Advance Analysis

Rational Number:

The rational no's are those real no's which can be expressed as the ratio of two integers. we denote it by Q.
Accordingly

$$Q = \{x / x = \frac{p}{q} \text{ where } p \in Z, q \in Z\}$$

Notice that each integer is also a rational number since, for example, $5 = \frac{5}{1}$
Hence

$$Z \subseteq Q$$

The sum, product, difference and quotient (except by 0) of two rational numbers is again a rational number.

NOTE: Notice the following relationship b/w the above number

$$N \subset Z \subset Q \subset R$$

The difference and quotient of two natural no's need not be a natural number.
Some irrational numbers are
$\sqrt{3}, \pi, \sqrt{3}$ etc.

Proper Subset:

'B' is a proper subset of 'A' if
$B \subset A$ and $B \neq A$

Q: If A is a subset of the null set ϕ then $A = \phi$

Sol.
The null set ϕ is a subset of every set. In particular $\phi \subseteq A$. By hypothesis $A \subseteq \phi$
So $\phi \subseteq A$ and $A \subseteq \phi$
$A = \phi$

with p. [illegible]

Function:

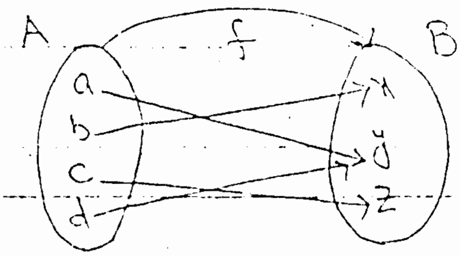
Suppose that to each element in a set 'A' there is assigned a unique element of a set 'B'.

Ex:

Let $A = \{a, b, c, d\}$

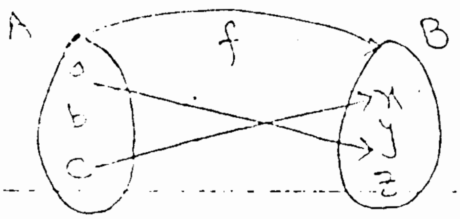
$B = \{x, y, z\}$

Let $f: A \to B$ be defined by



is a function.

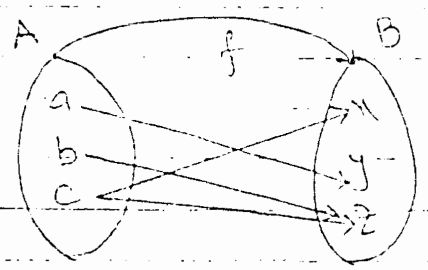
Ex:



is not a function.

No, there is nothing assigned to element $b \in A$.

Ex:



is not a function.

No, two elements x and z of B are assigned with only one element c of A.

In a function, only one element can be assigned to an element in the domain.

3

One-one function:

Let f map A into B. Then f is called a one-one function if different element in B are assinged to different element in A

ie ~~if~~ $f: A \to B$ is one-one

if $f(a) = f(a') \Rightarrow a = a'$

Ex:

$A = \{a, b, c, d, e\}$

$B = \{a, b, c, \_\_, y, z\}$

$f(a) = r$, $f(b) = a$, $f(c) = s$, $f(d) = r$, $f(e) = e$

Sol: f is not a one-one function since f assign ~~different~~ ~~image~~ r to both a and d.

NOTE: Note that in order for a function to be one-one, it must assign different image points to different element in the domain.

Onto Function:

Let f be a function of A into B

Then range $f(A)$ of the function f is subset of B, that is, $f(A) \subset B$. If $f(A) = B$, that is, if every member of B appears as the image of at least one element of A then we say that f is a function of 'A' onto 'B'

Ex: Let the function

$f: \mathbb{R} \to \mathbb{R}$ defined as

$f(x) = x^2$

is not onto

Since 3 and 3 has same square 9.

Equivalence Set:

A set 'A' is said to be equivalent to set 'B' if there exist a bijective mapping from A to B.

e.g

$A = \{1, 2, 3\}$, $B = \{a, b, c\}$

there exist $f : A \to B$ s.t

$f(1) = a$, $f(2) = b$, $f(3) = c$

Obviously f is bijective.

Hence 'A' is equivalence to B.

we write it $A \sim B$.

THEOREM: Relation '$\sim$' is an equivalence relation.

Proof: Reflexive:

For any non-empty 'A' there exist a identity mapping

$I : A \to A$ defined as

$I(x) = x \quad \forall \; x \in A$

Then obviously I is bijective and hence $A \sim A$.

Symmetric:

Let $A \sim B$, then there exist bijective function $f : A \to B$

Since f is bijective

So

$f^{-1} : B \to A$ is also a bijective function. Thus

$B \sim A$.

Transitive:

Let $A \sim B$ and $B \sim C$

then by def. there exist bijective mapping

$f : A \to B$ and $g : B \to C$

Consider the composition of f and g

$$gof : A \to C \quad \text{defined as}$$

$$gof(x) = g(f(x)) \quad \forall\ x \in A$$

Obviously gof is a bijective.
Thus '$\sim$' is an equivalence relation.
which Proves.

Equivalence Classes:

Infinite Set:
A set is said to be infinite if it is equivalent to its proper set.
e.g

$N = \{1, 2, 3, \ldots\}$
$A = \{2, 4, 6, \ldots\}$
$A \subset N$
and $N \sim A$

Since there exist
$f : N \to A$ such that

$f(x) = 2x \quad \forall\ x \in N$
which is bijective function.
$\Rightarrow N \sim A$

Denumerable Set:

A set is said to be denumerable if it is equivalent to set of natural numbers. e.g

$$A = \{2, 4, 6, \ldots\}$$

is a denumerable set.

V.Imp

Countable Set:

A set is said to be countable if it is either finite or denumerable.

e.g. $A = \phi$

$B = \{1, 2, 3\}$

Uncountable Set:

A set is said to be uncountable if it is neither finite nor denumerable.

e.g. $A = [0, 1]$ and $\mathbb{R}$ (set of real no.)

Example:

i) Every infinite sequence of distinct elements is denumerable.

Sol.

Let $A = \{a_1, a_2, a_3, \ldots\}$

then there exist

$f : N \longrightarrow A$ defined as

$$f(n) = a_n \quad \forall \; n \in N$$

Obviously, 'A' (infinite seq. of distinct elements) is denumerable.

Since there is one-one correspondence b/w A and N.

ii) $N \times N$ is denumerable set.

Sol:

As

$$N\times N = \{(1,1), (1,2), (1,3), \ldots$$
$$(2,1), (2,2), (2,3), \ldots$$
$$(3,1), (3,2), (3,3), \ldots$$
$$\ldots \}$$

Can be rearranged as

$$N\times N = \{(1,1), (1,2), (2,1), (3,1), (2,2), (1,3), (1,4), \ldots \}$$

is set of members of infinite sequence of distinct element. Hence it is denumerable.

iii) Let $M = \{0\} \cup N$

i.e $M = \{0,1,2,3,\ldots\}$

Show that $M \times M$ is denumerable?

Sol:

~~As~~ ~~we~~ Now every $a \in N$ can be written ~~use~~ ~~know~~ ~~the~~ ~~following~~ uniquely in the form

$$a = 2^r(2s+1) \qquad r, s \in M$$

Consider the function

$$f : N \longrightarrow M\times M \quad \text{defined as}$$
$$f(a) = (r, s) \quad \text{where } a = 2^r(2s+1)$$

Then f is one-one and onto

Hence $M \times M$ is denumerable.

NOTE: $N\times N$ is subset of $M\times M$

$Z = \{0, \pm 1, \pm 2, \ldots\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

iv) Set of integer is denumerable

Sol:

Consider the function

$f : N \to Z$ defined by

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Obviously, f is both one-one and onto.

Thus

$$N \sim Z$$

Hence Z is denumerable

Imp.

THEOREM: Every infinite set contains a denumerable subset.

Proof: Let A be infinite set.

Define $f : \mathcal{A} \to A$ as

*(where $\mathcal{A}$ is a collection of non-empty subset of A)

as

$f(A) = a_1$ for some $a_1 \in A$

$f(A \setminus \{a_1\}) = a_2$ for some $a_2 \in (A \setminus \{a_1\})$

$f(A \setminus \{a_1, a_2\}) = a_3$ for some $a_3 \in (A \setminus \{a_1, a_2\})$

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which shows that the set

$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ is a subset of A, that can be considered to be the member of infinite sequence of distinct element which is denumerable

So every infinite set contains a denumerable subset.

$^*\mathcal{A} = 2^A$

CHOICE FUNCTION:

choice function is a function which assigns the collection of subsets of given set to some element of that set.

THEOREM: Every subset of denumerable set is either finite or denumerable OR Every subset of denumerable set is Countable.

Proof:

Let $A = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ and B be a subset of A.

If $B = \phi$, then B is finite. (Nothing to Prove ~~ϕ is countable~~ ~~(finite)~~)

If B is non-empty then

Let a_{n_1} be the first element of B.

a_{n_2} be the second " " "

a_{n_3} be the third " Then $B = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$

If indexing set $\{n_1, n_2, n_3, \ldots\}$ of elements of B is finite then B is finite, otherwise B will denumerable.

Hence B will be either finite or denumerable or B is countable.

THEOREM: Every subset of a countable set is countable.

Proof.

Let A be countable set then there are two possibilities

1) A is finite.

2) A is denumerable.

Let B the subset of A.

1) Suppose A is finite then B is also finite (Since subset of a finite set is finite). Hence B will countable.

2) Suppose A is denumerable then there are again two possibilities that (its every subset

By Theorem

will be finite or denumerable) hence it will be countable.

v. Imp

THEOREM: Set of real number $\mathbb{R}$ is infinite.

Proof: Let $A =]-\frac{\pi}{2}, \frac{\pi}{2}[$ be proper subset of $\mathbb{R}$

Consider the following function

$$f : A =]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$$

defined as

$$f(x) = \tan x \qquad \forall\ x \in]-\frac{\pi}{2}, \frac{\pi}{2}[= A$$

Obviously, f is one-one and onto.

Hence

$$A =]-\frac{\pi}{2}, \frac{\pi}{2}[\sim \mathbb{R}$$

which show that $\mathbb{R}$ is equivalent to its proper subset.

Thus $\mathbb{R}$, by def., is a infinite set.

Ex: Concentric circles are given by

$$C_1 = \{(x,y) : x^2 + y^2 = a^2\}$$

$$C_2 = \{(x,y) : x^2 + y^2 = b^2\} \quad \text{where } 0 < a < b$$

Established geometrically one-one correspondence b/w C_1 & C_2

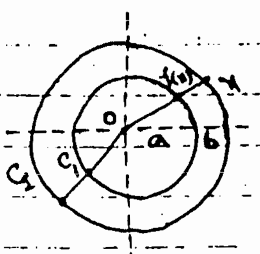
(i.e show that $C_1 \sim C_2$)

Sol.

Consider the function

$$f : C_2 \longrightarrow C_1$$

defined as



$f(x)$ = Pt. of intersection of line joining x to O; and C_1. where $x \in C_2$

Obviously, f is both one-one and onto.

Hence $C_2 \sim C_1$

or $C_1 \sim C_2$

THEOREM: Union of denumerable family of pairwise disjoint denumerable set is denumerable.

Proof:

Let $\{A_1, A_2, \ldots, A_n, \ldots\}$ be the denumerable family of pairwise disjoint denumerable set and

$$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$$
$$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots\}$$
$$\vdots$$

we show that $\bigcup_{i \in N} A_i$ is denumerable.

Then

$$\bigcup_{i \in N} A_i = \{a_{11} \rightarrow a_{12}, a_{13}, \ldots$$
$$a_{21}, a_{22}, a_{23}, \ldots$$
$$a_{31}, a_{32}, a_{33}, \ldots$$
$$\ldots\}$$

$\bigcup_{i \in N} A_i$ can now rearrange as (which is seq. of distinct element)

$$= \{a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, \ldots\}$$

which shows that $\bigcup_{i \in N} A_i$ is denumerable

As Required.

NOTE: Seq. of distinct elements provides denumerable set.

THEOREM: Set of rational number Q is denumerable.

Proof:

The set of rational no. Q can be written as

$$Q = Q^- \cup \{0\} \cup Q^+$$

where Q^- = Set of $-$ve rational no.

Q^+ = Set of $+$ve rational no.

Now

$$Q^+ = \left\{\frac{p}{q} : p, q \in N\right\}$$

$\Rightarrow (p_1, q_1) = (p_2, q_2)$ $f(p_1/q_1) = f(p_2/q_2)$

$\Rightarrow p_1 = p_2 \quad q_1 = q_2 \Rightarrow \dfrac{p_1}{q_1} = \dfrac{p_2}{q_2}$

$1 = 1 \quad 2 = 2$ $\quad f\left(\frac{p}{q_1}\right) \neq f\left(\frac{p_1}{q_1}\right)$

$\frac{1}{2} = \frac{1}{2}$

and

Consider a function

$$f : \mathbb{Q}^+ \longrightarrow \mathbb{N} \times \mathbb{N} \text{ defined as}$$

$$f(p/q) = (p, q) \quad \forall \ \frac{p}{q} \in \mathbb{Q}^+$$

Obviously f is both 1-1 and onto which show that

$$\mathbb{Q}^+ \sim \mathbb{N} \times \mathbb{N}$$

and hence $\mathbb{Q}^+$ is denumerable.

(As $\mathbb{N} \times \mathbb{N}$ is denumerable).

Moreover, it can be proved that $\mathbb{Q}^-$ is denumerable by defining the following function $g : \mathbb{Q}^- \longrightarrow \mathbb{Q}^+$

as
$$g(-p/q) = \frac{p}{q} \quad \forall \ -\frac{p}{q} \in \mathbb{Q}^-$$

Then $\mathbb{Q}^- \sim \mathbb{Q}^+$ Hence $\mathbb{Q}^-$ is denumerable.

Consequently,

$$\mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+ = \mathbb{Q} \text{ is denumerable}$$

since union of denumerable set is denumerable set. Imp

THEOREM: Unit Interval $[0, 1]$ is not denumerable.

PROOF:

Every element of the set $A = [0, 1]$ can be expressed as $0.a_1a_2a_3 \ldots$ (infinite decimal places)

where right side decimal place

$$a_i \in \{0, 1, 2, \ldots, 9\}$$

Suppose that A is denumerable.

i.e. $A = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$

and Let

$$x_1 = 0.a_{11}\, a_{12}\, a_{13} \ldots$$

$$x_2 = 0.a_{21}\, a_{22}\, a_{23} \ldots$$

$x_3 = 0.a_{31}\, a_{32}\, a_{33} \cdots$

$\vdots$

$x_n = 0.\, a_{n1}\, a_{n2}\, a_{n3} \cdots$

$\vdots$

Now consider the following number belonging to A such that

$x = 0.b_1 b_2 b_3 \cdots$ such that

$b_1 \neq a_{11}$, $b_1 \neq 0$

$b_2 \neq a_{22}$, $b_2 \neq 0$

$b_3 \neq a_{33}$, $b_3 \neq 0$

$\vdots \qquad \vdots$

$b_n \neq a_{nn}$, $b_n \neq 0$

which show that $x \neq x_1$, $x \neq x_2$, $x \neq x_3$, ... $x \neq x_n$

i.e $x \neq x_n$ for $n \in N$.

Hence $x \notin A$ which contradicts the fact that $x \in A$.

So $A = [0,1]$ is not denumerable.

which Proves.

CORROLLARY: Any interval $[a,b]$, $a<b$ $\forall a,b \in R$ is non denumerable

Proof: OR $[0,1] \sim [a,b]$ where $[0,1]$ is non-denumerable.

To show $[a,b]$ is non-denumerable we will show $[0,1]$ is equivalent to $[a,b]$.

i.e $[0,1] \sim [a,b]$

Consider the following function

$f : [0,1] \rightarrow [a,b]$ defined as

$$f(x) = a + (b-a)x \quad \forall\ x \in [0,1]$$

Obviously, f is both 1-1 and onto.

Hence $[0,1] \sim [a,b]$

which shows that $[a,b]$ is non denumerable.

v. Imp

Show that

i) $[0,1] \sim]0,1[$

ii) $[0,1] \sim]0,1]$

iii) $[0,1] \sim [0,1[$

Sol. $[0,1] \sim]0,1[$

we can write

$$[0,1] = \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\} \cup A$$
$$]0,1[= \{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\} \cup A$$

where

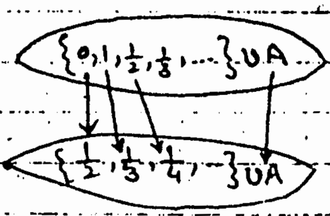
$$A = [0,1] - \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$$

$$A = (0,1) - \{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$$

Consider the function

$$f : [0,1] \longrightarrow]0,1[$$

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+2} & \text{if } x = \frac{1}{n}, \; n = 1,2,3\ldots \\ x & \text{if } x \in A \end{cases}$$



Obviously, f is both 1-1 and onto.

Hence

$$[0,1] \sim]0,1[$$

ii) $[0,1] \sim]0,1]$

we can write

$$[0,1] = \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\} \cup A$$
$$]0,1] = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\} \cup A$$

where

$$A = [0,1] = \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$$
$$A =]0,1] - \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$$

we consider the function

15

$f : [0,1] \longrightarrow]0,1]$

as

$\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup A$

$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup A$

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n},\ n = 1,2,3,\ldots \\ x & \text{if } x \in A \end{cases}$$

Obviously, f is both 1-1 and onto.

Hence

$$[0,1] \sim]0,1]$$

iii) $[0,1] \sim [0,1[$

we can write as

$[0,1] = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup A$

$[0,1[= \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup A$

where

$A = [0,1] - \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

$A = [0,1[- \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

we consider the func

$f : [0,1] \longrightarrow [0,1[$

defined as

$\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup A$

$\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup A$

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n},\ n = 1,2,3,\ldots \\ x & \text{if } x \in A \end{cases}$$

Obviously, f is both 1-1 and onto.

Hence

$$[0,1] \sim [0,1[$$

which Required

Ex: Prove that $A \sim A \times \{1\}$ for $A = \{a, b, c\}$.

Sol:

Define a function

$$f : A \longrightarrow A \times \{1\} \text{ as}$$

$$f(x) = (x, 1) \quad \forall \ x \in A$$

Obviously, f is both 1-1 and onto.

so

$$A \sim A \times \{1\}$$

CARDINAL NUMBER: (Generalization of Real no.)

Let A be any set equivalent to B i.e $A \sim B$. we already know that "$\sim$" is an equivalence relation. If C is a collection of sets in which the relation "$\sim$" is defined then, the equivalence class generated by any set $A \in C$ is called the Cardinal Numbers of A and denoted by $\#(A)$.

REMARKS: The cardinality of the following set

$$\phi, \{1\}, \{1, 2\}, \{a, b, c\}, \ldots$$

is considered to be $0, 1, 2, 3, \ldots$

which show that the cardinality of every finite set is equal to the number of elements in that set.

It may also be noted that $\#(\mathbb{N}) = \aleph$

The symbol $\aleph_0$ (aleph-null) is also used for cardinality of $\mathbb{N}$.

Moreover, the cardinality of unit interval $[0, 1]$ is denoted by

$$\#([0, 1]) = C$$

Cardinal Numbers:

Defination: Let A be any set and α denotes the collection of sets which are equivalent to A, then α is called Cardinal number.

Finite Cardinal Numbers.

As the cardinality of every finite set, is the number of elements of that set so these cardinal numbers are finite cardinal numbers.

Remark.

The set A which is equivalent to unit interval $[0,1)$ has cardinality C and is said to have "Power of Contium".

Arithmatic Of Cardinals:

i) Addition Of Cardinals:

If α, β cardinals of sets A and B respec.

where $A \cap B = \phi$

Then

$$\alpha + \beta = \#(A \cup B)$$

i.e

$$\#(A) + \#(B) = \#(A \cup B)$$

ii) Multiplication Of Cardinals:

If $\#(A) = \alpha$ and $\#(B) = \beta$

where

$$A \cap B = \phi$$

(The condition $A \cap B = \phi$ is not necessary for multiplication.)

Then

$$\alpha\beta = \#(A \times B)$$

Example:

Let $A = \{1,2,3,4,\ldots,10\}$
and
$B = \{11,12,13,14,15\}$
Show that
i) $\#(A) + \#(B) = \#(A \cup B)$
ii) $\#(A) \cdot \#(B) = \#(A \times B)$

Sol:
i) $\#(A) = 10 = \alpha$
$\#(B) = 5 = \beta$

Then
$$\alpha + \beta = \#(A \cup B) = O(A \cup B)$$
$$= \#\{1,2,3,\ldots,15\}$$
$$\alpha + \beta = 15$$
i.e
$$\#(A) + \#(B) = \#(A \cup B)$$
$$\#(A \times B) = O(A \times B)$$
ii) $\alpha\beta = \#(A \times B) = 50$
$$= O(A \times B) \quad \#(A)\,\#(B) = 50$$
$$= 50$$
$$\#(A)\,\#(B) = \#(A \times B)$$

Remark. It may be noted that the addition and multiplication of finite cardinal numbers corresponds to ordinary add. and mul. of natural numbers.

Example. (Show that cancellation laws doesn't hold for cardinals).
Let $A = \{1,3,5,7,\ldots\}$
$B = \{2,4,6,8,\ldots\}$

Then
$\#(A) = a \quad \because A \sim N$
$\#(B) = a \quad \because B \sim N$

we know that:

$$\#(A \cup B) = \#(A) + \#(B)$$
$$= a + a$$

Since $A \cup B \sim N$

$\therefore \#(A \cup B) = a$

$\Rightarrow a = a + a$ —— ①

Now consider

$$\#(A \times B) = \#(A) \cdot \#(B)$$

$$\#(\mathbb{N} \times \mathbb{N}) = \#(A)\#(B)$$

As $A \sim \mathbb{N}$ & $B \sim \mathbb{N}$ $\Rightarrow A \times B \sim \mathbb{N} \times \mathbb{N}$

$$\#(\mathbb{N}) = \#(A) \cdot \#(B)$$

$a = a \cdot a$ —— ② From ① & ②

it shows that cancellation law of cardinals are not valid.

NOTE: It can be observed in the above example that $a \neq 0$, $a \neq 1$ even $a + a = a$, $a \cdot a = a$

Moreover

$$a + 1 = a$$
$$a + 2 = a$$
$$a + 3 = a$$
$$\vdots$$
$$a + a = a$$

Now

$$a \cdot 1 = a$$
$$a \cdot 2 = a$$
$$\vdots$$
$$a \cdot a = a$$

$a + a = a = 1 + a \Rightarrow a =$

$a \cdot a = a = 1 \cdot a$

$\not\Rightarrow a = 1$

which show that cancellation Property of addition and multiplication of real numbers do not hold in cardinals.

THEOREM:

For any cardinals α, β and γ we have

i) $\alpha + (\beta+\gamma) = (\alpha+\beta)+\gamma$ (Associative Pr.)

ii) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

iii) $\alpha+\beta = \beta+\gamma$

iv) $\alpha\beta = \beta\alpha$

v) $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$

Proof: Let A, B and C be pairwise disjoint sets such that

$$\#(A) = \alpha$$
$$\#(B) = \beta$$
$$\#(C) = \gamma$$

i) L.H.S $= \alpha+(\beta+\gamma)$

$$= \#[A \cup (B \cup C)]^{*}$$

$$= \#[(A \cup B) \cup C]$$

$$= (\alpha+\beta)+\gamma$$

$$= R.H.S$$

* using associative property of union.

ii) $\cancel{\text{LHS}}$ $\alpha(\beta\gamma)$

$= \#[A\times(B\times C)]$

Now define

$f : A\times(B\times C) \longrightarrow (A\times B)\times C$

as

$f(a,(b,c)) = ((a,b),c) \quad \forall\ (a,(b,c)) \in A\times(B\times C)$

Obviously,

f is both 1-1 and onto.

So

$A\times(B\times C) \sim (A\times B)\times C$

So that

$\#[A\times(B\times C)] = \#[(A\times B)\times C]$

$\alpha(\beta\gamma) = (\alpha\beta)\gamma$

iii) $\alpha+\beta = \beta+\alpha$

$L.H.S = \alpha+\beta$

$= \#(A\cup B)$

As $A\cup B = B\cup A$

$= \#(B\cup A)$

$= \beta+\alpha$

$= R.H.S$

iv) $\alpha\beta = \beta\alpha$

Consider

$\alpha\beta = \#(A\times B)$

Let

$f : A\times B \longrightarrow B\times A$

defined as

$f(a,b) = (b,a) \quad \forall\ (a,b) \in A\times B$

Obviously, f is both 1-1 & onto

so

$$A \times B \sim B \times A$$

i.e $\# (A \times B) = \# (B \times A)$

$$\alpha\beta = \beta\alpha$$

v) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

Proof:

$$
\begin{aligned}
L.H.S &= \alpha(\beta + \gamma) \\
&= \# [A \times (B \cup C)] \\
&= \# [(A \times B) \cup (A \times C)]^{*} \\
&= \# (A \times B) + \# (A \times C) \\
&= \alpha\beta + \alpha\gamma \\
&= R.H.S
\end{aligned}
$$

* using distributive property of cartesian product over union.

Ex: $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$

$$
\begin{aligned}
(\alpha + \beta)\gamma &= \# [(A \cup B) \times C] \\
&= \# [(A \times C) \cup (B \times C)]^{*} \\
&= \# (A \times C) + \# (B \times C)
\end{aligned}
$$

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

which proves.

Exponent Of Cardinals:

Let α be the cardinality of A and β be cardinality of B
i.e

$$\alpha = \#(A) \text{ and } \beta = \#(B)$$

Then

$$\beta^{\alpha} = \#(B^{A})$$

where B^A denotes the collection of all possible function from A to B.

Example:

$$A = \{a, b, c\} \quad , \quad B = \{0, 1\}$$

Then:

$$\#(A) = 3 \quad \& \quad \#(B) = 2$$

Then

$$f_1 = \{(a,0), (b,0), (c,0)\}$$

$$f_2 = \{(a,0), (b,0), (c,1)\}$$

$$f_3 = \{(a,0), (b,1), (c,0)\}$$

$$f_4 = \{(a,1), (b,0), (c,0)\}$$

$$f_5 = \{(a,0), (b,1), (c,1)\}$$

$$f_6 = \{(a,1), (b,1), (c,0)\}$$

$$f_7 = \{(a,1), (b,0), (c,1)\}$$

$$f_8 = \{(a,1), (b,1), (c,1)\}$$

$$\#(B^A) = 8$$

It can be observed that

$$8 = 2^3$$

i.e

$$\#(B^A) = \#(B)^{\#A}$$

Remark:

In fact all properties of exponents of real no. valid for cardinals as well.

i.e

i) $\alpha^{\beta}\alpha^{\gamma} = \alpha^{\beta+\gamma}$ v. Imp

ii) $(\alpha^{\beta})^{\gamma} = \alpha^{(\beta\gamma)}$

iii) $(\alpha\beta)^{\gamma} = \alpha^{\gamma}\beta^{\gamma}$

where

α, β, γ are the cardinal numbers of sets A, B, C respectively.

Proof: 1) $\alpha = \#(A)$
$\beta = \#(B)$
$\gamma = \#(C)$ and $B \cap C = \phi$

Then

α^{β} and α^{γ} will be cardinality of A^B and A^C resp.

So $\alpha^{\beta}.\alpha^{\gamma}$ will be cardinality of $A^B \times A^C$.

Similarly, $\alpha^{\beta+\gamma}$ will be cardinality of $A^{B \cup C}$

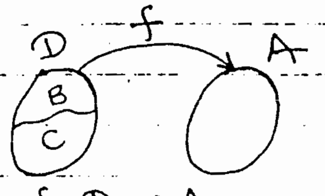
Now define

$$F : A^{B\cup C} \longrightarrow A^B \times A^C$$

as

$$F(f) = (f_{|B}, f_{|C})$$

where $f_{|B}$ and $f_{|C}$ are the restriction of f to B and C.



$f : D \to A$

$f_{|B} : B \to A$

$f_{|B}(x) = f(x) \ \forall \in D$

Restriction func.

Obviously, F is both 1-1 and onto

So

$$A^{B\cup C} \sim A^B \times A^C$$

Hence

$$\#(A^{B\cup C}) = \#(A^B \times A^C)$$

$$\#(A)^{\#(B\cup C)} = \#(A^B) \cdot \#(A^C)$$

$$\#(A)^{\#(B)+\#(C)} = \#(A)^{\#(B)} \cdot \#(A)^{\#(C)}$$

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$$

which Proves.

Exercise :1

Let S be the set of pts on cartesian plane, with rational components (coordinates) then show that S is denumerable

Sol: Since S is set of pts on cartesian plane with rational components i.e

$$S = \{(p,q) \,,\; p,q \in \mathbb{Q}\}$$

As we know that

$$\mathbb{Q}\times\mathbb{Q} = \{(p,q) \,,\; p,q \in \mathbb{Q}\}$$

$$\Rightarrow \quad S \sim \mathbb{Q}\times\mathbb{Q}$$

$\mathbb{Q}\times\mathbb{Q}$ is denumerable because $\mathbb{Q}\times\mathbb{Q} \sim \mathbb{N}\times\mathbb{N}$

But

$$\mathbb{N}\times\mathbb{N} \sim \mathbb{N}$$

so

$$\mathbb{Q}\times\mathbb{Q} \sim \mathbb{N} .$$

And

$$S \sim \mathbb{Q}\times\mathbb{Q} \sim \mathbb{N}$$

Thus

$$S \sim \mathbb{N}$$

$\Rightarrow$ S is denumerable.

Imp

Exercise: 2

Let P be the collection of all polynomials

$$P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

with integral coefficients then show that P is denumerable.

Sol: For each ordered pair (n,m) of natural numbers,

let $P_{(n,m)}$ be the set of polynomials,

$P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m \quad , m \neq 0$
such that
$|a_0| + |a_1| + |a_2| + \dots + |a_m| = n$

Then of Course $P_{(n,m)}$ is (m is degree & n is sum of coef.)
finite.
and hence
$P = \bigcup_{i \in \mathbb{N} \times \mathbb{N}} P_i$, being countable
union of countable set, is countable.
Since P is not finite
So P is denumerable

Imp
Ex:3) A real number r is called an algebraic number if r is a sol. of a polynomial equation

$P(x) = a_0 + a_1 x + \dots + a_n x^n = 0$
with integral coefficients.
Prove that the set A of algebraic numbers is denumerable.

Sol: Let $E = \{ P_1(x) = 0, P_2(x) = 0, P_3(x) = 0, \dots \}$
which shows that E is denumerable
(i.e set of all polynomial with integral coeff is denumerable)

Now define
$A_i = \{ x \mid x \text{ is solution of } P_i(x) = 0 \}$

Since a polynomial of degree n can have at most n roots, so each A_i is finite

Therefore

$A = \bigcup_{i \in N} A_i$ being countable union of countable sets, is countable

Accordingly, A is countable and, since A is not finite

~~xxxxxxx~~ Therefore A is denumerable

Exercise:4)

Let X be any set and C(X) be collection of characteristics function defined on X.

we show that

$$C(X) \sim 2^X$$

Sol:

Charateristics function χ is defined with the help of subset, say A of X, as

$$\chi_A : X \longrightarrow [0,1]$$

$$\chi_A(x) = \begin{cases} 1 & , x \in A \\ 0 & x \notin A \end{cases} \quad \forall x \in X$$

Define a function

$$f : C(X) \longrightarrow 2^X$$

as

$$f(\chi_A) = A \quad , A \subseteq X \quad \forall \chi_A \in C(X)$$

that is f mapps each ~~subset A of X~~ ~~xxxx~~ characteristics func of A (relative to X) into subset A of X.

Obviously, f is both 1-1 & onto.

Hence $C(X) \sim 2^X$

Exercise: 5

Show that

$a + n = a$ where $a = \#(IN)$

n is any finite cardinal.

Sol:

Let $A = \{a_1, a_2, a_3, \dots, a_n\}$

Consider the func

$$f : IN \cup A \longrightarrow IN$$

$$IN \cup A = \{a_1, a_2, \dots, a_n, 1, 2, 3, \dots\}$$
$$IN = \{1, 2, \dots, n, n+1, n+2, \dots\}$$

$$f(x) = \begin{cases} n & \text{if } x = a_n \\ n+i & \text{if } x = i, \; i = 1, 2, 3, \dots \end{cases}$$

Obviously, f is both 1-1 and onto.

So

$$IN \cup A \sim IN$$
$$\Rightarrow \#(IN \cup A) = \#(IN)$$
$$\Rightarrow \#(IN) + \#(A) = \#(IN)$$
$$a + n = a$$

which proves.

Exercise: 6

Show that

$a + 1 = a$, where $a = \#(IN)$

Sol:

Let $A = \{a\}$

Consider a function

$$f : IN \cup A \longrightarrow IN$$
$$IN \cup A = \{a, 1, 2, 3, \dots\}$$
$$IN = \{1, 2, 3, 4, \dots\}$$

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ n+1 & \text{if } x = n \ , \ n = 1,2,3,\ldots \end{cases}$$

Obviously, f is both 1-1 and onto.

So $\mathbb{N} \cup A \sim \mathbb{N}$

$\Rightarrow \#(\mathbb{N} \cup A) = \#(\mathbb{N})$

$\Rightarrow \#(\mathbb{N}) + \#(A) = \#(\mathbb{N})$

$a + 1 = a$ where $\#(\mathbb{N}) = a$ & $\#(A) = 1$

which Proves.

v.Imp 497

Schröeder Bernstein Theorem:

If $X \supset Y \supset X_1$ and $X \sim X_1$ then

$$X \sim Y$$

Proof:

Since $X \sim X_1$

therefore there exist a function f from X to X_1, which is both 1-1 and onto.

Since $Y \subset X$

Then the restriction of f from X to Y,

$f(Y) = Y_1 \subset X_1$

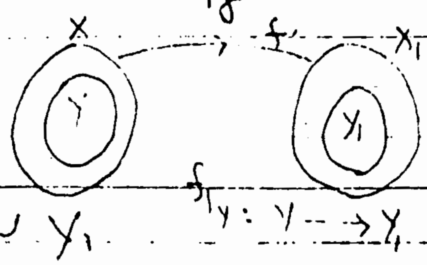
and $Y \sim Y_1$ ($\because f|_Y$ is 1-1 & onto)

So we have

$X \supset Y \supset X_1 \supset Y_1$

and

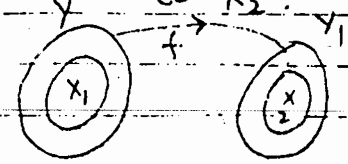
$X \sim X_1 \ , \ Y \sim Y_1$



Now since $X_1 \subset Y$ and $Y \sim Y_1$

So f is again a bijective function from X_1 to X_2

$\Rightarrow f(X_1) = X_2 \subset Y_1$



and $X_1 \sim X_2$

i.e

$$X \supset Y \supset X_1 \supset Y_1 \supset X_2$$

such that

$$X \sim X_1 ,\ Y \sim Y_1 ,\ X_1 \sim X_2$$

Continuing Process in the same way, we have

$$X \supset Y \supset X_1 \supset Y_1 \supset X_2 \supset Y_2 \supset X_3 \cdots$$

such that

$$X \sim X_1 ,\ Y \sim Y_1 ,\ X_1 \sim X_2 ,\ Y_1 \sim Y_2 ,\ X_2 \sim X_3 , \cdots$$

Let $B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2 \cap \cdots$

Now X and Y can be written as

$$X = (X - Y) \cup (Y - X_1) \cup (X_1 - Y_1) \cdots \cup B$$

$$Y = (Y - X_1) \cup (X_1 - Y_1) \cup (Y_1 - X_2) \cdots \cup B$$

Since

$$X \sim X_1 \text{ and } Y \sim Y_1$$

So using the same function f, we can observe that

$$(X - Y) \sim (X_1 - Y_1)$$

Similarly

$$(X_1 - Y_1) \sim (X_2 - Y_2)$$

and

$$(X_2 - Y_2) \sim (X_3 - Y_3) \cdots$$

and

Obviously
$$(Y - X_1) \sim (Y - X_1)$$
$$(Y_1 - X_2) \sim (Y_1 - X_2)$$

Hence considering $X = X_0$, $Y = Y_0$ we can define a function
$$g : X \longrightarrow Y$$
as

$$g(x) = \begin{cases} f(x) & \text{if } x \in (X_m - Y_m) \\ x & \text{if } x \in (Y_m - X_{m+1}) \end{cases}$$

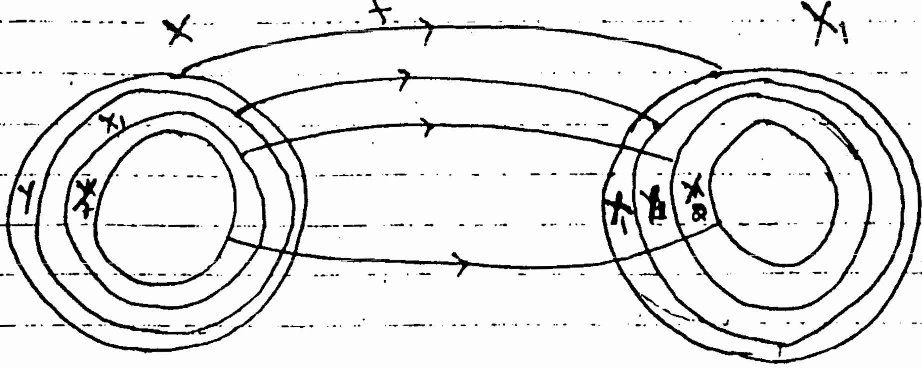
$m = 0, 1, 2, \ldots$

Obviously g is both 1-1 & onto. Hence
$$X \sim Y$$

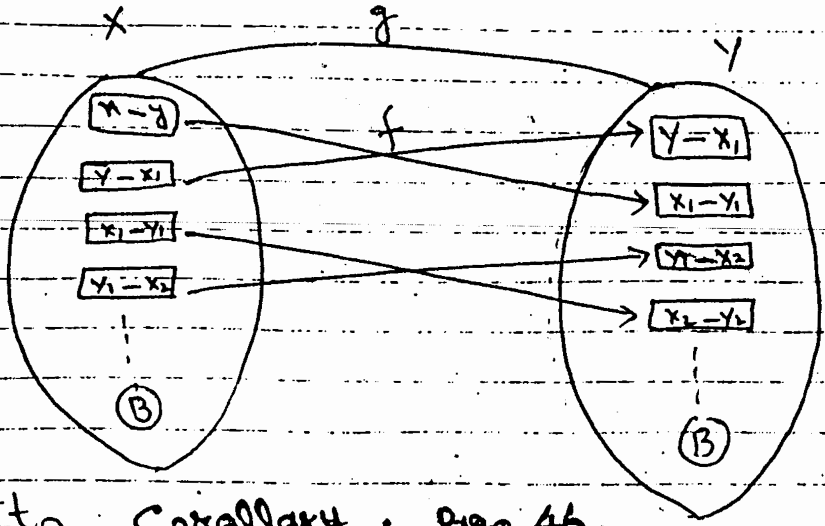
Other statement.

If $A \leq B$ & $B \leq A$ then $A \sim B$.

Hence for cardinal number α & β
$$\alpha \leq \beta \;\&\; \beta \leq \alpha$$
$$\Rightarrow \alpha = \beta$$



$$X \supset Y \supset X_1 \supset Y_1 \supset X_2 \ldots$$



Its Corollary : Page 46.

~~(COROLLARY)~~ v. Imp Theorem $\#(\mathbb{R}) = c = C$

y97 $C = 2^{a}$

Proof: To show $C \leq 2^{a}$

Define a function

$$f : \mathbb{R} \longrightarrow 2^{Q} \text{ as}$$

$$f(a) = \{x \in Q : x < a\} \quad \forall\, a \in \mathbb{R}$$

then f will be 1-1 as for $a, b \in \mathbb{R}$, $a < b$

then there exist rational no. r such that

$r < b$ $\qquad a < r < b$

which show that

$$r \in f(b)$$

and $\quad r \notin f(a)$

$\Rightarrow f(a) \neq f(b)$

which show that

$$\mathbb{R} \preceq 2^{Q}$$

or

$$\#(\mathbb{R}) \leq \#(2^{Q})$$

Since Q is denumerable i.e $Q \sim \mathbb{N}$ $\therefore \#(Q) = a$

$$C \leq 2^{a} \quad \text{———} \quad (1)$$

Conversly:
Consider $C(\mathbb{N})$ collection of all characteristics function defined on $\mathbb{N}$, which is equivalent to $2^{\mathbb{N}}$
i.e $C(\mathbb{N}) \sim 2^{\mathbb{N}}$
or $\#(C(\mathbb{N})) = 2^{a}$

Let $F : C(\mathbb{N}) \longrightarrow [0,1]$
defined as

$$F(f) = 0.f(1)\,f(2)\,f(3) \ldots\ldots \quad \forall\ f \in C(\mathbb{N})$$

F is one-one as
$f \neq g$

$$\Rightarrow F(f) \neq F(g)$$

i.e

$$C(\mathbb{N}) \preceq [0,1]$$

or

$$\#(C(\mathbb{N})) \leq \#([0,1])$$

$\because [0,1] \sim \mathbb{R}$
where $\#([0,1]) = C$
& $\#(C(\mathbb{N})) = 2^{a}$

$$2^{a} \leq C \quad \text{———} \quad (2)$$

From (1) and (2)

$$C = 2^{a}$$

As Required

Continum Hypothesis:

There does not exist any cardinal say α, such that

$$a < \alpha < c$$

$\mathbb{N} \preceq \mathbb{R}$
$\mathbb{N} \not\sim \mathbb{R}$
$\Rightarrow \mathbb{N} \prec \mathbb{R}$
$a < c$

v. Imp
Exercise: Prove that
y97

$$c \cdot c = c$$

or $\quad c^2 = c \quad$ where $c = \#[0,1]$

Proof: Let $A = [0,1]$
and

$$x, y \in [0,1]$$

Then x and y uniquely can be expressed in terms of an infinite decimal

$$x = 0 \cdot x_1 x_2 x_3 \cdots$$

or $\quad y = 0 \cdot y_1 y_2 y_3 \cdots$

which contains infinite many non-zero digits.

Let $f : A \times A \longrightarrow A$
defined as

$$f(x, y) = 0 \cdot x_1 y_1 x_2 y_2 x_3 y_3 \cdots$$

Obviously, f is one-one

$$\Rightarrow A \times A \preceq A$$

$$\Rightarrow \#(A \times A) \leq \#(A)$$

$$\Rightarrow c \times c \leq c$$

or

$$c^2 \leq c \quad \text{--- (1)}$$

On other hand, Cardinality of A can be proved to be maximum equal to the cardinality of $A \times A$ as

$$\{(0,x) / x \in A = [0,1]\}$$

is a subset of $A \times A$ and, is equivalent to A.

i.e. $A \sim \{(0,x)/x \in A\} \subseteq A \times A$

$$A \preceq A \times A$$

or $\#(A) \leq \#(A \times A)$

or $C \leq C^2$ —— ②

From ① and ②

$$C = C^2$$

v. Imp

Exercise: If α and β are cardinals then show that

i) $\alpha^{\beta} \cdot \alpha^{\gamma} = \alpha^{\beta+\gamma}$

ii) $\alpha \leq \beta$ ($\Rightarrow A \preceq B$)
$\Rightarrow \alpha^{\gamma} \leq \beta^{\gamma}$

iii) $\gamma^{\alpha} \leq \gamma^{\beta}$

Sol.

i) Already Proved

ii) $\alpha \leq \beta \Rightarrow \alpha^{\gamma} \leq \beta^{\gamma}$

Proof. Let $\alpha = \#(A)$
$\beta = \#(B)$

and $A \subseteq B$ $\therefore \alpha \leq \beta$

Moreover, Let

$$\#(C) = \gamma$$

Let $f \in A^C$

i.e $f : C \to A$

Since

$A \subseteq B$

So we can define the extension of f from C to B.

i.e $f : C \to B$

Hence A^C is subset of B^C.

which shows that

$$A^C \leq B^C$$

$$\Rightarrow \#(A^C) \leq \#(B^C)$$

or

$$\#(A)^{\#(C)} \leq \#(B)^{\#(C)}$$

or

$$\alpha^\gamma \leq \beta^\gamma$$

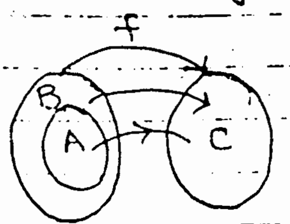
which Proves

iii) $\gamma^\alpha \leq \gamma^\beta$

Proof.

Consider the function f belonging to C^A.

Then corresponding to each function $f \in C^A$ we can define a function, say f', to be extension of f to B.



$C^A \subseteq C^B$

i.e

$$f' : B \to C$$

which shows that there is 1–1 correspondence b/w the class C^A and a subset of C^B

i.e

$$C^A \prec C^B$$

$\Rightarrow$

$$\#(C^A) \leq \#(C^B)$$

or

$$\gamma^\alpha \leq \gamma^\beta$$

As Required.

Defination:

If A and B are two sets such that the set A is equivalent to a subset of B, then we say that A 'precedes' B and is denoted by

$$A \preceq B$$

If $A \not\sim B$

then $A \prec B$

i.e A strictly precedes B.

Moreover,

$\#(A) = \alpha$, $\#(B) = \beta$

and $A \preceq B$

Then we say

$$\alpha \leq \beta$$

Also if

$$A \prec B$$

then

$$\alpha < \beta$$

*Observation:

If $\#(A) = \alpha$

$\#(B) = \beta$

~~and &~~ $\#(C) = \gamma$

and

$$A < B < C$$

then

$$A < C$$

$\Rightarrow \alpha \leq \beta \leq \gamma$

$\Rightarrow \alpha \leq \gamma$

which shows that laws of inequality of real numbers, also hold in cardinal numbers.

Example:

$\mathbb{N}$ the set of natural numbers, is a subset of set of real numbers $\mathbb{R}$, so

$N \preceq \mathbb{R}$ But $N \not\sim \mathbb{R}$

$\Rightarrow \mathbb{N} \prec \mathbb{R}$

and consequently

$$a < c$$

*NOTE: If $A \preceq B$ & $B \preceq C$ then $A \preceq C$

$\Rightarrow$ if $\#(A) = \alpha$, $\#(B) = \beta$, $\#(C) = \gamma$ and $\alpha \leq \beta$, $\beta \leq \gamma$ then $\alpha \leq \gamma$

Proof: $A \preceq B \Rightarrow \exists$ a func f s.t $f: A \to B$ is one-one. Similarly $B \preceq C \Rightarrow \exists$ a func $g: B \to C$ is one-one.

Now $g \circ f: A \to C$ is also one-one

So $A \preceq C$

or $\alpha \leq \gamma$

V. Imp

THEOREM: (Cantor's Theorem)

For any set A

$A < 2^A$ and hence for any Cardinal no. α, $\alpha < 2^\alpha$ where $\#(A) = \alpha$

Proof: Define a function

$$f : A \longrightarrow 2^A$$

such that

$$f(a) = \{a\} \quad \forall\, a \in A.$$

obviously, f is 1-1 but not onto.

(× Since every singelton is a subset of A. Therefore it can be observed that set is equivalent to collection of all singelton in A which is subset of 2^A ×)

Since A is equivalent to a subset of 2^A

Therefore

$$A \leq 2^A \quad \text{——} \quad (1)$$

Now we show that

$$A \nsim 2^A$$

Suppose that there exist a function

$$f : A \longrightarrow 2^A$$

which is both 1-1 & onto

Define a set

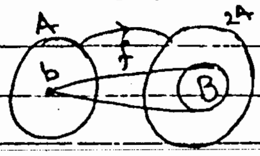
$$B = \{x \in A : x \notin f(x)\}$$

Since $B \subseteq A$ $\qquad \because A \subseteq 2^A$.

So $B \in 2^A$

Moreover, f is supposed to be onto. So there exist an element say $b \in A$ such that

$$f(b) = B$$



Consider the following two possibilities for b

i) $b \in B$

$$\Rightarrow {}^* b \notin f(b) = B$$

A contradiction.

ii) $b \notin B$

$$\Rightarrow b \in f(b) = B$$

A contradiction

which shows that there does not exist any function from A to 2^A which is both one-one and onto.

i.e.,

$$A \not\sim 2^A \quad \text{——} \quad (2)$$

From (i) and (ii)

$$A \prec 2^A$$

$$\Rightarrow \alpha < 2^\alpha$$

which Proves

* b does not belong to its image

B has those elements which does not belong to its image

v. Imp

Exercise:

Prove that
$$aC = C$$

Sol: Let $Z = \{0, \pm 1, \pm 2, \ldots\}$
and
$$A = [0,1]$$

and consider a function
$$f : Z \times A \longrightarrow \mathbb{R}$$
defined as
$$f(i, a) = i + a \qquad , i \in Z , a \in A$$

We show that f is $1-1$,
by considering
$$f(i,a) = f(j,b)$$
$$\Rightarrow \quad i + a = j + b$$
$$\Rightarrow \quad i = j \quad , \quad a = b$$
$$\Rightarrow \quad (i,a) = (j,b)$$
Hence
f is one-one.
Obviously, f is onto.

Hence $Z \times A \sim \mathbb{R}$

Since $R \sim]-\frac{\pi}{2}, \frac{\pi}{2}[\sim]0,1[$

$$\Rightarrow \quad R \sim]0,1[$$

$$\Rightarrow \quad \#(\mathbb{R}) = C$$

Also $Z \sim N$
$$\Rightarrow \quad \#(Z) = a \qquad \because \#(N) = a$$

$$\#(A) = \#([0,1])$$
$$\#(A) = c$$

$$Z \times A \sim \mathbb{R}$$
$$\#(Z \times A) = \#(\mathbb{R})$$
or
$$\#(Z) \cdot \#(A) = \#(\mathbb{R})$$
i.e
$$a \cdot c = c$$

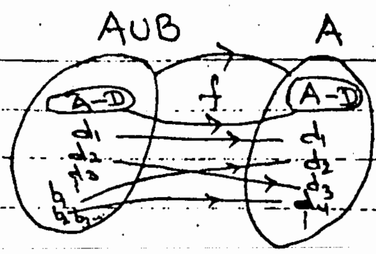
Which Proves.

v.Imp

Theorem:
$$\alpha + \beta = \beta$$
where
β is any infinite cardinal
and $a = \#(\mathbb{N})$

Proof. Let A be an infinite set
and
$B = \{b_1, b_2, b_3, \ldots\}$ a denumerable set such that
$$A \cap B = \phi$$
we show that
$$A \cup B \sim A$$
Let
$D = \{d_1, d_2, d_3, \ldots\}$ be a denumerable subset of A
Define a function
$$f : A \cup B \longrightarrow A$$
as

$$f(x) = \begin{cases} x & x \in A-D \\ d_{2n-1} & x \in d_n \\ d_{2n} & x \in b_n \end{cases}$$



which shows that f is both 1-1 and onto.

Hence $A \cup B \sim A$

$$\Rightarrow \# (A \cup B) = \# (A)$$

$$\# (A) + \# (B) = \# (A)$$

$$\beta + \alpha = \beta$$

or

$$\alpha + \beta = \beta$$

As required.

Exercise:

i) Prove that $[0,1] \sim \mathbb{R} =]-\infty, +\infty[$

Missing

$$(\alpha^{\beta})^{\gamma} = \alpha^{\beta\gamma}$$

&

$$(\alpha\beta)^{\gamma} = \alpha^{\gamma}\beta^{\gamma}$$

Corollary.

If $X \preceq Y$ and $Y \preceq X$
Then

$$X \sim Y$$

Proof.

$$X \preceq Y \Rightarrow X \sim Y_1 \subseteq Y$$

$$Y \preceq X \Rightarrow Y \sim X_1 \subseteq X$$

Since $Y \sim X_1$ so there exist a fun.
$f: Y \to X_1$ which is both one-one and onto.

Since $Y_1 \subseteq Y$
then the restriction of f to Y_1 is also one-one.

Hence Y_1 is equivalent to a subset, say X_2, of X_1.

i.e. $Y_1 \sim X_2$ where $X_2 \subseteq X_1$

So we have
$$X \supseteq X_1 \supseteq X_2$$
which shows that

$$X \sim X_1$$

and

$$X_1 \sim Y$$

$$\Rightarrow X \sim Y$$

which Proves.

(2)

SET THEORY

:- PATIALLY AND TOTALLY ORDERED SET

:- SIMILAR SETS

:- WELL ORDERED SETS

:- ORDINAL NUMBERS

:- AXIOM OF CHOICE

:- ZORN'S LEMMA

(SECTION - I)

CHAPTER : 2

PATIALLY OR TOTALLY ORDERED SET :

Partial Order: If A is a non-empty set and R be the relation defined in A such that

i) R is reflexive
i.e xRx $\forall x \in A$

ii) R is Anti-Symmetric
i.e xRy and $yRx \Rightarrow x = y$

iii) R is Transitive
i.e xRy and $yRz \Rightarrow xRz$

NOTE:

If R is a partial order of set then we write $a \leq b$ and read a preceed b.

• Example:

Let C be the collection of sets and define a relation R as ARB if $A \subseteq B$

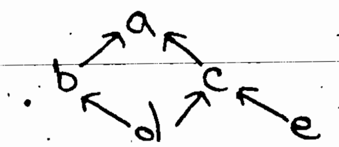
Sol.

Example: 1)

Let $A = \{a, b, c, d, e\}$ then the partial order is defined as $x \leq y$ if $x = y$ or if ~~one~~ we can go from x to y by following line indicated diagram

Sol:

We can observe



$b \preceq a$

$d \preceq b$

also $d \preceq a$

$e \preceq c$

$c \preceq a$

also $e \preceq a$

This diagram is called diagraph and also called "Hasse daiagram"

Example: 3)

Let $A = \{2, 3, 4, 5, 6\}$ and R be defined as $x R y$ if "x divides y".

Sol:

Then

$$R = \{(2,2), (3,3), (4,4), (5,5), (6,6), (2,4), (2,6), (3,6)\}$$

Example: 4)

In a set of natural no. $\mathbb{N}$ define an order R as $x R y$ if $x \leq y$. Then R is a partial order and is called the natural order.

Sol:

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

i) R is reflexive since every natural no. is equal to less than itself. i.e $x \leq x \;\; \forall x \in \mathbb{N}$

$\Rightarrow \; x R x \;\; \forall x \in \mathbb{N}$ —— (1)

ii) R is Anti symmetric ~~since~~ if $x R y$ & $y R x$ then by def. $x \leq y$ & $y \leq x$

$\Rightarrow \; x = y$

So R is Anti-symmetric —— (2)

iii) Now R is transitive if $x R y$ and $y R z$
then by def. $x \leq y$ and $y \leq z$
$\Rightarrow x \leq y \leq z$
$\Rightarrow x \leq z$ — (3)
So R is transitive.
From (1), (2), (3)
R is Partial order.

TOTAL ORDER IN A SET:

Def: A relation R which is partial order defined in a non-empty set is said to a "totally order" if for $a, b \in A$
either $a \leq b$ or $b \leq a$ or $a = b$

Example:
The natural order defin in $\mathbb{N}$ is total order?
Sol: when we have already prove that set of natural no. are partial order. Now for $x, y \in \mathbb{N}$
either $x \leq y$ or $y \leq x$ or $x = y$
So natural order in $\mathbb{N}$ is total order.

Inverse Order:

Def: Let R be a partial order defined in a set A then R^{-1} is also a partial order defined in A. i.e R^{-1} is a partial order then
$R^{-1} = \{(y, x) ; (x, y) \in R\}$

i) Reflexive: $x R x \ \forall x \in A$
$\Rightarrow x R^{-1} x \ \forall x \in A$

ii) Anti-Symmetric: $x R y$ & $y R x \Rightarrow x = y$
$\Rightarrow y R^{-1} x$ & $x R^{-1} y \Rightarrow y = x$

iii) Transitive:
$x R y$ & $y R z \Rightarrow x R z$
$\Rightarrow y R^{-1} x$ & $z R^{-1} y \Rightarrow z R^{-1} x$
So inverse of a partial order is a partial order.

Exercise:

Show that the order R defined by x divides y, in set $A=\{1,2,3,4,5,6\}$ is a partial order?

Sol:

So Here

$R=\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,2),(1,3),(1,4),(1,5),(1,6),(2,4),(2,6),(3,6)\}$

i) Reflexive: Every element belong to A divides itself

i.e $x|x \quad \forall\ x \in A$

$\Rightarrow xRx \quad \forall\ x \in A$

ii) Anti-Symmetric:

Let xRy and yRx

then by def. $x|y$ and $y|x$

$\Rightarrow x=y$

iii) Transitive:

Let xRy and yRz

then by def. $x|y$ and $y|z$

then $x|z$

$\Rightarrow xRz$

So R is Partial Order. Since it have satisfied all three condition.

Ex: Let $A=\{a,b,c,d,e\}$ and

$R=\{(a,a),(b,b),(c,c),(d,d),(e,e),(e,c),(c,a),(d,b),(b,a),(d,a),(d,c)\}$

Sol:

Then $R^{-1}=\{(a,a),(b,b),(c,c),(d,d),(e,e),(c,e),(a,c),(b,d),(a,b),(a,d),(c,d)\}$

i) Reflexive: Every element of A related with itself in R^{-1}

i.e $xR^{-1}x \quad \forall\ x \in A$

ii) Anti-Symmetric:

there is no $aR^{-1}c$ for $cR^{-1}a$ in R^{-1}.

So R^{-1} is anti-symmetric.

iii) Transitive:

In R^{-1} $\quad dR^{-1}b$ & $bR^{-1}a \Rightarrow dR^{-1}a$

So R^{-1} is partial order.

Ordered Set: If A is a non-empty set and R is a partial order defined in A, then A is said to be partially ordered set and is denoted by (A, R) or $(A, \preceq)$.

Totally Ordered set is defined similarly.

NOTE:

If in a ordered set an element preceed other but not equal to other i.e

If $a \preceq b$ but $a \neq b$:

then $a \prec b$ read as a strictly preceed b.

similarly $b \succ a$ means b strictly dominates a.

It may also noted that relation R total order in a set A, if for $a, b \in A$

$a \prec b$ or $b \prec a$ or $a = b$

Example:

Let A and B be two totally ordered set then cartesian product $A \times B$, can be totally ordered as follows:

$$(a, b) \prec (a', b')$$

If $a \prec a'$

or If $a = a'$ and $b < b'$

The order defined above is called Lexicographical order i.e Order given for the words in any English dictionary.

Example:

Let A and B be the totally ordered set. Then show that the order defined in $A \times B$ is not total order when R is defined as

$$(a, b) \preceq (a', b')$$

if $a \preceq a'$ & $b \preceq b'$ $\forall (a, b), (a', b') \in A \times B$

Sol: The relation R in $A \times B$ is not totally ordered since in $A \times B$ those order pairs are not comparable for which either $a \preceq a'$ or $b \preceq b'$.

NOTE:

The order defined above is called "Product order".

First And Last Element:

Let A be an ordered set (either totally ordered or Partially Ordered), an element $a \in A$ is said to be the "first element of A".

if $a \preceq x \quad \forall \; x \in A$

"i.e a never dominates any element of A."

And an element $a \in A$ is said to be the "Last Element" of A.

if $x \preceq a \quad \forall \; x \in A$

"i.e a must dominates every element of A."

Example:

Let $M = \{2, 4, 8, 16, \ldots\}$ and define an order in M as $x \preceq y$ if "x divides y".

First element = ? & Last element = ?

Sol.

The first element of M is 2.

The Last element of M does not exist.

$S = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$

Example: Let $S = \{x ; x \in Q, 0 \leq x \leq 1\}$ be an ordered set with natural order.

Find First and Last Element ?

Sol.

The first element of S is 0.

The last element of S is 1.

If $T = \{x : x \in Q, 0 < x < 1\}$

Then first and last elements of T does not exist.

~~Ex~~ **Example:** Let $S = \{x : x \in Q,\ 2 \le x^2 \le 3\}$
Find 1st and Last element?

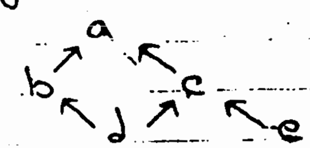
Sol: Here
$$S = \{x : x \in Q,\ \sqrt{2} \le x \le \sqrt{3}\}$$
S has no first and Last element.
Because $\sqrt{2}, \sqrt{3} \notin Q$ (rational no.).

Maximal And Minimal Element Of An Ordered Set:

Maximal Element: If A is an ordered set then an element $a \in A$ is said to be maximal element of A
if $a \preceq x$ then $a = x$
i.e a never ~~xxxx~~ preceed any other element of A.

Minimal Element: If A is an ordered set then an element $b \in A$ is said to be minimal element of
if $x \preceq b \Rightarrow b = x$
i.e b never dominetes any other element of A.

Example: Let $w = \{a, b, c, d, e\}$ be ordered by following diagram in the usual way:



Sol:
First Element = no
Last Element = a
Minimal Element = d, e
Maximal Element = a

یہاں First Ele. کوئی نہیں کیونکہ w میں کوئی بھی ایسا Element نہیں جو سب کو Preceed کرے۔ 'a' Last Ele. اس لیے ہے کیونکہ یہ w کے تمام Ele. کو dominet کرتا ہے۔ d, e Minimal Ele. اس لیے ہیں کیونکہ یہ w کے کسی بھی دوسرے ele. کو domenat نہیں کرتے۔ a max. ele اس لیے ہے کیونکہ یہ w کے کسی بھی دوسرے ele. کو Preceed نہیں کرتا۔

NOTE: A minimal element should not be a first element ~~[illegible]~~ converse is ~~[illegible]~~

Example:

Let $A = \{2, 3, 4, 5, \ldots\}$ with the order defined as $x R y$ if x divides y.

Sol:

First element = no

Last element = no

Minimal element = $\{2, 3, 5, 11, \ldots\}$ (Prime numbers)

Maximal element = no

Example:

Let $A = \{2, 3, 4, 5, 6, 8, 9, 10\}$ and order is defined as "x is multiple of y".

Sol:

First element = no

Last element = no

Minimal element =

Lower Bound:

If 'S' is an ordered set and 'A' is a subset of S, then an element $x \in S$ is said to be lower bound of A if $x \preceq a \quad \forall\ a \in A$

Infimum: An element x of an ordered set S is said to be infimum of $A \subseteq S$ if x is the last element of set of lower bounds of A.

Upper Bound: If S is an ordered set and A is a subset of S, then an element $x \in S$ is said to be Upper Bound of A if

$$a \preceq x \quad \forall\ a \in A$$

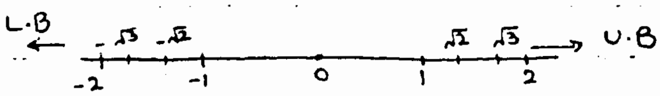
Suprimum: An element x of an ordered set S is said to be Suprimum of $A \subseteq S$ if x is the first element of set of upper bounds of A.

Example: 1)

A = Set of rational no.

$B = \{x \in A : 2 < x^2 < 3\}$

It may be noted that the set B have both upper and lower bounds but no infimum and suprimum.



or

$B = \{x \in A : \sqrt{2} < x < \sqrt{3}\} \cup \{x \in A : -\sqrt{3} < x < -\sqrt{2}\}$

$-\sqrt{3}$ سے کم تمام نمبر Lower bounds ہوں گے۔

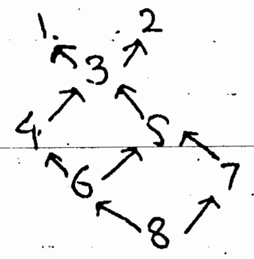
$\sqrt{3}$ سے بڑے تمام نمبر upper bounds ہوں گے۔

$\begin{cases} B \subseteq A \\ x \in A \\ b \preceq x \quad \forall b \in B \end{cases}$ upper bound

$\begin{cases} B \subseteq A \\ x \in A \\ x \preceq b \quad \forall b \in B \end{cases}$ Lower bound

Example: 2

$A = \{1,2,3,4,5,6,7,8\}$
$B = \{4,5,6\}$
R is defined in A as



Sol: Lower Bound (B): 8, 6
Upper Bound (B): 1, 2, 3
Infimum : 6
Suprimum : 3

THEOREM: In an ordered set first and last elements are unique.

Proof:

i) (For first element)

Let a, b be two first element of an ordered set A. Then considering 'a' to be the first element

So $a \preceq x \quad \forall x \in A$

In particular $a \preceq b$ —— (i)

Similarly, b being first element of A

So $b \preceq x \quad \forall x \in A$

In particular $b \preceq a$ —— (ii)

From (i) and (ii)

$a = b$

Hence Ist element of an ordered set A is unique.

which Proves.

ii) (For Last element)

Let c, d be two last element of an ordered set A. Then condering c to be the last element

So $x \preceq c \quad \forall x \in A$

In particular $d \preceq c$ —— (iii)

Similarly, d being last element of A

So $x \preceq d \quad \forall x \in A$

In particular $c \preceq d$ —— (iv)

From (iii) and (iv)

$c = d$

So last element of an ordered set is also unique.

As required.

Observation: (I think this obs. is not Corre)

①: If 'a' is first element of an ordered set then a is the only minimal element.

②: If 'b' is last element of an ordered set then 'b' is the only maximal element.

Proof: ① Let 'a' and 'b' be two minimal element (Maximal) of A.

Since 'a' is first element of A

so $a \leq x \;\forall\; x \in A$

In particular $a \leq b$

Since 'b' is the minimal element of A, i.e it does not dominate any element of A if it does then it is itself.

So $a = b$

Hence 'a' is the only minimal element of A.

Proof: ② Let c and d be two maximal element of A.

Since c is last element of A

So $x \leq c \;\forall\; x \in A$

In particular, $d \leq c$

Since d is the maximal element

So $c = d$

Hence c is the only maximal element of A

Which Proves.

THEOREM.

(1): In a totally ordered set, the maximal element is unique.
(2): In a totally ordered set, the minimal element is unique.

Proof: (1) Let a, b be two maximal elements of a totally ordered set A.
Since A is totally ordered and $a \neq b$
so let $a \leq b$
Moreover, a is maximal element as well.
So $a = b$
Hence a totally ordered set A can not have two or more maximal element. (i.e. It has unique maximal element.

Proof (2): Let c, d be two minimal element of A. Since A is totally ordered and $c \neq d$
so let $c \leq d$
Moreover, d is minimal element as well.
So $c = d$
Hence a totally ordered set has unique minimal element.
which Proves.

THEOREM: Every finite partial ordered set has at least one maximal and at least one minimal element.

Proof:
Let 'A' be a finite partial ordered set and $a_1 \in A$ which is not maximal element.
So there exists an element $a_2 \in A$ such that
$$a_1 \preceq a_2$$
Again let a_2 is not maximal element then there exist an element $a_3 \in A$ such that
$$a_2 \preceq a_3$$

If a_3 is not maximal element then we can find an element $a_4 \in A$ such that

$$a_3 \leq a_4$$

Continuing the process, since A is finite, there exist $a_n \in A$ such that

$$x \leq a_n \Rightarrow x = a_n$$

i.e a_n is maximal element.

(2): Similarly, there exist at least one minimal element.

Proof (2) Let A be a finite partial ordered set and $a_1 \in A$ which is not minimal element. So there exist an element $a_2 \in A$ s.t

$$a_1 \succeq a_2$$

Again let a_2 is not minimal element then there exist an element $a_3 \in A$ s.t

$$a_2 \succeq a_3$$

If a_3 is not minimal element then we can find an element $a_4 \in A$ s.t

$$a_3 \succeq a_4$$

Continuing the process, since A is finite, there exist $a_n \in A$ s.t

$$x \succeq a_n \Rightarrow a_n = x$$

i.e a_n is minimal element.

Remark:

A set which have first and last element must be totally ordered but converse is not true.

Similar Sets:

Two ordered sets A and B are said to be similar if there exist a mapping

$$f : A \longrightarrow B$$

such that

i) f is bijective.

ii) for $a_1, a_2 \in A$

$$a_1 \preceq a_2 \quad \text{iff} \quad f(a_1) \preceq f(a_2)$$

NOTE: It may be noted that the function defined above is called "Similarity Mapping" and it is an equivalence relation.

Example.

$$N = \{1, 2, 3, 4, \ldots\}$$
$$E = \{2, 4, 6, 8, \ldots\}$$

Define a function

$$f : N \longrightarrow E \text{ as}$$
$$f(x) = 2x \quad \forall\, x \in N$$

i) f is one-one & onto.

ii) for $1, 2 \in N$

$$1 \preceq 2 \quad \text{iff} \quad f(1) \preceq f(2)$$

Hence N is similar to E.

NOTE: It may be noted that if we take

$$M = \{-1, -2, -3, \ldots\}$$

then $(M, \leq)$ is not similar to $(N, \leq)$.

Since as $1 \leq 2$, $1, 2 \in N$

$$f(1) \nleq f(2)$$
$$\text{i.e. } (-1 \nleq -2)$$

However $(M, \geq)$ is similar to $(N, \geq)$.

THEOREM:

Let $f: A \to B$ be a similarity mapping from an ordered set A to another ordered set B, the element $a \in A$ is the last element of A iff $f(a)$ is the last element of B.

Proof: (Necessary Cond.)

Since 'a' is the last element of A

then $x \preceq a \quad \forall \; x \in A$

Since $f: A \to B$ is a similarity mapping

then by def.

$$f(x) \preceq f(a) \quad \forall \; f(x) \in B$$

$\Rightarrow f(a)$ is the last element of B.

(Sufficient Cond.)

If $f(a)$ is last element of B and

Suppose $f(a) = b$ then

$$y \preceq b \quad \forall \; y \in B \qquad \because f \text{ is bijective func. So is } f^{-1}$$

$$\Rightarrow f^{-1}(y) \preceq f^{-1}(b)$$

$$\Rightarrow f^{-1}(y) \preceq a \quad \forall \; f^{-1}(y) \in A$$

So 'a' is last element of A.

As Required.

THEOREM:

Let $f: A \to B$ be a similarity mapping then $a \in A$ is minimal or maximal if and only if $f(a)$ is minimal or maximal element of B.

Proof: Let 'a' be a minimal element of A. If $x \in A$

then $x \preceq a \Rightarrow x = a$

i.e. $x \npreceq a$

$\Rightarrow f(x) \npreceq f(a)$ ($\because f$ be similarity func.)

Since f is bijective So

$$B = \{ b / b = f(x), x \in A \}$$

Thus

$$f(x) \npreceq f(a) \quad \forall \; f(x) \in B$$

Hence $f(a)$ is a minimal element of B.

Conversly: Let $f(a) \in B$ be a minimal element of B. Then

$$f(x) \nless f(a) \quad \forall f(x) \in B$$

$$\Rightarrow \quad x \nless a \quad \forall x \in A \quad \because f \text{ is similar map.}$$

then 'a' is minimal element of A.

Proof: (For maximal Element)

Let C be a maximal element of A.

if $x \in A$

then $C \preceq x \Rightarrow x = C$

i.e $C \nless x \quad \forall x \in A$

$$\Rightarrow \quad f(C) \nless f(x) \quad (\because f \text{ be similarity func.})$$

Since f is bijective. So

$B = \{d / d = f(x), x \in A\}$

Thus

$$f(c) \nless f(x) \quad \forall f(x) \in B$$

Hence $f(c)$ is a maximal element of B.

Conversly: Let $f(c) \in B$ be a maximal element of B. Then

$$f(c) \nless f(x) \quad \forall f(x) \in B$$

$$\Rightarrow \quad c \nless x \quad \forall x \in A$$

Hence c is maximal element of A.

As Required

CHAP : 3

WELL ORDERED SETS AND ORDINAL NUMBERS

Def.: If A is an ordered set, then A is said to be well ordered set, if every subset of A has the first element.

Remarks:

①: If A is well ordered set then A is a totally ordered set as well as if $a, b \in A$ then $\{a,b\} \subseteq A$, By def. of well ordered set, $\{a,b\}$ must have first element, say, 'a' then $a \leq b$. which shows that every pair of element in A, is comparable.

Hence A is totally ordered set.

(It may be noted that a totally ordered set may not be well ordered.)

②: A subset of a well ordered set is always well ordered.

③: If two sets A and B are similar and if A is well ordered, then B is also well ordered.

Example:

The set of natural number N, with natural order, is well ordered.

Principle Of Mathematical Induction:

Let S be a subset of N, the natural number, with the following properties.

1) $1 \in S$

2) $n \in S \Rightarrow n+1 \in S$

Then

$$S = N$$

Theorem: (1) 64

Principle Of Transfinite Induction:

(Generalization Of Mathematical Induction):

Let S be a subset of a well ordered set A such that

1) $a_0 \in S$, where a_0 is the first element of A

2) $s(a) \subseteq S \Rightarrow a \in S$

where $s(a) = \{x \in A ; x < a\}$ (set of elements in A which strictly precede a).

Then $S = A$

Proof: suppose (1) & (2) satisfied and

Let $S \neq A$

i.e. Let $T = A \setminus S$ then

T is non empty.

Let t_0 be the first element of T. (Since A is well ordered.)

(Consider $s(t_0)$

i.e. $s(t_0) = \{x \in A : x < t_0\}$)

Let

$x \in s(t_0)$

then $x < t_0$

Since t_0 is supposed to be the first element of T

So $x \notin T$

$\Rightarrow x \in S$

$\Rightarrow s(t_0) \subseteq S$

using (ii)

$t_0 \in S$

a contradiction, as $t_0 \in T$ and $t_0 \notin S$

Hence

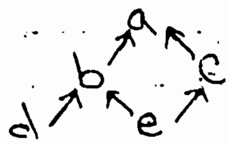
$S = A$

As required.

Limit Elements:

Defination: If A is a well ordered set we say that an element $a \in A$, is the immediate successor of an element $b \in A$ if there does not exist $c \in A$ such that
$$b < c < a$$
It is also to be noted that in such case, b is said to be the immediate predecessor of a.

Example: Let $A = \{a, b, c, d, e\}$ be ordered as follows:



Sol: Here b is an immediate successor of both d and e and e is an immediate predecessor of both b and c.

THEOREM ②: Let A be a well ordered set and let $S(A)$ be the collection of all initial segments of A then there is a mapping
$$f : A \longrightarrow S(A)$$
defined by
$$f(a) = s(a) \quad \forall\, a \in A$$
is a similarity mapping.

Proof: The function f is obviously onto.
Now we show that f is one-one
~~Let~~ $\cancel{x \neq y}$
Consider $s(x)$ and $s(y) \in S(A)$.
Since A is well ordered (total ordered).
Let $x \neq y$ So either $x < y$ or $y < x$
Let $x < y$ * Then by def. of initial segment
$\Rightarrow$ * $x \notin s(x)$ & $x \in s(y)$

x-- Since x, y are arbitrary elements, so f preserve order for all elements of A.
Hence f is an order preserving bijective mapping.

66

$\Rightarrow s(x) \neq s(y)$

Hence f is one-one

Now we have to show that

$x \leq y$

$\Rightarrow s(x) \subseteq s(y)$

Let $a \in s(x)$

then $a < x$

but $x \leq y$

$\Rightarrow a < y$

$\Rightarrow a \in s(y)$

$\Rightarrow s(x) \subseteq s(y)$ i.e $f(x) \leq f(y)$

Now suppose $x \not\leq y$

i.e. $y < x$

then $y \in S(x)$

by def of int. seg.

$y \notin S(y)$

$\Rightarrow S(x) \nsubseteq S(y)$

In other words

$x \leq y$ iff $S(x) \subseteq S(y)$

x---

Thus f is similarity mapping.

THEOREM 3: Let A be a well ordered set and B be a subset of A.
If $f: A \to B$ is a similarity mapping then $a \leq f(a) \quad \forall a \in A$

Proof:

Let $D = \{x \in A ; f(x) < x\}$ or $x \not\leq f(x)$ if D is empty Th. is true.

Suppose $D \neq \phi$

Since A is the well ordered set.

Let d_0 be first element of D.

So $d_0 \in D \Rightarrow f(d_0) < d_0$ (by structure of D)

Since f is similarity mapping

So $f(f(d_0)) < f(d_0)$

$\Rightarrow f(d_0) \in D$ but $f(d_0) < d_0$ & $d_0 \in D$

A contradiction as d_0 was supposed to be the first element of D.

Hence D is empty set.

OR

$a \leq f(a) \quad \forall a \in A$

THEOREM ④: If A and B are well ordered sets and A is similar to B then there exist a unique similarity mapping from A to B.

Proof: Let $f : A \longrightarrow B$

and $g : A \longrightarrow B$

be two similarity mapping such that

$$f(x) \neq g(x) \quad \text{for some } x \in A$$

Since B is well ordered, so

either $f(x) \prec g(x)$ or $g(x) \prec f(x)$

Suppose $f(x) \prec g(x)$ ——— (1)

Since g is a similarity mapping

So $g^{-1} : B \longrightarrow A$ is also similarity mapping.

Now $$g^{-1}(f(x)) \prec g^{-1}(g(x))$$

$$\Rightarrow (g^{-1} o f)x \prec x = (g^{-1} o g)x$$

But

$g^{-1} o f : A \longrightarrow A$ (The product of two similar map. is similar map.)

is also similarity mapping.

~~but~~ and

$$(g^{-1} o f)x \prec x$$

A contradiction as $g^{-1} o f$ is a similarity mapping from A to A. (by last th.) also A is well-ordered $\forall a \in A$

which implies

$$a \preceq (g^{-1} o f)(a) \qquad \forall a \in A$$

Hence

$$f(x) = g(x) \qquad \forall x \in A$$

Thus

$$f = g$$

* and $s(a)$ be an initial seg. of A, for some $a \in A$. 23

THEOREM: (5)

A well ordered set can not be similar to any of its initial segments.

Proof: Let A be a well ordered set. Suppose A is similar to $S(a)$, $a \in A$ such that

$f : A \longrightarrow s(a)$ is a similarity mapping.

Then for $x \in A$, $f(x) \in s(a)$

In particular $f(a) \in s(a)$

$\Rightarrow f(a) < a$

A contradict to the fact that

$S(a) \subseteq A$ and $A \simeq s(a)$

$\Rightarrow a \leq f(a) \quad \forall\, a \in A$

Hence the set A can not be similar to any of its initial segment.

THEOREM: If A and B are well ordered sets then either A is similar to B or A is shorter than B or A is longer than B.

Proof: Construct

$S = \{x \in A \,/\, s(x) \simeq s(y),\ y \in B\}$

$T = \{y \in B \,/\, s(y) \simeq s(x),\ x \in A\}$

then using Lemma (3) we have the following possibilities

i) $S = A$ and $T = B$

ii) $S = A$ and T is an initial segment of B.

iii) $T = B$ and S is an initial segment of A.

iv) S is an initial segment of A and T is an initial segment of B.

Now using lemma (4)
Consider the above all possibilities.

i) $S = A$ and $T = B$
$S \simeq T \Rightarrow A \simeq B$
i.e A is similar to B.

ii) $S \simeq T \Rightarrow$ A is similar to an initial segment of B.
i.e A is shorter than B.

iii) $S \simeq T \Rightarrow$ B is similar to an initial segment of A.
i.e B is shorter than A.

iv) $S \simeq T \Rightarrow$ An initial segment, say, $s(a)$ of A is similar to an initial segment, say, $s(b)$ of B.
or

$$s(a) \simeq s(b)$$

$$\Rightarrow a \in S$$

Since its initial seg. $s(a)$ is similar to an inial segment $\underset{\sim}{s(b)}$ of B.

But

$$a \notin S(a)$$

Since a cannot belong to its own initial segment.

Hence this case is impossible.

Hence the theorem.

Lemm 1: Let A be a well ordered set and S be a subset of A, with the (mapping) property that

$$* \; a \preceq b \;, b \in S$$
$$\Rightarrow a \in S$$

Then either $A = S$ or S is an initial segment of A.

Proof: Let $A \setminus S$ be non-empty.

Then $A \setminus S$ will be well ordered set.

Let a_0 be the first element of $A \setminus S$

where $a_0 \notin S$.

we show that

$$S = s(a_0)$$

if $x \in s(a_0)$ i.e $x \prec a_0$

then $x \notin A \setminus S$

i.e $x \in S$.

Hence $s(a_0) \subset S$

Now suppose

$$y \notin s(a_0) \qquad y \not\prec a_0$$

So $a_0 \preceq y$

But $y \in S$ and $a_0 \preceq y$ using *

$\Rightarrow a_0 \in S$

A contradict the fact that $a_0 \notin S$.

Hence $y \notin S$

Thus

$$S \subseteq s(a_0)$$

Consequently,

$$S = s(a_0)$$

which Required.

Ex: Two different initial segment of a well ordered set can not be similar?

Sol: Let A be a well ordered and $s(a)$ and $s(b)$ be two initial segment of A such that

$$s(a) \simeq s(b) \quad , \text{i.e } a \neq b$$

Since A is well order so either $a < b$ or $b < a$

So let $a < b$

then $a \in s(b)$

If $x \in s(a)$

then $x < a$ and $a < b$

$\Rightarrow x < b$

which shows that

$x \in s(b)$

$\Rightarrow s(a) \subset s(b)$

Thus $s(b)$ becomes similar to its initial segment, a contradiction, that a well ordered set can not be similar to any of its initial segment.

Lemma: 2) Let A and B be two well ordered set then an initial segment $s(a)$ of A is similar to a unique initial segment $s(b)$ of B.

Proof: Let $b, b' \in B$ such that $b' \neq b$

and $s(a) \simeq s(b)$

and $s(a) \simeq s(b')$

which shows that

$s(b) \simeq s(b')$

A contradiction, that two distinct initial segment of any well ordered set can not be similar.

Hence

$$s(b) = s(b')$$

i.e initial segment $s(a)$ of A is similar to unique initial segment $s(b)$ of B.

Ex: Let A and B be two well ordered sets and initial segment $s(a)$ of A is similar to an initial segment $s(b)$ of B then every initial segment of $s(a)$ is similar to an initial of $s(b)$.

Proof: Since $s(a) \simeq s(b)$, so there exist a similarity mapping

$$f : s(a) \longrightarrow s(b)$$

Let $a' < a$

we show that

$$s(a') \simeq s(b') \quad \text{where } b' < b$$

Since f is bijective and order preserving mapping so

$$s(a') \text{ will be similar to } f(s(a')) \qquad \text{Hence } s(a') \simeq f(s(a'))$$

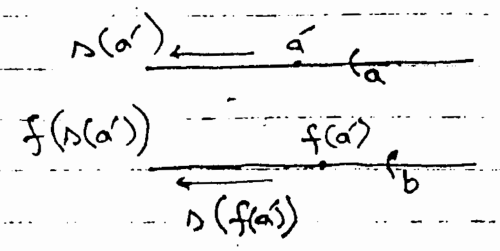
Considering the restriction of f to $s(a')$.

Hence b' can be considered to be equal to $f(a')$.

$$\text{Hence} \quad s(a') \simeq s(b')$$

NOTE:

$$f(s(a')) = s(f(a'))$$



If A & B two well ordered sets such that initial seg. of A similar to an init. seg. of B then each initial seg. of A similar to each init. seg. of B.

Lemma 3)

Let A and B be two well ordered sets and $S = \{x \in A \,/\, s(x) \simeq s(y),\ y \in B\}$

Then either $S = A$ or S is an initial segment of A.

Proof: Let $x \in S$ and $x' < x$

Then $s(x')$ similar to an initial segment of B. (by* last Ex.)

In other words

$$x' < x \text{ and } x \in S$$

$$\Rightarrow x' \in S \qquad \text{Then } S = A \text{ or } S \text{ is an I.S of A.}$$

[Use the Lemma

If A be a well ordered set and let S be subset of A with property that

$$a \leq b \text{ and } b \in S \Rightarrow a \in S$$

Then $S = A$ or S is an initial segment of A.]

Lemma (4): Let A and B be two well ordered sets and $S = \{x \in A \,/\, s(x) \simeq s(y),\ y \in B\}$

$$T = \{y \in B \,/\, s(y) \simeq s(x),\ x \in A\}$$

Then

$$S \simeq T.$$

Proof: Define a function

$$f : S \longrightarrow T$$

defined by $f(x) = y$ such that

$$s(x) \simeq s(y), \quad \forall x \in S,\ y \in B$$

i) Since an initial segment of S, can not similar more than one initial segment of T.

ii) Again no two distinct initial segment of S can be similar to an initial segment of T.

Hence f is one-one.

iii) Now, by construction of T, each element of T is image of some element of S.

(i.e. for each initial segment $s(y)$ of T is similar to some initial segment $s(x)$ of S).

To show f is order preserving.

Let $x' < x$, where $x, x' \in S$

then $s(x) \simeq s(y)$ for some $y \in B$

and so there exist ~~a~~ similarity mapping

$$\phi : s(x) \longrightarrow s(y)$$

Since $x' < x$

so $x' \in s(x)$

Then using a result proved earlier.

$$s(x') \simeq \phi(s(x')) = s(\phi(x')) = s(y')$$

$$s(x') \simeq s(y') \Rightarrow y' < y$$

so by def. of f

$$y' = f(x')$$

Hence

$$x' < x \Rightarrow f(x') < f(x)$$

i.e f is similarity mapping.

Hence

$$S \simeq T$$

THEOREM: Let A be a well ordered set and $\mathcal{A}$ be a class of initial segments of A, then there is initial segment, which is shorter than all the other initial segment.

Proof:

As we know that $A \simeq S(A)$ (by theorem 1)

(where $S(A)$ is collection of all initial segment)

i.e $\mathcal{A} \simeq \mathcal{A}$ $\because \mathcal{A} \subset S(A)$

Since A is well ordered. $S(A)$ is also well ordered.

So: there exist an initial segment in $\mathcal{A}$ which is shorter than all initial segments of A.

$A_1 \subset A_2$ next Page.

$A_1 \simeq A_1 \subset A_2$

NOTE: Initial segment of first element is empty.

Proof: As we know that
$$A \simeq S(A) \quad (\text{by Th.}(1))$$
($S(A)$ is collection of all initial seg. of A)

Then $A \simeq \mathcal{A}$ $\ni \mathcal{A} \subset S(A)$

Since A is well ordered and $S(A)$ is also well ordered.

Consequently, $\mathcal{A}$, a subset of $S(A)$, has a first element $s(a)$.

Therefore
$$s(a) \subset s(x)$$
for any other initial seg. $s(x) \in \mathcal{A}$.

which Proves.

Theorem:
Let $\mathcal{A}$ be a collection of pairwise disjoint non-similar well ordered sets. Then there is a set A_0 in $\mathcal{A}$ which is shorter than all other sets in $\mathcal{A}$.

Proof: Let $B \in \mathcal{A}$ and $\mathcal{B}$ be the class of members of $\mathcal{A}$ shorter than B. i.e
$$\mathcal{B} = \{A \in \mathcal{A} \,;\, A \text{ is shorter than } B\}$$
If $\mathcal{B}$ is empty then B is the set we are looking for.

If $\mathcal{B}$ is non-empty then let $A \in \mathcal{B}$

i.e A is shorter than B.

Let C be the collection of initial seg. of B so that each member of C is similar to some member of B.

Now, using the previous theorem, C will be well ordered (order defined by set inclusion) and will have an initial seg. which is shorter than all other initial segments in C.

Corresponding B will contain a well ordered set, say B, which will shorter than all other sets belonging to B.

Since B was taken to be arbitrary so B can also be considered to be the member of A which is shorter than all other members of A.

ORDINALS: (Ordinal Numbers)

Definition: Let A be a well ordered set and C_A be the class of all ordered sets similar to A. Then C_A is said to be the Ordinality of A.

Ordinal number of $A = \text{ord}(A)$

The ordinal numbers of the following sets
$\{\}$, $\{1\}$, $\{1,2,3\}$, ...
are $0, 1, 3,$ .....

However,

$\text{ord}(\mathbb{N}) = \omega$
$\text{ord}(\mathbb{Z}) = \bar{\pi}$
$\text{ord}(\mathbb{Q}) = \eta$

There are two types of ordinal numbers.
i) Finite
ii) Transfinite

Def.: If $\lambda = \text{ord}(A)$
$\mu = \text{ord}(B)$
then $\lambda < \mu$
If A is shorter than B
and $\lambda > \mu$
If A is longer than B.
and $\lambda = \mu$
If A is similar to B.

THEOREM: If $\lambda = \text{ord}(A)$ and $\mu < \lambda$ then there is unique initial segment $s(a)$ of a such that $\mu = \text{ord}(S(a))$

Proof: Since $\lambda = \text{ord}(A)$
and Let $\mu = \text{ord}(B)$
Since $\mu < \lambda$
So B is similar to some a unique initial segment $s(a)$ of A so that

For $\mu, \nu \in s(\lambda)$
Let $f(\mu) = f(\nu)$
$s(a) = s(b)$ $\quad s(a), s(b) \in S(A)$
$ord(s(a)) = ord(s(b))$ $\quad \mu = ord(s(a))$
$\Rightarrow \mu = \nu$ $\quad \nu = ord(s(b))$

for each $s(a) \in S(A)$ there exist some ordinal no., say, μ less than λ s.t. $f(\mu) = s(a)$ for $a \in A$

$\therefore f$ is one-one

$\mu = ord(B) = ord(s(a))$ $\Rightarrow f$ is onto.

$\mu = ord(s(a))$

which proves.

THEOREM: Let λ be an (initial segment) an ordinal number and $s(\lambda)$ be the collection of ordinal numbers less than λ. Then

$$Ord(s(\lambda)) = \lambda$$

Proof: Let $\lambda = ord(A)$

As we have already $A \simeq S(A)$

So $ord(S(A)) = \lambda$

To show

$$ord(s(\lambda)) = \lambda$$

It is sufficient to show

$$s(\lambda) \simeq S(A)$$

Define

$$f : s(\lambda) \longrightarrow S(A) \quad \text{as}$$

$$f(\mu) = s(a) \quad \text{if } \mu = ord(s(a))$$
$$\forall \; \mu < \lambda \;, \; s(a) \in S(A)$$

* Obviously, f is both one-one and onto.
we show that f is order preserving
Let $\mu, \eta \in s(\lambda)$
such that
$$\mu < \eta$$
and
$$\mu = ord(s(a))$$
$$\eta = ord(s(b))$$
where $s(a), s(b) \in S(A)$

Then by def. of inequality of ordinal numbers,

$s(a)$ must be shorter than $s(b)$.

i.e $s(a) \leq s(b)$

or $f(u) \leq f(\eta)$

Hence

f is the similarity mapping from $s(\lambda)$ to $S(A)$.

Thus

$$\text{ord}(s(\lambda)) = \lambda$$

Remark:

Since the collection of ordinal number is well ordered so every ordinal number has immediate predecessor. However, there are some non-zero ordinal numbers.

e.g ω, which do not have immediate predecessors.

Such ordinal no. are called Limit Ordinal Number or Limit Number.

THEOREM: $\lambda+1$, is the immediate successor of λ.

Proof: Let u be the immediate successor of λ. we show that

$$u = \lambda + 1$$

Here $s(u) = s(\lambda) \cup \{\lambda\}$ $\because \lambda$ is Immediate Predessor

$\Rightarrow$ $\text{ord}(s(u)) = \text{ord}(s(\lambda) \cup \{\lambda\})$

$\text{ord}(s(u)) = \text{ord}(s(\lambda)) + \text{ord}\{\lambda\}$

Since $\text{ord}(s(\gamma)) = \gamma$ for any ordinal γ.

$\Rightarrow$ ~~ord~~ $u = \lambda + 1$

Hence $\lambda+1$ is the immediate successor of λ.

Limit Element:

The ordinal is said to be Limit Element, such that there is no ordinal which is immidiate predecessor of that ordinal number.

Addition Of Ordinals:

Let $\lambda = \text{ord}(A)$

$\mu = \text{ord}(B)$

where A and B are disjoint well ordered set. we defince

$$\lambda + \mu = \text{ord}(A \cup B)$$

$$\lambda + \mu = \text{ord}(A; B)$$

where $(A;B)$ means that $A \cup B$, the elements of A are written before the elements of B and this order is must to maintain. which shows that

$$\lambda + \mu = \text{ord}(A; B) \neq \text{ord}(B; A) = \mu + \lambda$$

Example:

$A = \{a_1, a_2, a_3, \ldots, a_n\}$

$B = \{1, 2, 3, \ldots\} = \mathbb{N}$

then

$$\text{ord}(A) = n \quad \text{and} \quad \text{ord}(B) = \omega$$

$$\text{ord}(B \cup A) = \text{ord}(B) + \text{ord}(A)$$

$$\omega + n = \text{ord}\{1, 2, 3, \ldots ; a_1, a_2, \ldots, a_n\}$$

Here

$\{1, 2, 3, \ldots ; a_1, a_2, \ldots, a_n\}$ show that $\mathbb{N}$ is similar to initial segment of $(B;A)$.

i.e $\omega + n > \omega$ ——— (1)

Now

$n + \omega = \text{ord}\{a_1, a_2, \ldots, a_n; 1, 2, 3, \ldots\}$ is similar to $\{1, 2, 3, \ldots, n, n+1, n+2, \ldots\} = \mathbb{N}$

i.e

$$n + \omega = \omega \qquad \because \text{ord}(\mathbb{N}) = \omega$$

From (1)

$$\omega + n > \omega = n + \omega$$ (i.e addition of ordinals are not commutative)

$$\omega + n > n + \omega$$

Additive Identity.

Consider any ordinal λ to the ordinality of well ordered set A and '0' be that of $B = \{\}$

$$\lambda + 0 = ord\,(A \cup \phi)$$
$$= ord\,(A)$$
$$= \lambda$$

Moreover,

$$0 + \lambda = ord\,(\phi \cup A)$$
$$= ord\,(A)$$
$$= \lambda$$

Hence $\lambda + 0 = \lambda = 0 + \lambda$

i.e '0' is the additive identity of ordinal numbers.

Associate Property w.r.t '+' of Ordinals:

$$\lambda + (\mu + \eta) = (\lambda + \mu) + \eta$$

Let λ, μ and η be the ordinal numbers of mutually disjoint well ordered set A, B and C respectively

Then

$$\lambda + (\mu + \eta) = ord\,(A) + ord\,(B;C)$$
$$= ord\,(A;(B;C))$$
$$= ord\,((A;B);C)$$
$$= ord\,(A;B) + ord\,(C)$$
$$\lambda + (\mu + \eta) = (\lambda + \mu) + \eta$$

So Associative Law hold.

Multiplication Of Ordinals:

Let A and B be well ordered sets (disjoint) and $A \times B$ be ordered as

$$(a, b) \leq (a', b')$$

if $b < b'$ and if $b = b'$ then $a < a'$.

This order is called "Inverse Lexicographical order".

we define ordinality

$$\text{ord}(A \times B) = \lambda \mu$$

where $\lambda = \text{ord}(A)$

and $\mu = \text{ord}(B)$

It may be noted that the multiplication of ordinals is not commutative

If $A = \{1, 2, 3\}$, $B = \{a, b\}$

$$A \times B = \{(1,a), (2,a), (3,a), (1,b), (2,b), (3,b)\}$$

Example:

The multiplication of ordinals is not commutative in general

sol

Consider $A = \{1, 2, 3, \ldots\} = \mathbb{N}$

and $B = \{a, b\}$

Then

$$\text{ord}(A) = \omega$$

$$\text{ord}(B) = 2$$

Now

$$\omega \cdot 2 = \text{ord}(A \times B)$$

$$= \text{ord}\{(1,a), (2,a), (3,a), \ldots, (1,b), (2,b), (3,b), \ldots\}$$

It can we observed that

$\mathbb{N} = \{1, 2, 3, \ldots\}$ is similar to $\{(1,a), (2,a), (3,a), \ldots\}$

Since $\{(1,a), (2,a), (3,a), \ldots\}$ is an initial segment of $\{(1,a), (2,a), (3,a), \ldots, (1,b), (2,b), (3,b), \ldots\}$

So $\omega 2 > \omega$ — (i) $\because \text{ord}(\mathbb{N}) = \omega$.

Now

$$2\omega = \text{ord}\,(B \times A)$$
$$= \text{ord}\,\{(a,1),(b,1),(a,2),(b,2),\ldots\}$$

Here

$\{(a,1),(a,2),(b,1),(b,2),\ldots\}$ is similar to
$\{1, 2, 3, 4, \ldots\} = \mathbb{N}$

Hence

$$2\omega = \omega \quad \text{——— (ii)}$$

From (i) and (ii)

$$\omega 2 > \omega = 2\omega$$

i.e. $\omega 2 > 2\omega$

Ex. Show that

$$\omega . n \neq n . \omega$$

when $\omega = \text{ord}\,(\mathbb{N})$

$n = \text{ord}\,\{a_1, a_2, \ldots, a_n\}$

Prove that

$$\omega \cdot 1 = \omega = 1 \cdot \omega$$

Proof:

we know that

$$\omega = \text{ord}(\mathbb{N}) = A$$
$$1 = \text{ord}\{a\} = B$$

Consider

$$\omega \cdot 1 = \text{ord}(A \times B)$$
$$= \text{ord}\{(1,a),(2,a),(3,a),\ldots\}$$

It can be observed that

$\{(1,a),(2,a),(3,a),\ldots\}$ is similar to

$\{1, 2, 3, \ldots\} = \mathbb{N}$

So

$$\omega \cdot 1 = \omega \quad \text{---(1)} \qquad \because \text{ord}(\mathbb{N}) = \omega$$

Now

$$1 \cdot \omega = \text{ord}(B \times A)$$
$$= \text{ord}\{(a,1),(a,2),(a,3),\ldots\}$$

Here $\{(a,1),(a,2),(a,3),\ldots\}$ is similar to

$\{1, 2, 3, \ldots\} = \mathbb{N}$

So

$$1 \cdot \omega = \omega \quad \text{---(2)}$$

From (1) and (2)

$$\omega \cdot 1 = \omega = 1 \cdot \omega$$

As required

Observation:

For ordinals λ, μ, η

i) $\lambda(\mu\eta) = (\lambda\mu)\eta$

ii) $\lambda(\mu+\eta) = \lambda\mu + \lambda\eta$

Proof: i) Let

$$\lambda = \text{ord}(A)$$
$$\mu = \text{ord}(B)$$
$$\eta = \text{ord}(C)$$

Then

$$\lambda(\mu\eta) = \text{ord}(A) \cdot \text{ord}(B \times C)$$
$$= \text{ord}[A \times (B \times C)]$$

But $A \times (B \times C) = (A \times B) \times C$

$$\lambda(\mu\eta) = \text{ord}\,[(A\times B)\times C]$$
$$= \text{ord}\,(A\times B)\,.\,\text{ord}\,(C)$$
$$\lambda(\mu\eta) = (\lambda\mu)\,\eta$$

As Required.

ii) $\lambda(\mu+\eta) = \lambda\mu + \lambda\eta$ (Left Distributive Prop.)

But Right Distributive Prop.

Proof:

Let $\lambda = \text{ord}\,(A)$
$\mu = \text{ord}\,(B)$
$\eta = \text{ord}\,(C)$

Then

$$\lambda(\mu+\eta) = \text{ord}\,(A)\,.\,\text{ord}\,(B;C)$$
$$= \text{ord}\,[A\times(B;C)]$$

But

$$A\times(B;C) = (A\times B);(A\times C)$$

$$\lambda(\mu+\eta) = \text{ord}\,[(A\times B);(A\times C)]$$
$$= \text{ord}\,(A\times B) + \text{ord}\,(A\times C)$$
$$\lambda(\mu+\eta) = \lambda\mu + \lambda\eta$$

As Required.

Remark:

It may be noted that Right Distributive property of multiplication over addition does not hold in general. e.g

$$(1+1)\,\omega = 2\omega$$
but $2\omega = \omega$

So

$$(1+1)\,\omega = \omega \quad \text{——} \quad (1)$$

Now

$$1\omega + 1\omega = 1\,(\omega+\omega)$$
$$= \omega+\omega$$

$$= \omega\cdot 1 + \omega\cdot 1$$
$$= \omega(1+1)$$
$$1\omega + 1\omega = \omega 2 \quad \text{———} \quad (2)$$

From (1) and (2)

$$(1+1)\omega \neq 1\omega + 1\omega$$

Structure Of Ordinals:

$0, 1, 2, 3, ----, \omega$; (ω is Ist limit ordinal)

$\omega+1, \omega+2, -----, \omega+\omega = \omega 2$ ($\omega 2$ is 2nd " ")

$\omega 2+1, \omega 2+2, ---, \omega 2+\omega = \omega 3$ ($\omega 3$ is 3rd " ")

$- - - - - - - -$

$- - - - - - - -$

$- - - - - - - - - = \omega\cdot\omega = \omega^2$

$0, 1, 2, ---, \omega, ----, \omega 2, ---, \omega 3, ------ \omega\cdot\omega = \omega^2$

$\omega^2+1, \omega^2+2, ----, \omega^2+\omega, -----, \omega^2+\omega^2 = \omega^2 2$

$- - - - - - - -$

$- - - - - - - - - - - = \omega^2\omega = \omega^3$

$(\omega)^\omega ----((\omega)^\omega)^\omega ---- \varepsilon_0 + \varepsilon_0+1, \varepsilon_0+2, -----$

(SECTION — I)

Form (i) + (ii)

$(1+1)\omega = 1\omega + 1\omega$

Zorn's Lemma:-

If A is partially ordered set in which every totally ordered subset has an upper bound in A, then A has a maximal element.

Well-ordering Theorem (Zermelo):-

Every non-empty set can be well ordered.

Proof:-

Let A be any non-empty set and $\mathcal{A}$ be the family of all well ordered subsets of A i.e. if $W \in \mathcal{A}$, then $W = (B, \leq)$ where B is a subset with order '$\leq$' defined in B.

Give partial order to $\mathcal{A}$ as if $W_1, W_2 \in \mathcal{A}$, then $W_1 \leq W_2$ if $W_1 \subseteq W_2$

$\mathcal{A} = \{B \subseteq A, \text{'}\leq\text{'} \text{ defined in } B\}$

$W_i = (B_i, \leq)$

$\mathcal{A} = \{W_1, W_2, W_3, \ldots\}$

i.e. partial order defined by "set inclusion".

Let $\{W_i\}_{i \in I}$ be the totally ordered subset of $\mathcal{A}$.

Then family of sets $\{B_i\}_{i \in I}$ ordered by the set inclusion is totally ordered.

Define

$W = (B, \leq)$

where $B = \bigcup_i B_i$

Consider $a, b \in B$, Then $\exists$ B_i, B_j s.t.
$a \in B_i$, $b \in B_j$

Since $\{B_i\}_{i \in I}$ is totally ordered, so one member must be contained in other.

Let $B_i \subseteq B_j$

Then $a, b \in B_j$

Now $a \leq b$ if $a \leq b$ in B_j

So $(W, \leq)$ is well-ordered subset of A and therefore belongs to $\mathcal{A}$.

Also W is an upper bound of $\{W_i\}_{i \in I}$ so in $\mathcal{A}$ every totally ordered subset has an upper bound in $\mathcal{A}$.

Using Zorn's lemma,

$\mathcal{A}$ contains a maximal element, say, W^*

We show that $W^* = A$

If $W^* \neq A$, let $a \in A \setminus W^*$

Consider the set $\{ W^* ; \{a\} \}$ which is well ordered and so belongs to $\mathcal{A}$, a contradiction as W^* was supposed to be a maximal element.

Hence,

$W^* = A$

Problem:-

If B is a partially ordered set then B has a totally ordered subset, say A_0, which is not proper subset of any other totally ordered subset of B.

Proof:-

Let $\mathcal{B}$ be the family of totally ordered subset of B. Give a partial order to $\mathcal{B}$ by set inclusion.

Let $\{B_i\}_{i\in I}$ be totally ordered subset of $\mathcal{B}$. Let $A = \bigcup_{i\in I} B_i$, then A is totally ordered subset of B.

Define an order in A as

if $a, b \in A$ then $\exists$ $i, j \in I$ s.t. $a \in B_i$, $b \in B_j$

Since $\{B_i\}_{i\in I}$ is totally ordered, then either B_i is contained in B_j or B_j is contained in B_i.

Let $B_i \subset B_j$

$\Rightarrow$ $a, b \in B_j$

Now $a \leq b$ if $a \leq b$ in B_j

Thus $A \in \mathcal{B}$.

Also A is an upper bound of $\{B_i\}_{i\in I}$.

Using Zorn's Lemma,

$\mathcal{B}$ has a maximal element, say, A_0.

Hence A_0 is a totally ordered subset of B, which is not contained in any other totally ordered subset of B.

Problems:-

Every vector space has basis.

Proof:-

Let V be a non-zero vector space. Let A be the family of all linearly independent subsets of V. A is non-empty for if v_1 is a non-zero vector, then $\{v_1\}$ is linearly independent.

Give a partial order to A by set inclusion. Let $\{B_i\}_{i\in I}$ be a totally ordered subset of A.

Taking $B = \bigcup_{i\in I} B_i$

We show that B is linearly independent.

Suppose the contrary i.e. let B be linearly dependent subset of V

i.e. $x_1, x_2, x_3, \ldots, x_n \in B$ s.t.

$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ for at least one $c_i \neq 0$

Since B is superset of all B_i, $i \in I$, so $\exists$ integers $i_1, i_2, \ldots, i_n$ s.t.

$x_1 \in B_{i_1}$, $x_2 \in B_{i_2}$, $\ldots$, $x_n \in B_{i_n}$

Since $\{B_i\}_{i\in I}$ is a totally ordered so one of B_i's is contained in some other member of $\{B_i\}_{i\in I}$.

Let $x_1, x_2, \ldots, x_n \in B_{i_1}$ which shows that B_{i_1} is linearly dependent, a contradiction to the supposition that $\{B_i\}_{i\in I}$ is family of linearly independent subsets of V.

Since B is linearly independent and is upper bound of $\{B_i\}_{i\in I}$, also then it belongs to A.

... maximal element.

Let B^* be a maximal element of $\mathcal{A}$.
We show that B^* is a basis for V.
Let $B^* = \{b_1, b_2, b_3, \ldots, b_n\}$ and $v \in V$
Then the set $\{v, b_1, b_2, \ldots, b_n\}$ is linearly dependent.
Let $c_0 v + c_1 b_1 + c_2 b_2 + \cdots + c_n b_n = 0$. where $c_0 \neq 0$
so

$$v = -\frac{c_1}{c_0} b_1 - \frac{c_2}{c_0} b_2 \cdots - \frac{c_n}{c_0} b_n$$

Thus every element v of V can be expressed in linear combination of members of B^*.
Thus B^* spans V.
Hence
B^* is a basis.

Choice function:-

Let $\{A_i\}_{i\in I}$ be a non-empty class of non-empty subsets of X. Then the function f defined on $\{A_i\}_{i\in I}$ is said to be a <u>Choice function</u> if

$$f(A_i) = a_i \quad , \; a_i \in A_i \quad , \; \forall\, i \in I$$

Cartesian Product of non-empty class of non-empty sets:-

Let $\{A_1, A_2, A_3, \dots, A_n\}$ be a finite class of non-empty sets, then each choice function f, defines a unique n-tuple

$$(f(A_1), f(A_2), f(A_3), \dots, f(A_n))$$

The set of <u>all</u> choice functions defined on $\{A_i\}_{i=1}^{n}$ defines the Cartesian product of $A_1, A_2, A_3, \dots, A_n$ denoted by

$$A_1 \times A_2 \times A_3 \times \dots \times A_n = \bigtimes_{i=1}^{n} A_i$$

Remark:- Generalization.

Axiom of Choice:-

Cartesian product of a non-empty class of non-empty sets is not empty.

or <u>equivalently</u>, there exists a Choice function for any non-empty class of non-empty sets.

Remarks:-

The axiom of choice is equivalent to the following Zermelo's Postulate.

Let $\{A_i\}_{i \in I}$ be any non-empty class of disjoint non-empty sets. Then there exists a subset B of $\bigcup_i A_i$, such that

$(B \cap A_i)$, $\forall\, i \in I$, contains exactly one element.

Proof:-

Let $\{A_i\}_{i \in I}$ be a non-empty class of disjoint non-empty sets and let f be a choice function defined on $\{A_i\}_{i \in I}$.

Set

$$B = \{ f(A_i) : i \in I \}$$

Then $B \cap A_i = \{ f(A_i) \}$ $\forall\, i \in I$ consists of exactly one element.

Since A_i's are disjoint and f is a choice function. Hence axiom of choice provides Zermelo's Postulate.

Conversely,

Let $\{A_i\}_{i \in I}$ be any non-empty class of non-empty sets.

Then $A_i^* = \{A_i\} \times \{i\}$

Obviously, $\{A_i^*\}_{i \in I}$ will be class of disjoint sets, as:

for $i \neq j$, $\{A_i\} \times \{i\} \neq \{A_j\} \times \{j\}$ even $A_i = A_j$

So By Zermelo's Postulate,

there exists a subset B of $\bigcup_i A_i^*$ such that $B \cap A_i^*$ consists of exactly one element.

Then $a_i \in A_i$ and π function f defined on $\{A_i\}_{i \in I}$ as $f(A_i) = a_i$ will be a choice function.

Hence Zermelo's Postulate provides the axiom of choice.

Section II

Set Function:

A function f is said to be a set func if its domain is a collection of sets.

e.g If M is the collection of set.

Then

$$f : M \longrightarrow \{-1, 1\}.$$

defined as

$$f(A) = \begin{cases} -1 & \text{if } o(A) = \infty \\ 1 & \text{if } o(A) < \infty \end{cases}$$

Non-negative Set Function:

A set function is said to be non-negative if its range is $[0, \infty]$ (set of non-negative real no)

e.g

$$f : M \longrightarrow [0, \infty]$$

defined as

$$f(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{otherwise} \end{cases} \quad \forall\, A \in M$$

Additivity of A Set Function: (non-ve set functn)

A set function $f : M \longrightarrow [0, \infty]$ is said to be additive if for, $A, B \in M$ disjoint sets and $A \cup B = M$

$$f(A \cup B) = f(A) + f(B)$$

e.g

$$\mu : M \longrightarrow [0, \infty]$$

defined as

$$\mu(I) = l(I) \qquad \forall\ I \in M$$

where M is collection of open and closed interval.

we ~~___~~ shows that (for additivity of μ)

$$\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$$

Hence

$$I_1 = [a,b) \quad , \quad I_2 = [b,c)$$

$$I_1 \cup I_2 = [a,c)$$

$$l(I_1) = b-a \quad , \quad l(I_2) = c-b$$

$$l(I_1 \cup I_2) = c-a$$

$\Rightarrow$

$$l(I_1 \cup I_2) = l(I_1) + l(I_2)$$

$$\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$$

So μ is an additive set function.

Finitely Additive Set Function:

The set function $f : M \longrightarrow [0,\infty]$ is said to be finitely additive if

$$f\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} f(A_i) \qquad \text{For disjoint } A_i \in M$$

Countably Additive Set Function:

σ- (σ-additive function):

If M is a collection of sets and $f : M \longrightarrow [0,\infty]$ is said to be countably additive (or σ- additive) if, for mutually disjoint sets

$$A_1, A_2, A_3, \ldots, A_i, \ldots \in M$$

and $\bigcup_i A_i \in M$

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} f(A_i)$$

e.g

$$M = 2^X$$

$$\mu : M \longrightarrow [0,\infty]$$

defined as

$$\mu(A) = o(A) \qquad \forall A \in M$$

then for mutually disjoint sets
$A_1, A_2, A_3, \ldots \in M$
and $\bigcup_i A_i \in M$

we show that $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$

set
$\mu(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_i \cup \cdots) = O(A_1) + O(A_2) + \cdots + OA_i + \cdots$

$$\mu(\bigcup_i A_i) = \sum_i^{\infty} O(A_i)$$

$$\mu(\bigcup_i A_i) = \sum_i^{\infty} \mu(A_i)$$

$\Rightarrow \mu$ is a Countably additive set func.

Sub-additive Set Function:

A set function $f: M \to [0,\infty]$ is said to be Sub-additive set function if for $A, B \in M$ and $A \cup B \in M$

$$f(A \cup B) \leq f(A) + f(B)$$

Similarly Countably Subadditivity of functions.

σ-Sub additive Set Function:

A set function $f: M \to [0,\infty]$ is said to be σ-Sub additive if for
$A_1, A_2, \ldots \in M$ and
$\bigcup_i A_i \in M$

$$f(\bigcup_i A_i) \leq \sum_i f(A_i)$$

Example: (Sub-additive)

Taking $M = 2^X$
$\mu: M \to [0,\infty]$
defined as
$\mu(A) = O(A) \quad \forall A \in M$

Theorem: for $A, B \in M$, $A \cup B \in M$

$$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

$$\mu(A \cup B) = o(A \cup B)$$
$$\leq o(A) + o(B)$$
$$= \mu(A) + \mu(B)$$
$$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

Remark: It may be noted that the sub-additivity of a set function is the generalization of additivity of a set function.

Premeasure

A set function $f: M \to [0, \infty]$ is said to be Premeasure if
$$f(\phi) = 0$$

e.g

Let M be the collection of all open intervals and
$$f: M \to [0, \infty] \text{ defined as}$$
$$f(I) = l(I) \quad \forall \; I \in M$$

Here
$$f(]a,a[) = f(\phi) = l(]a,a[)$$
$$= 0$$
$$f(\phi) = 0$$

Thus f is Premeasure.

Outer Measure:

A set function $f: 2^X \to [0, \infty]$ is said to be outer measure if

i) $f(\phi) = 0$

ii) for $A_1, A_2, A_3, \dots \in 2^X$, $A \subseteq \bigcup_i A_i$
$$f(A) \leq \sum_i f(A_i)$$

Monotone Set Function:

A set function $f : M \to [0,\infty]$ is said to be monotone, for $A \subseteq B$
$$\Rightarrow \quad f(A) \leq f(B)$$

Remarks:

1) Every outer measure is a premeasure but the converse may not true.
2) An outer measure is a monotone σ ~~E~~ — sub additive.

Proof: i) For $A, B \in 2^X$ and $A \subseteq B$
then $f(A) \leq f(B)$

ii) For $A_1, A_2, A_3, \ldots \in 2^X$
and $A \subseteq \bigcup_i A_i$
$$f(A) \leq f\left(\bigcup_i A_i\right)$$
$$\leq \sum_i f(A_i)$$

Hence
$$f(A) \leq \sum_i f(A_i)$$

So an outer measure is a monotone ~~E~~ — sub additive.

Example: Define
$$f : 2^X \to [0,\infty]$$
defined as
$$f(A) = o(A) \quad \forall\ A \in 2^X$$

Here
i) $f(\phi) = o(\phi) = 0$
ii) Let $A_1, A_2, A_3, \ldots \in 2^X$, $A \subseteq \bigcup_i A_i$
$$f(A) = o(A) \leq o\left(\bigcup_i A_i\right)$$
$$\leq \sum_i o(A_i)$$

$$= \sum_i f(A_i)$$

$$\Rightarrow \quad f(A) \leq \sum_i f(A_i)$$

Hence f is outer measure.

2) Define

$$\mu : 2^X \longrightarrow [0,\infty]$$

as

$$\mu(A) = \begin{cases} 0 & \text{if } A = \phi \\ 1 & \text{if } A \neq \phi \end{cases}$$

Here

i) $\mu(\phi) = 0$

ii) Let $A_1, A_2, A_3, \ldots \in 2^X$ and $A \subseteq \bigcup_i A_i$

a) if A is empty, then

$$\mu(A) = 0$$

Now there may exist one A_i which is not empty so

$$\sum_i \mu(A_i) \geq 0$$

$$\Rightarrow \quad \mu(A) \leq \sum_i \mu(A_i)$$

b) if A is non-empty, then

$$\mu(A) = 1$$

Since A is non-empty and is subset of $\bigcup_i A_i$, so there exists at least one A_i, which is non-empty

$$\sum_i \mu(A_i) \geq 1$$

$$\Rightarrow \quad \mu(A) \leq \sum_i \mu(A_i)$$

Hence μ is an outer measure.

3) Define

$$\Delta : 2^X \longrightarrow [0,\infty]$$

as

$$\Delta(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{otherwise} \end{cases}$$

Here

i) $\Delta(\phi) = 0$ Since empty set is countable.

ii) Let $A_1, A_2, A_3, \dots \in 2^X$ and $A \subseteq \bigcup_i A_i$

a) If A is countable

then $\Delta(A) = 0$

Since $A \subseteq \bigcup_i A_i$, and A is countable, then there may exist one A_i which is not countable.

So

$$\sum_i \Delta(A_i) \geq 0$$

$$\Rightarrow \quad \Delta(A) \leq \sum_i \Delta(A_i)$$

b) If A is un-countable

then $\Delta(A) = 1$

Since $A \subseteq \bigcup_i A_i$ and A is not countable so there exist at least one A_i which is not countable.

Hence

$$\sum_i \Delta(A_i) \geq 1 \; (= \Delta A)$$

$$\Rightarrow \Delta(A) \leq \sum_i \Delta(A_i)$$

Hence Δ is an outer measure.

* Since $A \subseteq \cup_i A_i$, and $y \notin A$, so there may exist one A_i s.t $y \in A$

4) Define

$$\mu : 2^X \longrightarrow [0, \infty]$$

as

$$\mu(A) = \begin{cases} 0 & , \text{if } y \notin A \\ 1 & , \text{if } y \in A \end{cases}$$

Here

i) $\mu(\phi) = 0$, $y \notin A$ for fixed $y \in X$

ii) Let $A_1, A_2, A_3, \ldots \in 2^X$ and $A \subseteq \cup_i A_i$, $y \notin$

If $y \notin A$

then

$$\mu(A) = 0 \quad \text{—— (i)}$$

* therefore there is ~~at least~~ may exist one A_i s.t $y \notin A_i$ ~~in~~ $\cup_i A_i$

$$\sum_i \mu(A_i) \geq 0 \quad \text{—— (ii)}$$

From (i) and (ii)

$$\mu(A) \leq \sum_i \mu(A_i)$$

b) If $y \in A$

then $\mu(A) = 1$ —— (iii)

Since $y \in A$ and $A \subseteq \cup_i A_i$

So there exists at least one $y \in A_i$ in $\cup_i A_i$

i.e $\sum_i \mu(A_i) \geq 1$ —— (iv)

From (iii) and (iv)

$$\mu(A) \leq \sum_i \mu(A_i).$$

Hence μ is an outer measure.

Sum Of Two Outer Measure:

Let f and g be two outer measure from 2^X to $[0, \infty]$.

Define

$$f+g : 2^X \longrightarrow [0, \infty]$$

as

$$(f+g)A = f(A) + g(A)$$

Consider

i) $(f+g)(\phi) = f(\phi) + g(\phi)$

$$= 0 + 0 = 0$$

ii) Let $A_1, A_2, A_3, \text{---} \in 2^X$, $A \subseteq \bigcup_i A_i$

$$\begin{aligned}(f+g)(A) &= f(A) + g(A) \\ &\leq \sum_i f(A_i) + \sum_i g(A_i) \\ &\leq \sum_i \{f(A_i) + g(A_i)\} \\ &= \sum_i (f+g) A_i\end{aligned}$$

i.e

$$(f+g)(A) \leq \sum_i (f+g) A_i$$

Scalar Multiplication Of An Outer Measure:

Let f be an outer measure defined

$$f : 2^X \longrightarrow [0, \infty]$$ and c be a non-negative real number.

Define a function

$$cf : 2^X \longrightarrow [0, \infty)$$

as

$$(cf)(A) = c(f(A))$$

It can be proved that cf is also an outer measure as,

i) $(cf)(\phi) = c[f(\phi)]$

Since f is outer measure.

$$f(\phi) = 0$$

then $cf(\phi) = C(0)$

$cf(\phi) = 0$

ii) Let $A_1, A_2, A_3, \ldots \in 2^X$ and $A \subseteq \bigcup_i A_i$

then

$$(cf)(A) = c[f(A)]$$
$$\leq c \sum_i f(A_i)$$
$$= \sum_i cf(A_i)$$
$$= \sum_i (cf)(A_i)$$

i.e

$$(cf)(A) \leq \sum_i (cf)(A_i)$$

Difference Of Two Outer Measure:

Let f and g be two outer measure from 2^X to $[0, \infty]$

defined as

$$(f-g)(A) = f(A) - g(A)$$

If

$f(A) - g(A) < 0$ then function is not outer measure.

Ex: Give example of two outer measures f and g so that $f-g$ is not an outer measure i.e $(f-g) \geq 0$

Sol:

Let $f: M \to [0, \infty]$ and $g: M \to [0, \infty]$

defined as

$$f(A) = \begin{cases} 0 & , A = \phi \\ 1 & , A \neq \phi \end{cases} \qquad g(A) = \begin{cases} 0 & , A \text{ is Countable} \\ 1 & \text{otherwise} \end{cases}$$

i) $(f-g)(\phi) = f(\phi) - g(\phi)$

$= 0 - 0 = 0$

ii) Let $A_1, A_2, A_3, \_\_\_ \in M$, $A \subseteq \bigcup_i A_i$

$$(f-g)A = f(A) - g(A)$$

Since f and g are outer measures then we know

$$f(A) \le \sum_i f(A_i)$$

$$g(A) \le \sum_i g(A_i)$$

then

$$(f-g)(A) \le \sum_i f(A_i) - \sum_i g(A_i)$$

This is not always true.

Lebesgue Outer Measure:

Let A be any subset of $\mathbb{R}$ and $\{I_n\}$ be a countable collection of open intervals, which covers A i.e

$$A \subseteq \bigcup_n I_n$$

$\{I_n\}$ covers A, $\{I_n\}$ set cover$_A$

But, $m^*(A) = \text{Inf}\{\sum_n l(I_n)$

for $n \in 2$

sin

Define

$$m^* : 2^{\mathbb{R}} \longrightarrow [0, \infty].$$

as

$$m^*(A) = \text{Inf}\{\sum_n l(I_n) \; ; \; A \subseteq \bigcup_n I_n\}$$

or

$$m^*(A) = \text{Inf} \sum_n l(I_n) \quad , \quad A \subseteq \bigcup_n I_n$$

we show that m^* is an outer measure.

i) $m^*(\phi) = 0$

It may be noted that

$$\phi \subseteq \bigcup_n I_n \quad \text{where} \quad I_n =]a_n, b_n[_{n=1}^{\infty}$$

$$m^*(\phi) = \inf\{\sum_n l(I_n) \; ; \; \phi \subseteq \bigcup_n I_n\}$$

$$= 0$$

ii) Let $A, A_1, A_2, A_3, \text{---} \in 2^{\mathbb{R}}$ and $A \subseteq \bigcup_i A_i$

Also $\{I_{n,i}\}$ be the countable collection of open intervals which cover A_i

$$\Longrightarrow A_i \subseteq \bigcup_{n}^{\infty} A\{I_{n,i}\}$$

inf $\quad e_{h_i}$

Then $m^*(A_i) = \inf\left\{\sum_n l(I_{n,i})\ ;\ A_i \subseteq \bigcup_n I_{n,i}\right\}$
Now corresponding to each A_i, we can choose $\varepsilon > 0$ such that

$$\cancel{\inf}\sum_n l(I_{n,i}) < m^*(A_i) + \frac{\varepsilon}{2^i} \quad \text{——} \ (1)$$

As $A \subseteq \bigcup_i A_i$ and $A_i \subseteq \bigcup_n I_{n,i}$
so

$$A \subseteq \bigcup_i \bigcup_n I_{n,i}$$

So by def. of m^*

$$m^*(A) = \inf\left\{\sum_i \sum_n l(I_{n,i})\ ;\ A \subseteq \bigcup_i \bigcup_n I_{n,i}\right\}$$

$m^*(A) \le \sum_i \sum_n l(I_{n,i}) \quad \therefore m^* = \inf$

$$\le \sum_i \left(m^*(A_i) + \frac{\varepsilon}{2^i}\right) \quad \text{using } (1)$$

$$= \sum_i m^* A_i + \varepsilon \sum_i \frac{1}{2^i}$$

$$= \sum_i m^* A_i + \varepsilon$$

where $\sum_i \frac{1}{2^i}$ is G.P

and it sum is equal to 1.
i.e $\sum_i \frac{1}{2^i} = 1$

So

$$m^*(A) \le \sum_i m^* A_i + \varepsilon$$

Since ε was taken to be arbitrary
so $m^*(A) \le \sum_i m^*(A_i)$

Hence m^* is an outer measure and is called Lebesgue Outer Measure.

$$\sum_i \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$|r| = \frac{1}{2} < 1 \qquad S_n = \frac{1-r}{a}$$

$$S_n = \frac{1-\frac{1}{2}}{\frac{1}{2}}$$

$$= \frac{\cancel{\frac{1}{2}}\,1}{\cancel{\frac{1}{2}}}$$

$$S_n = 1$$

Lemma: 1)

ℓ-outer measure of a finite closed interval, is its length.

Proof: Let $[a,b]$ any finite closed interval. Then for each $\varepsilon>0$, the intervals $]a-\varepsilon, b+\varepsilon[$ contains $[a,b]$.

i.e

$[a,b] \subset]a-\varepsilon, b+\varepsilon[$

therefore

$$m^*[a,b] \leq \ell(]a-\varepsilon, b+\varepsilon[)$$
$$= b-a+2\varepsilon$$

Since ε was taken to be arbitrary so

$$m^*[a,b] \leq b-a \quad \text{——— (i)}$$

To show $m^*[a,b] \geq b-a$

Let $\{I_n\}$ be the countable collection of open intervals covering $[a,b]$.

Then using Hein Borel Theorem, there exist a finite sub-collection $\{I_{n_i}\}$ of open interval, which covers $[a,b]$.

i.e $\quad [a,b] \subseteq \bigcup_{i=1}^{p} I_{n_i}$

Since $a \in [a, b]$ and $[a,b] \subseteq \bigcup_{i=1}^{p} I_{n_i}$

so 'a' belongs to some interval in $\{I_{n_i}\}$
Let

$$a \in (a_1, b_1)$$

If $b_1 \leq b$ then $b_1 \in [a,b]$
and there exist an open interval in $\{I_{n_i}\}$, say (a_2, b_2) such that $a_2 < b_1 < b_2$

If $b_2 \leq b$ then $b_2 \in [a,b]$ $\Rightarrow \cancel{b_2 \in (a,b)}$
and there exist an open interval in $\{I_{n_i}\}$, say (a_3, b_3) such that $a_3 < b_2 < b_3$

Continuing the process, we have

$$a_i < b_{i-1} < b_i$$

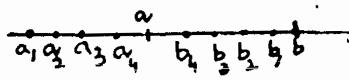
Since $\{I_{n_i}\}$ is finite sub collection of $\{I_n\}$, so this process must terminate after finit many steps. i.e

$$\exists \ (a_k, b_k) \text{ in } \{I_{n_i}\}$$

Such that

$$a_k < b \leq b_k$$

Now



$$\sum_n l(I_n) \geq \sum_{i=1}^{k} l(a_i, b_i)$$

$$= \sum_{i=1}^{k} (b_i - a_i)$$

$$= (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_k - a_k)$$

$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1)$$

$$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \cdots - (a_2 - b_1) - a_1$$

$$> b_k - a_1 \qquad \because (a_i - b_{i-1} < 0) \quad i = 2, 3, \ldots, k$$

$$\sum_n l(I_n) > b - a \qquad \text{——— (I)}$$

$\because a_k < b < b_k.$

$$\Rightarrow m^*[a,b] = \inf\left\{\sum_n l(I_n) : [a,b] \subseteq \bigcup_n I_n\right\}$$

$m^*[a,b] \geq b-a$ using (I)

—— (ii)

From (i) and (ii)

$$m^*[a,b] = b-a$$

which Required.

Lemma:2

L-outer measure of an arbitrary finite interval is its length.

Proof: Let I be an arbitrary finite interval.

* As

$$m^*(A) = \inf\left\{\sum_n l(I_n) ; A \subseteq \bigcup_n I_n\right\}$$

$\Rightarrow$ for $\varepsilon > 0$

$$\sum_n l(I_n) \leq m^*(A) + \varepsilon \quad *$$

For any $\varepsilon > 0$, we can find closed interval $J \subseteq I$ such that

$$l(I) \leq m^*(J) + \varepsilon$$

$$\begin{aligned} l(I) - \varepsilon &\leq m^*(J) \\ &\leq m^*(I) \\ &\leq m^*(\bar{I}) \\ &= l(\bar{I}) \\ &= l(I) \end{aligned}$$

(use monotonesty $I \subseteq \bar{I}$ → closed interval)

(since $l[a,b] = b-a$ & $l\,]a,b[= b-a$)

i.e

$$l(I) - \varepsilon \leq m^*(I) \leq l(I)$$

Since ε was arbitrary (very very small $\to 0$)

so

$$l(I) \leq m^*(I) \leq l(I)$$

$$m^*(I) = l(I)$$

which Required.

Lemma: 3

L-outer measure of any infinite interval, is its length.

Proof: Let I be any infinite interval.

For any real number λ, we can find finit closed interval $J \subset I$

Such that $l(J) = \lambda$

$$\Rightarrow \quad m^*(I) > m^*(J) \quad \text{(use monotonesty)}$$
$$= l(J) \quad \text{(using Lemma 1)}$$
$$= \lambda$$

i.e $\quad m^*(I) > \lambda$

Since λ was taken arbitrary, so

$$m^*(I) = \infty = l(I)$$

i.e

$$m^*(I) = l(I)$$

which Required.

Proposition:

L-outer measure of any interval, is its length.

Proof: There are three possible cases.

i) when interval is finite closed (Lemma 1)

ii) when interval is arbitrary finite (Lemma 2)

iii) when interval is infinite. (Lemma 3)

which Proves.

Remark:

Let m^* be $\mathcal{L}$– outer measure and $A, B \in 2^{\mathbb{R}}$ such that $A \subseteq B$ then $m^*(A) \leq m^*(B)$

Proof: Let $\{I_n\}$ be a countable collection of open intervals, which covers B, then

$$m^*(B) = \operatorname{Inf}\{\sum_n l(I_n) ;\ B \subseteq \bigcup_n I_n\}$$

$$\geq \operatorname{Inf}\{\sum_n l(I_n) ;\ A \subseteq \bigcup_n I_n\} \quad \because A \subseteq B$$

$$= m^*(A)$$

i.e $m^*(B) \geq m^*(A)$

or $m^*(A) \leq m^*(B)$

Hence m^* is monotone.

Proposition: Let $\{A_i\}$ be a countable family of subsets of $\mathbb{R}$

then $m^*(\bigcup_i A_i) \leq \sum_i m^*(A_i)$

or ($\mathcal{L}$-outer measure is σ– subadditive)

Proof: If $\mathcal{L}$– outer measure of any A_i, is infinite, then there is nothing to prove.

If every A_i, is finite, then let $\{I_{n,i}\}$ be the countable collection of open interval that cover A_i. i.e $A_i \subseteq \bigcup_n I_{n,i}$

Then

$$m^*(A_i) = \operatorname{Inf}\{\sum_n l(I_{n,i}) ;\ A_i \subseteq \bigcup_n I_{n,i}\}$$

Then for $\varepsilon > 0$ (corresponding to each A_i) we have

$$\sum_n l(I_{n,i}) < m^*(A_i) + \frac{\varepsilon}{2^i} \quad \text{—(I)}$$

Since $A_i \subseteq \bigcup_n I_{n,i} \Rightarrow \bigcup_i A_i \subseteq \bigcup_i \bigcup_n I_{n,i}$

Now

$$m^*(\cup_i A_i) = \operatorname{Inf}\left\{ \sum_i \sum_n l(I_{n,i}) \; ; \; \bigcup_i A_i \subseteq \bigcup_i \bigcup_n I_{n,i} \right\}$$

$$\Rightarrow \quad m^*(\cup_i A_i) \le \sum_i \sum_n l(I_{n,i})$$

$$< \sum_i \left(m^*(A_i) + \frac{\varepsilon}{2^i} \right)$$

$$= \sum_i m^*(A_i) + \varepsilon \sum_i \frac{1}{2^i}$$

$$= \sum_i m^*(A_i) + \varepsilon$$

i.e

$$m^*(\cup_i A_i) \le \sum_i m^*(A_i) + \varepsilon$$

Since ε was taken to be arbitrary

so $\quad m^*(\cup_i A_i) \le \sum_i m^*(A_i)$

As Required.

Remarks:

1) L-outer measure of a singleton is zero.

Proof: Let $A = [a, a] = \{a\}$, $a \in \mathbb{R}$

Then

$$m^*(A) = l(A)$$

Since L-outer measure of finite closed interval is its length.

So $\quad m^*(A) = l(A) = l([a, a])$

$$= a - a$$

$$m^*(A) = 0$$

2) L-outer measure of a countable set is zero.

Proof:

Let $A = \{a_1, a_2, a_3, \ldots\}$ $\quad a_i \in \mathbb{R}, \forall i$

Then

$$A = \cup_i \{a_i\} = \bigcup_i A_i$$

Each A_i is singleton set.

 $m^*(\cup_i A_i) \leq \sum_i m^*(A_i)$

$$m^*(A) = m^*(\cup_i A_i)$$
$$\leq \sum_i m^*(A_i)$$

$$= 0 + 0 + 0 + \cdots = 0$$

Since L– outer measure of singelton is zero.

$$m^*(A) \leq 0 \quad \text{—— (1)}$$

But by def. of m^*

$$m^*(A) \geq 0 \quad \text{—— (2)}$$

From (1) and (2)

$$m^*(A) = 0$$

Ex:

Prove that $[0,1]$ is not Countable.

Sol:

Let $A = [0,1]$ be Countable.

then

$$m^*(A) = 0$$

Since L– outer measure of a Countable set is zero.

But

$$m^*(A) = l(A)$$

Since L– outer measure of finite closed is its length.

$$m^*(A) = l(A) = l([0,1])$$
$$= 1 - 0$$
$$m^*(A) = 1$$

a Contradiction the supposition that $[0,1]$ is countable.

Hence

$[0,1]$ is not Countable.

Proposition: Let $A \subseteq \mathbb{R}$, then given $\varepsilon > 0$

i) there exist an open set O such that $A \subseteq O$ and
$$m^*(O) \le m^*(A) + \varepsilon$$

Proof: If $m^*(A) = \infty$.

Then result is obvious, as taking,
$$O = \mathbb{R}$$
$$m^*(O) = \infty$$
$$\Rightarrow \quad m^*(O) = m^*(A) + \varepsilon$$

If A is finite (i.e $m^*(A)$ is finite)

then, let $\{I_n\}$ be the countable collections of open intervals, covering A

i.e $A \subseteq \bigcup_n I_n$

$$m^*(A) = \operatorname{Inf}\{\textstyle\sum_n l(I_n)\ ;\ A \subseteq \bigcup_n I_n\}$$

Now for $\varepsilon > 0$

$$\sum_n l(I_n) \le m^*(A) + \varepsilon \quad \text{——} \ (1)$$

Putting

$$O = \bigcup_n I_n \quad \text{but} \quad A \subseteq O$$

Then

$$A \subseteq \bigcup_n I_n$$

Hence

$$\begin{aligned} m^*(O) &= m^*(\textstyle\bigcup_n I_n) \\ &\le \sum_n m^*(I_n) && \because m^*(\textstyle\bigcup_i A_i) \le \sum_i m^*(A_i) \\ &= \sum_n l(I_n) && m^*(\text{[illegible]}) = \text{[illegible]} \\ &\le m^*(A) + \varepsilon && \text{using } (1) \end{aligned}$$

i.e

$$m^*(O) \le m^*(A) + \varepsilon$$

$g \in \mathcal{G}_\delta$

ii) There is a set $G \in \mathcal{G}_\delta$ (Countable intersection of open set)

Such that $A \subseteq G$

and $m^*(A) = m^*(G)$

Proof:

For $\varepsilon_n > 0$, $n \in \mathbb{N}$

There exist an open set V_n

Such that $A \subseteq V_n$

and

$$m^*(V_n) \leq m^*(A) + \varepsilon_n \quad \text{— (2)} \qquad \text{— using (1)}$$

Putting $\varepsilon_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$

Put $G = \bigcap_n V_n$ then $G \in \mathcal{G}_\delta$.

and $A \subseteq G$

$$\Rightarrow m^*(A) \leq m^*(G) \quad \text{— (3)} \qquad \text{(by monotonesty)}$$

Now also $G \subseteq V_n \quad \forall n$

$$\Rightarrow m^*(G) \leq m^*(V_n) \qquad \text{(by monotonesty)}$$

$$\leq m^*(A) + \varepsilon_n$$

$$m^*(G) \leq m^*(A) + \frac{1}{n} \qquad \forall n$$

Taking limint $n \to \infty$

$$\lim_{n\to\infty} m^*(G) \leq \lim_{n\to\infty} m^*(A) + \lim_{n\to\infty} \frac{1}{n}$$

$$m^*(G) \leq m^*(A) \quad \text{— (4)}$$

From (3) and (4)

$$m^*(A) = m^*(G)$$

which Required

Exercise:

$\mathcal{L}$ – outer measure is translation invarient?

i.e If $A \subseteq R$

$$m^*(A+x) = m^*(A) \quad , x \in \mathbb{R}$$

Sol:

If A is countable

then $m^*(A) = 0 = m^*(A+x)$

since 'A' is countable then $(A+x)$ is also countable and $\mathcal{L}$ – outer measure of countable set is zero.

If A is un-countable, then let $\{I_n\}$ be the countable collection of open intervals which cover A.

$$m^*(A) = \operatorname{Inf}\{\sum_n l(I_n) : A \subseteq \bigcup_n I_n\}$$

Consider

$$m^*(A+x) = \operatorname{Inf}\{\sum_n l(I_n+x) : (A+x) \subseteq \bigcup_n (I_n+x)\}$$

as

$$l(I_n+x) = l(I_n)$$

So

$$\sum_n l(I_n+x) = \sum_n l(I_n)$$

$$m^*(A+x) = m^*(A)$$

As Required

Exercise:

$\mathcal{L}$ – outer measure is not one – one.

Sol:

Counter Example:

If A is Countable

then $m^*(A) = 0$

and $m^*(A+x) = 0$

i.e Different elements but same image

So $\mathcal{L}$ – outer measure is not one-one.

Exercise:

If $m^*(B) = 0$

Then $m^*(A \cup B) = m^*(A)$

where $A, B \subseteq \mathbb{R}$

Sol:

As

$$m^*(A \cup B) \leq m^*(A) + m^*(B) \quad \left(\text{use Sub-additivity of function}\right)$$
$$= m^*(A) + 0$$
$$= m^*(A)$$

i.e

$$m^*(A \cup B) \leq m^*(A) \quad \text{——— (1)}$$

Since $A \subseteq A \cup B$

$$m^*(A) \leq m^*(A \cup B) \quad \text{——— (2)} \quad (\text{by monotonesty})$$

From (1) and (2)

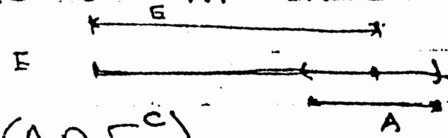
$$m^*(A \cup B) = m^*(A)$$

which Required.

$\mathcal{L}$– Measurable Sets:

(m*–mble Sets):

Let $E \subseteq \mathbb{R}$ and m^* be $\mathcal{L}$–outer-measure. Then E is said to be m^*–mble if for every $A \subseteq \mathbb{R}$



$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

Remarks:

1): Since

$$A = (A \cap E) \cup (A \cap E^c)$$
$$\Rightarrow m^*(A) = m^*[(A \cap E) \cup (A \cap E^c)]$$
$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$$

So to prove E m^*–mbl.

It is sufficient to prove

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

2): If 'E' is m^*-mble, then for any $A \subseteq R$

$$m^*(A \cap E^c) + m^*(A \cap (E^c)^c)$$

$$= m^*(A \cap E^c) + m^*(A \cap E) \qquad \because (E^c)^c = E$$

$$= m^*(A)$$

Hence E^c is also L-measurable.

3): '$\mathbb{R}$' is L-measurable set.

As for any $A \subseteq R$

$$m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^c)$$

$$= m^*(A) + m^*(A \cap \phi) \qquad \because \mathbb{R}^c = \phi$$

$$= m^*(A) + m^*(\phi)$$

$$= m^*(A) + 0$$

Since 'ϕ' is countable so its L-outer measure is zero. i.e $m^*(\phi) = 0$

Then

$$m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^c) = m^*(A)$$

Using Remark (2)

$\mathbb{R}^c = \phi$ is also m^*-able.

Lemma: Let E_1 and E_2 be two m^*-mble. Then $E_1 \cup E_2$ is also m^*-mble.

Proof: Since E_1 and E_2 is m^*-mble, then for $A \subseteq \mathbb{R}$

$$m^*(A \cap E_1^c) = m^*((A \cap E_1^c) \cap E_2) + m^*((A \cap E_1^c) \cap E_2^c)$$

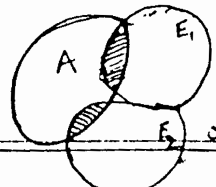
Also $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)$

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_1^c \cap E_2)$$

$\Rightarrow$ $m^*[A \cap (E_1 \cup E_2)] = m^*[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)]$.

$$m^*(A \cap (E_1 \cup E_2)) \leq m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2)$$

$\because m^*(A \cup B) \le m^*(A) + m^*(B)$



or in Real line. 25

Consider (to prove it to be a L. m-able set i.e $(E_1 \cup E_2)$)

$$m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

$$\le m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap (E_1^c \cap E_2^c))$$

$$= m^*(A \cap E_1) + m^*(A \cap E_1^c) \quad \text{By def of } m^*\text{-able}$$

$$= m^*(A)$$

i.e

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

Thus

$E_1 \cup E_2$ is m^*-mble.

As Required.

Lemma: If $m^*(E) = 0$ then E is m^*-mble set

Proof: Let $A \subseteq \mathbb{R}$ then

$$A \cap E \subseteq E$$

$$\Rightarrow m^*(A \cap E) \le m^*(E) \quad \text{(by monotonesty)}$$

$$= 0$$

i.e $m^*(A \cap E) \le 0$ —— (1)

But

$m^*(A \cap E) \geq 0$ —— (2)

From (1) and (2)

$$m^*(A \cap E) = 0$$

Moreover,

$$A \cap E^c \subseteq A$$

$$\Rightarrow m^*(A \cap E^c) \le m^*(A)$$

$$\Rightarrow m^*(A \cap E^c) + m^*(A \cap E) \le m^*(A)$$

Since $m^*(A \cap E) = 0$.

$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \rightarrow (4)$$

Hence

E is m^*-mble.

$\because m^*(A) \le m^*(A \cap E) + m^*(A \cap E^c) \rightarrow (3)$ (always holds)

from (3) & (4) $\Rightarrow m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$.

Lemma: If E_1 and E_2 are m^*-mble, then $E_1 \cap E_2$ and $E_1 \setminus E_2$ are also m^*-mble.

OR

Proof: Since E_1 and E_2 are m^*-mble, so are E_1^c and E_2^c, which shows that $E_1^c \cup E_2^c$ is also m^*-mble. Which further shows that

$$(E_1^c \cup E_2^c)^c \text{ is also } m^*\text{-mble.}$$

Consequently,

$$(E_1^c \cup E_2^c)^c = (E_1^c)^c \cap (E_2^c)^c$$
$$= E_1 \cap E_2$$

is m^*-mble.

Ex: If E is m^*-mble, so is $(E+x)$.

Sol: To prove this, we use the following results

i) $A \cap (E+x) = [(A-x) \cap E] + x$

ii) $A \cap (E+x)^c = [(A-x) \cap E^c] + x$

Since m^* is translation invarient and E is m-able So

$$m^*(A) = m^*(A-x)$$
$$= m^*[(A-x) \cap E] + m^*[(A-x) \cap E^c]$$

$\because E$ is m^*-mble

$$= m^*\left[[(A-x)\cap E]+x\right] + m^*\left[[(A-x)\cap E^c]+x\right]$$

$$m^*(A) = m^*(A\cap(E+x)) + m^*(A\cap(E+x)^c)$$

using (i) & (ii)

Hence $E+x$ is m^*-mble.

Lemma. If $A \subseteq \mathbb{R}$ and $E_1, E_2, E_3, \ldots, E_n$ be countable and disjoint are m^*-mble sets.

Then
$$m^*\left(A\cap\left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A\cap E_i)$$

Proof: Using principle of Mathematical Induction

B) For $n=1$

$$L.H.S = m^*(A\cap E_1)$$

$$R.H.S = \sum_{i=1}^{1} m^*(A\cap E_i)$$

$$= m^*(A\cap E_1)$$

$$L.H.S = R.H.S$$

which Provides, basis for induction.

Induction) Let $m^*\left(A\cap\left(\bigcup_{i=1}^{k} E_i\right)\right) = \sum_{i=1}^{k} m^*(A\cap E_i)$ —— (1)

where $1 \le k < n$.

Since E_i is m^*-mble for all i,

So

$$m^*\left(A\cap\left(\bigcup_{i=1}^{k+1} E_i\right)\right) = m^*\left[\left[A\cap\left(\bigcup_{i=1}^{k+1} E_i\right)\right]\cap E_{k+1}\right]$$

$$+ m^*\left[\left[A\cap\left(\bigcup_{i=1}^{k+1} E_i\right)\right]\cap E_{k+1}^c\right]$$

$$= m^*\left[A\cap E_{k+1}\right] + m^*\left[A\cap\left(\bigcup_{i=1}^{k} E_i\right)\right]$$

$$*\left(\bigcup_{i=1}^{k+1} E_i \cap E_{k+1}^c = \bigcup_{i=1}^{k+1} E_i \cap \bigcup_{i=1}^{k} E_i = \bigcup_{i=1}^{k} E_i\right)*$$

$$= m^*(A \cap E_{k+1}) + \sum_{i=1}^{k} m^*(A \cap E_i) \quad \text{using } ①$$

$$= \sum_{i=1}^{k+1} m^*(A \cap E_i)$$

i.e

$$m^*\left(A \cap \left(\bigcup_{i=1}^{k+1} E_i\right)\right) = \sum_{i=1}^{k+1} m^*(A \cap E_i)$$

Which, by M.I, shows that

Exercise:

$E_1 \setminus E_2$ is m^* – mble.

Sol:

Now consider

$$E_1 \setminus E_2 = E_1 \cap E_2^c$$

Since E_1 and E_2^c are m^* – mble.

So $E_1 \setminus E_2$ being intersection of two m^* – mble sets, is m^* – mble

Exercise:

Show that every singleton is m^* – mble.

Sol.

Since $\mathcal{L}$ – outermeasure of every singleton is Zero.

So every singleton is m^* – mble.

Lemma: Let $A \subseteq \mathbb{R}$, and $E_1, E_2, E_3, \ldots$ be the countable class of disjoint m^*-mble subsets of $\mathbb{R}$

Then

$$m^*(A \cap (\bigcup_i E_i)) = \sum_i m^*(A \cap E_i)$$

Proof:

$$A \cap (\bigcup_i E_i) \supseteq A \cap (\bigcup_{i=1}^{n} E_i) \quad \text{for finite } n$$

$$\Rightarrow m^*(A \cap (\bigcup_i E_i)) \geq m^*(A \cap (\bigcup_{i=1}^{n} E_i))$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i)$$

Since L.H.S above, is free from n.
so taking limit $n \to \infty$

$$m^*(A \cap (\bigcup_i E_i)) \geq \sum_i m^*(A \cap E_i) \quad \text{--- (1)}$$

Moreover,

$$A \cap (\bigcup_i E_i) = \bigcup_i (A \cap E_i)$$

$$\Rightarrow m^*(A \cap (\bigcup_i E_i)) = m^*(\bigcup_i (A \cap E_i))$$

$$\leq \sum_i m^*(A \cap E_i) \quad \text{--- (2)}$$

(using σ-sub additivity)

From (1) and (2)

$$m^*(A \cap (\bigcup_i E_i)) = \sum_i m^*(A \cap E_i)$$

Lemma: Let $E_1, E_2, E_3, \ldots,$ be mutually disjoint, m^*-mble sets. Then $\bigcup_i E_i$ is also m^*-mble.

Proof: Let $E = \bigcup_i E_i$ and $F_n = \bigcup_{i=1}^{n} E_i$ for some finite n

As, union of finite number of m^*-mble is also m^*-mble

So F_n is m^*-mble. Thus for any subset A of $\mathbb{R}$.

we have

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c) \quad \text{—— (1)}$$

Now

$$F_n \subseteq E$$

$$\Rightarrow E^c \subseteq F_n^c$$

$$\Rightarrow A \cap E^c \subseteq A \cap F_n^c$$

$$\Rightarrow m^*(A \cap E^c) \leq m^*(A \cap F_n^c) \quad \text{(by monotonesty)}$$

So equ (1)

$$\Rightarrow m^*(A) \geq m^*(A \cap F_n) + m^*(A \cap E^c)$$

$$= m^*(A \cap (\bigcup_{i=1}^{n} E_i)) + m^*(A \cap E^c)$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^c) \quad \text{from above}$$

Taking Limit $n \to \infty$

$$m^*(A) \geq \sum_i m^*(A \cap E_i) + m^*(A \cap E^c)$$

$$= m^*(A \cap (\bigcup_i E_i)) + m^*(A \cap E^c)$$

$$= m^*(A \cap E) + m^*(A \cap E^c)$$

Hence

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

Hence $E = \bigcup_i E_i$ is m^*-mble.

* Then $B_1, B_2, \ldots B_n, \ldots$ are mutually disjoint, m^*-mble sets. This implies that $\cup_i B_i$ is also m^*-mble. And $\cup_i B_i = \cup_i E_i$

Lemma: Let $E_1, E_2, E_3, \ldots$ be countable class of m^*-mble sets, then $\cup_i E_i$ is also m^*-mble.

Proof.

Set
$$B_1 = E_1$$
$$B_2 = E_2 \setminus E_1$$
$$B_3 = E_3 \setminus (E_1 \cup E_2)$$
$$\cdots\cdots\cdots$$
$$B_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i\right)$$
$$\cdots\cdots\cdots$$

* Then B_i for all i will be m^*-mble, which are mutually disjoint.

Thus using previous lemma,
$$\bigcup_i B_i = \bigcup_i E_i \quad , \text{ is } m^*\text{-mble.}$$

Remark: Intersection of countable class of m^*-mble, is m^*-mble.

Proof. Let $E_1, E_2, E_3, \ldots$ be countable class of m^*-mble sets. Then $E_1^c, E_2^c, E_3^c, \ldots$ will also be m^*-mble.

So
$$\cup_i E_i^c \text{ , will also be } m^*\text{-mble.}$$
which shows that
$$(\cup_i E_i^c)^c = \cap_i (E_i^c)^c \text{ will also be } m^*\text{-mble.}$$
$$\text{i.e } \bigcap_i E_i \text{ is } m^*\text{-mble.}$$

Lemma: The interval $]a,\infty[$ is m^*-mble.

Proof: Let $A \subseteq \mathbb{R}$

To prove $]a,\infty[$ m^*-mble, it is sufficient to show

$$m^*(A) \geq m^*(A\cap]a,\infty[) + m^*(A\cap]a,\infty[^c)$$

If $m^*(A) = \infty$
then there is nothing to prove.

Now let $m^*(A)$ be finite.

Then for $\varepsilon>0$ there is a countable collection $\{I_n\}$ of open intervals covering A, such that

$$\sum_n l(I_n) \leq m^*(A)+\varepsilon \quad \text{—— (1)}$$

Let $I_n' = I_n \cap]a,\infty[$

and $I_n'' = I_n \cap]-\infty,a]$

Then

$$I_n = I_n' \cup I_n''$$

where

I_n' and I_n'' are mutuall disjoint intervals. $(m^*(I_n) = m^*(I_n') + m^*(I_n''))$ m^* of any interval is its length.

Also $l(I_n) = l(I_n') + l(I_n'')$

so

$$m^*(I_n) = m^*(I_n') + m^*(I_n'') \quad \text{—— (2)}$$

Now $A\cap]a,\infty[\subseteq (\bigcup_n I_n)\cap]a,\infty[\quad \because A \subseteq \bigcup_n I_n$

$$A\cap]a,\infty[\subseteq \bigcup_n [I_n \cap]a,\infty[]$$
$$= \bigcup_n I_n'$$

similarly

$$A \cap]-\infty,a] \subseteq \bigcup_n (I_n \cap]-\infty,a])$$
$$= \bigcup_n I_n''$$

$m^*(A \cap]a,\infty[) \le m^*(\cup_n I_n')$ by σ-subadditivity of m^*

$\le \sum_n m^*(I_n')$ —— (3)

similarly

$m^*(A \cap]-\infty,a]) \le m^*(\cup_n I_n'')$

$\le \sum_n m^*(I_n'')$ —— (4)

using (3) and (4)

$$m^*(A \cap]a,\infty[) + m^*(A \cap]-\infty,a]) \le \sum_n m^*(I_n') + \sum_n m^*(I_n'')$$

$$= \sum_n \left(m^*(I_n') + m^*(I_n'')\right)$$

$$= \sum_n m^*(I_n)$$

$$= \sum_n l(I_n)$$

$$\le m^*(A) + \varepsilon$$

Since ε was taken to be arbitrary,

so

$$m^*(A \cap]a,\infty[) + m^*(A \cap]-\infty,a]) \le m^*(A)$$

i.e

$$m^*(A) \ge m^*(A \cap]a,\infty[) + m^*(A \cap]-\infty,a])$$

Hence $]a,\infty[$ is m^*-mble.

which Required.

35

Corollary. 1)

$]-\infty, a]$ is m^*-mble?

Proof:

Since $]a, \infty[$ is m^*-mble.

Then
$$(]a,\infty[)^c =]-\infty, a]$$
is also m^*-mble.

Corollary. 2) $]-\infty, b[$ is m^*-mble.

Proof.

As
$$]-\infty, b[=]-\infty, b] \setminus \{b\}$$

Since $]-\infty, b]$ and $\{b\}$ are both m^*-mble

So $]-\infty, b[$ being difference of two m^*-mble set, is m^*-mble.

As required.

Corollary: 3) $]a, b[$ is m^*-mble

Proof: As
$$]a,b[=]-\infty, b[\cap]a, \infty[$$

where both $]-\infty, b[$ and $]a, \infty[$ are m^*-mble.

So $]a, b[$ being intersection of two m^*-mble sets, is m^*-mble.

which Proves.

Exercise: Let A be the set of rational numbers b/w 0 and 1. Let $\{I_n\}$ be finite collection of open interval, covering A. Then

$$\sum_n l(I_n) \geq 1$$

Sol:

$A \subseteq]0,1[$

$A \subseteq \bigcup_n I_n$

Since $A \subseteq]0,1[\subseteq \bigcup_n I_n$

$$m^*]0,1[\leq m^*(\bigcup_n I_n)$$

$$\leq \sum_n m^*(I_n)$$

$$= \sum_n l(I_n)$$

$\Rightarrow$

$$\sum_n l(I_n) \geq m^*]0,1[= 1$$

Hence

$$\sum_n l(I_n) \geq 1$$

σ-Algebra:

Def: Collection of subsets of non-empty set X, is said to σ-algebra if

i) ϕ and X ∈ σ-algebra

ii) $E_1, E_2, E_3, \dots \in$ σ-algebra then $\bigcup_i E_i \in$ σ-algebra

iii) If $E \in$ σ-algebra then $E^c \in$ σ-algebra

THEOREM: The collection M^* of all m^*-mble subset of $\mathbb{R}$ is σ-algebra.

Proof.

i) Let $A \subseteq \mathbb{R}$ and A is arbitrary. then

$$m^*(A\cap\mathbb{R}) + m^*(A\cap\mathbb{R}^c)$$
$$= m^*(A\cap\mathbb{R}) + m^*(A\cap\phi)$$
$$= m^*(A) + m^*(\phi)$$
$$= m^*(A)$$

which shows that $\mathbb{R}$ is m^*-mble. and $\mathbb{R}\in M^*$

Moreover,

$$m^*(A\cap\phi) + m^*(A\cap\phi^c)$$
$$= m^*(\phi) + m^*(A\cap\mathbb{R})$$
$$= 0 + m^*(A)$$
$$= m^*(A)$$

So $\mathbb{R}$ and $\phi \in M^*$

ii) Let $E_1, E_2, E_3, \ldots \in M^*$

Since $E_1, E_2, E_3, \ldots$, are m^*-mble

So $\cup_i E_i$ will also be m^*-mble

and

hence $\cup_i E_i \in M^*$

iii) Let $E \in M^*$

then E will be m^*-mble.

so for $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A\cap E) + m^*(A\cap E^c)$$
$$= m^*(A\cap E^c) + m^*(A\cap E)$$

$\Rightarrow$ E^c is m^*-mble.

or $E^c \in M^*$

i.e $E^c \in M^* \quad \forall\ E\in M^*$

So M^* is σ-algebra.

Proposition:
Let C be a family of subsets of X. Then there exists the smallest σ-algebra containing C.

Proof: Let τ be family of σ-algebras containing C.

Let $B = \bigcap_{A \in \tau} A$

Of course B contains C.
And B is σ-algebra as

i) $X \in A$, $\forall A \in \tau$
(A's are σ-algebras)
$\Rightarrow X \in \bigcap_{A \in \tau} A = B$

$\Rightarrow X \in B$

Moreover,
$\phi \in A$, $\forall A \in \tau$
$\Rightarrow \phi \in \bigcap_{A \in \tau} A = B$

$\Rightarrow \phi \in B$

$\Rightarrow \phi$ and X (itself) belongs to the set B.

ii) Let $E_1, E_2, E_3, \dots \in B$
$\Rightarrow E_1, E_2, E_3, \dots \in A$, $\forall A \in \tau$
$\Rightarrow \bigcup_i E_i \in A \quad \forall A \in \tau$ $\because$ A is σ-algebra
$\Rightarrow \bigcup_i E_i \in \bigcap_{A \in \tau} A = B$

$\Rightarrow \bigcup_i E_i \in B$

iii) Let $E \in B$
$\Rightarrow E \in A \quad \forall A \in \tau$
$\Rightarrow E^c \in A \quad \forall A \in \tau$
$\Rightarrow E^c \in \bigcap_{A \in \tau} A = B$
$\Rightarrow E^c \in B$

Hence B is σ-algebra containing C.

Let $\mathcal{U}$ be σ-algebra containing C, which is contained in B.

Since

$$B = \bigcap_{A\in\tau} A \Rightarrow B \subseteq A, \; \forall A \in \tau$$

$$\Rightarrow B \subseteq \mathcal{U} \in \tau$$

But $\mathcal{U} \subseteq B$

Hence $B = \mathcal{U}$

i.e. B is the required σ-algebra.

Borel Field:

A σ-algebra defined on $X = \mathbb{R}$ generated by family of open sets, is called "Borel Field". Its member are called "Borel sets.

Let the Borel field be denoted by $\mathcal{B}$. Then we have the following preposition:-

1) Every Borel set is m^*-mble.

Proof: Since $(a,\infty) \in M^*$ & $(-\infty, b) \in M^*$ $\forall a, b \in \mathbb{R}$ where M^* is collection of all m^*-mble subset of $\mathbb{R}$

$(a,b) = (-\infty, b) \cap (a, \infty) \in M^*$

as M^* is closed under countable intersection.

Hence

Every open interval is m^*-mble.

we shows that every open set is m^*-mble:-

As countable union of m^*-mble is, m^*-mble.

(as every open set can be considered

* — So every open set is m^*-mble.

to be countable union of open intervals).
*Moreover, every closed set, being Complement of open set, is also m^*-mble.

Lebesgue Measure.

Def:
The restriction $m^*/_{M^*} = m$ is called Lebesgue Measure.
Where M^* is collection of all m^*-mble subsets of $\mathbb{R}$.

Lemma:- If $E_1, E_2 \in M^*$
Then
$$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$$

Proof: Since $E_1, E_2 \in M^*$.
So
$$E_1 \cup E_2 \in M^* \text{ and } E_1 \cap E_2 \in M^*$$
Then for any $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \quad \text{——} (1)$$
and
$$m^*(A) = m^*(A \cap E_2) + m^*(A \cap E_2^c) \quad \text{——} (2)$$

Replacing 'A' by $E_1 \cup E_2$ in (1), we have

$$m(E_1 \cup E_2) = m[(E_1 \cup E_2) \cap E_1] + m[(E_1 \cup E_2) \cap E_1^c]$$

$$m(E_1 \cup E_2) = m(E_1) + m(E_2 \cap E_1^c) \quad \text{——} (3)$$

Adding $m(E_1 \cap E_2)$ in eq (3) on both side,

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2 \cap E_1^c) + m(E_1 \cap E_2)$$
$$= m(E_1) + m(E_2) \quad , E_1 \in M^*$$

Hence
$$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$$

Corollary: If E_1 and E_2 are mutually disjoint m^*-mble sets.

Then
$$m(E_1 \cup E_2) = m(E_1) + m(E_2)$$

Proof: Since E_1 and E_2 are m^*-mble sets. So $E_1 \cup E_2$ and $E_1 \cap E_2$, being countable union and intersection resp., are also m^*-mble.

Then for $A \subseteq \mathbb{R}$

$$m(A) = m(A \cap E_1) + m(A \cap E_1^c) \quad \text{—} ①$$
$$m(A) = m(A \cap E_2) + m(A \cap E_2^c) \quad \text{—} ②$$

Replace A by $E_1 \cup E_2$ in ①

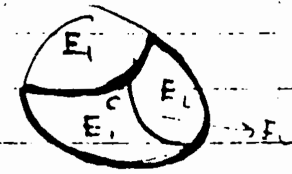
$$m(E_1 \cup E_2) = m[(E_1 \cup E_2) \cap E_1] + m[(E_1 \cup E_2) \cap E_1^c]$$
$$= m(E_1 \cup \phi) + m(\phi \cup E_2 \cap E_1^c)$$
$$= m(E_1) + m(\phi \cup E_2) \quad \because E_1 \,\&\, E_2 \text{ are mutually disjoint}$$
$$= m(E_1) + m(E_2) \quad \therefore E_1 \cap E_2 = \phi, \; E_2 \cap E_1^c = E_2$$

i.e
$$m(E_1 \cup E_2) = m(E_1) + m(E_2)$$

which Required

$m[(E_1 \cup E_2) \cap E_1^c]$ Let $E_1 \cup E_2 = E$

$= m[E \cap E_1^c] = m(E_2)$



Proposition:

If $E_1, E_2, E_3, \ldots, E_n \in M^*$ and are mutually disjoint. Then

$$m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i)$$

Proof:

Proposition:

For any sequence $\{E_i\}$ of m^*-mble

Then

i) $m(\bigcup_i E_i) \leq \sum_i m(E_i)$

ii) $m(\bigcup_i E_i) = \sum_i m(E_i)$ for mutually disjoint E_i's

Proof: i)

Relation (i) is just the replacement of σ-subadditivity of m^* as

$$m(\bigcup_i E_i) = m^*(\bigcup_i E_i) \leq \sum_i m^*(E_i)$$

$$m(\bigcup_i E_i) \leq \sum_i m^*(E_i)$$

$$= \sum_i m^*(E_i)$$

i.e

$$m(\bigcup_i E_i) \leq \sum_i m(E_i)$$

Proof: (ii) If E_i's are mutually disjoint.

$$\bigcup_i E_i \supseteq \bigcup_{i=1}^{n} E_i$$

$$\Rightarrow m(\bigcup_i E_i) \geq m(\bigcup_{i=1}^{n} E_i) \quad \text{(by monotonocity)}$$

$$= \sum_{i=1}^{n} m(E_i) \quad \text{(from Last Proof P-42)}$$

Taking Limit $n \to \infty$ on both sides

$$m(\bigcup_i E_i) \geq \lim_{n\to\infty} \left(\sum_{i=1}^{n} m(E_i)\right)$$

$$m(\bigcup_i E_i) \geq \sum_i m E_i \quad \text{———} \; ①$$

Moreover

$$m(\cup_i E_i) \leq \sum_i m(E_i) \quad \text{(as proved earlier)} \quad \text{---} (2)$$

From (1) and (2)

$$m(\cup_i E_i) = \sum_i m(E_i)$$

Proposition: If $\{E_i\}$ is seq of m^*-mble and $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots\ldots$

Then

$$m(\bigcup_i E_i) = \lim_{n\to\infty} m(E_n)$$

Proof: Set

$$B_1 = E_1$$

$$B_2 = E_2 \setminus E_1$$

$$B_3 = E_3 \setminus E_2$$

$$\ldots\ldots$$

$$B_n = E_n \setminus E_{n-1}$$

$$\ldots\ldots$$

We have the following:-

i) All B_i's are m^*-mble

ii) $B_i \cap B_j = \phi \quad i \neq j, \forall i, j$

iii) $\bigcup_{i=1}^{n} B_i = E_n$

iv) $\bigcup_i B_i = \bigcup_i E_i$

From (iii)

$$m(\bigcup_{i=1}^{n} B_i) = m(E_n)$$

Taking Limit $n \to \infty$

$$\lim_{n\to\infty} m(\bigcup_{i=1}^{n} B_i) = \lim_{n\to\infty} m(E_n)$$

$$m(\bigcup_i B_i) = \lim_{n\to\infty} m(E_n)$$

$$m(\bigcup_i E_i) = \lim_{n\to\infty} m(E_n)$$

from (iv) Proved

$$\lim_{n\to\infty} \sum_{i=1}^{n} m(B_i) = \lim_{n\to\infty} m(E_n)$$

i.e

$$\sum_i m(B_i) = \lim_{n\to\infty} m(E_n) \qquad \text{——— (1)}$$

Now from (iv)

$$m(\cup_i B_i) = m(\cup_i E_i)$$

$$\sum_i m(B_i) = m(\cup_i E_i)$$

Put in (1) we get

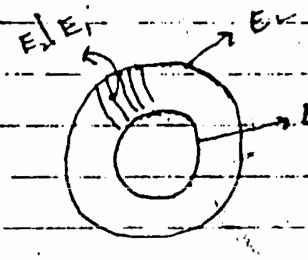
$$m(\cup_i E_i) = \lim_{n\to\infty} m(E_n)$$

Lemma: If E_1 and E_2 are m^*-mble and $E_1 \subseteq E_2$

Then

$$m(E_2 \setminus E_1) = m(E_2) - m(E_1)$$

Proof: Here we can write



$$E_2 = E_1 \cup (E_2 \setminus E_1)$$

and

$$E_1 \cap (E_2 \setminus E_1) = \phi$$

Thus

$$m(E_2) = m(E_1) + m(E_2 \setminus E_1)$$ since E_1 and $E_2\setminus E_1$ are disjoint

$$\Rightarrow \quad m(E_2 \setminus E_1) = m(E_2) - m(E_1)$$

As Required.

$E_2 = \{1,2,3,4,5,6,7,8,9,10\}$, $E_1 = \{2,4,6,8,10\}$

$E_2 \setminus E_1 = \{1,3,5,7,9\}$

$E_1 \cup (E_2 \setminus E_1) = \{1,2,3,4,5,6,7,8,9,10\} = E_2$

$\therefore \ E_2 = E_1 \cup (E_2 \setminus E_1)$

Proposition: If $\{E_i\}$ is a seq. of m^*-mble sets and $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$
and

$m(E_1) < \infty$ (or E_1 is of finite measure)

Then $m(\bigcap_i E_i) = \lim_{n\to\infty} m(E_n)$

Proof:

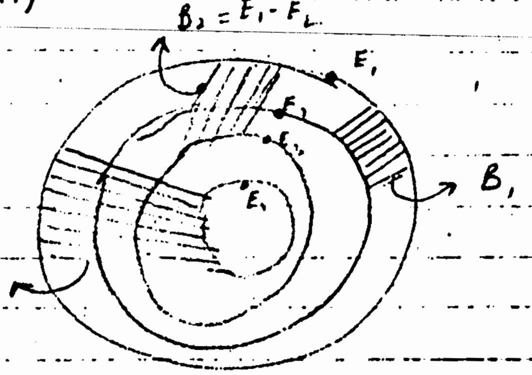
Let $B_1 = E_1 \setminus E_2$

$B_2 = E_1 \setminus E_3$

$B_3 = E_1 \setminus E_4$

$\ldots\ldots$

$B_n = E_1 \setminus E_{n+1}$



we have the following:

i) Each B_i is m^*-mble.

ii) $B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots$

iii) $m(\bigcup_i B_i) = \lim_{n\to\infty} m(B_n)$ (using Previous Proposition)

iv) $\bigcup_i B_i = \bigcup_i (E_1 \setminus E_{i+1})$

$= \bigcup_i (E_1 \cap E_{i+1}^c)$

$\cancel{= E_1 \cap \bigcup_i E_{i+1}^c}$

$\cancel{= E_1 \cap (\bigcap_i E_{i+1}^c)}$

$\bigcup_i B_i = E_1 \setminus \bigcap_i E_{i+1}$

$= \bigcup_i (E_1 \cap E_{i+1}^c)$

$= E_1 \cap \bigcup_i (E_{i+1})^c$

$= E_1 \cap (\bigcap_i E_{i+1})^c$

$= E_1 \setminus \bigcap_i E_{i+1}$

v) $B_n = E_1 \setminus E_{n+1}$

From (iv), we have

$m(\bigcup_i B_i) = m(E_1 \setminus \bigcap_i E_{i+1})$

$m(\bigcup_i B_i) = m(E_1) - m(\bigcap_i E_{i+1})$ —— ⓐ

From (v)

$$m(B_n) = m(E_1 \setminus E_{n+1})$$

$$m(B_n) = m(E_1) - m(E_{n+1}) \quad \because E_1 \supseteq E_2, E_3, \ldots$$

———— (b)

Taking $\lim\limits_{n\to\infty}$

$$\lim_{n\to\infty} m(B_n) = m(E_1) - \lim_{n\to\infty} m(E_{n+1})$$

OR $\quad m(\cup_i B_i) = m(E_1) - \lim\limits_{n\to\infty} m(E_{n+1}) \quad$ (using (iii))

OR

$$\cancel{m(E_1)} - m(\cap_i E_{i+1}) = \cancel{m(E_1)} - \lim_{n\to\infty} m(E_{n+1})$$

— using (a)

$$\Rightarrow \quad m(\cap_i E_{i+1}) = \lim_{n\to\infty} m(E_{n+1})$$

$$\Rightarrow \quad m(\cap_i E_i) = \lim_{n\to\infty} m(E_n)$$

Remark:

The equality proved above does not hold if $m(E_1) = \infty$

Counter Example.

Let $E_n = (n, \infty) \quad n = 1, 2, 3, \ldots$ be a sequence of open intervals.

Then

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots \supseteq E_n \supseteq \ldots$$

$$m(E_1) = m(1, \infty) = \infty$$

(since m^*-mble of an open interval is its length).

$$m(E_n) = m(n, \infty) = \infty$$

$\Rightarrow$

$$\lim_{n\to\infty} m(E_n) = \infty \quad \text{———— (1)}$$

But $\bigcap_i E_i = \phi$

and

$$m(\bigcap_i E_i) = m(\phi)$$
$$= 0 \quad \text{---} (2)$$

$E_1 \cap E_2 \cap \cdots \cap E_n = E_n$

$E_1 \cap E_2 \cdots \cap E_n = (n, \infty)$

$\lim_{n\to\infty} (E_1 \cap E_2 \cap \cdots \cap E_n) = \lim_{n\to\infty} (n, \infty)$

$\bigcap_i E_i = (\infty, \infty)$

$= \phi$

From (1) and (2)

$$m(\bigcap_i E_i) \neq \lim_{n\to\infty} m(E_n)$$

Non-measurable Set:

Consider the unit interval $[0,1)$.

Defince additive modulus 1 in $[0,1)$ as

$$x \oplus y = \begin{cases} x+y & \text{if } x+y<1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

Remarks:

1) $[0, 1)$ is closed under $\oplus$.
2) $\oplus$ is both commutative and associative.

Defination:

Let x and $y \in [0,1)$.

Then x is said to be equivalent to y denoted by $x \sim y$, if $(x-y)$ is a rational number.

$\because E_1$ & E_2 are m^*-mble. then E_1+y & E_2+y-1 is also m^*-mble

$\Rightarrow E_1+y_1 = E_1 \oplus y$

& $E_2+y-1 = E_2+y$ is m^*-mble.

Lemma:

If E is any m^*-mble subset of $[0,1)$ then for $y \in [1,0)$, $E \oplus y$ will also be m^*-mble.

and

$$m(E \oplus y) = m(E)$$

or prove that it is translation invarient.

Proof. Let $E_1 = E \cap [0, 1-y)$

and

$$E_2 = E \cap [1-y, 1)$$

Then E_1 and E_2 being intersection of m^*-mble set, is m^*-mble.

It can be observed that

$$E = E_1 \cup E_2$$

$$E_1 \oplus y = E_1 + y$$

$$E_2 \oplus y = E_2 + y - 1$$

Next $E = E_1 \cup E_2$

$$\Rightarrow E \oplus y = (E_1 \oplus y) \cup (E_2 \oplus y)$$

Since

E_1 and E_2 are disjoint sets.

So $E_1 \oplus y$ and $E_2 \oplus y$ are also disjoint.

Now

$$m(E \oplus y) = m(E_1 \oplus y) + m(E_2 \oplus y)$$

$$= m(E_1 + y) + m(E_2 + y - 1)$$

$$= m(E_1) + m(E_2)$$

$$= m(E_1 \cup E_2) \quad \because E_1 \text{ \& } E_2 \text{ are disjoint.}$$

i.e.

$$= m(E)$$

$$m(E \oplus y) = m(E).$$

Which Proves.

Remarks:-

1) '$\sim$' is an equivalence relation. So '$\sim$' partition $[0,1)$ into mutually disjoint equivalence classes.

2) If x and y belong to two disjoint equivalence classes then they differ by an irrational number.

3) By axiom of Choice, there exist a subset of $[0,1)$ consisting of exactly one element from each equivalence class.
In fact this set, say P, is non-measurable.

Proof: Let $\{r_0, r_1, r_2, \ldots\}$ be the set of rational number belonging to $[0,1)$ with $r_0 = 0$

Construct $P_i = P \oplus r_i \quad , i = 1, 2, 3, \ldots$
then

$$P_0 = P$$

and all P_i's will be mutually disjoint.
i.e

$$P_i \cap P_j = \phi \quad , i \neq j$$

as: Let

$$x \in P_i \cap P_j$$

$\Rightarrow$

$$x \in P_i \text{ and } x \in P_j$$

there exist r_i and r_j such that

$$x = p_i + r_i$$
$$x = p_j + r_j \qquad p_i, p_j \in P$$

$$\Rightarrow \quad p_i + r_i = p_j + r_j$$

$$p_i - p_j = r_j - r_i \quad (= \text{rational no.})$$

$\Rightarrow$ p_i and p_j differ by a rational no.
A contradiction because the members of P differs by an irrational.

Moreover,

$$P_i \subseteq [0,1) \quad \forall i$$

$$\Rightarrow \quad \cup_i P_i \subseteq [0,1) \quad \text{——(1)}$$

Let $x \in [0,1)$

x belongs to some equivalence class
i.e x is equivalent to some element, say p of P, which shows that x and p differ by a rational no., say r_i,
i.e

$$x - p = r_i$$

or $x = p + r_i$

$$\Rightarrow \quad x \in P_i$$

$$\Rightarrow \quad [0,1) \subseteq P_i$$

$$\Rightarrow \quad [0,1) \subseteq \cup_i P_i \quad \text{——(2)}$$

From (1) and (2)

$$\cup_i P_i = [0,1)$$

Let P be m'ble then using previous lemma $P \oplus r_i$ for $i = 1, 2, 3, \ldots$ will also be m'ble

Consider

$$m[0,1) = m(\cup_i P_i) = \sum_i m(P_i)$$

$$1 = \sum_i m(P_i)$$

i.e

$$\sum_i m(P_i) = 1$$

But $\quad m(P) \geq 0$

which shows that $\sum_i m(P_i) \neq 1$

a contradiction.

Hence

P is non-measurable

Ex.

Let E be a mble subset of P. Then show that $m(E)=0$

Sol:

Construct

$$E_i = E \oplus r_i \quad , i = 1,2,3,\ldots$$

as P_i's were constructed earlier.

Obviously,

all E_is are mutually disjoint mble subsets of $[0,1)$

and

$$m(E_i) = m(E) \quad \forall i \quad \text{——} \ (1)$$

Now

$$\bigcup_i E_i \subseteq \bigcup_i P_i \qquad \because E \subseteq P \subseteq [0,1)$$

$$E \oplus r_i \subseteq P \oplus r_i$$

$$\bigcup_i E_i \subseteq \bigcup_i P_i$$

or

$$\bigcup_i E_i \subseteq [0,1)$$

$$\Rightarrow m\left(\bigcup_i E_i\right) \leq m[0,1)$$

or

$$\sum_i m(E_i) \leq 1.$$

or

$$\sum_i m(E) \leq 1 \qquad \text{using } (1)$$

But

$$m(E) \geq 0$$

$$\Rightarrow m(E) = 0$$

As Required.

Measurable Function:

Proposition:

Let f be an extended real valued function with mble domain. Then the following statement are equivalent.

i) For every real no. α $\{x : f(x) > \alpha\}$ is mble

ii) " " " " " $\{x : f(x) \geq \alpha\}$ " "

iii) " " " " " $\{x : f(x) < \alpha\}$ " "

iv) " " " " " $\{x : f(x) \leq \alpha\}$ is mble

NOTE: we show that (i) $\Leftrightarrow$ (iv), (ii) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (ii)

Proof: i) $\Leftrightarrow$ iv)

Let D be the domain of f

Let (i) holds.

then

$$\{x : f(x) \leq \alpha\} = D \setminus \{x : f(x) > \alpha\}$$

Since

D and $\{x : f(x) > \alpha\}$ are mble

So

$\{x : f(x) \leq \alpha\}$ being difference of mble sets, is mble.

i.e (i) $\Rightarrow$ (iv)

Similarly, we can write

(iv) $\Rightarrow$ (i)

i.e Let (iv) holds. $\{x : f(x) \leq \alpha\}$ is m-able

then

$$\{x : f(x) > \alpha\} = D \setminus \{x : f(x) \leq \alpha\}$$

Since

D and $\{x : f(x) \leq \alpha\}$ are mble.

So

$\{x : f(x) > \alpha\}$ being difference of mble sets is mble.

So (i) $\Leftrightarrow$ (iv)

Proof: (ii) $\Leftrightarrow$ (iii)

ii) $\Rightarrow$ iii)

Let D be the domain of f.

And ii) holds.

Then

$$\{x : f(x) < \alpha\} = D \setminus \{x : f(x) \geq \alpha\}$$

Since

D and $\{x : f(x) \geq \alpha\}$ are mble.

So is

$$\{x : f(x) < \alpha\}$$

i.e ii) $\Rightarrow$ iii)

Now iii) $\Rightarrow$ ii)

Let iii) holds.

Then

$$\{x : f(x) \geq \alpha\} = D \setminus \{x : f(x) < \alpha\}$$

Since

D and $\{x : f(x) < \alpha\}$ are mble

So is

$$\{x : f(x) \geq \alpha\}$$

i.e iii) $\Rightarrow$ ii)

So (ii) $\Leftrightarrow$ (iii)

Now To Show (i) $\Leftrightarrow$ (ii)

Let (i) holds.

Consider the set

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \tfrac{1}{n}\}$$

Since $\{x : f(x) > \alpha\}$ $\forall$ real no. α is mble

So

$\{x : f(x) > \alpha - \frac{1}{n}\}$ will also be mble for all $n = 1, 2, 3, \ldots$

Hence $\{x : f(x) \geq \alpha\}$ being countable intersection of mble sets, is mble.

i.e i) $\Rightarrow$ ii)

ii) $\Rightarrow$ i)

Let ii) holds.

Then

$$\{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f(x) \geq \alpha + \frac{1}{n}\}$$

Since $\{x : f(x) \geq \alpha\}$ is mble for every real no.

So

$\{x : f(x) \geq \alpha + \frac{1}{n}\}$ is mble $\forall$ $n = 1, 2, 3, \ldots$

Hence

$\{x : f(x) > \alpha\}$ being countable union of mble sets, is mble

i.e. ii) $\Rightarrow$ i)

i) $\Leftrightarrow$ ii)

Which Required.

Defination:

(Measurable Function)

An extended real valued function f defined on mble domain, is said to be mble, if and only if it satisfies one of the condition described in the previous proposition.

THEOREM:

If c is a constt. and f & g are real valued measurable functions, with mble domain.

Then

i) $f + c$ ii) cf iii) $f + g$

iv) $f - g$ v) f^2 vi) $|f|$

vii) fg

are mble functions

Proof i): For any real no. α the set

$$\{x : (f+c)x > \alpha\} = \{x : f(x)+c > \alpha\}$$
$$= \{x : f(x) > \alpha - c\}$$

Since f is mble

So $\{x : f(x) > \alpha\}$ is mble for any real no. α.

Hence so is $\{x : f(x) > \alpha - c\}$ ($\because (\alpha - c)$ is also real no.)

Hence

$\{x : (f+c)x > \alpha\}$ is mble for any real no. α.

Thus $f+c$ is mble

ii) $(cf)x = c(f(x))$

Consider the set for any real no. α

$$\{x : cf(x) > \alpha\} = \begin{cases} \{x : f(x) > \frac{\alpha}{c}\}, & c>0 \\ \{x : f(x) < \frac{\alpha}{c}\}, & c<0 \end{cases}$$

Since f is given to be mble.

So

$\{x : f(x) < \frac{\alpha}{c}\}$ & $\{x : f(x) > \frac{\alpha}{c}\}$ are mble

i.e

$\{x : c(f(x)) > \alpha\}$ is mble.

Hence the function cf is mble.

iii) $f+g$ is mble function.

Proof: Let α be any real no.

Consider

$$A = \{x : (f+g)(x) > \alpha\}$$
$$= \{x : f(x) + g(x) > \alpha\}$$

$A = \{x : f(x) > \alpha - g(x)\}$

for $x \in A$

$f(x) > \alpha - g(x)$

By an axiom of Archmedian $\exists$ a real number r such that

$$f(x) > r > \alpha - g(x)$$

i.e

for $x \in A$, we have $f(x) > r$

and $g(x) > \alpha - r$ for some r

or $x \in A \Rightarrow x \in \{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}$

$\Rightarrow A \subseteq \bigcup_r [\{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}]$

We now show that

$$\bigcup_r \{\{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}\} \subseteq A$$

Let

$x \in \bigcup_r \{\{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}\}$

$\Rightarrow x \in \{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}$ for some r

$\Rightarrow x \in \{x : f(x) > r\}$

and

$x \in \{x : g(x) > \alpha - r\}$ for some r

$\Rightarrow f(x) > r$ and $g(x) > \alpha - r$ for some r

or

$f(x) > r > \alpha - g(x)$ for some r

or

$f(x) > \alpha - g(x)$

or $\quad f(x)+g(x) > \alpha$

$\Rightarrow \quad x \in A$

Hence

$$\bigcup_r \left[\{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}\right] \subseteq A$$

or $\quad A = \bigcup_r \left[\{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}\right]$

Since f and g are given to be mble, so the set A being countable union of mble sets, is mble.

which further shows that $(f+g)$ is also mble.

iv) $f-g$ is mble function.

Proof:
Since f & g are mble so is $(-1)g$ and hence $\quad$ since cg is mble.

$f+(-1)g$ is also mble.

Thus $f+(-1)g = f-g$ is mble.

v)

Proof:
Consider real no. $\alpha \geq 0$
and
$A = \{x : f^2(x) > \alpha\} = \{x : [f(x)]^2 > \alpha\}$

$$A = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}$$

Since f is given to be mble.
So set A being union of mble set's, is mble.

For $\alpha < 0$

$$A = \{x : f^2(x) > \alpha\} = D$$

Since D is given to be mble,
so A is mble.
Hence f^2 is mble.
Thus f^2 is mble.

vi) $|f|$ is mble

Proof: Consider $\alpha \geq 0$

Let $A = \{x : |f|(x) > \alpha\}$

$\Rightarrow A = \{x : |f(x)| > \alpha\}$

$$= \{x : f(x) > \alpha\} \cup \{x : f(x) < -\alpha\}$$

which shows that set A, being union of mble sets, is mble.

Now for $\alpha < 0$

$$A = \{x : |f(x)| > \alpha\} = D$$

Since D is given to be mble
So A is mble.
Hence
$\{x : |f(x)| > \alpha\}$ is mble for any $\alpha \in \mathbb{R}$.
Thus
$|f|$ is mble.

vii) fg is mble

Proof:
As $fg = \frac{1}{4}\{(f+g)^2 - (f-g)^2\}$

Since f and g are given to be mble.
So $f+g$, $f-g$ are mble.
Also their square i.e. $(f+g)^2$, $(f-g)^2$ are

mble.

Difference of two mble sets.

i.e

$$(f+g)^2 - (f-g)^2 \text{ is mble.}$$

Hence

$$\frac{1}{4}\{(f+g)^2 - (f-g)^2\} \text{ is also mble.}$$

Thus fg is mble function.

Ex: If f is a real-valued mble function then show that for an extended real number α

$$\{x : f(x) = \alpha\} \text{ is also mble.}$$

Sol:

Since f is mble, so for finite α the sets

$$\{x : f(x) \geq \alpha\} \ \& \ \{x : f(x) \leq \alpha\}$$

are also mble.

Consequently

$$\{x : f(x) \geq \alpha\} \cap \{x : f(x) \leq \alpha\}$$

$$= \{x : f(x) = \alpha\}$$

is also mble for finite α.

Now

If $\alpha = \infty$

then $$\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \geq n\}$$

Since $\{x : f(x) \geq n\}$ is mble $\forall n$, as f is mble.

So $\{x : f(x) = \infty\}$ being countable intersection of mble sets, is mble

Now

If $\alpha = -\infty$

Then

$$\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \leq -n\}$$

Since f is mble

So

$\{x : f(x) \leq -n\}$ will also be mble.

Consequently,

the set $\{x : f(x) = -\infty\}$ being countable intersection of mble sets, will also be mble.

Theorem: For any open interval $]a,b[$,

$f^{-1}(]a,b[)$ is mble, where f is mble

Proof:

Here

$$f^{-1}(]a,b[) = f^{-1}(]a,\infty[\cap]-\infty,b[)$$

$$= f^{-1}(]a,\infty[) \cap f^{-1}(]-\infty,b[)$$

$$= \{x : f(x) > a\} \cap \{x : f(x) < b\}$$

Since f is mble

So

$\{x : f(x) > a\}$ & $\{x : f(x) < b\}$ are also mble.

Consequently,

$f^{-1}(]a,b[)$ is also mble

(being intersection of two mble set)

Corallary : 1

A continous function defined on mble domain, is mble

Proof: For continous function, f is open if and only if f^{-1} is open.

In the above theorem f mble $\Rightarrow f^{-1}$ mble
Thus continous function is mble.

Corallary : 2 Constant
~~Countable~~ function is mble.

Proof:
Since constant function is continous, so constant func is mble (as continous function is mble).

Theorem: Every function defined on a set of measure zero, is mble.

Proof: Let f be defined on E
such that
$$m(E) = 0$$
To show f is mble.
Consider
$$\{x : x \in E, f(x) > \alpha\} \subseteq E$$
$$\Rightarrow m\{x : x \in E, f(x) > \alpha\} \leq m(E) = 0$$
i.e
$$m\{x : x \in E, f(x) > \alpha\} \leq 0$$
But
$$m\{x : x \in E, f(x) > \alpha\} \geq 0 \quad \text{always}$$
So
$$m\{x : x \in E, f(x) > \alpha\} = 0$$

But every set of measure zero, is always mbl, so

$$\{x : x \in E,\ f(x) > \alpha\} \text{ is also mble}$$

Thus

$$f \text{ is mble.}$$

Exercise: Let f be an extended real valued function and B be any open set. Then f is mble iff $f^{-1}(B)$ is mble.

Sol.

Since B is open, so B can be considered countable union of open intervals.

i.e

$$f^{-1}(B) = f^{-1}(\cup_n I_n) \quad \text{where } I_n \text{ are open intervals}$$

$$= \cup_n f^{-1}(I_n)$$

Since f is mble, so $f^{-1}(I_n)$ is mble for each n $\because f^{-1}$[any open interval] is measurable.

Thus $f^{-1}(B) = \cup_n (f^{-1}(I_n))$, being countabl union of mble sets, is mble.

Conversé: Let inverse image of an open set is mble and $\alpha \in \mathbb{R}$

then the set $]\alpha, \infty[$ is an open set and

$$f^{-1}(]\alpha, \infty[) \text{ is mble.}$$

i.e

$$\{x : f(x) > \alpha\} \text{ is mble.}$$

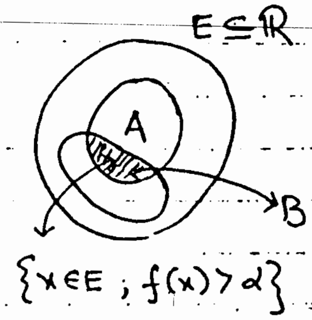
Hence f is mble.

Which Proves.

Theorem:

Let f be mble function defined on mble set E.
If A is mble subset of E, then the restriction of f to A, is also mble.

Proof:



Let α be any real number and

$$B = \{x : x \in A, f(x) > \alpha\}$$

or $B = \{x : f_{|A}(x) > \alpha\}$

Now

$$B = A \cap \{x : x \in E, f(x) > \alpha\}$$

Since f is mble, so $\{x \in E : f(x) > \alpha\}$ will also be mble.

Moreover,

A is given to be mble, so B, being intersection of mble sets, will also be mble.

Hence $f_{|A}$ is mble.

Theorem:

Let f be defined on mble set E, which is countable union of mble sets E_k (i.e $E = \bigcup_k E_k$)

If the restriction of f to each E_k is mble, then f is mble on E.

Proof: Consider the set

$\{x \in E : f(x) > \alpha\}$ for any real no. α.

which can be written as

$$\{x \in E : f(x) > \alpha\} = \bigcup_k \{x \in E_k : f(x) > \alpha\}$$

or $\{x \in E : f(x) > \alpha\} = \bigcup_k \{x \in E : f_{|E_k}(x) > \alpha\}$

Since restriction of f on each E_k is mble, so $\{x \in E_k : f(x) > \alpha\}$ will be mble for each k (using last th.)

Hence $\{x \in E : f(x) > \alpha\}$ being countable union of mble sets, is mble

Thus f is mble on E.

Theorem: If f is mble function defined on E and if g is another function defined on E such that

$$f = g \quad (a.e) \quad \text{almost everywhere}$$

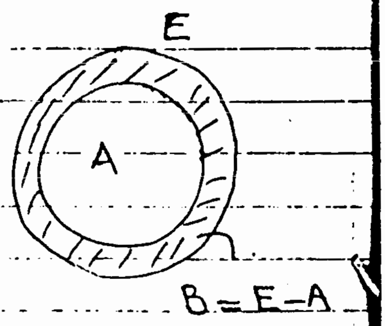
i.e

$$m\{x \in E \cdot f(x) \neq g(x)\} = 0$$

Then g is also mble on E.

Proof: Let

$$A = \{x \in E \ f(x) \neq g(x)\} \subseteq E$$



By def of (a.e)

$$m(A) = 0$$

Let

$$B = E - A$$

Since E is given to be mble and A, being of measure zero, is also mble

So B, being difference of two mble sets, is mble.

Moreover

$$E = B \cup A$$

Now

$$f(x) = g(x) \quad \forall\ x \in B$$

where B is mble subset of E.

Therefore,

restriction of f to B is mble

i.e g is mble on B.

Also g is mble on A, because function defined on set of measure Zero, is mble

Hence g will be mble on

E = A∪B (using Previous Th.)

Theorem:

The constant function defined on mble set, is mble.

Proof: Let $f(x) = c \quad \forall\ x \in E$

then for any $\alpha \in \mathbb{R}$

$$A = \{x \in E : f(x) > \alpha\} = \begin{cases} E & , \text{ if } \alpha < c = f(x) \\ \phi & \text{ if } \alpha \geq c \end{cases}$$

Here

E is given to be mble

and

ϕ is always mble

So A will be mble

Hence f is mble

Step Function:

Defn:

Let $[a, b[$ be an interval. Subdivide this interval as

$$a = c_1 < c_2 < c_3 < \cdots < c_i < \cdots < c_n = b$$

$$\left(\text{i.e } \underbrace{[c_1, c_2[}_{\delta_1}, \underbrace{[c_2, c_3[}_{\delta_2}, \ldots \underbrace{[c_i, c_{i+1}[}_{\delta_i}, \ldots, \underbrace{[c_{n-1}, c_n[}_{\delta_{n-1}} \right)$$

Let $\delta_i = [c_i, c_{i+1}[$

Then

$$[a, b[= \bigcup_{i=1}^{n-1} \delta_i$$

Define a function from $[a, b[$ as

$$f(x) = \lambda_i \quad , \text{ if } x \in \delta_i$$

Then f is called a step function.

THEOREM:

Step function defined above is mble.

Proof: Here each $\delta_i = [c_i, c_{i+1}[$ is mble $\forall \ i = 1, 2, 3, \ldots, n-1$

Moreover,

restriction of f to each δ_i is contant function, so $f_{/\delta_i}$ is mble

Thus f will be mble on

$$[a, b[= \bigcup_{i=1}^{n-1} \delta_i$$

(Since f is defined on $[a,b[$, which is coutable union of mble set δ_i. If $f_{/\delta_i}$ is mble then f is mble.)

Defn:

Let f & g be any real-valued functions, The suprimum of f & g, define on same domain denoted by $f \vee g$, is defined as

$$(f \vee g)(x) = \max \{f(x), g(x)\}$$ ($\vee$ OR, $\wedge$ And)

The infimum of f & g is defined as

$$(f \wedge g)(x) = \inf \{f(x), g(x)\}$$

Defn:

Let $\{f_i\}$ be a sequence of real-valued function defined on same domain, the suprimum of $\{f_i\}$, denoted by $\sup\{f_i\}$ is defined as

$$(\sup_i f_i)(x) = \sup_i \{f_i(x)\}$$

The infimum of $\{f_i\}$ is defined similarly.

Proposition: If f and g are mble functions with the same domain, then $f \vee g$ and $f \wedge g$ are mble.

Proof: Since f and g mble functions.
So for any $\alpha \in \mathbb{R}$,
The sets
$$A = \{x : f(x) > \alpha\}$$
and
$$B = \{x : g(x) > \alpha\}$$ are mble
Now
$$C = \{x : (f \vee g)(x) > \alpha\}$$
$$= \{x : \max.\{f(x), g(x)\} > \alpha\}$$

Let $x \in C$

$\Rightarrow \{x : \max.\{f(x), g(x)\} > \alpha\}$

$\Leftrightarrow \{x : f(x) > \alpha\}$ or $\{x : g(x) > \alpha\}$

$\Leftrightarrow x \in A$ or $x \in B$

$\Leftrightarrow x \in A \cup B$

$\Rightarrow C = A \cup B$ $\left(\begin{array}{l} x \in A \\ \Leftrightarrow x \in B \end{array} \right)$ $A = B$

Now, the set C being union of mble set is mble.
Hence
$f \vee g$ is mble.

ii) $f \wedge g$ are mble.

Proof:

Since f and g are mble.
So for any $\alpha \in \mathbb{R}$,
The sets

$$A = \{x : f(x) > \alpha\}$$

and

$$B = \{x : g(x) > \alpha\}$$ are mble

Now

$$E = \{x : (f \wedge g)(x) > \alpha\}$$
$$= \{x : \min\{f(x), g(x)\} > \alpha\}$$

Let $x \in E$

$\Rightarrow \{x : \min\{f(x), g(x)\} > \alpha\}$

$\Leftrightarrow \{x : f(x) > \alpha\}$ and $\{x : g(x) > \alpha\}$

$\Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow x \in A \cap B$

$\Rightarrow E = A \cap B$

Now, the set E, being intersection of mble sets, is mble.

Hence

$f \wedge g$ is mble.

which proves.

Proposition:- If $\{f_i\}$ is a sequence of mble functions defined on the same domain then

$\sup_i f_i$ & $\inf_i f_i$ are also mble.

Proof

Since each f_i is mble so for any $\alpha \in \mathbb{R}$

$A_i = \{x : f_i(x) > \alpha\}$ will also be mble for $i = 1, 2, 3, \ldots$

Let $B = \{x : (\sup_i f_i)(x) > \alpha\}$

$B = \{x : \sup_i (f_i(x)) > \alpha\}$

we show that

$B = \bigcup_i A_i$

Let $x \in B$

then

$\sup_i (f_i(x)) > \alpha$

$\Leftrightarrow x \in A_i$ for some i

$\Leftrightarrow x \in \bigcup_i A_i$

Hence & $A_i \subseteq B$.

$B = \bigcup_i A_i$

i.e the set B, being countable union of mble sets is mble.

Thus $\sup_i f_i$ is mble.

ii) $\operatorname{Inf}_i f_i$ is mble.

Proof: Since each f_i is mble so for any real no. α

$$A_i = \{x : f_i(x) > \alpha\}$$ is also mble for $i = 1, 2, 3, \ldots$

Let
$$B = \{x : (\operatorname{Inf}_i f_i)(x) > \alpha\}$$

$$B = \{x : \operatorname{Inf}_i (f_i(x)) > \alpha\}$$

We show that
$$B = \bigcap_i A_i$$

Let $x \in B$

$\Leftrightarrow \quad \operatorname{Inf}_i (f_i(x)) > \alpha$

$\Leftrightarrow \quad x \in A_i$ for all i

$\Leftrightarrow \quad x \in \bigcap_i A_i$

Hence $B = \bigcap_i A_i$

i.e. B, being countable intersection of mble sets, is mble.

Thus $\operatorname{Inf}_i f_i$ is mble.

which Proved.

Limit Superior & Limit Inferior

Defination: Let $\{x_i\}$ be a sequence of real numbers and let

$$a_1 = \sup\{x_1, x_2, x_3, \ldots\}$$

$$a_1 = \sup_{i \geq 1}\{x_i\}$$

$$a_2 = \sup_{i \geq 2}\{x_i\}$$

$$a_3 = \sup_{i \geq 3}\{x_i\}$$

$$\vdots$$

$$a_n = \sup_{i \geq n}\{x_i\}$$

$$\vdots$$

Moreovele, let

$$b_1 = \inf\{x_1, x_2, x_3, \ldots\} = \inf_{i \geq 1}\{x_i\}$$

$$b_2 = \qquad = \inf_{i \geq 2}\{x_i\}$$

$$b_3 = \qquad = \inf_{i \geq 3}\{x_i\}$$

$$\vdots$$

$$b_n = \qquad = \inf_{i \geq n}\{x_i\}$$

$$\vdots$$

It can be obreve that

$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq$$

and Obviously

$$b_1 \leq b_2 \leq b_3 \leq \ldots \leq b_n \leq$$

Then the limit superior of $\{x_i\}$, dgnoted by $\overline{\lim}\{x_i\}$, is defined as

$$\overline{\lim}(x_i) = \inf_K a_K = \inf_K \sup_{i \geq K}\{x_i\}$$

Similarly, limit inferior of $\{x_i\}$, denoted by $\underline{\lim}(x_i)$, is defined as

$$\underline{\lim}(x_i) = \sup_K b_K = \sup_K \inf_{i \geq K} \{x_i\}$$

Remarks:

i): Let $\{x_i\}$ be a sequence of real numbers, whose limit exists
then

$$\underline{\lim}(x_i) = \lim(x_i) = \overline{\lim}(x_i)$$

ii): In general

$$\inf(x_i) = b_1$$

$$\inf_i(x_i) \leq \underline{\lim}(x_i) \leq \overline{\lim}(x_i) \leq \sup(x_i)$$

Limit Inferior & Limit Superior of seq. of function.

Def.

Let $\{f_i\}$ be a sequence of real valued functions then

$\overline{\lim} f_i$ is defined as

$$(\overline{\lim} f_i)(x) = \overline{\lim}(f_i(x)) \qquad \forall\ i$$

Similarly

$$(\underline{\lim} f_i)(x) = \underline{\lim}(f_i(x))$$

Proposition: If $\{f_i\}$ is a sequence of mble function defined on the same domain then $\overline{\lim} f_i$ & $\underline{\lim} f_i$ are also mble.

Proof: Let $g_K = \sup_{i \geq K} \{f_i\}$ $\qquad K = 1, 2, 3, \cdots$

* g_K, being ~~a~~ Infimum of seq. of mble func., is mble. i.e $\{g_K\}$ is a seq. of mble function.

168

Since each f_i is given to be mble
* So each g_K is mble.
Moreover,
$\inf_K g_K$, being Infimum of mble functions, is also mble.

Thus

$$\overline{\lim} f_i = \inf_K g_K = \inf_K \sup_{i \geq K} (f_i)$$

is also mble.

Similarly Let

$$h_K = \inf_{i \geq K} \{f_i\} \qquad K = 1, 2, 3,$$

Since each f_i is given to be mble then $\inf_i \{f_i\}$ will also be mble for all i
So each h_K is mble.

Moreover, $\sup_K h_K$, being Supremum of mble function, is also mble.

Thus

$$\underline{\lim} f_i = \sup_K h_K$$

$$\underline{\lim} f_i = \sup_K \inf_{i \geq K} \{f_i\}$$

is also mble.

Characteristics Function.

Definition: Let $A \subseteq \mathbb{R}$, then the function defined on $\mathbb{R}$ with respect to A is said to be Characteristics function, denoted by χ_A, is defined as

$$\chi_A = \begin{cases} 0 & ; \; x \notin A \\ 1 & ; \; x \in A \end{cases}$$

Observation:

i) $\chi_{A\cap B} = \chi_A \cdot \chi_B$

ii) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$

iii) $\chi_{A^c} = 1 - \chi_A$

Proof i) If $x \in A \cap B$

~~then~~

$x \in A$ & $x \in B$

Hence

$$\chi_{A\cap B}(x) = 1$$

and $\chi_A(x) = 1$ & $\chi_B(x) = 1$

or $(\chi_A \cdot \chi_B)(x) = [\chi_A(x)] \cdot [\chi_B(x)]$

$$= 1 \cdot 1$$

$$(\chi_A \cdot \chi_B)(x) = 1$$

i.e

$$\chi_{A\cap B} = \chi_A \cdot \chi_B = 1 \quad \text{for } x \in A\cap B$$

If $x \notin A \cap B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow \chi_{A \cap B}(x) = 0$

And

$$(\chi_A \cdot \chi_B)(x) = [\chi_A(x)][\chi_B(x)]$$

$$(\chi_A \cdot \chi_B)(x) = \begin{cases} 1 \cdot 0 = 0 & \text{if } x \in A, x \notin B \\ 0 \cdot 0 = 0 & \text{if } x \notin A, x \notin B \\ 0 \cdot 1 = 0 & \text{if } x \notin A, x \in B \end{cases} = 0$$

i.e

$$\chi_{A \cap B} = \chi_A \chi_B = 0 \quad \forall \; x \notin A \cap B$$

Thus

$$\chi_{A \cap B} = \chi_A \cdot \chi_B \quad \forall \; x$$

ii) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$

Proof.

If $x \in A \cup B$

then

$x \in A$ or $x \in B$

Hence

$$\chi_{A \cup B}(x) = 1$$

if $x \in A$ & $x \notin B$

$$(\chi_A + \chi_B - \chi_{A \cap B})(x)$$

$$= \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x)$$

$= 1 + 0 - 0$
$= 1$

i.e

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B} \quad \text{for } x \in A,\ x \notin B$$

If $x \notin A$, $x \in B$

and

$(\chi_A + \chi_B - \chi_{A\cap B})(x)$

$= \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$

$= 0 + 1 - 0$
$= 1$

i.e $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ for $x \notin A$, $x \in B$

If $x \in A$ & $x \in B$

$(\chi_A + \chi_B - \chi_{A\cap B})(x)$

$= \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$

$= 1 + \cancel{1} - \cancel{1}$
$= 1$

i.e $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ for $x \in A$, $x \in B$

If $x \notin A\cup B$

then

$x \notin A$, $x \notin B$

$\Rightarrow \chi_{A\cup B}(x) = 0$

$(\chi_A + \chi_B - \chi_{A\cap B})(x)$

$= \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$

$= 0 + 0 - 0$

$= 0$

i.e

$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$

Thus

$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B} \quad \forall\ x$

iii) $\chi_{A^c} = 1 - \chi_A$

Proof: If $x \in A^c$

then

$x \notin A$

$\Rightarrow \chi_{A^c}(x) = 1$

and $1 - \chi_A(x) = 1 - 0 = 1$

i.e $\chi_{A^c} = 1 - \chi_A$ for $x \in A^c$

If $x \notin A^c$

then $x \in A$

$\Rightarrow \chi_{A^c}(x) = 0$

and $1 - \chi_A = 1 - 1 = 0$

i.e $\chi_{A^c} = 1 - \chi_A$ for $x \notin A^c$

Hence $\chi_{A^c} = 1 - \chi_A \quad \forall\ x$

Proposition:

The characteristics function χ_A is mble iff A is mble.

Proof:

Let χ_A is mble
then

$$A = \{x : \chi_A(x) = 1\}$$

Since χ_A is supposed to be mble

So the A is mble.

Conversly, Let A be mble, then for any real no. α

$$C = \{x : \chi_A(x) > \alpha\} = \begin{cases} \mathbb{R} & \text{if } \alpha < 0 \\ A & \text{if } 0 \leq \alpha < 1 \\ \phi & \text{if } \alpha \geq 1 \end{cases}$$

Which shows that, for every possibility of C ($\mathbb{R}$ is mble, A is supposed to be mble and ϕ is always mble)

$\Rightarrow$ C is mble.

$\Rightarrow$ χ_A is mble.

Which Proves

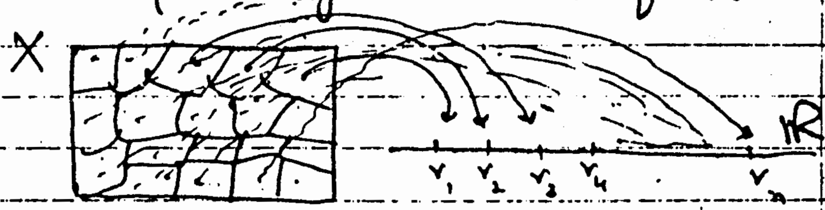
Simple Function:

A real valued function (defined on non-empty set X) is said to be simple function, if the set of all images, is finite, e.g

the characteristics function.

Remark: Let S be the simple function defined on set X, then S partitions X, into mutually disjoint subsets of X.

Proof



Akhtar 16.4.00 sunday

fig, which is Construct by "Naveed"

Let the set of images $(S(x))$ of S be $\{\alpha_1, \alpha_2, \text{——}, \alpha_n\}$

and $E_i = \{x \in X : S(x) = \alpha_i\}$, $i = 1, 2, 3, \text{---}, n$

It can be shown that

$$E_i \cap E_j = \phi \qquad , \qquad i \neq j$$

by taking $x \in E_i \cap E_j$

$$\Rightarrow x \in E_i \ \& \ x \in E_j$$

or

$$S(x) = \alpha_i \quad , \quad S(x) = \alpha_j$$

$$\Rightarrow \alpha_i = \alpha_j \quad \text{where } i \neq j$$

Contradiction, which is Impossible.

Hence ~~$\nexists$ $x \in E_i \cap E_j$ or~~ $E_i \cap E_j = \phi$, $i \neq j$

$$X = \bigcup_{i=1}^{n} E_i$$

Let $x \in X$

then $S(x) = \alpha_i$ for some i

$\Rightarrow x \in E_i$ for some i

$$\Rightarrow X \subseteq \bigcup_{i=1}^{n} E_i$$

But each E_i is subset of X.

so

$$\bigcup_{i=1}^{n} E_i \subseteq X$$

Hence

$$X = \bigcup_{i=1}^{n} E_i$$

Proposition(2):

For simple function, as defined in Remark 1, can be expressed as

$$S = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

Proof

we have to show

$$S(x) = \left(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\right)(x) \quad \forall x \in X$$

Let $x \in X$

then

$x \in E_i$ for some i

$$\Rightarrow \chi_{E_i}(x) = 1$$

Now $S(x) = \alpha_i \quad \forall x \in X$

Moreover,

$$\left(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\right)(x) = \alpha_1 (\chi_{E_1}(x)) + \alpha_2 (\chi_{E_2}(x)) + \cdots + \alpha_i (\chi_{E_i}(x)) + \cdots + \alpha_n (\chi_{E_n}(x))$$

$$= \alpha_1 0 + \alpha_2 0 + \cdots + \alpha_i 1 + \cdots + \alpha_n \cdot 0$$

i.e $\left(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\right)(x) = \alpha_i \quad \forall\, x \in X$

$$s(x) = \alpha_i \quad \forall\, x \in X$$

Thus $s = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$

Proposition: A simple function $s = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ (defined in remark 1) is mble iff each E_i is mble.

Proof: Let s be mble, then for each α_i

$E_i = \{x : s(x) = \alpha_i\}$ will be mble (?)

Conversly,
Let each E_i be mble.
Then χ_{E_i} will also be mble for all $i = 1, 2, 3, \ldots, n$

Consequently $\sum_{i=1}^{r} \alpha_i \chi_{E_i} = s$ will also be mble. Since linear combination of

Reimann Integration:

Let f be bounded function defined on the interval $[a, b]$

Let $a = x_0 < x_1 < x_2 < \dots < x_n = b$ be subdivision of $[a,b]$, over all possible subdivision of $[a,b]$, consider the

$$S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$$

&

$$s = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i$$

where $\quad M_i = \operatorname{Sup} f(x) \quad x_{i-1} \leq x \leq x_i$

and $\quad m_i = \operatorname{Inf} f(x) \quad x_{i-1} \leq x \leq x_i$

Infimum of S is defined to be Reimann Upper integral, denoted by,

$$R \overline{\int_a^b} f(x)\,dx = \inf S$$

Moreover,

Suprimum of s is defined to be Reiemann lower Integral, denoted by,

$$R \underline{\int_a^b} f(x)\,dx = \sup s$$

Observation:

1) In general

$$R \overline{\int_a^b} f(x)\,dx \geq R \underline{\int_a^b} f(x)\,dx$$

2). If Reimann upper and lower integral coincides, then f is said to be Reimann integral and we write

$$R\int_a^b f(x)\,dx$$

Remark.

Recalling the definition of step function.

Let ψ be a step function defined on interval $[a,b]$ as

$$\psi(x) = c_i \qquad x_{i-1} < x \leq x_i \qquad i = 1,2,\dots,n$$

for some subdivision of $[a,b]$ and set of constant c_i

Practically, one can defines integration of ψ as $\int_a^b \psi(x)\,dx = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$

Comparing the above definition of integral with the definition of Reimann integral

$$R\overline{\int_a^b} f(x)\,dx = \inf \int_a^b \psi(x)\,dx$$

for all $\psi \geq f$

Similarly

$$R\underline{\int_a^b} f(x)\,dx = \sup \int_a^b \psi(x)\,dx$$

$\forall\ \psi \leq f$

Defination. Let ϕ be a simple function having it's representation

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

This represtation is called canonical if all a_i are non-zero and distinct.

Moreover, the sets

$$A_i = \{x : \phi(x) = a_i\}$$

should be disjoint.

If cannonical representation of a simple function ϕ is

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

and if it vanishes outside the set of finite measure. Then the integral of ϕ, is defined as

$$\int \phi(x)\,dx = \sum_{i=1}^{n} a_i m(A_i)$$

This integral is some time also written as integral

$\int \phi$. If E is any mble set then

$$\int_E \phi = \int \phi \chi_E$$

Remark:

It is often convinient to use the representation of simple function, which is not in cannonical representation.

Lemma:

Moreover, let

Let $A_a = \{x : \phi(x) = a\}$

Then

$$A_a = \bigcup_{a_i = a} E_i \qquad m(A_a) = m\Big(\bigcup_{a_i=a} E_i\Big) = \sum_{a_i = a} m E_i$$

$$\Rightarrow \quad a\,m(A_a) = \sum_{a_i = a} a\, m(E_i)$$

Now

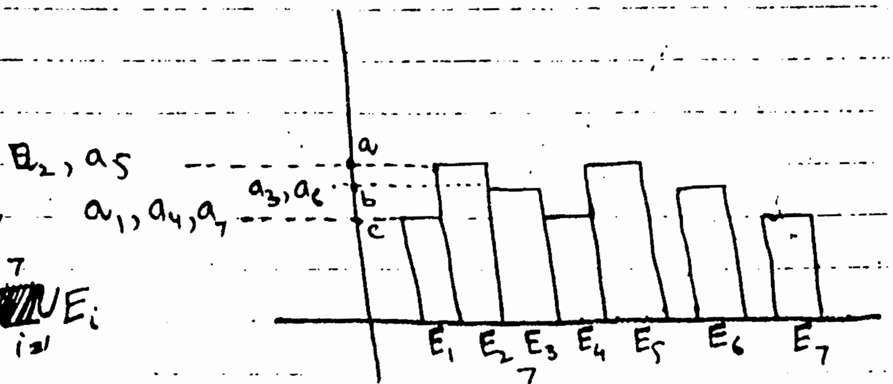
$$\int \phi(x)\,dx = \sum_a a\, m(A_a) = \sum_{i=1}^{n} a_i\, m(E_i)$$

$A_a = E_2 \cup E_5$

$A_b = E_3 \cup E_6$

$A_c = E_1 \cup E_4 \cup E_7$

$A_a \cup A_b \cup A_c = \bigcup_{i=1}^{7} E_i$



$$\phi = \sum_{i=1}^{7} a_i \chi_{E_i}$$

Proposition.

Let ϕ and ψ be two simple func. which vanish outside the set of finite measure, then

$$\int (a\phi + b\psi) = a\int\phi + b\int\psi$$

and

if $\phi \geq \psi$ (a.e)

then

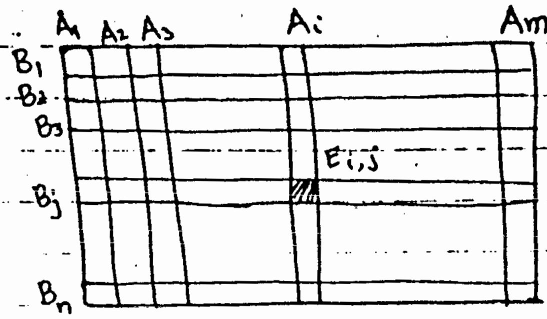
$$\int\phi \geq \int\psi$$

Proof: Let $\phi = \sum_{i=1}^{m} a_i \chi_{A_i}$

$$\psi = \sum_{j=1}^{n} b_j \chi_{B_j}$$

be the cannonical representations of ϕ & ψ.

Let A_0 & B_0 be the sets where ϕ & ψ vanish respectively.



Let

$$E_{i,j} = A_i \cap B_j$$

$i = 1, 2, 3, \ldots, m$

$j = 1, 2, 3, \ldots, n$

then $E_{i,j}$'s are mutually dijoint and mble sets.

Moreover, $X = \bigcup_{i=1}^{m} A_i = \bigcup_{j=1}^{n} B_j = \bigcup_{i,j} E_{i,j}$

(Note: For convenience, we may omit the upper limit of Σ or $\cup$).

Now $\phi = \sum_i a_i \chi_{A_i}$

$= \sum_i a_i \left(\sum_j \chi_{E_{i,j}} \right)$ $\qquad \because A_i = \bigcup_{j=1}^{n} E_{i,j}$

$$\phi = \sum_{i,j} a_i \chi_{E_{i,j}} \quad \text{——— (1)}$$

Similarly,

$\psi = \sum_j b_j \chi_{B_j}$

$= \sum_j b_j \left(\sum_i \chi_{E_{i,j}} \right)$

$$\psi = \sum_{i,j} b_j \chi_{E_{i,j}} \quad \text{——— (2)}$$

If $x \in E_{i,j}$

then $(\phi + \psi)(x) = \phi(x) + \psi(x)$

$= a_i + b_j$

Therefore

$$(\phi + \psi) = \sum_{i,j} (a_i + b_j) \chi_{E_{i,j}} \qquad \because X = \bigcup_{i,j} E_{i,j}$$

(Note: This representation may not be cannonical

using Previous Lemma

$$\int (\phi + \psi) = \sum_{i,j} (a_i + b_j) m(E_{i,j})$$

$$= \sum_{i,j} a_i m(E_{i,j}) + \sum_{i,j} b_j m(E_{i,j})$$

$$\int(\phi+\psi) = \int\phi + \int\psi \quad \text{——③}$$ by using ① and ②

Moreover,

$$a\phi = a\sum_{i=1}^{m} a_i \chi_{A_i}$$

$$a\phi = \sum_{i=1}^{m} a a_i \chi_{A_i}$$

$$\int(a\phi) = \sum_{i=1}^{m} (a a_i)\, m(A_i)$$

$$= a\sum_{i=1}^{m} a_i\, m(A_i)$$

$$\int(a\phi) = a\int\phi \quad \text{——④}$$

Similarly,

$$\int(b\psi) = b\int\psi \quad \text{——⑤}$$

So from ③, ④ & ⑤

$$\int(a\phi+b\psi) = \int(a\phi) + \int(b\psi)$$

$$= a\int\phi + b\int\psi$$

which Proves.

~~Now~~ We show that

$$\int\phi \geq \int\psi$$

Since $\phi \geq \psi$ (a.e)

$$\Rightarrow \int (\phi - \psi) \geq 0$$

$$\int (\phi - \psi) \geq 0$$

Thus $\int \phi - \int \psi \geq 0$

or $\int \phi \geq \int \psi$

which Completes the Proof.

Proposition.
Let f be a bounded function and E be a mble set of finite measure then
for simple function ϕ and ψ

$$\inf_{\psi \geq f} \int \psi = \sup_{\phi \leq f} \int \phi$$

iff

f is mble.

Proof.
We first show the sufficiency of the condition. i.e.
Let f be mble and bounded by M. i.e

$$|f(x)| \leq M$$

or

$$f(x) \geq -M$$

and

$$f(x) \leq +M$$

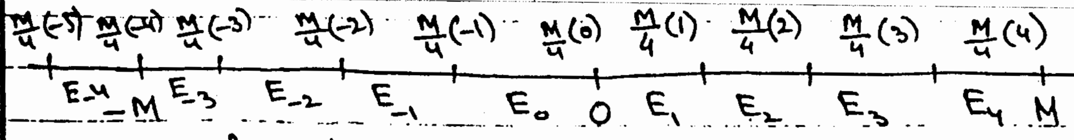
Consider the sets

$$E_k = \{x : \frac{M}{n}(k-1) < f(x) \le \frac{M}{n}(k)\}$$

$1-n \le k \le n$

$n=4$

*



for $k=n$

$$E_n = \{x : \frac{M}{n}(-n-1) < f(x) \le \frac{M}{n}(n)\}$$

* $\frac{M}{n}(-n-1)$

$\frac{M}{n}(k-1)$

Proposition: If f and g are bounded mble functions define on set of finite measure then

1: $$\int_E (af+bg) = a\int_E f + b\int_E g$$

2: If $f = g$ (a.e)
then
$$\int_E f = \int_E g$$

3: i) If $f \leq g$ (a.e)
then
$$\int_E f \leq \int_E g$$

ii) and hence $$\left|\int_E f\right| \leq \int_E |f|$$

4: If $a \leq f(x) \leq b$
then
$$a\,m(E) \leq \int_E f \leq b.m(E)$$

5: If A & B are disjoint set of finite measure
then
$$\int_{A\cup B} f = \int_A f + \int_B f$$

Proof. ① we will prove that

i) $$\int_E (af) = a\int_E f$$

ii) $$\int_E (f+g) = \int_E f + \int_E g$$

i) If ψ is a simple function, so is $a\psi$ and if $a>0$ and $\psi \geq f$ then

$$\int_E (af) = \inf_{a\psi \geq af} \int_E (a\psi) = a \inf_{\psi \geq f} \int_E \psi$$

$$\int_E (af) = a \cdot \int_E f \qquad \text{———} \; (a')$$

If $a<0$ and $\psi \leq f$ then $a\psi \geq af$

$$\int_E (af) = \inf_{a\psi \geq af} \int_E (a\psi) = \inf_{\psi \leq f} a \int_E \psi$$

$$= a \sup_{\psi \leq f} \int_E \psi = a \inf_{\phi \geq f} \int_E \phi \quad \text{as } f \text{ is mble}$$

$$\int_E (af) = a \int_E f \qquad \text{———} \; (b')$$

ii) Let ψ_1 & ψ_2 be the simple functions such that

$$\psi_1 \leq f \;\; \& \;\; \psi_2 \leq g$$

then

$$\psi_1 + \psi_2 \leq f + g$$

$$\int_E (f+g) \geq \int_E (\psi_1 + \psi_2) = \int_E \psi_1 + \int_E \psi_2$$

i.e

$$\int_E (f+g) \geq \int_E \psi_1 + \int_E \psi_2$$

Taking supremum

$$\int_E (f+g) \geqslant \sup_{\psi_1 \le f} \int_E \psi_1 + \sup_{\psi_2 \le g} \int_E \psi_2$$

$$= \inf_{\phi_1 \geq f} \int_E \phi_1 + \inf_{\phi_2 \geqslant g} \int_E \phi_2$$ as f & g are mble.

$$= \int_E f + \int_E g$$

i.e

$$\int_E (f+g) \geqslant \int_E f + \int_E g \quad \text{———} (\mathrm{I})$$

Again let ψ_1 & ψ_2 be simple functions such that

$$\psi_1 \geqslant f \quad \& \quad \psi_2 \geqslant g$$

$$\psi_1 + \psi_2 \geqslant f + g$$

$$\Rightarrow \int_E (f+g) \le \int_E (\psi_1 + \psi_2) = \int_E \psi_1 + \int_E \psi_2$$

Taking infimum

$$\int_E (f+g) \le \inf_{\psi_1 \geqslant f} \int_E \psi_1 + \inf_{\psi_2 \geqslant g} \int_E \psi_2$$

$$= \int_E f + \int_E g \quad \text{———} (\mathrm{II})$$

From (I) & (II)

$$\int_E (f+g) = \int_E f + \int_E g$$

Thus

$$\int_E (af+bg) = a\int_E f + b\int_E g$$

2) If $f=g$ (a.e)
then
$$\int_E f = \int_E g$$

Proof:

If $f = g$ (a.e)
then
$$(f-g) = 0 \quad (a.e)$$

If ψ is a simple function
such that
$$\psi \geq (f-g)$$

$$\Rightarrow \quad \psi \geq 0 \quad (a.e)$$

$$\Rightarrow \int_E \psi \geq 0 \qquad \because \ \phi \geq \psi \ (a.e) \quad \int \phi \geq \int \psi$$

Now taking infimum

$$\inf_{\psi \geq (f-g)} \int_E \psi \geq 0$$

$$\Rightarrow \int_E (f-g) \geq 0$$

$$\Rightarrow \int_E f - \int_E g \geq 0$$

$$\Rightarrow \int_E f \geq \int_E g \quad \text{——— III}$$

Let ϕ be a simple function such that

$$\phi \leq (f - g)$$

then

$$\phi \leq 0 \quad (a.e)$$

then $\int_E \phi \leq 0$

Taking supremum

$$\sup \int_E \phi \leq 0$$

$$\phi \leq (f-g)$$

$$\Rightarrow \inf \int_E \psi \leq 0 \qquad (f-g) \text{ is mble}$$

$$\psi \geq f - g$$

$$\Rightarrow \int_E (f-g) \leq 0$$

$$\Rightarrow \int_E f - \int_E g \leq 0$$

or $\int_E f \leq \int_E g$ ——— IV

From III and IV

$$\int_E f = \int_E g$$

3) If $f \leq g$ (a.e) then $\int_E f \leq \int_E g$ ~~31-12-99~~

III) Proof: If $f \leq g$ (a.e) 17-01-99

then

$$(f-g) \leq 0 \quad (a.e)$$

Let ψ be a simple function such that

$$\psi \leq (f-g)$$

then

$$\psi \leq 0 \quad (a.e)$$

$$\Rightarrow \quad \int_E \psi \leq 0$$

Taking Supremum

$$\sup \int_E \psi \leq 0$$

$$\psi \leq (f-g)$$

or

$$\inf \int \phi \leq 0 \quad \text{(using}$$

$$\phi \geq (f-g)$$

$$\int_E (f-g) \leq 0$$

or

$$\int_E f - \int_E g \leq 0$$

$$\Rightarrow \quad \int_E f \leq \int_E g$$

which Proves.

5) If A & B are disjoint set of finite measure then

$$\int_{A\cup B} f = \int_A f + \int_B f$$

Proof:

$$\int_{A\cup B} f = \int f \chi_{A\cup B}$$

$$= \int f(\chi_A + \chi_B) \qquad \because A\cap B = \phi$$

$$= \int (f\chi_A + f\chi_B)$$

$$= \int f\chi_A + \int f\chi_B \qquad \because$$

$$= \int_A f + \int_B f$$

which Proves.

III) ii)

$$\left|\int_E f\right| \leq \int_E |f|$$

Proof:

Consider three possibilities of f

1) $f \geq 0$

then

$$\int_E f \geq 0$$

$$\left|\int_E f\right| = \int_E f \quad \text{——— (i)}$$

Also

$$f \geq 0$$

$$\Rightarrow |f| = f$$

$$\int_E |f| = \int_E f \quad \text{—(ii)}$$

From (i) & (ii)

$$\Rightarrow \left[\int_E f\right] = \int_E |f|$$

2) $f < 0$

then $\int_E f < 0$

$$\Rightarrow \left|\int_E f\right| = -\int_E f \quad \text{--- (iii)}$$

$f < 0$

$\Rightarrow |f| = -f$

$$\int_E |f| = \int_E (-f)$$

$$\int_E |f| = -\int_E f \quad \text{--- (iv) (use Result 1)}$$

$$\Rightarrow \left|\int_E f\right| = \int_E |f|$$

3) $f(x) \geq 0$ for some $x \in E$
and $f(x) < 0$ for remaining $x \in E$.

i.e

Let $E_1 = \{x \in E : f(x) \geq 0\}$

$E_2 = \{x \in E : f(x) < 0\}$

Now

$$\int_E f = \int_{E_1 \cup E_2} f \qquad E_1 \cap E_2 = \phi$$

$$\Rightarrow \left|\int_E f\right| = \left|\int_{E_1 \cup E_2} f\right|$$

$$= \left|\int_{E_1} f + \int_{E_2} f\right| \qquad \text{use (5)}$$

$$\leq \left|\int_{E_1} f\right| + \left|\int_{E_2} f\right|$$

$$= \int_{E_1} |f| + \int_{E_2} |f|$$

$$\left|\int_E f\right| \le \int_{E_1} |f| + \int_{E_2} |f|$$

$$= \int_{E_1 \cup E_2} |f|$$

$$\left|\int_E f\right| = \int_E |f|$$

which Proves.

4) If $a \le f(x) \le b$

then

$$am(E) \le \int_E f \le b\,m(E)$$

Proof:

$$a \le f(x) \le b$$

$$\Rightarrow \int_E a \le \int_E f(x) \le \int_E b$$

$$\Rightarrow a.m(E) \le \int_E f \le b.m(E)$$

which Proves.

Integral of non-negative function: 19-01-2000

Let F be a non-negative mble function defined on mble set E.
Then we define

$$\int_E f = \sup_{\psi \le f} \int_E \psi$$

where ψ is bdd. mble function such that $m\{x : \psi(x) \ne 0\}$ is finite
i.e ψ vanishes out side the set of finite measure

Theorems:

If f and g are non-negative mble functions defined on E.

Then

i) $\int_E cf = c\int_E f \quad , c>0$

ii) $\int_E (f+g) = \int_E f + \int_E g$

$\Rightarrow \int_E (af+bg) = a\int_E f + b\int_E g \quad$ for $a,b\geq 0$

iii) if $f\leq g$ (a.e)

then $\int_E f \leq \int_E g$

iv) For $A\subseteq B$

$\int_A f \leq \int_B f$

Proof. i)

If $f \geq 0$ (a.e)
then $\int_E f \geq 0$

Proof: Let we first prove that
$f \leq g$ on E

$\Rightarrow \int_E f \leq \int_E g$

Let ϕ and ψ be bounded mble functions that vanishes outside the set of finite measure such that
$$\phi \leq f \quad \text{and} \quad \psi \leq g$$
then
$$\{\phi : \phi \leq f\} \subseteq \{\psi : \psi \leq g\} \quad \because f \leq g$$
$$\Rightarrow \{\int \phi : \phi \leq f\} \subseteq \{\int \psi : \psi \leq g\}$$
or $\int_E \phi \leq \int_E \psi$

Taking supremum,
$$\sup_{\phi \leq f} \int_E \phi \leq \sup_{\psi \leq g} \int_E \psi$$
$$\Rightarrow \int_E f \leq \int_E g$$

Which shows that, in particular,
If $f \geq 0$
$$\int_E f \geq 0$$

Moreover, if $f \geq 0$ (a.e) on E
then
$$\int_E f \geq 0$$

iii)

Proof: as $f \leq g$ (a.e) on E
so
$(g-f) \geq 0$ (a.e) on E
$$\Rightarrow \int_E (g-f) \geq 0$$

Consider $\int_E g = \int_E (f+g-f)$
$$= \int_E f + \int_E (g-f)$$
$$\geq \int_E f + 0 \qquad \because \int_E (g-f) \geq 0$$
$$= \int_E f$$
i.e
$$\int_E g \geq \int_E f$$
which proves.

iv)

Proof: As f is given to be non-negative msble func so are $f.\chi_A$ & $f.\chi_B$.

Moreover, $A \subseteq B$

$$\Rightarrow f\chi_A \leq f\chi_B$$

$\because f\chi_A(x) = 0,\ x \notin A$ & $x \in B$

but $f\chi_B(x) = f(x)$

If $x \notin A$ & $x \in B$ then $f\chi_A \leq f\chi_B$

Thus

$$\int f\chi_A \leq \int f\chi_B$$

$$\int_A f \leq \int_B f$$

Bounded Convergence Th: 21-01-99

Def: If $\{f_n\}$ is seq. of bounded mble func and $\lim f_n = f$ then

$$\int_E f = \underline{\lim} \int_E f_n$$

\## Lemma: (Fatou's Lemma)

If $\{f_n\}$ is a seq. of non-negative mble functions such that

$$f_n \longrightarrow f \quad (a.e) \quad \text{on } E.$$

Then

$$\underline{\lim}\, f_n = f$$

$$\int_E f \leq \underline{\lim} \int_E f_n$$

Proof:

without loss of generality, it can be assume that the convergence is everywhere as the integral over sets of measure are zero

under this assumption it can be proved that

$$\int_E \lim\inf f_n \leq \lim\inf \int f_n$$

It is obvious that f being limit of seq of mble func is mble

Moreover, $f \geq 0$

$$\Rightarrow f = \lim_n f_n \geq 0$$

Let ϕ be bounded function which vanishes outside the set of finite measure and

$$\phi \leq f$$

Let

$$\phi_n = \min(\phi, f_n)$$

Hence ϕ_n being minimum of mble function will be mble.

Moreover the function ϕ_n remains bound by the bounds for ϕ

Moreover, for any element

$x \in F^c$

$$\phi(x) = 0$$

Since $f_n(x) \geq 0$

therefore for $x \in F^c$

$$\phi_n(x) = \min\{\phi(x), f_n(x)\} = 0$$

i.e ϕ_n vanishes outside the set F.

Also $\lim_n \phi_n = \lim_n \min\{\phi, f_n\}$

$$= \min\{\phi, f\}$$

$$\lim_n \phi_n = \phi \qquad \because \phi \leq f$$

Since the seq. $\{\phi_n\}$ becomes a seq of bounded mble functions which vanishes outside the set of finite measure

(P.T.O)

Therefore by bounded Convergence Th.

$$\int_F \phi = \lim_n \int_F \phi_n \quad \text{—— (a)}$$

or

$$\underline{\lim} \int_F \phi_n = \overline{\lim} \int_F \phi_n = \int_F \phi = \lim_n \int_F \phi_n$$

In Particular

$$\lim_n \int_E \phi(n) \leq \underline{\lim} \int_F \phi_n$$

Since by def.

$$\phi_n \leq f_n \quad \forall n$$

There

$$\int_F \phi = \lim \int_F \phi_n \leq \underline{\lim} \int_F \phi_n \leq \underline{\lim} \int_E f_n$$

$$\Rightarrow \int_F \phi \leq \underline{\lim} \int_E f_n \quad \text{—— (b)} \quad (\because F \subset E)$$

Now $\int_E \phi = \int_{F \cup F^c} \phi = \int_F \phi + \int_{F^c} \phi = \int_F \phi \qquad \because \int_{F^c} \phi = 0$

$$\Rightarrow \int_E \phi = \int_F \phi$$

So by (b)

$$\int_E \phi \leq \underline{\lim} \int_E f_n$$

Taking Sup.

$$\operatorname{Sup} \int_E \phi \leq \operatorname{Sup} \underline{\lim} \int_E f_n$$

$$\operatorname{Sup}_{\phi \leq f} \int_E \phi \leq \underline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f \leq \underline{\lim} \int_E f_n$$

Which Proves.

Monotone Convergence Theorem: 22-01-00

If $\{f_n\}$ is an increasing sequence of non-negative mble function and
$$Lim\, f_n = f$$
then
$$\int f = Lim \int f_n \quad \text{Domain is defined over } E.$$
or
$$\int Lim_n\, f_n = Lim_n \int f_n$$

Proof: By preceding theorem
$$\int f \leq \underline{Lim} \int f_n \quad \text{—— (a)} \quad \text{by Fatou's Lemma}$$
Moreover,
$$f_n \leq f \quad (\text{as } \{f_n\} \text{ is increasing seq.})$$
$$\Rightarrow \quad \int f_n \leq \int f$$
Taking limit sup
Then
$$\overline{Lim}_n \int f \leq \int f \quad \text{—— (b)}$$
From (a) & (b)
$$\int f = Lim_n \int f_n$$

Example: (counter example)
Let $\{f_n\}$ be a seq. of non-negative mble functions where
$$f_n : \mathbb{R} \longrightarrow \mathbb{R}$$
defined as
$$f_n(x) = \frac{1}{n} \chi_{(n,\infty)}(x) \quad \forall\, n = 1, 2, 3, \ldots$$
Sol: * Obviously $\{f_n\}$ is monotonically decreasing seq. with
$$Lim_n\, f_n = 0$$

we show that

$$\int \lim_n f_n \neq \lim f_n$$

Proof:

As $\lim_n f_n = 0$

$\therefore \int \lim f_n = 0$ ——— (a)

Now $\int f_n = \int \frac{1}{n} \chi_{(n,\infty)}$

$= \frac{1}{n} \int \chi_{(n,\infty)}$

$= \frac{1}{n} \; m(n,\infty)$

$\int f_n = \infty$

$\Rightarrow$

$\lim_n \int f_n = \lim_n (\infty)$

$= \infty$ ——— (b)

From (a) & (b)

$$\int \lim_n f_n \neq \lim_n \int f_n$$

* $f_1(x) = \frac{1}{1} \chi_{(1,\infty)}(x)$

Counter Example:

Let $\{f_n\}$ be a sequence of non-negative mble functions, where

$$f_n : [0,\infty) \longrightarrow \{0,1\}$$

defined as

$$f_n(x) = \begin{cases} 1 & , \quad x \in [n,\infty) \\ 0 & , \quad \text{otherwise or } x \in [0,n) \end{cases}$$

Sol:

It is to be noted that f_n can be expressed as

$$f_n = \chi_{A_n} \qquad A_n \in [n,\infty)$$

Now $\int f_n = \int \chi_{A_n}$ — $\int$ of a characteristics func is measure of the set on which char. func. is defined.

$$= m(A_n)$$

$$= m[n,\infty)$$

$$\int f_n = \infty$$

$$\Rightarrow \lim_n \int f_n = \lim_n (\infty)$$

$$= \infty \quad \text{——— (a)}$$

(Value of func. decreasing)

Let $x \in [0,\infty)$

Then there can be choosen an integer $w > 0$ such that

$$x \in [0,w)$$

and $x \notin [w,\infty)$

which shows that

$$f_w(x) = \chi_{[w,\infty)}(x)$$

$$= 0$$

$f_w(x) = \chi_{[w,\infty)}(x) = 0$

Moreover, for any $n \geq w$

$$x \in [0,w) \subseteq [0,n)$$

Thus

$$f_n(x) = \chi_{[n,\infty)}(x) = 0 \qquad \forall\, n \geq w, \quad x \in [0,n)$$

$\Rightarrow \lim_n f_n = 0$

$a_1, a_2, a_3, \ldots \to a$
if we leave finite a_n then still
$a_{n+1}, a_{n+2} \ldots \to a$

$\Rightarrow \int \lim_n f_n = 0$ — (b)

From (a) & (b)

$$\int \lim_n f_n \neq \lim_n \int f_n$$

which Proves

Example: (counter) 24-01-00

Let $E = (0,1)$

and $\{f_n\}$ be a seq. of non-negative mble functions defined on E as

$$f_n(x) = \begin{cases} n(n+1) & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right) \\ 0 & \text{otherwise} \end{cases}$$

Sol: Obviously, $\{f_n\}$ is a sequence of non-negative mble functions, which are monotonically decreasing and $\lim_n f_n = 0$

$\Rightarrow \int_E \lim_n f_n = 0$ — (1)

$f_1(x) = 1(2) = 2 \quad x \in \left(\frac{1}{2}, 1\right)$
$f_2(x) = 2(3) = 6 \quad x \in \left(\frac{1}{3}, \frac{1}{2}\right)$
$f_3(x) = 3(4) = 12 \quad x \in \left(\frac{1}{4}, \frac{1}{3}\right)$
$\vdots \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$

Now Putting

$$E_n = \left(\frac{1}{n+1}, \frac{1}{n}\right) \quad \forall\, n = 1, 2, 3, \ldots$$

$$\int_E f_n = \int_{E_n \cup E_n^c} f_n$$

$$= \int_{E_n} f_n + \int_{E_n^c} f_n \qquad E_n \,\&\, E_n^c \text{ are disjoints}$$

Integrable Function:

Def: A non-negative mble function f, defined on mble set E, is said to be integrable, if

$$\int_E f < \infty$$

26-01-00

Proposition: If f & g are non-negative mble functions, defined on mble set E, f is integrable on E and $g \leq f$ on E then g is also integrable over E and

$$\int_E (f-g) = \int_E f - \int_E g$$

Proof:

$\because g \leq f$

$\therefore f-g \geq 0$

Since $g \leq f$

So

$$f-g \geq 0 \quad \text{defined on } E$$

Also

$$\int_E f = \int_E \{(f-g)+g\}$$

$$\int_E f = \int_E (f-g) + \int_E g \quad ?$$

or

$$\int_E (f-g) = \int_E f - \int_E g$$

Moreover,

$$\int_E f = \int (f-g) + \int_E g$$

shows that in L.H.S 'f' is integrable, so L.H.S is finite

Consequently, R.H.S is also finite i.e each term in R.H.S is finite

Hence $\int_E g$ is finite or g is integrable over E.

which Proves

Example: 1)

Define a function

$$f : (0,1] \longrightarrow \mathbb{R}$$

as

$$f(x) = \frac{1}{x} \quad \forall\, x \in (0,1]$$

It can be observe that f being continous, is mble. Moreover, f is non-negative. We show that f is not integrable.

Sol. For any positive integer k, we have

$$\frac{1}{x} \geqslant \sum_{n=1}^{k} n\, \chi_{\left(\frac{1}{n+1}, \frac{1}{n}\right]}(x) \qquad k = 1, 2, 3, \ldots$$

$f\left(\frac{1}{2}\right) = 2$

$f\left(\frac{1}{3}\right) = 3$

$1\chi_{\left(\frac{1}{2}, 1\right]} + 2\chi_{\left(\frac{1}{3}, \frac{1}{2}\right]} + 3\chi_{\left(\frac{1}{4}, \frac{1}{3}\right]} + \cdots + k\chi_{\left(\frac{1}{k+1}, \frac{1}{k}\right)}$

So
$$\int_0^1 \frac{1}{x} \geqslant \int_0^1 \sum_{n=1}^{k} n\chi_{\left(\frac{1}{n+1}, \frac{1}{n}\right]}(x)$$

$$= \sum_{n=1}^{k} \int_0^1 n\chi_{\left(\frac{1}{n+1}, \frac{1}{n}\right]}(x)$$

$$= \sum_{n=1}^{k} n\, m\left(\frac{1}{n+1}, \frac{1}{n}\right]$$

$$= \sum_{n=1}^{k} n\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \sum_{n=1}^{k} n\left(\frac{1}{n(n+1)}\right)$$

$$= \sum_{n=1}^{k} \frac{1}{n+1} \qquad \forall\, k = 1, 2, 3, \ldots$$

$$\int_E f_n = \int_{E_n} f_n + 0$$

$$\int_E f_n = \int_E f_n \chi_E$$

$$= \int_E n(n+1) \chi_{E_n} = n(n+1) \int_E \chi_{E_n}$$

$$= n(n+1)\, m(E_n)$$

$$\int_E f_n = n(n+1) \left(\frac{1}{n(n+1)}\right) = 1$$

$$\Rightarrow \lim_n \int_E f_n = 1 \quad \text{—— (2)}$$

From (1) & (2):

$$\int_E \lim_n f_n \neq \lim_n \int_E f_n$$

which proves.

Corollary.

If $\{u_n\}$ is a seq. of non-negative mble functions and

$$f = \sum_{n=1}^{\infty} u_n$$

Then $$\int f = \sum_{n=1}^{\infty} \int u_n$$

Proof: Let $$f_n = \sum_{i=1}^{n} u_i$$

(i.e sequence of Partial sum)

Then $\{f_n\}$ is monotonically increasing sequence and

$$\lim_n f_n = \sum_{i=1}^{\infty} u_i$$

$$= f$$

using Monotone Convergence Th.

$$\int \lim_n f_n = \int f = \lim_n \int f_n$$

$$= \lim_n \int \sum_{i=1}^{n} u_i$$

$$= \lim_n \sum_{i=1}^{n} \int u_i$$

$$= \sum_{i=1}^{\infty} \int u_i$$

which Proves.

Proposition.

Let f be a non-negative mble func. and $\{E_i\}$ be a sequence of mutually disjoint mble sets such that

$$E = \bigcup_i E_i$$

then

$$\int_E f = \sum_i \int_{E_i} f$$

Proof: If f is non-negative mble function so are

$$f\chi_{E_i} \qquad \left(\because \int f\chi_{E_i} = \int_{E_i} f\right)$$

Now $f\chi_E = f\chi_{\cup E_i}$

$$f\chi_E = f\left(\chi_{E_1} + \chi_{E_2} + \chi_{E_3} + \cdots\right)$$

$$f\chi_E = \sum_i f\chi_{E_i}$$

$$\int_E f = \int f\chi_E$$

$$= \int \sum_i (f\chi_{E_i})$$

$$\int_E f = \underbrace{\sum_i \int f\chi_{E_i}}_{\text{using above corollary}} = \sum_i \int_{E_i} f$$

i.e $\int_0^1 \frac{1}{x} \geq \sum_{n=1}^{k} \frac{1}{n+1} \quad \forall\ k = 1, 2, 3, \ldots$

Thus

$$\int_0^1 \frac{1}{x} = \infty$$

Hence f is non-integrable.

Remark: Similarly $\int_1^{\infty} \frac{1}{x} = \infty$

i.e function $f : [1, \infty) \to \mathbb{R}$
defined as

$$f(x) = \frac{1}{x} \quad \forall\ x \in [1, \infty)$$

Show that f is not integrable.

Example: 2)

Define $f: \mathbb{R} \to \{0,1\}$

as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Sol. i.e

$$f = \chi_{\mathbb{Q}}$$

Then

$$\int_{\mathbb{R}} \chi_{\mathbb{Q}} = 1 \, (m(\mathbb{Q}))$$

$$= 1 \, (0)$$

since $\mathbb{Q}$ is countable $\therefore m(\mathbb{Q}) = 0$

$$\int \chi_{\mathbb{Q}} = 0$$

$$\Rightarrow \int \chi_{\mathbb{Q}} < \infty$$

Thus $f = \chi_{\mathbb{Q}}$ is integrable

Example: 3)

If $f: \mathbb{R} \to \{0,1\}$

$$f(x) = \begin{cases} 1 & , \; x \in \mathbb{Q}^c \\ 0 & \; x \in \mathbb{Q} \end{cases}$$

Sol.

Here $f = \chi_{\mathbb{Q}^c}$

$$\int \chi_{\mathbb{R}} = \int \chi_{\mathbb{Q} \cup \mathbb{Q}^c}$$

$$= \int \chi_{\mathbb{Q}} + \int \chi_{\mathbb{Q}^c}$$

$$\Rightarrow \int \chi_{\mathbb{R}} = 1 \, m(\mathbb{Q}) + 1 \, m(\mathbb{Q}^c)$$

$$\infty = 0 + \int \chi_{\mathbb{Q}^c}$$

$$\Rightarrow \int \chi_{\mathbb{Q}^c} = \infty$$

$\Rightarrow f = \chi_{\mathbb{Q}^c}$ is not integrable.

Remark :

χ_A is integrable iff $m(A) < \infty$

because

$$\int \chi_A = 1 \cdot m(A)$$

$$\Rightarrow \int \chi_A < \infty \text{ iff } m(A) < \infty$$

General Lebesgue Integral: $\overset{\text{And}}{\wedge}_{\text{min}}$ $\overset{\text{OR}}{\vee}_{\text{max}}$

Def: 28-01-00

Let f be any real valued function, we define +ve part of f, denoted by f^+, as

$$f^+ = f \vee 0$$

i.e $f^+(x) = \max\{f(x), 0\} \quad \forall x$

Similarly -ve part of f, denoted by f^-, as

$$f^- = f \wedge 0$$

i.e $f^-(x) = \min\{f(x), 0\} \quad \forall x$

Imp

Proposition: 28-01-00

A non-negative integrable / mble function f defined on set E, is finite on E (a.e)

Proof: Let $A = \{x \in E : f(x) = \infty\}$

Then,

for every +ve integer n,

$$0 \leq n\chi_A \leq f$$

$$\Rightarrow 0 \leq \int n\chi_A \leq \underset{<\infty}{\int f} \quad \because f \text{ is integrable}$$

$$\Rightarrow 0 \le n\, m(A) \le \int f < \infty \quad \forall n$$

$$\Rightarrow 0 \le m(A) \le \frac{1}{n}\int f \quad \forall n$$

$0 \le m(A)$ $\quad m(A) \le \frac{1}{n}\int f = 0 \quad \forall n$

$$m(A) \le 0$$

$\Rightarrow m(A) = 0$

i.e

f is finite on E (a.e).

Proposition:

Let $\{f_n\}$ be a sequence of non-negative mble functions defined on $\mathbb{R}$ such that

$$f_n \to f \quad (a.e)$$

or $\lim_n f_n = f \quad (a.e)$

and

suppose that $\int f_n \to \int f < \infty$

Then

for each mble set A

$$\int_A f_n \to \int_A f$$

i.e

$$\lim_n \int_A f_n = \int_A f$$

Proof: Put

$$g_n = f_n \chi_A$$

i.e

$g_n(x) = f_n(x) \quad$ if $x \in A$

and

$g_n(x) = 0 \quad$ if $x \notin A$

It is obvious that g_n is non-negative mble function $\forall\, n$.

Now $\lim\limits_{n} g_n = \lim\limits_{n}(f_n \chi_A)$

$$= (\lim_{n} f_n)\chi_A$$

$$\lim_{n} g_n = f\chi_A \quad (a.e) \qquad \because \lim_{n} f_n = f \; (a.e)$$

or

$$\lim_{n} g_n = \int \chi_A f = g \text{ (say)} \quad (a.e)$$

i.e $\quad \lim\limits_{n} g_n = g \quad (a.e)$

each $g_n \leq f_n$
show that

$$g \leq f \quad (a.e)$$

which further shows that

$$\int g \leq \int f$$

Moreover

$$0 \leq g_n \leq f_n \quad \forall\, n$$

$$\Rightarrow \quad \int g_n \leq \int f_n \quad \forall\, n$$

Also $\{f_n - g_n\}$ being sequence of non-negative mble functions, shows that

$$\lim_{n}(f_n - g_n) = \lim_{n} f_n - \lim_{n} g_n$$

$$= (f - g) \quad (a.e)$$

using Fatou's Lemma for $\{g_n\}$ we have

$$\int g \leq \underline{\lim} \int g_n \qquad \text{—— (1)}$$

Also $g = f\chi_A$

and

$g_n = f_n \chi_A$

Show that

$$\int g = \int_A f$$

and

$$\int g_n = \int_A f_n \quad \text{--- } (1')$$

So (1) can be write as

$$\int_A f \le \underline{\lim} \int_A f_n \quad \text{--- } (2)$$

using again (1) for $\{f_n - g_n\}$ we have

$$\int_A (f - g) \le \underline{\lim} \int_A (f_n - g_n)$$

$$\int f - \int g \le \underline{\lim} \int f_n - \overline{\lim} \int g_n$$

Now $f \geq f\chi_A$

$$\Rightarrow \int f \geq \int_A f$$

$$\Rightarrow \int f - \int g \geq \int_A f - \int g$$

$$\Rightarrow \int_A f - \int g \le \int f - \int g \le \underline{\lim} \int f_n - \overline{\lim} \int g_n$$

or $\Rightarrow$

$$\int_A f - \int g \le \underline{\lim} \int f_n - \overline{\lim} \int g_n$$

or $0 \le \underline{\lim} \int f_n - \overline{\lim} \int g_n$

$$= \underline{\lim} \int f_n - \overline{\lim} \int_A f_n$$

by (1')

$$\Rightarrow \int_A f \geq \overline{\lim} \int f_n \quad \text{--- } (3)$$

From (2) & (3)

$$\int_A f = \lim_n \int f_n$$

which required.

(64)

SECTION – II

(using definition of a limit of sequence)

2 From the definition of convergence of sequences of functions, it can be observed that, uniform convergence $\Rightarrow$ pointwise convergence $\Rightarrow$ convergence (a.e)

★ ~~fortunately~~ the converse of above is not true in general

3 Uniform convergence provides convergence in measure as follows

Let $\{f_n\}$ converges uniformly to f. Then for $\varepsilon > 0$ and sufficiently large n,

$$A_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\} = \{\ \}$$

or $m(A_n) = 0$ for sufficiently large n.

$\lim_n m(A_n) = 0$. Hence by defn $\{f_n\}$ ~~$\to f$~~ ~~it~~ is convergence in measure to f.

4 Pointwise convergence and consequently convergence almost everywhere, of seq. of mble function, need not imply the convergence in measure. As follows.

Let $\{f_n\}$ be a seq of mble functions where $f_n : \mathbb{R} \to (0,1)$ defined as

$$f_n(x) = \begin{cases} 1 & , x \in (n, n+1) \\ 0 & \text{otherwise} \end{cases}$$

i.e. $f_n = \chi_{(n, n+1)}$

Convergence in measure.

If we are given a sequence of real-valued function $\{f_n\}$ then the convergence of the seq $\{f_n\}$ to a function f can be interpretted in the following ways.

(i) $\{f_n\}$ converges uniformly to f.
(ii) $\{f_n\}$ converges point wise to f.
(iii) $\{f_n\}$ converges (a.e.) to f.

There is still another way to define the convergence of a sequence of functions in measure space which is very important and is frequently used in the theory of probability.

This convergence is referred to as convergence in measure or convergence in probability.

A ~~function~~ seq of $\{f_n\}$ of mble functions defined on set E is said to converge in measure to a mble function f. if for each real no $\epsilon > 0$

$$m(\{x : E : |f_n(x) - f(x)| \geq \epsilon) \to 0.$$

Observation

A sequence $\{f_n\}$ of mble functions converges in measure to a mble function f iff given $\epsilon > 0$ $\exists$ a +ve integer N s.t $\forall n \geq N$, $m(\{x : E : |f_n(x) - f(x)| \geq \delta\}) < \epsilon$

It is clear that if $n \to \infty$
$f_n(x) = 0 \quad \forall x \in R$, as in that case no element is contained in $(n, n+1]$ (which will become $(\infty, \infty]$)

Thus $f_n \to 0$ pointwise
and consequently $f_n \to 0$ (a.e) on R.
Now for $\varepsilon > 0$

$$\lim m(\{x \in R : |f_n(x) - 0| \geq \varepsilon)$$

$$= \lim_{n\to\infty} m(\{x \in R : |f_n(x)| \geq \varepsilon).$$

$$= \lim_{n\to\infty} m(n, n+1]$$

$$= \lim_{n\to\infty} (1) = 1$$

This shows that f_n does not converges to zero in measure. (While it converges to zero pointwise
Similarly it can be shown that if $\{f_n\}$ is χ i.e

$$f_n = \chi_{[n, \infty)}$$

Then $\{f_n\}$ converges pointwise to zero but does not converges in measure to zero
☆ (On the next Page).

Convergence in measure neither implies pointwise convergence nor convergence in measure.

Let $\{f_n\}$ be a seq of mble functions where

$$f_n : [0, 1) \to \{0, 1\}, \text{ defined as}$$

$$f_n = \chi_{(k2^{-j}, (k+1)2^{-j})} \quad 0 \leq k \leq 2^j \text{ and } n = k + 2^j$$

for some integer k & j.

For example $n = 1 = 0 + 2^0$
and our interval becomes $\left(0, \frac{1}{1}\right)$

$n = 2 = 0 + 2^1 \Rightarrow$ $\left[0, \frac{1}{2}\right]$

$n = 3 = 1 + 2^1 \Rightarrow$ $\left(\frac{1}{2}, 1\right]$

$n = 4 = 0 + 2^2 \Rightarrow$ $\left[0, \frac{1}{4}\right)$

$n = 5 = 1 + 2^2$ $\left[\frac{1}{4}, \frac{1}{2}\right)$

$n = 6 = \cancel{N} 2 + 2^2$ $\left[\frac{1}{2}, \frac{3}{4}\right)$

$n = 7 =$ $\left[\frac{3}{4}, 1\right]$

(★) $f_n \Rightarrow 0$ as $n \to \infty$
Since $\chi_{[\infty, \infty]} = \chi_{\{\}} = 0$

Here also $f_n \longrightarrow 0$ pointwise because for each f_n if we exclude ~~this~~ set $(R - [n, \infty))$ $[n, \infty)$ from R the remaining set of terms are mapped to zero under f_n i.e they converge to zero under f_n. Such a set of terms can be found for each n hence we say that $f_n \to 0$ pointwise.
and consequently $f_n \to 0$ (a.e). on R

Now for $\varepsilon > 0$

$$\lim m\left(x \in R : |f_n(x) - 0| \geq \varepsilon\right)$$

$$= \lim m\left(x \in R : |f_n(x)| \geq \varepsilon\right)$$

$$= \lim_{n \to \infty} m\,[n, \infty] = \cancel{m(\ldots)} \infty$$
$$= \infty$$

i.e which not zero.

$f_n : [0,1) \to \{0,1\}$

$f_n = \chi_{[k2^{-j},(k+1)2^{-j})}$

Obviously $2^{-j} \leq \frac{2}{n}$

Moreover $m(\{x : |f_n(x) - 0| \geq \varepsilon\}) = 2^{-j}$

$\leq \frac{2}{n}$

$\lim_n m(\{x : |f_n(x) - 0| \geq \varepsilon\}) \leq \lim_n \frac{2}{n}$

$= 0$

$\Rightarrow$ f_n converges to zero in measure. It can be observed that $\{f_n\}$ doesn't converge to any point $x \in [0,1)$ as there are infinite many pts for which f_n is zero and there are still infinite many pts for which f_n is 1.

However the following proposition provides an important relation between convergence in measure and convergence (a.e).

Proposition

Let $\{f_n\}$ be a sequence of mble functions defined on a set E of finite measure and sequence $\{f_n\}$ converges (a.e) to a mble function f. Then $\{f_n\}$ converges to f in measure.

Proof $\Rightarrow$ For any $\varepsilon > 0$, define

$A_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$

(P T O)

and
$$B_n = \bigcup_{k=n}^{\infty} A_k \longrightarrow (1)$$

Obviously the sequence $\{B_n\}$ is decreasing seq and $\bigcap_n B_n \subseteq A$ where

$$A = \{x \in E : f_n(x) \not\to f(x)\}.$$

As $f_n \to f$ (a.e) given Therefore
$$m(A) = 0$$

Thus
$$m(\bigcap_n B_n) \leq m(A)$$
$$m(\bigcap_n B_n) \leq 0$$
$$\Rightarrow m(\bigcap_n B_n) = 0 \quad \text{because } m(\bigcap_n B_n) \geq 0$$

$$\lim_n m(B_n) = 0$$

$\boxed{\text{If } E_n \text{ is decreasing seq with } m(E_1) < \infty \quad m(\bigcap_i E_i) = 0 \quad \lim_n = m(E_n).}$

(1) $\Rightarrow$ As $A_n \subseteq B_n$
so
$$m(A_n) \leq m(B_n)$$

Taking limit
$$\lim_n m(A_n) \leq \lim_n m(B_n)$$
$$\lim_n m(A_n) \leq 0$$
but
$$\lim_n m(A_n) \geq 0$$

Hence $\lim_n m(A_n) = 0$
Which fulfill the condition of convergence

in measure.

Proposition

Let $\{f_n\}$ be a sequence of mble functions which converges in measure, to f. Then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges to f (a.e).

Proof

The convergence of $\{f_n\}$ to f in measure, gives for positive $\varepsilon > 0$ and $n \geq N$ for some integer N.

$$m(\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon$$

(Using the definition of limit of seq) i.e

$$\left(\lim m(\{x : E : |f_n(x) - f(x)| \geq \varepsilon) \longrightarrow 0 \right)$$
$$\lim_{n\to\infty} m = 0$$

construct a subsequence $\{f_{n_k}\}$ of $\{f_n\}$

s.t $m\left(\left\{x \in E : |f_{n_k}(x) - f(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$

, $k = 1, 2, 3$

Let

$$A_k = \left\{x \in E : |f_{n_k}(x) - f(x)| \geq \frac{1}{2^k}\right\}, k = 1, 2, 3$$

$$B_j = \bigcup_{k=j}^{\infty} A_k$$

and $A = \bigcap_j B_j$

Obviously $\{B_j\}$ is decreasing and

$A \subseteq B_j \quad \forall j$

If $x \notin A$ then $\exists$ a +ve integer j, $x \notin B_j$ and so

$|f_{n_k}(x) - f(x)| < \frac{1}{2^k}$ holds for $k = j, j+1, j+2$

which shows that

$\{f_{n_k}\}$ converges to f at each point

$x \notin A$ Since $A \subseteq B_j, \forall j$

Therefore $m(A) \leq m(\bigcup_{k=j} A_k)$

$$\leq \sum_{k=j} m(A_k)$$

$$< \sum_{k=j} \frac{1}{2^k}$$

$\Rightarrow m(A) = 0$

Hence

$\{f_{n_k}\}$ converges to f (a.e.)