} = 0$$

Which show that $\vec{A}$ is Conservative force field.

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Now we find Scalar potential function for $\vec{A}$

Since

$$\nabla \phi = \vec{A}$$

$$\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = (y^2 \cos x + z^3)\hat{i} + (2y \sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$$

$\Rightarrow$ By comparing we get

$$\frac{\partial \phi}{\partial x} = y^2 \cos x + z^3 \quad \text{——— I)}$$

$$\frac{\partial \phi}{\partial y} = 2y \sin x - 4 \quad \text{——— II)}$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 + 2 \quad \text{——— III)}$$

Integrating I), II), III) we

we get

$$\phi = y^2 \sin x + z^3 x + f(y, z)$$

$$\phi = y^2 \sin x - 4y + g(x, z)$$

$$\phi = xz^3 + 2z + h(x, y)$$

these agree if we choose

$$f(y, z) = -4y + 2z$$

$$g(x, z) = z^3 x + 2z$$

$$h(x, y) = y^2 \sin x - 4y$$

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Hence

$$\phi = y^2 \sin x + xz^2 - 4y + 2z + \text{Constant}$$ Ans

# Q#16 prove that each of the following vector field is Conservative and find the Scalar potential in each case.

i) $\vec{A} = r^2 \vec{r}$ ii) $\vec{A} = \frac{\vec{r}}{r^3}$

## Solution

Note

i) $\vec{A} = r^2 \vec{r}$

ractice

$$\vec{A} = (x^2+y^2+z^2)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\vec{A} = (x^2+y^2+z^2)x\hat{i} + (x^2+y^2+z^2)y\hat{j} + (x^2+y^2+z^2)z\hat{k}$$

Vector field is conservative

if $\nabla \times \vec{A} = 0$

Now

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2+z^2)x & (x^2+y^2+z^2)y & (x^2+y^2+z^2)z \end{vmatrix}$$

$$\nabla \times \vec{A} = (\cancel{2yz} - \cancel{2yz})\hat{i} + \hat{j}(\cancel{2xz} - \cancel{2xz}) + \hat{k}(\cancel{2xy} - \cancel{2xy})$$

$$\nabla \times \vec{A} = 0$$

Hence $\vec{A}$ is conservative vector field.

⇒ Now we find scalar function $\phi$

As $\nabla \phi = \vec{A}$

We know that

Learn → $$\frac{d\phi}{dr} = \nabla \phi \cdot \hat{r}$$

Hence

$$\frac{d\phi}{dr} = \vec{A} \cdot \hat{r} \qquad \because \vec{A} = r^2 \vec{r}$$

$$\frac{d\phi}{dr} = r^2 \vec{r} \cdot \hat{r} \qquad \because \hat{r} = \frac{\vec{r}}{r}$$

$$\frac{d\phi}{dr} = r^2 . \vec{r} . \frac{\vec{r}}{r} \Rightarrow \frac{r^2 . r^2}{r}$$

$$\frac{d\phi}{dr} = r^3$$ Integrating w.r.t r

We get

$$\phi = \frac{r^4}{4} + \text{constant}$$ Ans

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**Q#17** Show that $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\, dz$ is an exact differential of a scalar function $\phi$ and find $\phi$.

[illegible] $\vec{A}\cdot d\vec{r}$ [illegible] $\vec{A}$

**Solution:** Here we have

$$\vec{A} = (2x\cos y + z\sin y)\hat{i} + (xz\cos y - x^2\sin y)\hat{j} + (x\sin y)\hat{k}$$

$\Rightarrow$ For Exact differential

$$\nabla \times \vec{A} = 0$$

Now

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x\cos y + z\sin y) & (xz\cos y - x^2\sin y) & x\sin y \end{vmatrix}$$

$$\nabla\times\vec{A} = \hat{i}(\cancel{x\cos y} - \cancel{x\cos y}) + \hat{j}(\sin y - \sin y) + \hat{k}(-2x\cancel{\sin y} + z\cancel{\cos y} - z\cancel{\cos y} + \dots$$

$$\nabla\times\vec{A} = 0$$

Now we find Scalar function $\phi$.

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As $\int \vec{A} \cdot d\vec{r} = \int (2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + (x\sin y\, dz)$

**Note**
اس ٹین کے Method والا کرنا زیادہ آسان ہے کیونکہ ہمیں Ans دیا ہوا ہے

$$\int \vec{A} \cdot d\vec{r} = \int d\left(x^2\cos y + zx\sin y + xz\sin y + x^2\cos y + xz\sin y\right)$$

**Note**
جو دیا ہوا ہو وہی لکھنا

Hence

$$\phi = x^2\cos y + xz\sin y + \text{constant} \quad \underline{\text{Ans}}$$

---

**Q#18** If $\phi = x^3y + 2y$, evaluate $\int_C \phi\, d\vec{r}$ along the following paths C:

i) the straight line $y = 3x - 2$ from $(1,1)$ to $(2,4)$.

**Solution:**

$$\phi = x^3y + 2y$$

Let $\quad d\vec{r} = dx\hat{i} + dy\hat{j}$

Now

$$\int_C \phi\, d\vec{r} = \int_C (x^3y + 2y)(dx\hat{i} + dy\hat{j})$$

$$= \hat{i}\int_C (x^3y + 2y)dx + \hat{j}\int_C (x^3y + 2y)dy \quad \text{—(I)}$$

As $\quad y = 3x - 2 \Rightarrow dy = 3dx$

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$x$ Varies from 1 to 2

Using Values in I)

$$\int_C \phi\, d\vec{r} = \hat{i}\int_1^2 x^3(3x-2)\,dx + 2(3x-2)\,dx + \hat{j}\int_1^2 [x^3(3x-2) + 2(3x-2)](3\,dx)$$

$$= \hat{i}\int_1^2 (3x^4 - 2x^3 + 6x - 4)\,dx + \hat{j}\int_1^2 (3x^4 - 2x^3 + 6x - 4)(3\,dx)$$

$$= \hat{i}\int_1^2 (3x^4 - 2x^3 + 6x - 4)\,dx + \hat{j}\int_1^2 (9x^4 - 6x^3 + 18x - 12)\,dx$$

$$= \hat{i}\left|\frac{3x^5}{5} - \frac{2x^4}{4} + \frac{6x^2}{2} - 4x\right|_1^2 + \hat{j}\left|\frac{9x^5}{5} - \frac{6x^4}{4} + \frac{18x^2}{2} - 12x\right|_1^2$$

$$= \hat{i}\left[\left(\frac{96}{5} - 8 + 12 - 8\right) - \left(\frac{3}{5} - \frac{1}{2} + 3 - 4\right)\right]$$

$$+ \hat{j}\left[\left(\frac{288}{5} - \frac{48}{2} + 36 - 24\right) - \left(\frac{9}{5} - \frac{3}{2} + \frac{9}{2} - 12\right)\right]$$

$$= \hat{i}\left(\frac{96}{5} \cancel{-4} - \frac{3}{5} + \frac{1}{2} - 3 \cancel{+4}\right) + \hat{j}\left(\frac{288}{5} - 24 + 36 - 24 - \frac{9}{5} + \frac{3}{2} - \frac{9}{2}\right.$$

$$= \hat{i}\left(\frac{96}{5} - \frac{3}{5} + \frac{1}{2} - 3\right) + \hat{j}\left(\frac{288}{5} - \frac{9}{5} + \frac{3}{2} - \cancel{48} + \cancel{48}\right.$$

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$$\int_C \phi \, d\vec{r} = \hat{i}\left(\frac{192-6+5-30}{10}\right) + \hat{j}\left(\frac{576-18+15}{10}\right)$$

$$\int_C \phi \, d\vec{r} = \hat{i}\left(\frac{161}{10}\right) + \hat{j}\left(\frac{573}{10}\right)$$

Hence

$$\int_C \phi \, d\vec{r} = \frac{161}{10}\hat{i} + \frac{573}{10}\hat{j} \quad \text{Ans}$$

ii) the parabola $y = x^2$ from $(1,1)$ to $(2,4)$.

**Solution:** $\phi = x^3 y + 2y$

Let $d\vec{r} = dx\hat{i} + dy\hat{j}$

Now

$$\int_C \phi \, d\vec{r} = \int_C (x^3y+2y)dx\hat{i} + (x^3y+2y)dy\hat{j}$$

$$= \hat{i}\int_C (x^3y+2y)dx + \hat{j}\int (x^3y+2y)dy$$

Since $y = x^2 \Rightarrow dy = 2x\,dx$

and $y$ varies from $1 \to 2$.

using values in I)

$$\int_C \phi \, d\vec{r} = \hat{i}\int_1^2 (x^3 \cdot x^2 + 2x^2)dx + \hat{j}\int_1^2 (x^3 \cdot x^2 + 2x^2)($$

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$$\int \phi\, d\vec{r} = \hat{i}\int_1^2 (x^5+2x^2)\,dx + \hat{j}\int_1^2 (2x^6+4x^3)\,dx$$

$$= \hat{i}\left|\frac{x^6}{6}+\frac{2x^3}{3}\right|_1^2 + \hat{j}\left|\frac{2x^7}{7}+\frac{4x^4}{4}\right|_1^2$$

$$= \hat{i}\left[\left(\frac{64}{6}+\frac{16}{3}\right)-\left(\frac{1}{6}+\frac{2}{3}\right)\right] + \hat{j}\left[\left(\frac{256}{7}+16\right)-\left(\frac{2}{7}-1\right)\right]$$

$$= \hat{i}\left(\frac{64+32-1+4}{6}\right) + \hat{j}\left(\frac{256-2+119}{7}\right)$$

$$= \hat{i}\left(\frac{\cancel{99}^{\,33}}{\cancel{6}_{\,2}}\right) + \hat{j}\left(\frac{373}{7}\right)$$

Hence

$$\int_C \phi\, d\vec{r} = \frac{33}{2}\hat{i} + \frac{373}{7}\hat{j} \quad \underline{\text{Ans}}$$

**Q#19** If $\phi = 2xy^2z + x^2y$ evaluate $\int_C \phi\, d\vec{r}$ where C

i) is the curve $x=t$ $y=t^2$ $z=t^3$ from $t=0$, $t=1$

Date:

**Solution:** $\phi = 2xy^2z + x^2y$

Let $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Now

$$\int_C \phi \, d\vec{r} = \int_C (2xy^2z + x^2y)(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \hat{i}\int_C (2xy^2z + x^2y)\,dx + \hat{j}\int_C (2xy^2z + x^2y)\,dy \quad \text{— I)}$$

$$+ \hat{k}\int_C (2xy^2z + x^2y)\,dz$$

Since $x = t \Rightarrow dx = dt$

$y = t^2 \Rightarrow dy = 2t\,dt$

$z = t^3 \Rightarrow dz = 3t^2\,dt \quad , \; 0 \le t \le 0$

By using values in I)

$$= \hat{i}\int_0^1 (2\cdot t\cdot t^4\cdot t^3 + t^2 t^2)\,dt + \hat{j}\int_0^1 (2 t\, t^4\, t^3 + t^2 t^2)(2t)$$

$$+ \hat{k}\int_0^1 (2\cdot t\cdot t^4\cdot t^3 + t^2\cdot t^2)(3t^2)\,dt$$

$$= \hat{i}\int_0^1 (2t^8 + t^4)\,dt + \hat{j}\int_0^1 (2t^8 + t^4)(2t)\,dt$$

$$+ \hat{k}\int_0^1 (2t^8 + t^4)(3t^2)\,dt$$

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$$\int_C \phi \, d\vec{r} = \hat{i}\int_0^1 (2t^8 + t^4)\,dt + \hat{j}\int_0^1 (4t^9 + 2t^5)\,dt + \hat{k}\int_0^1 (6t^{10} + 3t^6)\,dt$$

$$= \hat{i}\left|\frac{2t^9}{9} + \frac{t^5}{5}\right|_0^1 + \hat{j}\left|\frac{4t^{10}}{10} + \frac{2t^6}{6}\right|_0^1 + \hat{k}\left|\frac{6t^{11}}{11} + \frac{3t^7}{7}\right|_0^1$$

$$= \hat{i}\left(\frac{2}{9} + \frac{1}{5}\right) + \hat{j}\left(\frac{2}{5} + \frac{2}{6}\right) + \hat{k}\left(\frac{6}{11} + \frac{3}{7}\right)$$

$$= \hat{i}\left(\frac{10+9}{45}\right) + \hat{j}\left(\frac{12+10}{30}\right) + \hat{k}\left(\frac{42+33}{77}\right)$$

$$= \hat{i}\left(\frac{19}{45}\right) + \hat{j}\left(\frac{22}{30}\right) + \left(\frac{75}{77}\right)\hat{k}$$

Hence

$$\int_C \phi \, d\vec{r} = \frac{19}{45}\hat{i} + \frac{11}{15}\hat{j} + \frac{75}{77}\hat{k} \quad A$$

ii) consisting of the line segment $C_1$ from $(0,0,0)$ to $(1,0,0)$, the line segment $C_2$ from $(1,0,0)$ to $(1,1,0)$ and the line segment $C_3$ from $(1,1,0)$ to $(1,1,1)$.

**Solution:** $\phi = 2xy^2z + x^2y$

Let $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Now

$$\int_C \phi \, d\vec{r} = \hat{i}\int_C (2xy^2z + x^2y)\,dx + \hat{j}\int_C (2xy^2z$$

$$+\hat{k}\int_C (2xy^2z + x^2y)\,dz$$

Here we have line segments $C_1$, $C_2$ and $C_3$

Therefore

$$\int_C \phi \, d\vec{r} = \int_{C_1} \phi \, d\vec{r} + \int_{C_2} \phi \, d\vec{r} + \int_{C_3} \phi \, d\vec{r}$$

## Along Line Segment $C_1$:

from $(0,0,0)$ to $(1,0,0)$

Here $y = 0$, $dy = 0$ | $z = 0$, $dz = 0$ | $x$ varies from $0$ to $1$

Hence

$$\int_{C_1} \phi \, d\vec{r} = \int_0^1 (2x(0)(0) + x^2(0))\,dx$$

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$$\int_{C_1} \phi \, d\vec{r} = 0$$

## Along Line Segment $C_2$:

from $(1,0,0)$ to $(1,1,0)$

Here $x = 1$, $dx = 0$ | $z = 0$, $dz = 0$ | $y$ varies from 0 to 1

Hence

$$\int_{C_2} \phi \, d\vec{r} = \hat{j}\int_0^1 (0 + (1)^2 y)\, dy = \hat{j}\int_0^1 y \, dy$$

$$= \hat{j}\int_0^1 y \, dy = \hat{j} \left| \frac{y^2}{2} \right|_0^1 \Rightarrow \frac{1}{2}\hat{j}$$

## Along Line Segment $C_3$:

from $(1,1,0)$ to $(1,1,1)$

Here

$x = 1$, $dx = 0$ | $y = 1$, $dy = 0$ | $z$ varies from 0 to 1

Hence

$$\int_{C_3} \phi \, d\vec{r} = \hat{k}\int_0^1 \left(2(1)(1)^2 z + (1)^2(1)\right) dz$$

$$= \hat{k}\int_0^1 (2z + 1)\, dz$$

$$= \hat{k} \left| \frac{2z^2}{2} + z \right|_0^1 = 2\hat{k}$$

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→ By using values in II)

$$\int_C \phi \, d\vec{r} = 0 + \frac{1}{2}\hat{j} + 2\hat{k}$$

Hence $$\int_C \phi \, d\vec{r} = \frac{1}{2}\hat{j} + 2\hat{k} \text{ Ans}$$

Note diff Q Same Emple#20

## Q#20

If $\vec{A} = 2y\hat{i} - z\hat{j} + x\hat{k}$. evaluat $\int_C \vec{A} \times d\vec{r}$ along the curve defines

$x = \cos t$, $y = \sin t$, $z = 2\cos t$

from $t = 0$ to $t = \frac{\pi}{2}$.

**Solution:** $\vec{A} = 2y\hat{i} - z\hat{j} + x\hat{k}$

$\int_C \vec{A} \times d\vec{r} = ?$

Now $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\vec{A} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$\vec{A} \times d\vec{r} = \hat{i}(-z\,dz - x\,dy) + \hat{j}(x\,dx - 2y\,d$$
$$+ \hat{k}(2y\,dy + z\,dx)$$

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Therefore

$$\int_C \vec{A} \times d\vec{r} = \hat{i}\int_C (-z\,dz - x\,dy) + \hat{j}\int_C (x\,dx - 2y\,dz)$$

$$+ \hat{k}\int_C (2y\,dy + z\,dx) \quad \text{--- (I)}$$

Since

$x = \cos t$ | $y = \sin t$ | $z = 2\cos t$

$dx = -\sin t\,dt$ | $dy = \cos t\,dt$ | $dz = -2\sin t\,dt$

$$0 \leq t \leq \frac{\pi}{2}$$

$\Rightarrow$ By using values in I)

$$\int_C \vec{A} \times d\vec{r} = \hat{i}\int_0^{\pi/2} (-2\cos t)(-2\sin t)dt - \cos t(\cos t)dt$$

$$+ \hat{j}\int_0^{\pi/2} \cos t(-\sin t)dt - 2\sin t(-2\sin t) +$$

$$\hat{k}\int_0^{\pi/2} 2\sin t(\cos t)dt + 2\cos t(-\sin t)dt$$

$$= \hat{i}\int_0^{\pi/2} \left[4\sin t\cos t - \left(\frac{1+\cos 2t}{2}\right)\right] dt +$$

$$\hat{k}\int_0^{\pi/2} \left[-\sin t\cos t + 4\left(\frac{1-\cos 2t}{2}\right)\right] dt$$

$$+\hat{k}\int_0^{\pi/2}\left(2\sin t\cancel{\cos t}-2\sin t\cancel{\cos t}\right)dt$$

$$\int_C \vec{A}\times d\vec{r}=\hat{i}\left|\frac{4\sin^2 t}{2}-\frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\right|_0^{\pi/2}$$

$$+\hat{j}\left|-\frac{\sin^2 t}{2}+2\left(t-\frac{\sin 2t}{2}\right)\right|_0^{\pi/2}$$

$$=\hat{i}\left|2\sin^2 t-\frac{t}{2}-\frac{\sin 2t}{4}\right|_0^{\pi/2}+$$

$$\hat{j}\left|-\frac{\sin^2 t}{2}+2t-\sin 2t\right|_0^{\pi/2}$$

$$\int_C \vec{A}\times d\vec{r}=\hat{i}\left(2-\frac{\pi}{4}-0\right)+\hat{j}\left(-\frac{1}{2}+\pi-0\right)$$

Hence

$$\int_C \vec{A}\times d\vec{r}=\left(2-\frac{\pi}{4}\right)\hat{i}+\left(\pi-\frac{1}{2}\right)\hat{j}\quad \text{An}$$

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# Topic: Vector Analysis

# Normal Surface Integral

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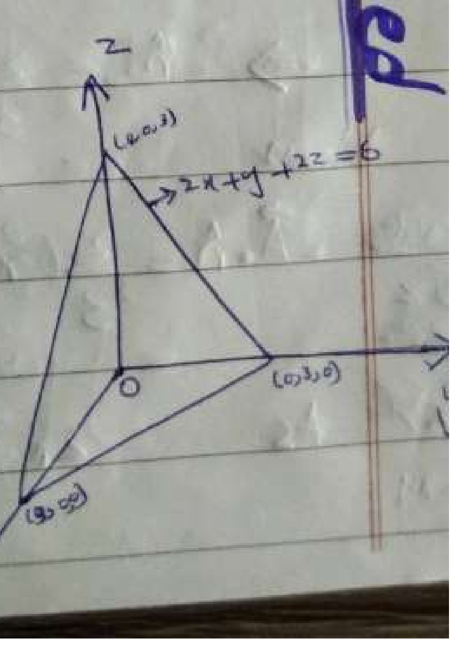
**Q#21** If $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ evaluate $\iint \vec{A}\cdot\hat{n}\,ds$, Where $S$ is the Surface $S$ of the plane $2x+y+2z=6$ in the first octant

**Solution:** $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$

plane $2x+y+2z=6$

We have to find

$$\iint_S \vec{A}\cdot\hat{n}\,ds = ?$$




Take projection in $xy$-plane

So $ds = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

Hence

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_{S} \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \text{--- (I)}$$

As

$$\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$$

Now

$$\text{Surface} = 2x+y+2z = 6$$

$\because$

$$\vec{n} = \nabla(\text{Surface})$$

$$\vec{n} = \nabla(2x+y+2z-6)$$

$$\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$$

So $\qquad \because \hat{n} = \dfrac{\vec{n}}{|n|}$

$$\hat{n} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{(2)^2+(1)^2+(2)^2}}$$

$$\hat{n} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}} \Rightarrow \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

or

$$\Rightarrow \hat{n} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Now

Note

$$\vec{A}\cdot\hat{n} = \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz \quad \text{--- (II)}$$

As $\qquad 2x+y+2z = 6$

$$z = \frac{6-y-2x}{2}$$

$\Leftarrow$ Using in (II) we have

$$\vec{A}\circ\hat{n} = \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}y\left(\frac{6-y-2x}{2}\right)$$

$$\vec{A}\circ\hat{n} = \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{2}{3}(6y-y^2-2xy)$$

$$= \frac{\cancel{2x}+\cancel{2y^2}-\cancel{2x}+12y-\cancel{2y^2}-4xy}{3}$$

$$\vec{A}\circ\hat{n} = \frac{12y-4xy}{3}$$

{اب $\vec{A}\circ\hat{n}$ $xy$ plane میں آ گیا}

Now

$$\boxed{\hat{n}\circ\hat{k} = \frac{2}{3}}$$

$\Rightarrow$ using values in (I)

$$\iint_S \vec{A}\circ\hat{n}\,ds = \iint_R \frac{12y-4xy}{3}\,\frac{dx\,dy}{\left|\frac{2}{3}\right|}$$

$$= \iint_R \frac{2(6y-2xy)}{3}\cdot\frac{3}{2}\,dx\,dy$$

کیونکہ

variable val

$$= \iint_R y(6-2x)\,dx\,dy$$

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$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R y(6-2x)\,dy\,dx \qquad \text{(III)}$$

limit اب نکالی ہیں plane give ہے

As

$$2x+y+z=6$$

In $xy$-plane $z=0$

$$\Rightarrow 2x+y=6$$

For $x$-Intercept put $y=0$

$$\Rightarrow 2x=6$$

$$\boxed{x=3}$$

چونکہ y کی values variable میں لائی ہے اس لیے یہاں درج کی

Also $y=6-2x$

So, $x$ varies from $0$ to $3$

and $y$ varies from $0$ to $(6-2x)$

Using in III) becomes

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \int_0^3\int_0^{(6-2x)} y(6-2x)\,dy\,dx$$

y Just limit لگائی

$$= \int_0^3 \left| \frac{y^2}{2}(6-2x) \right|_0^{(6-2x)}$$

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$$\iint_S \vec{A}\cdot\hat{n}\,ds = \int_0^3 \left|\frac{y^2}{2}\right|_0^{(6-2x)} (6-2x)\,dx$$

$$= \int_0^3 \frac{(6-2x)^2}{2}(6-2x)\,dx$$

$$= \int_0^3 \frac{(6-2x)^2}{\cancel{2}}\cdot \cancel{2}(3-x)\,dx$$

$$= \int_0^3 (36+4x^2-24x)(3-x)\,dx$$

$$= \int_0^3 (108-36x+12x^2-4x^3-72x+24x^2)\,dx$$

$$= \int_0^3 (108-108x+36x^2-4x^3)\,dx$$

$$= \left| 108x - \frac{\overset{54}{\cancel{108}}x^2}{\cancel{2}} + \frac{\overset{12}{\cancel{36}}x^3}{\cancel{3}} - \frac{\cancel{4}x^4}{\cancel{4}}\right|_0^3$$

$$= \left| 108x - 54x^2 + 12x^3 - x^4 \right|_0^3$$

$$= 324 - 486 + 324 - 81 = 81$$

Hence

$$\iint_S \vec{A}\cdot\hat{n}\,ds = 81 \quad \underline{\text{Ans}}$$

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**Q#22** If $\vec{A} = x\hat{i} + y\hat{j} + (z^2-1)\hat{k}$. evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $S$ is the closed surface bounded by the planes $z=0$, $z=1$ and the cylinder $x^2+y^2=a^2$.

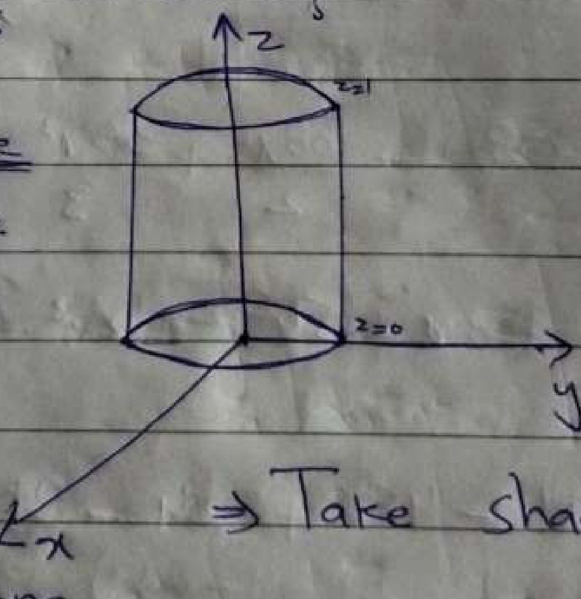
**Note**
... z=0 ... کیونکہ ... تک ... اور $x^2+y^2=a^2$ circle shape میں ہے

**Solution:** $\vec{A} = x\hat{i} + y\hat{j} + (z^2-1)\hat{k}$

$x^2+y^2=a^2 \quad , z=0 \quad , z=1$

We have to find

$$\iint_S \vec{A}\cdot\hat{n}\,ds = ?$$




**Note** اس کا shadow xz یا yz plane میں ہوگا xy plane میں نہیں ہوگا

$\Rightarrow$ Take shadow in xz-plane

So

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

(1)

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As surface $x^2+y^2=a^2$

$$\vec{n} = \nabla(\text{surface}) \Rightarrow \nabla(x^2+y^2-a^2)$$

$$\vec{n} = 2x\hat{i}+2y\hat{j}$$

$$\because \hat{n} = \frac{\vec{n}}{|n|}$$

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{(2x)^2+(2y)^2}} \Rightarrow \frac{\cancel{2}(x\hat{i}+y\hat{j})}{\cancel{2}\sqrt{x^2+y^2}}$$

$$\hat{n} = \frac{x\hat{i}+y\hat{j}}{\sqrt{a^2}} = \frac{x}{a}\hat{i}+\frac{y}{a}\hat{j}$$

Now

$$\vec{A}\cdot\hat{n} = \frac{x^2}{a}+\frac{y^2}{a} \Rightarrow \frac{x^2+y^2}{a} = \frac{a^{\cancel{2}}}{\cancel{a}}$$

$$\vec{A}\cdot\hat{n} = a$$

Now $\hat{n}\cdot\hat{j} = \frac{y}{a}$

using values in I)

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R a\,\frac{dx\,dz}{\left|\frac{y}{a}\right|}$$

hange of y Note

value

$$= a^2\iint_R \frac{1}{y}\,dx\,dz$$

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As $x^2+y^2=a^2$

$$y=\sqrt{a^2-x^2}$$

So

$$\iint_S \vec{A}\cdot\hat{n}\,ds = a^2\iint_R \frac{1}{\sqrt{a^2-x^2}}\,dz\,dx$$

$\{z$
given
$x$
find

For $x$-Intercept put $y=0$

$$x^2=a^2$$

$$x=\pm a$$

Also given $z=0$ to $z=1$

Hence

$$\iint_S \vec{A}\cdot\hat{n}\,ds = a^2\int_{-a}^{a}\int_{0}^{1} \frac{1}{\sqrt{a^2-x^2}}\,dz\,dx$$

$$= a^2\int_{-a}^{a}\left|\frac{1}{\sqrt{a^2-x^2}}z\right|_0^1 dx$$

$$= a^2\int_{-a}^{a}\frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$= a^2\left|\sin^{-1}\left(\frac{a}{x}\right)\right|_{-a}^{a}$$

$$= a^2\left|\sin^{-1}\left(\frac{a}{a}\right)-\sin^{-1}\left(\frac{a}{-a}\right)\right|$$

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<u>**Javeria Abbas Ahmad**</u>

$$\iint_S \vec{A}\cdot\hat{n}\,ds = a^2\left(\sin^{-1}(1) - \sin^{-1}(-1)\right) \qquad \because \sin^{-1}(1) = \frac{\pi}{2}$$

$$= a^2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = a^2\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^2\left(\frac{2\pi}{2}\right) = a^2\pi$$

Hence

$$\iint_S \vec{A}\cdot\hat{n}\,ds = a^2\pi \quad \underline{\text{Ans}}$$

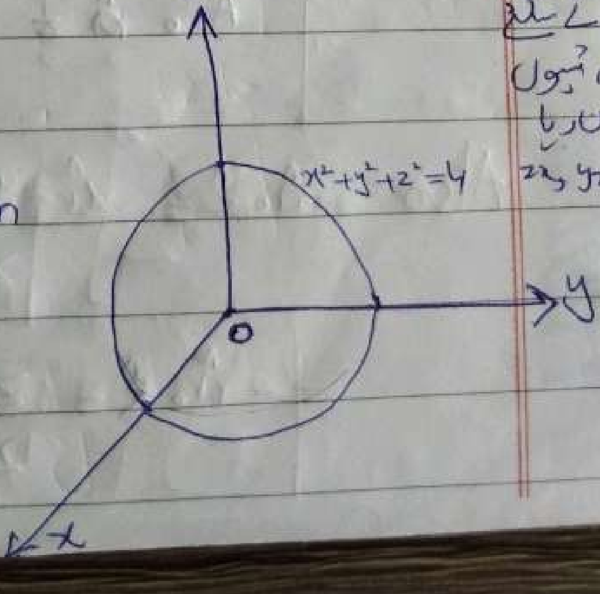
**Q#23** If $\vec{A} = \frac{\vec{r}}{r^2}$, evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $S$ is the surface of the sphere $x^2+y^2+z^2=4$.

<u>**Solution:**</u> $\vec{A} = \frac{\vec{r}}{r^2}$

$x^2+y^2+z^2=4$

We have to find $\iint_S \vec{A}\cdot\hat{n}\,ds = ?$

Take projection of sphere in $xy$-plane.




Note

Sphere

projection

plane

zx, yz

Date:

Hence

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \longrightarrow (1)$$

Now

$$\vec{n} = \nabla(\text{surface})$$

$$\vec{n} = \nabla(x^2+y^2+z^2-4)$$

$$\vec{n} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\because \hat{n} = \frac{\vec{n}}{|n|}$$

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}$$

$$\hat{n} = \frac{\cancel{2}(x\hat{i}+y\hat{j}+z\hat{k})}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{\sqrt{4}}$$

$$\Rightarrow \quad \hat{n} = \frac{\vec{r}}{2}$$

Now

$$\vec{A}\cdot\hat{n} = \frac{\vec{r}}{r^2}\cdot\frac{\vec{r}}{2} \Rightarrow \frac{r^2}{2r^2}$$

$$\boxed{\vec{A}\cdot\hat{n} = \frac{1}{2}}$$

Also $\hat{n}\cdot\hat{k} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{2}\cdot\hat{k}$

$$\boxed{\hat{n}\cdot\hat{k} = \frac{z}{2}}$$

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using values in I)

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_R \frac{1}{2} \frac{dx\,dy}{\left|\frac{z}{2}\right|}$$

$$= \iint_R \frac{1}{|z|}\, dx\,dy$$

As $x^2+y^2+z^2=4$

$$z = \pm\sqrt{4-x^2-y^2}$$

$$|z| = \sqrt{4-x^2-y^2}$$

$$= \iint_R \frac{1}{\sqrt{4-(x^2+y^2)}}\, dx\,dy \quad \text{--- II}$$

For limits

As $x^2+y^2+z^2=4$

In $xy$-plane $z=0$

$$x^2+y^2=4$$

$$\Rightarrow \quad x = \pm\sqrt{4-y^2}$$

for $y$ intercept put $x=0$

$$y^2=4$$

$$\boxed{y=\pm 2}$$

$\Rightarrow$ By using values in II)

We have

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Change into polar form

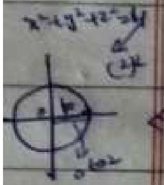
$$\iint_S \vec{A}\cdot\hat{n}\,ds = \int_{-2}^{2}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\sqrt{4-(x^2+y^2)}}\,dx\,dy$$

In polar coordinates

$x = r\cos\theta \quad , \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta \quad , \quad r^2 = x^2+y^2$

$0 \le \theta \le 2\pi$



$4 \qquad 0 \le r \le 2$

Hence

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \int_0^{2\pi}\int_0^{2} \frac{1}{\sqrt{4-r^2}}\,r\,dr\,d\theta$$

Base derivative

$$= \int_0^{2\pi}\int_0^{2} (4-r^2)^{-1/2}\,r\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}\int_0^{2} (4-r^2)^{-1/2}(-2r)\,dr\,d\theta$$

$-\frac{1}{2}+1$

$\frac{-1+2}{2}$

$$= -\frac{1}{2}\int_0^{2\pi} \left| \frac{(4-r^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right|_0^2 d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} \left| \frac{\sqrt{4-r^2}}{1/2} \right|_0^2 d\theta \Rightarrow -\int_0^{2\pi} (0-\sqrt{4})\,d\theta$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = -\int_0^{2\pi} -2\,d\theta \Rightarrow 2\int_0^{2\pi} d\theta$$

$$= 2\Big|\theta\Big|_0^{2\pi} \Rightarrow 2(2\pi) = 4\pi$$

Hence

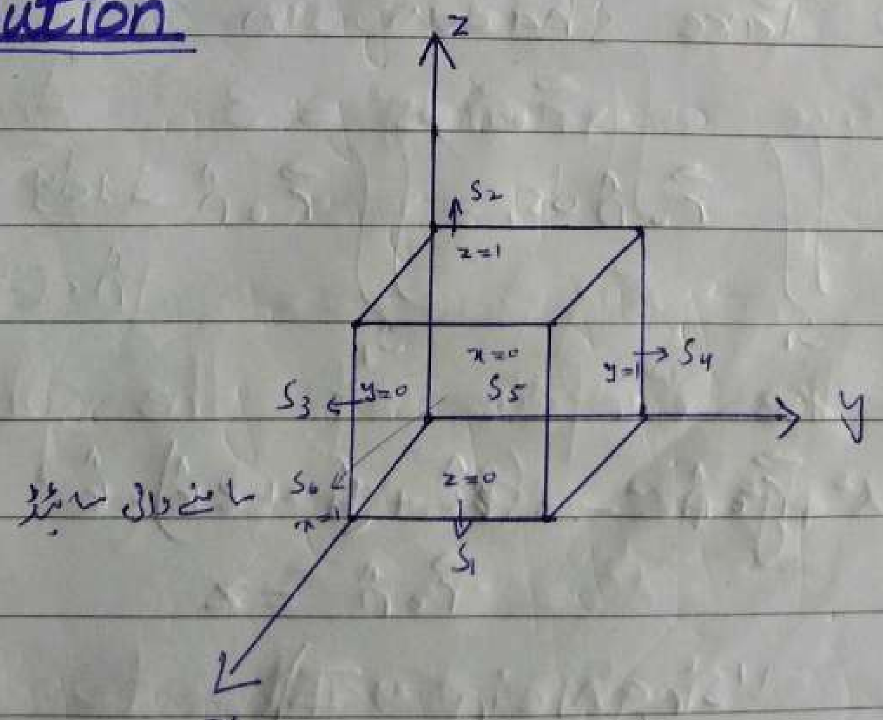
$$\iint_S \vec{A}\cdot\hat{n}\,ds = 4\pi \quad \underline{\text{Ans}}$$

**Q#24** Evaluate $\iint_S \vec{r}\cdot\hat{n}\,ds$ Where $S$ is

i) the surface of the unit cube bounded by the planes

$x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

**Solution**




Note: $\vec{A}$ کی جگہ $\vec{r}$ ہے اور یہیں $\vec{r}$ کی Value کا پتہ ہوتا ہے $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

Note: Routine بنا لیتے ہیں کہ z والے دونوں $S_1$ اور $S_2$ ہیں اسی طرح y والے دونوں $S_3$, $S_4$ اسی طرح باقی

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Since the unit cube have Six Surfaces therefore

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iint_{S_1} \vec{r}\cdot\hat{n}\,ds_1 + \iint_{S_2} \vec{r}\cdot\hat{n}\,ds_2 +$$

$$\iint_{S_3} \vec{r}\cdot\hat{n}\,ds_3 + \iint_{S_4} \vec{r}\cdot\hat{n}\,ds_4 + \iint_{S_5} \vec{r}\cdot\hat{n}\,ds_5 + \iint_{S_6} \vec{r}\cdot\hat{n}\,ds_6$$

Now **For $S_1$ $(z=0)$**

$\hat{n} = \frac{\vec{n}}{n} = \frac{\hat{k}}{1}$

$$\vec{n} = \nabla(z) = 1\hat{k}$$

$$\boxed{\hat{n} = \hat{k}}$$

Take shadow in xy-plane

$$\iint_{S_1} \vec{r}\cdot\hat{n}\,ds_1 = \iint_{S_1} \vec{r}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot(-\hat{k})|} \quad \text{---- II)}$$

Now

$$\vec{r}\cdot\hat{n} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{k}$$

$$\vec{r}\cdot\hat{n} = z$$

using in II)

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$$\iint_{S_1} \vec{r}\cdot\hat{n}\,ds_1 = \int_0^1\int_0^1 z\,\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$

$\because z=0$ in $S_1$

$$= \int_0^1\int_0^1 0\,\frac{dx\,dy}{1} = 0$$

$\Rightarrow$ By symmetry

$$\iint_{S_1} \vec{r}\cdot\hat{n}\,ds_1 = \iint_{S_3} \vec{r}\cdot\hat{n}\,ds_2 = \iint_{S_5} \vec{r}\cdot\hat{n}\,ds_5 = 0$$

Now **For $S_2$**

As $z = 1$

$$\hat{n} = \nabla(z-1) = \hat{k}$$

$$\boxed{\hat{n} = \hat{k}}$$

Now $\vec{r}\cdot\hat{n} = (x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k} = z$

Take shadow in xy-plane

$$\iint_{S_2} \vec{r}\cdot\hat{n}\,ds_2 = \iint_{S_2} z\,\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$

$\because Z = 1$ in $S_2$

$$= \int_0^1\int_0^1 1\,\frac{dx\,dy}{1}$$

Note

$\hat{3}$ mod [illegible]

Note

$S_3$ [illegible] $y=0$ [illegible] $x=0$ [illegible] Ans [illegible] 0 [illegible]

Not[illegible]

[illegible] $S_2$ [illegible] outwar[illegible] [illegible]

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$$\iint_{S_2} \vec{r}\cdot\hat{n}\,ds_2 = \int_0^1 |x|_0^1\,dy = \int_0^1 (1-0)\,dy$$

$$= \int_0^1 dy = |y|_0^1 = 1$$

By Symmetry

$$\iint_{S_2} \vec{r}\cdot\hat{n}\,ds_2 = \iint_{S_4} \vec{r}\cdot\hat{n}\,ds_4 = \iint_{S_6} \vec{r}\cdot\hat{n}\,ds_6 = 1$$

چونکہ
$S_4$ اور
$S_6$ میں
اور =1
ہے اس لیے
ان کا
1 ہو گا

By using values in I)

$$\iint_S \vec{r}\cdot\hat{n}\,ds = 0+1+0+1+0+1$$

$$= 3$$

Note
Even کے
1 آئے
اور odd
0 آئے گا
Ans

Hence $\iint_S \vec{r}\cdot\hat{n}\,ds = 3$ Ans

ii) The Surface of the sphere $x^2+y^2+z^2=a^2$

**Solution:** We want to evaluate $\iint_S \vec{r}\cdot\hat{n}\,ds$

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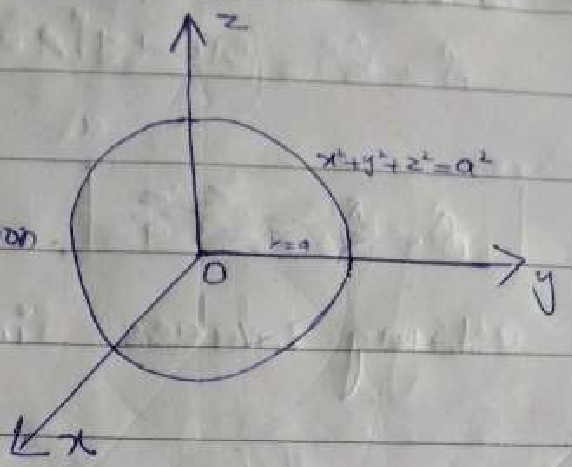
$$x^2+y^2+z^2 = a^2$$




$\Rightarrow$ Let projection of sphere in xy-plane

$$\Rightarrow \iint_S \vec{r}\cdot\hat{n}\, ds = \iint_S \vec{r}\cdot\hat{n} \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \text{— (I)}$$

$$\because \quad \vec{n} = \nabla(\text{surface})$$

$$\vec{n} = \nabla(x^2+y^2+z^2-a^2)$$

$$\vec{n} = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\because \hat{r} = \frac{\vec{r}}{|r|}$$

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}$$

$$\hat{n} = \frac{\cancel{2}(x\hat{i}+y\hat{j}+z\hat{k})}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{a^2}}$$

$$\hat{n} = \frac{\vec{r}}{a}$$

Now $\vec{r}\cdot\hat{n} = \vec{r}\cdot\dfrac{\vec{r}}{a} = \dfrac{r^2}{a}$

$$\vec{r}\cdot\hat{n} = \frac{x^2+y^2+z^3}{a} = \frac{a^2}{a} = a$$

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$$\boxed{\vec{r}\cdot\hat{n} = a}$$

Now $\quad \hat{n}\cdot\hat{k} = \left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)\cdot\hat{k}$

$$\boxed{\hat{n}\cdot\hat{k} = \frac{z}{a}}$$

By using Values in I)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S a\,\frac{dx\,dy}{\left|\frac{z}{a}\right|}$$

$$= a^2\iint_S \frac{1}{|z|}\,dx\,dy$$

چونکہ xy plane میں z نہیں ہے اس لیے value لی ہے

$\because\ x^2+y^2+z^2 = a$

$$z = \pm\sqrt{a^2-x^2-y^2}$$

$$|z| = \sqrt{a^2-x^2-y^2}$$

So

$$= a^2\iint_S \frac{1}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

or

In $xy$-plane $\quad z=0$

mits $\Rightarrow$ $\quad x^2+y^2 = a^2$

$$x = \pm\sqrt{a^2-y^2}$$

Also $\quad$ For $y$-Intercept $x=0$

$$y = \pm a$$

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$$\iint_S \vec{r}\cdot\hat{n}\,ds = a^2\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{1}{\sqrt{a^2-(x^2+y^2)}}\,dx\,dy$$

In polar Coordinates

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$r^2 = x^2+y^2$$

$$dx\,dy = r\,dr\,d\theta$$

Here $0 \le \theta \le 2\pi$ and $0 \le r \le a$

Hence

$$= a^2\int_0^{2\pi}\int_0^{a} \frac{1}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

$$= -\frac{a^2}{2}\int_0^{2\pi}\int_0^{a} (a^2-r^2)^{-1/2}(-2r)\,dr\,d\theta$$

$$= -\frac{a^2}{2}\int_0^{2\pi}\int_0^{a} (a^2-r^2)^{-1/2}(-2r)\,dr\,d\theta$$

$$= -\frac{a^2}{2}\int_0^{2\pi}\left|\frac{\sqrt{a^2-r^2}}{1/2}\right|_0^{a} d\theta$$

$$= -a^2\int_0^{2\pi} -a\,d\theta \Rightarrow a^3\int_0^{2\pi} d\theta$$

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$$\iint_S \vec{r}\cdot\hat{n}\,ds = a^3\left|\theta\right|_0^{2\pi} \Rightarrow a^3(2\pi - 0)$$

Hence

$$\iint_S \vec{r}\cdot\hat{n}\,ds = 2\pi a^3 \quad \text{Ans}$$

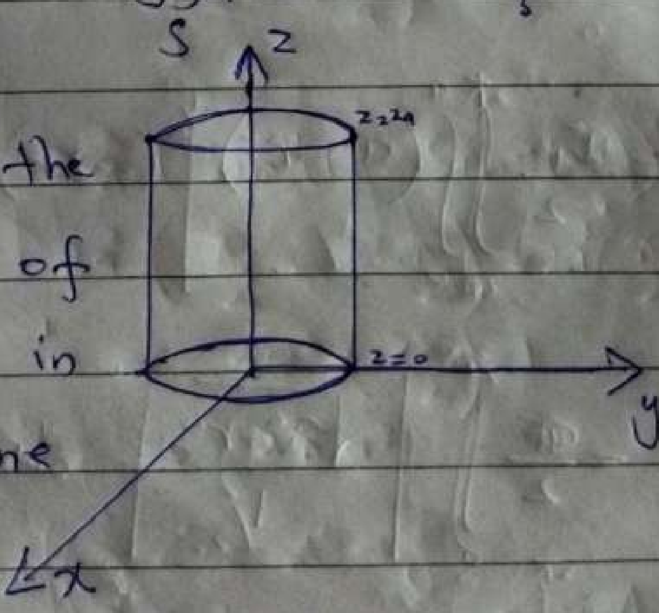
Same Q#22

**iii)** the curved surface of the cylinder $x^2+y^2=a^2$, $z=0$, $z=2a$.

**Solution:** $x^2+y^2=a^2$ and $z=0$ to $z=2a$

Here we have to find that

$$\iint_S \vec{r}\cdot\hat{n}\,ds = ?$$




$\Rightarrow$ Take the shadow of cylinder in $xz$-plane

So

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iint_S \vec{r}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} \quad \text{--- (1)}$$

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Now $\vec{n} = \nabla(\text{Surface})$

$$\vec{n} = \nabla(x^2+y^2-a^2)$$

$$\vec{n} = 2x\hat{i}+2y\hat{j}$$

$$\because \hat{n} = \frac{\vec{n}}{n}$$

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{2(x\hat{i}+y\hat{j})}{2\sqrt{x^2+y^2}}$$

$$\hat{n} = \frac{x\hat{i}+y\hat{j}}{\sqrt{a^2}} \Rightarrow \frac{x}{a}\hat{i}+\frac{y}{a}\hat{j}$$

Now

$$\vec{r}\cdot\hat{n} = (x\hat{i}+y\hat{j})\cdot\left(\frac{x}{a}\hat{i}+\frac{y}{a}\hat{j}\right)$$

$$\vec{r}\cdot\hat{n} = \frac{x^2}{a}+\frac{y^2}{a} \Rightarrow \frac{x^2+y^2}{a} = \frac{a^2}{a}$$

$\boxed{\vec{r}\cdot\hat{n} = a}$ Also $\boxed{\hat{n}\cdot\hat{j} = \frac{y}{a}}$

$\Rightarrow$ By using values in I)

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iint_S a\,\frac{dx\,dz}{\left|\frac{y}{a}\right|} \Rightarrow \iint_S \frac{a^2}{y}\,dx\,dz$$

$$\because \quad x^2+y^2 = a^2$$

$$y = \sqrt{a^2-x^2}$$

$$= a^2\iint \frac{1}{\sqrt{a^2-x^2}}\,dz\,dx$$

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As $x^2+y^2=a^2$

For $x$ intercept $y=0$

$\Rightarrow \quad x=\pm a$

Also given $z=0$ to $z=2a$

Hence

$$\iint_S \vec{r}\cdot\hat{n}\,ds = a^2\int_{-a}^{a}\int_{0}^{2a} \frac{1}{\sqrt{a^2-x^2}}\,dz\,dx$$

$$= a^2\int_{-a}^{a}\left|\frac{1}{\sqrt{a^2-x^2}}z\right|_0^{2a} dx$$

$$= 2a^3\int_{-a}^{a}\frac{1}{\sqrt{a^2-x^2}}\,dx = 2a^3\left|\sin^{-1}\frac{x}{a}\right|_{-a}^{a}$$

$$= 2a^3\left|\sin^{-1}\left(\frac{a}{a}\right) - \sin^{-1}\left(\frac{a}{-a}\right)\right.$$

$$= 2a^3\left(\sin^{-1}(1) - \sin^{-1}(-1)\right)$$

$$= 2a^3\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2a^3\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= 2a^3\left(\frac{2\pi}{2}\right)$$

$$= 2\pi a^3$$

Hence

$$\iint_S \vec{r}\cdot\hat{n}\,ds = 2\pi a^3 \quad \text{Ans}$$

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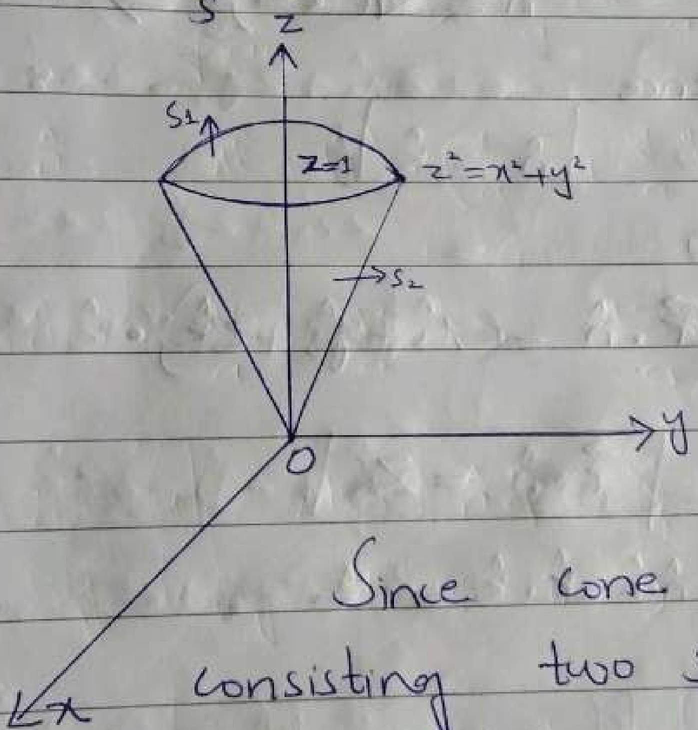
iv) The closed surface formed by the cone $z^2 = x^2+y^2$ and the plane $z=1$.

**Solution:** $z^2 = x^2+y^2$ and $z=1$

We have to find that

$$\iint_S \vec{r}\cdot\hat{n}\,ds$$




Since cone consisting two surfaces $S_1$ and $S_2$

there

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iint_{S_1} \vec{r}\cdot\hat{n}\,ds_1 + \iint_{S_2} \vec{r}\cdot\hat{n}\,ds_2$$

## For Surface $S_1$.

For Surface $S_1$ Let

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shadow is in $xy$-plane

Hence

$$\iint_{S_1} \vec{r}\cdot\hat{n}\, ds = \iint_{S_1} \vec{r}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \text{(II)}$$

Now

چونکہ $S_1$ کے $z=1$ ہے اس لیے یہاں سے $\hat{n}$ نکالیں گے یا direct بھی لکھ سکتے ہیں upward یعنی $\hat{k}$

$$\vec{n} = \nabla(\text{Surface})$$

$$\vec{n} = \nabla(z-1) \Rightarrow 1\hat{k}$$

Hence $\quad \because \hat{n} = \dfrac{\vec{n}}{|n|}$

$$\boxed{\hat{n} = \hat{k}}$$

Now

$$\vec{r}\cdot\hat{n} = (x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k}$$

$$\boxed{\vec{r}\cdot\hat{n} = z}$$

Also $\hat{n}\cdot\hat{k} = \hat{k}\cdot\hat{k} = 1$

Hence

$$\iint_{S_1} \vec{r}\cdot\hat{n}\,ds_1 = \iint z\,\frac{dx\,dy}{1} \quad \text{(I) be}$$

for $S_1$ $\quad z=1$

Now

$$z^2 = x^2+y^2$$

$$(1)^2 = x^2+y^2 \Rightarrow 1 = x^2+y^2$$

For $x$-Intercept $(y=0)$

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$$x = \pm 1$$

Also

$$y = \pm\sqrt{1-x^2}$$

$$\iint_{S_1} \vec{r}\cdot\hat{n}\,dS_1 = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1)\,dy\,dx$$

$\Rightarrow$ In polar coordinates

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$x^2+y^2=r^2$$

and $$dx\,dy = r\,dr\,d\theta$$

$$0 \le \theta \le 2\pi$$

Also $$0 \le r \le 1$$

Note
$dx\,dy = r\,dr\,d\theta$
OR
$dy\,dx = r\,dr\,d\theta$

Hence

$$\iint_{S_1} \vec{r}\cdot\hat{n}\,ds_1 = \int_0^{2\pi}\int_0^1 (1)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left|\frac{r^2}{2}\right|_0^1 d\theta = \int_0^{2\pi} \frac{1}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta \Rightarrow \frac{1}{2}\left|\theta\right|_0^{2\pi} = \frac{1}{\cancel{2}}(\cancel{2}\pi)$$

$$\iint_{S_1} \vec{r}\cdot\hat{n}\,ds_1 = \pi \quad \text{—— (A)}$$

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## For Surface $S_2$:

Let shadow is in $xy$-plane

Hence

$$\iint_{S_2} \vec{r}\cdot\hat{n}\, dS_2 = \iint_{S_2} \vec{r}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \text{— II)}$$

Note

Now

$$\vec{n} = \nabla(x^2+y^2-z^2)$$

$$\vec{n} = 2x\hat{i}+2y\hat{j}-2z\hat{k}$$

Now

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}-2z\hat{k}}{\sqrt{(2x)^2+(2y)^2+(-2z)}}$$

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}-2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

$\because x^2+y^2=z^2$

$$\hat{n} = \frac{\cancel{2}(x\hat{i}+y\hat{j}+z\hat{k})}{\cancel{2}\sqrt{z^2+z^2}}$$

$$\hat{n} = \frac{x\hat{i}+y\hat{j}-z\hat{k}}{\sqrt{2}\,z}$$

Now

$$\vec{r}\cdot\hat{n} = (x\hat{i}+y\hat{j}+z\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}-z\hat{k}}{\sqrt{2}\,z}\right)$$

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$$\vec{r}\cdot\hat{n} = \frac{x^2}{\sqrt{2}z} + \frac{y^2}{\sqrt{2}z} - \frac{z^2}{\sqrt{2}z}$$

$$\vec{r}\cdot\hat{n} = \frac{x^2 + y^2 - z^2}{\sqrt{2}z}$$

$$\because x^2 + y^2 = z^2$$

$$\vec{r}\cdot\hat{n} = \frac{z^2 - \cancel{z^2}}{\sqrt{2}z} = 0$$

Now

$$\boxed{\vec{r}\cdot\hat{n} = 0}$$

$$\hat{n}\cdot\hat{k} = \left(\frac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{2}z}\right)\cdot\hat{k}$$

$$\hat{n}\cdot\hat{k} = -\frac{1}{\sqrt{2}} \Rightarrow |\hat{n}\cdot\hat{k}| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

Now

$$\iint_{S_2} \vec{r}\cdot\hat{n}\, ds = \iint_{S_2} 0\, \frac{dx\,dy}{\frac{1}{\sqrt{2}}} = 0$$

Note

لگانے limit

کی ضرورت

نہیں کیونکہ

سے کچھ

آئے گا

Hence

$$\iint_{S_2} \vec{r}\cdot\hat{n}\, ds_2 = 0$$

$\Rightarrow$ By using in I) we get

$$\iint_S \vec{r}\cdot\hat{n}\, ds = \pi + 0 = \pi$$

Hence

$$\iint_S \vec{r}\cdot\hat{n}\, ds = \pi \quad \text{Ans}$$

Date:

**Q#25** If $\phi = 4x+3y-2z$, evaluate $\iint_S \phi \hat{n}\, ds$ where $S$ is the surface of the plane $2x+y+2z=6$ bounded by $x=0$, $y=0$ and $z=0$.

**Solution:** $\phi = 4x+3y-2z$

plane $\Rightarrow 2x+y+2z=6$

We want to show that

$$\iint_S \phi \hat{n}\, ds$$

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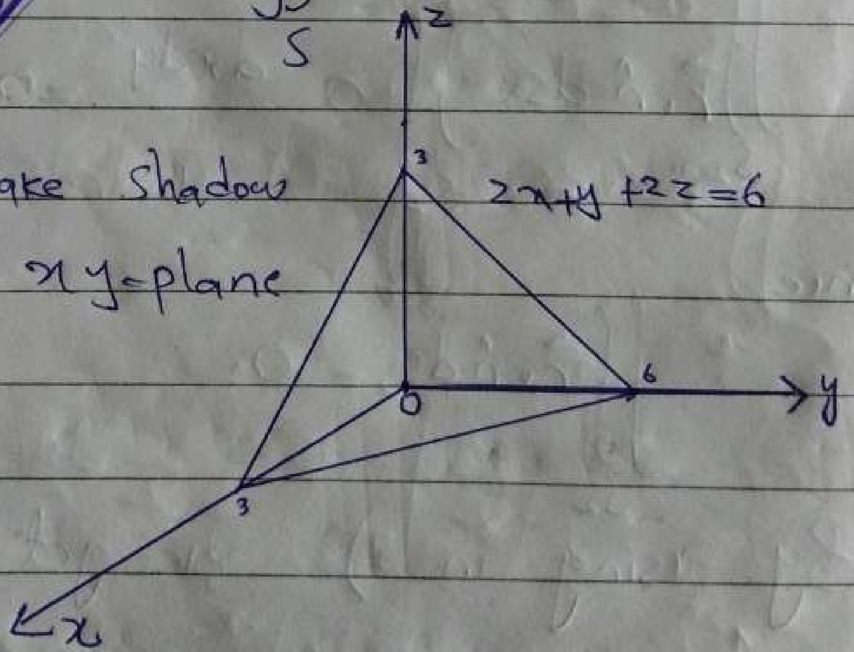
Take shadow in $xy$-plane




Hence

$$\iint_S \phi \hat{n}\, ds = \iint_S \phi \hat{n} \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \text{(I)}$$

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Now $\vec{n} = \nabla(\text{surface})$

$$\vec{n} = \nabla(2x + y + 2z)$$

$$\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$\because \hat{n} = \frac{\vec{n}}{|n|}$

$$\hat{n} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

Now

$$\hat{n} \cdot \hat{k} = \left(\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\right) \cdot \hat{k}$$

$$\hat{n} \cdot \hat{k} = \frac{2}{3}$$

$\Rightarrow$ By using values in I)

$$\iint_S \phi \, \hat{n} \, ds = \iint_S (4x + 3y - 2z)\left(\frac{2\hat{i} + \hat{j} + 2\hat{k}}{\cancel{3}}\right) \frac{dx\,dy}{\frac{2}{\cancel{3}}}$$

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{2} \iint_S (4x + 3y - 2z)\, dx\,dy$$

$\because 2x + y + 2z = 6$

$\Rightarrow$ $z = \frac{6 - 2x - y}{2}$

Hence

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{2} \iint_S \left(4x + 3y - \cancel{2}\left(\frac{6 - 2x - y}{\cancel{2}}\right)\right) dx\,dy$$

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$$\iint_S \phi \hat{n}\, ds = \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\iint_S (4x+3y-6+2x+y)\, dx\, dy$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\iint_S (6x+4y-6)\, dy\, dx$$

For limits

$\Rightarrow$ By $\quad 2x+y+2z=6$

$\because z=0$ (In xy-plane)

$\Rightarrow \quad 2x+y=6$

$y = 6-2x$

Also for $x$

$2x = 6 \Rightarrow \boxed{x=3}$

Hence

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\int_0^3\int_0^{(6-2x)} (6x+4y-6)\, dy\, dx$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\int_0^3 \left| 6xy + \frac{\overset{2}{\cancel{4}}y^2}{\cancel{2}} - 6y \right|_0^{(6-2x)} dx$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\int_0^3 \left( 6x(6-2x) + 2(6-2x)^2 - 6(6-2$$

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$$\iint_S \phi\, \hat{n}\, ds = \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\int_0^3 36x - 12x^2 + 2(36 + 4x^2 - 24x - 36 + 12x)\,dx$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\int_0^3 (\cancel{36x} - 12x^2 + 72 + 8x^2 - \cancel{48x} - 36 + \cancel{12x})\,dx$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\int_0^3 (8x^2 - 12x^2 + 72 - 36)\,dx$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\int_0^3 (-4x^2 + 36)\,dx$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\left| -\frac{4x^3}{3} + 36x \right|_0^3$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\left(-\frac{4(3)^3}{3} + 36(3)\right)$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{2}\left(-\frac{\overset{36}{\cancel{108}}}{\cancel{3}_1} + 108\right)$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{\cancel{2}}\left(\overset{36}{\cancel{72}}\right)$$

Hence

$$\iint_S \phi\, \hat{n}\, ds = 72\hat{i} + 36\hat{j} + 72\hat{k} \text{ Ans}$$

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Lengthy But easy.....

Date:

## Q#26

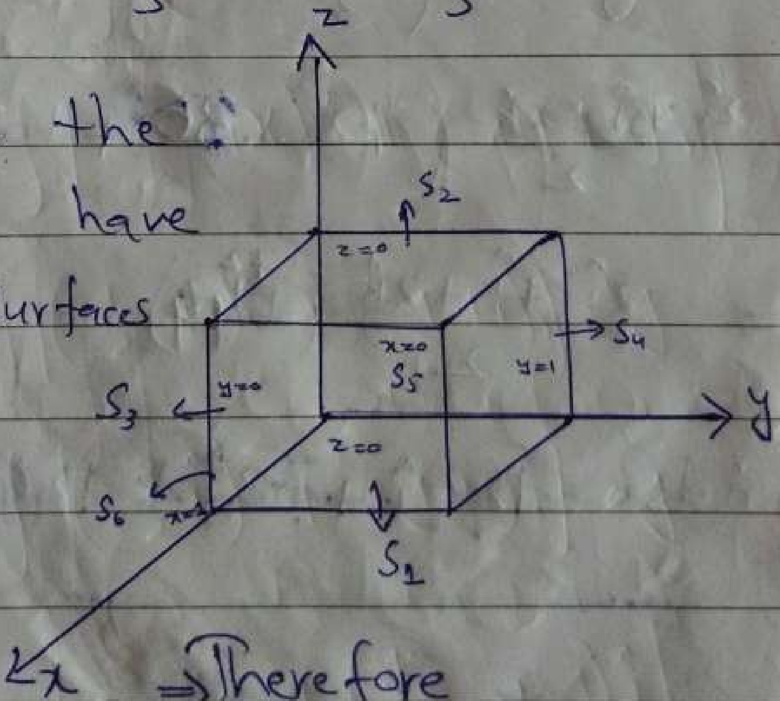
If $\vec{A} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$, evaluate $\iint_S \vec{A} \times d\vec{S}$ where $S$ is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

**Solution:** $\vec{A} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

We have to find that

$$\iint_S \vec{A} \times d\vec{s} = \iint_S \vec{A} \times \hat{n}\, ds$$

$\Rightarrow$ Since the cube have six surfaces:




$\Rightarrow$ Therefore

$$\iint_S \vec{A} \times d\vec{s} = \iint_{S_1} \vec{A} \times \hat{n}\, ds_1 + \iint_{S_2} \vec{A} \times \hat{n}\, ds_2 + \iint_{S_3} \vec{A} \times \hat{n}\, ds_3 +$$

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$$+\iint_{S_4}\vec{A}\times\hat{n}\,ds_4+\cdots\cdots\iint_{S_6}\vec{A}\times\hat{n}\,ds_6 \quad \text{--- (I)}$$

## For Surface $S_1$:

$$\hat{n}=-\hat{k} \quad \text{and} \quad z=0$$

Now

$$\vec{A}\times\hat{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ y+z & z+x & x+y\\ 0 & 0 & -1\end{vmatrix}$$

$$\vec{A}\times\hat{n}=\hat{i}(-z-x)+\hat{j}(-y-z)$$

For $S_1$ shadow in $xy$-plane

Hence

$$\iint_{S_1}\vec{A}\times\hat{n}\,ds_1=\int_0^1\int_0^1\left((-z-x)\hat{i}+(-y-z)\hat{j}\right)\frac{dx\,dy}{|\hat{k}\cdot(-$$

$$=\int_0^1\int_0^1(-z-x)\hat{i}+(-y-z)\hat{j}\,\frac{dx\,dy}{1}$$

$\because z=0$

$$=\int_0^1\int_0^1\left(-x\hat{i}-y\hat{j}\right)dx\,dy\Rightarrow\int_0^1\left|-\frac{x^3}{2}\hat{i}-xy\right.$$

$$=\int_0^1\left(-\frac{1}{2}\hat{i}-y\hat{j}\right)dy\Rightarrow\left|\frac{-y}{2}\hat{i}-\frac{y^3}{2}\hat{j}\right|_0^1$$

Note

shadow plane ... $\hat{n}$ ... $\hat{k}$ ... $xy$ ...

Hence **Written By: Javeria Abbas**

$$\iint_{S_1} \vec{A} \times \hat{n}\, ds_1 = -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}$$

## For Surface $S_2$:

$\hat{n} = \hat{k}$ and $z = 1$

Now

$$\vec{A} \times \hat{n} = (x+z)\hat{i} + (y+z)\hat{j}$$

Take shadow in xy-plane

Now

$$\iint_{S_2} \vec{A} \times \hat{n}\, ds_2 = \int_0^1\int_0^1 \left((x+z)\hat{i} + (y+z)\hat{j}\right)\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} \quad \because z=1$$

$$= \int_0^1\int_0^1 \left((x+1)\hat{i} + (y+1)\hat{j}\right) dx\,dy$$

$$= \int_0^1 \left|\left(\frac{x^2}{2} + x\right)\hat{i} + (xy + x)\hat{j}\right|_0^1 dy$$

$$= \int_0^1 \left[\left(\frac{1}{2} + 1\right)\hat{i} + (y+1)\hat{j}\right] dy$$

$$= \int_0^1 \left[\left(\frac{3}{2}\hat{i}\right) + (y+1)\hat{j}\right] dy$$

$$= \left|\frac{3}{2}y\hat{i} + \left(\frac{y^2}{2} + y\right)\hat{j}\right|_0^1$$

$$= \frac{3}{2}\hat{i} + \left(\frac{1}{2} + 1\right)\hat{j} \Rightarrow \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j}$$

Hence

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$$\iint_{S_2} \vec{A} \times \hat{n}\, ds_2 = \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j}$$

## For Surface $S_3$:

$$\hat{n} = -\hat{j} \quad \text{and} \quad y = 0$$

Now

$$\vec{A} \times \hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y+z & z+x & x+y \\ 0 & -1 & 0 \end{vmatrix}$$

$$\vec{A} \times \hat{n} = \hat{i}(x+y) + \hat{k}(-y-z)$$

Take shadow in $xz$-plane

چونکہ $xy$ میں پڑے گا سایہ نہیں بنے گا اس لیے $xz$ میں لیا -

Now

$$\iint_{S_3} \vec{A} \times \hat{n}\, ds = \int_0^1\int_0^1 \left((x+y)\hat{i} + (-y-z)\hat{k}\right) \frac{dx\,dz}{|\hat{j}\cdot(-\hat{j})|} \quad \because y=0$$

$$= \int_0^1\int_0^1 (x\hat{i} - z\hat{k})\, dx\, dz$$

$$= \int_0^1 \left(\frac{x^2}{2}\hat{i} - xz\hat{k}\right)_0^1 dz$$

$$= \int_0^1 \left(\frac{1}{2}\hat{i} - z\hat{k}\right) dz = \left|\frac{z}{2}\hat{i} - \frac{z^2}{2}\hat{k}\right|_0^1$$

$$= \frac{1}{2}\hat{i} - \frac{1}{2}\hat{k}$$

Hence **Javeria Abbas**

$$\iint_{S_4} \vec{A}\times\hat{n}\, dS_4 = \frac{1}{2}\hat{i} - \frac{1}{2}\hat{k}$$

For Surface $S_4$:

$\hat{n} = \hat{j}$ and $y = 1$

Now

$\vec{A}\times\hat{n} = (-x-y)\hat{i} + \hat{k}(y+z)$

Take shadow in $xz$-plane

Now

$$\iint_{S_4} \vec{A}\times\hat{n}\, ds = \int_0^1\int_0^1 \left((-x-y)\hat{i} + \hat{k}(y+z)\right)\frac{dx\,dz}{|\hat{j}\cdot\hat{j}|}$$

$\because y = 1$

$$= \int_0^1\int_0^1 \left((-x-1)\hat{i} + (1+z)\hat{k}\right) dx\,dz$$

$$= \int_0^1 \left|\left(-\frac{x^2}{2} - x\right)\hat{i} + (x + xz)\hat{k}\right|_0^1 dz$$

$$= \int_0^1 \left(\left(-\frac{1}{2} - 1\right)\hat{i} + (1+z)\hat{k}\right) dz$$

$$= \int_0^1 \left(\left(-\frac{3}{2}\hat{i}\right) + (1+z)\hat{k}\right) dz$$

$$= \left|-\frac{3}{2}z\hat{i} + \left(z + \frac{z^2}{2}\right)\hat{k}\right|_0^1$$

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$$\iint_{S_4} \vec{A} \times \hat{n}\, ds = \left(-\frac{3}{2}\hat{i} + \left(1+\frac{1}{2}\right)\hat{k}\right)$$

Hence

$$\iint_{S_4} \vec{A} \times \hat{n}\, ds = -\frac{3}{2}\hat{i} + \frac{3}{2}\hat{k}$$

## For Surface $S_5$:

$\hat{n} = -\hat{i}$ and $x = 0$

Now

$$\vec{A} \times \hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y+z & z+x & x+y \\ -1 & 0 & 0 \end{vmatrix}$$

$$\vec{A} \times \hat{n} = \hat{j}(x+y) + \hat{k}(z+x)$$

Take shadow in yz-plane

Now

$$\iint_{S_5} \vec{A} \times \hat{n}\, ds = \int_0^1\int_0^1 \left((x+y)\hat{j} + (z+x)\hat{k}\right) \frac{dy\, dz}{|\hat{i}\cdot(-\hat{i})|}$$

$\because x = 0$

$$= \int_0^1\int_0^1 (y\hat{j} + z\hat{k})\, dy\, dz$$

$$= \int_0^1 \left| \frac{y^2}{2}\hat{j} + yz\hat{k} \right|_0^1 dz = \int_0^1 \left(\frac{1}{2}\hat{j} + z\hat{k}\right) dz$$

$$= \left| \frac{z}{2}\hat{j} + \frac{z^2}{2}\hat{k} \right|_0^1 = \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

Hence

$$\iint_{S_5} \vec{A} \times \hat{n}\, ds_5 = \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

## For Surface $S_6$:

$\hat{n} = \hat{i}$ and $x = 1$

Now

$$\vec{A} \times \hat{n} = (-x-y)\hat{j} + \hat{k}(-z-x)$$

Take shadow in yz-plane

Now

$$\iint_{S_6} \vec{A} \times \hat{n}\, ds = \int_0^1\int_0^1 \left((-x-y)\hat{j} + (-z-x)\hat{k}\right)\frac{dy\,dz}{|\hat{i}\cdot\hat{i}|}$$

$\because x = 1$

$$= \int_0^1\int_0^1 \left((-1-y)\hat{j} + (-z-1)\hat{k}\right) dy\,dz$$

$$= \int_0^1 \left| \left(-y - \frac{y^2}{2}\right)\hat{j} + (-yz - y)\hat{k} \right|_0^1 dz$$

$$= \int_0^1 \left(\left(-1 - \frac{1}{2}\right)\hat{j} + (-z-1)\hat{k}\right) dz$$

$$= \int_0^1 \left(-\frac{3}{2}\hat{j} + (-z-1)\hat{k}\right) dz$$

$$= \left| -\frac{3}{2}z\hat{j} + \left(-\frac{z^3}{2} - z\right)\hat{k} \right|_0^1$$

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$$\iint_{S_6} \vec{A} \times \hat{n}\, dS_6 = -\frac{3}{2}\hat{j} + \left(-\frac{1}{2} - 1\right)\hat{k}$$

$$= -\frac{3}{2}\hat{j} - \frac{3}{2}\hat{k}$$

$\Rightarrow$ By using values in I)

We have

$$\iint_S \vec{A} \times d\vec{s} = \left(-\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \left(\frac{3}{2}\hat{i} + \frac{3}{2}\hat{j}\right)$$

$$+ \left(\frac{1}{2}\hat{i} - \frac{1}{2}\hat{k}\right) + \left(-\frac{3}{2}\hat{i} + \frac{3}{2}\hat{k}\right) +$$

$$\left(\frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) + \left(-\frac{3}{2}\hat{j} - \frac{3}{2}\hat{k}\right)$$

Hence

$$\iint_S \vec{A} \times d\vec{S} = \vec{0} \quad \text{Ans}$$

**Credit goes to:**

(0301.4695644)

**Prof: Fazal Abbas Sajid**

**Written By: Javeria Abbas Ahmed**

# Vector Analysis

## Topic: Volume Integral

Written By: Javeria Abbas Ahmed

**Q#27** If $\phi = z^2$, evaluate $\iiint_R \phi \, dv$ where $v$ is the volume of the region $R$ enclosed by the plane $x+y+z=a$ and the first octant.

**Solution:** $\phi = z^2$ and

$x+y+z=a$

We want to evaluate

$$\iiint_R \phi \, dv = ?$$

$\because dv = dz\,dy\,dx$

Hence

$$\iiint_R \phi \, dv = \iiint_R z^2 \, dz\,dy\,dx$$

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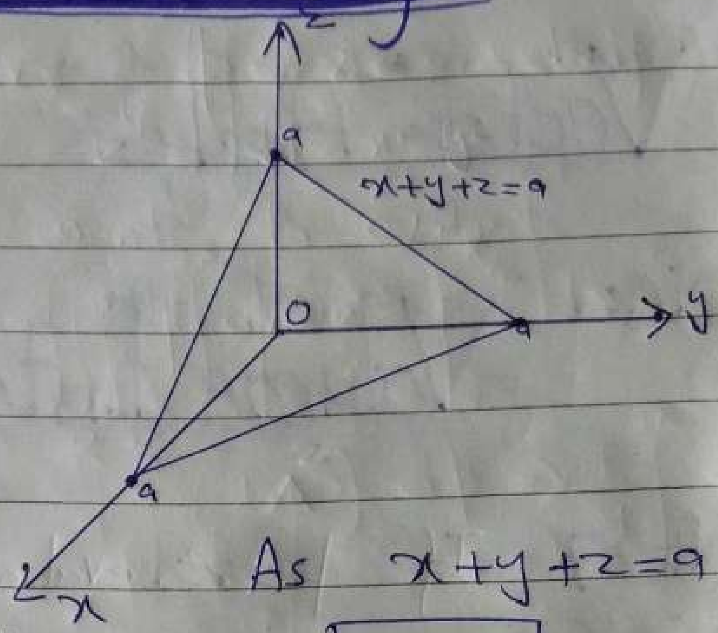


Note

As $x+y+z=a$

For limits: $\Rightarrow$ $\boxed{x=a}$

$\Rightarrow$ $\boxed{y=a-x}$

$\Rightarrow$ $\boxed{z=a-x-y}$

Hence

$$\iiint_R \phi \, dv = \int_0^a \int_0^{(a-x)} \int_0^{(a-x-y)} z^2 \, dz\, dy\, dx$$

$$= \int_0^a \int_0^{(a-x)} \left| \frac{z^3}{3} \right|_0^{(a-x-y)} dy\, dx$$

$$= \int_0^a \int_0^{(a-x)} \frac{(a-x-y)^3}{3} \, dy\, dx$$

$$= \int_0^a \left| \frac{(a-x-y)^4}{3(4)(-1)} \right|_0^{(a-x)} dx$$

Base ka derivative

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$$\iiint_R \phi \, dv = \int_0^a \left| \frac{(a-x-y)^4}{-12} \right|_0^{(a-x)} dx$$

$$= \int_0^a \left[ \frac{(a-x-a+x)^4}{-12} - \frac{(a-x-0)^4}{(-12)} \right] dx$$

$$\iiint_R \phi dv = \int_0^a \left[ 0 + \frac{(a-x)^4}{12} \right] dx \Rightarrow \int_0^a \frac{(a-x)^4}{12} dx$$

$$= \left| \frac{(a-x)^5}{12(5)(-1)} \right|_0^a \Rightarrow \left| \frac{(a-x)^5}{-60} \right|_0^a$$

$$= \frac{(a-a)^5}{-60} + \frac{(a-0)^5}{60} \Rightarrow 0 + \frac{a^5}{60}$$

Hence $\iiint_R \phi \, dv = \frac{1}{60} a^5$ Ans

Most Imp

Q#28 If $\phi = 2z$, evaluate $\iiint_R \phi \, dv$, where $R$ is the region bounded by $z = \sqrt{1-x^2-y^2}$ and $z = 0$

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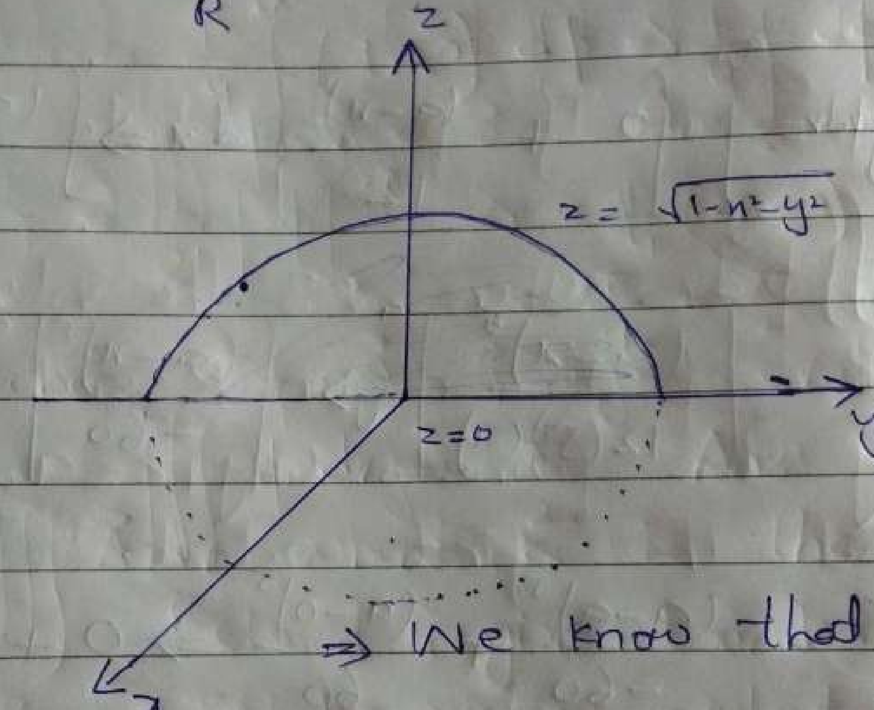
**Solution:** $\phi = 2z$

and $\quad z = \sqrt{1 - x^2 - y^2}$

We want to evaluate

$$\iiint_R \phi \, dv = ?$$

Note
$x^2+y^2+z^2=1$
$z=\sqrt{1-x^2-y^2}$
Sphere




Note
half
Sphere

$\Rightarrow$ We know that

$$\iiint_R \phi \, dv = \iiint_R \phi \, dz \, dy \, dx \quad \text{—— (I)}$$

As $\quad z = \sqrt{1-x^2-y^2}$

$\Rightarrow \quad x^2+y^2+z^2 = 1$

for limit $\quad \Rightarrow \quad x = \pm 1$

Also $\quad y = \pm\sqrt{1-y^2}$

$z = 0$ to $z = \sqrt{1-x^2-y^2}$ (Given)

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$\Rightarrow$ By using in Eq — I) we have

$$\iiint_R \phi \, dv = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^2 2z \, dz \, dy \, dx$$

In spherical coordinates

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Also $dv = dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

as

$0 \leq r \leq 1$

$0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq \phi \leq 2\pi$

$\leftarrow$ For half sphere

Hence

$$\iiint_R \phi \, dv = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 2(r\cos\theta) r^2 \sin\theta \, dr\,d\theta\,d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^3 \cos\theta\sin\theta \, dr\,d\theta\,d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2} \left| \frac{r^4}{4}\cos\theta\sin\theta \right|_0^1 d\theta\,d\phi$$

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$$\iiint_R \phi \, dv = \int_0^1 (2x+2)\, dx$$

$$= \left| \frac{2x^2}{2} + 2x \right|_0^1$$

$$= 1 + 2 = 3$$

Hence

$$\iiint_R \phi \, dv = 3 \quad \text{Ans}$$

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**Q#30** If $\vec{A} = \hat{j} - 2y\hat{k}$, evaluate $\iiint_R \vec{A}\, dv$ where R is the closed region bounded by the planes $x=0$, $y=0$, $z=0$, and $2x+2y+z=$

**Solution:** $\vec{A} = \hat{j} - 2y\hat{k}$

$$x=0, \quad y=0, \quad z=0$$

Also

$$2x+2y+z=4$$

We have to evaluate

$$\iiint_R \vec{A}\, dv = \iiint_R \vec{A}\, dz\, dy\, dx \quad \text{—— (I)}$$

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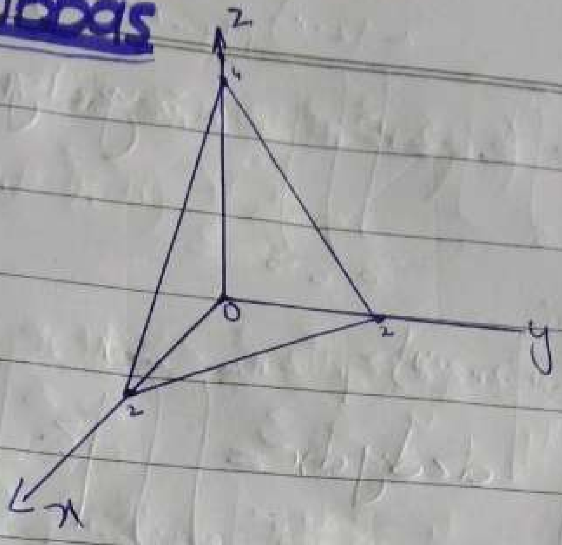


Now

$$\iiint_R \vec{A}\,dv = \iiint_R \left(\hat{j} - 2y\hat{k}\right) dz\,dy\,dx$$

$$= \hat{j}\iiint_R dz\,dy\,dx - \hat{k}\iiint_R 2y\,dz\,dy\,dx$$

For limits: $x = 2$

$$\Rightarrow 2y = 2(2-x) \Rightarrow y = (2-x$$

$$\Rightarrow z = (4-2x-2y)$$

Hence

$$= \hat{j}\int_0^2\int_0^{(2-x)}\int_0^{(4-2x-2y)} dz\,dy\,dx - \hat{k}\int_0^2\int_0^{(2-x)}\int_0^{(4-2x-2y)} 2y\,dz\,dy$$

Let

$$I_1 = \int_0^2\int_0^{(2-x)}\int_0^{(4-2x-2y)} dz\,dy\,dx \quad \text{and}$$

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$$I_x = \int_0^2\int_0^{(2-x)}\int_0^{(4-2x-2y)} 2y\,dz\,dy\,dx$$

Now

$$\int_0^2\int_0^{(2-x)}\int_0^{(4-2x-2y)} dz\,dy\,dx = \int_0^2\int_0^{2-x} \Big|z\Big|_0^{(4-2x-2y)} dy\,dx$$

$$= \int_0^2\int_0^{(2-x)} (4-2x-2y)\,dy\,dx$$

$$= \int_0^2 \left| 4y - 2xy - \frac{2y^2}{\cancel{2}}\right|_0^{(2-x)} dx$$

$$= \int_0^2 \left(4(2-x) - 2x(2-x) - (2-x)^2\right) d$$

$$= \int_0^2 \left(8 - \cancel{4x} - 4x + 2x^2 - 4 + \cancel{4x} - x^2\right)$$

$$= \int_0^2 \left(x^2 - 4x + 4\right) dx \Rightarrow \left|\frac{x^3}{3} - \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} + 4\right.$$

$$= \frac{8}{3} - \cancel{8} + \cancel{8} = \frac{8}{3}$$

Hence

$$\boxed{I_1 = \frac{8}{3}}$$

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$$I_2 = \int_0^2 \int_0^{(2-x)} \int_0^{(4-2x-2y)} 2y\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{(2-x)} \Big| 2yz \Big|_0^{(4-2x-2y)} dy\,dx$$

$$= \int_0^2 \int_0^{(2-x)} \left( 2y(4-2x-2y) \right) dy\,dx$$

$$= \int_0^2 \int_0^{(2-x)} \left( 8y - 4xy - 4y^2 \right) dy\,dx$$

$$= \int_0^2 \Big| \frac{\overset{4}{\cancel{8}}y^2}{\cancel{2}} - \frac{\overset{2}{\cancel{4}}xy^2}{\cancel{2}} - \frac{4y^3}{3} \Big|_0^{(2-x)} dx$$

$$= \int_0^2 \left( 4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3 \right) dx$$

$$= \int_0^2 \left( 4(4-4x+x^2) - 2x(4-4x+x^2) - \frac{4}{3} \left(8 - x^3 - 6x(2-x)\right) \right) dx$$

$$= \int_0^2 \left( 16 - 16x + 4x^2 - 8x + 8x^2 - 2x^3 - \frac{4}{3}\left(8 - x^3 - 12x + 6x^2\right) \right) dx$$

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$$= \int_0^2 \left( (-2x^3 + 12x^2 - 24x + 16) - \frac{4}{3}(-x^3 + 6x^2 - 12x + 8) \right) dx$$

$$= \left| -\frac{2x^4}{4} + \frac{12x^3}{3} - \frac{24x^2}{2} + 16x \right|_0^2 - \frac{4}{3} \left| -\frac{x^4}{4} + \frac{6x^3}{3} - \frac{12x^2}{2} + 8x \right|_0^2$$

$$= \left( -\frac{2(2)^4}{4} + 4(2)^3 - 12(2)^2 + 16(2) \right) - \frac{4}{3}\left( -\frac{(2)^4}{4} + \frac{6(2)^3}{3} - 6(2^2) + 8 \right.$$

$$= \left( -\frac{32}{4} + 32 - 48 + 32 \right) - \frac{4}{3}\left( -\frac{16}{4} + 16 - 24 + 16 \right)$$

$$= (-8 + 32 - 48 + 32) - \frac{4}{3}(-4 + 16 - 2$$

$$= 8 - \frac{4}{3}(4) = 8 - \frac{16}{3}$$

$$= \frac{24 - 16}{3} \Rightarrow \frac{8}{3}$$

Hence

$$I_2 = \frac{8}{3}$$

By using values in (II)

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$\Rightarrow$ We have

$$\iiint_R \vec{A}\, dv = \hat{j}\,\frac{8}{3} - \frac{8}{3}\,\hat{k}$$

$$= \frac{8}{3}(\hat{j}-\hat{k}) \quad \text{Ans}$$

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