Unit 8 Logic

EXERCISE 8

1. Four options are given against each statement. Encircle the correct option.

(i) Which of the following expressions is often related to inductive reasoning?
(a) based on repeated experiments (b) if and only if statements
(c) Statement is proven by a theorem (d) based on general principles

(ii) Which of the following sentences describe deductive reasoning?
(a) general conclusions from a limited number of observations
(b) based on repeated experiments
(c) based on units of information that are accurate
(d) draw conclusion from well-known facts

(iii) Which one of the following statements is true?
(a) The set of integers is finite
(b) The sum of the interior angles of any quadrilateral is always 180°
(c) $\frac{22}{7} \notin Q'$
(d) All isosceles triangles are equilateral triangles

(iv) Which of the following statements is the best to represent the negation of the statement "The stove is burning"?
(a) the stove is not burning.
(b) the stove is dim
(c) the stove is turned to low heat
(d) it is both burning and not burning.

(v) The conjunction of two statements p and q is true when:
(a) both p and q are false. (b) both p and q are true.
(c) only q is true. (d) only p is true

(vi) A conditional is regarded as false only when:

(a) antecedent is true and consequent is false.
(b) consequent is true and antecedent is false.
(c) antecedent is true only.
(d) consequent is false only.

(vii) Contrapositive of $q \to p$ is

(a) $q \to \sim p$ (b) $\sim q \to p$ (c) $\sim p \to \sim q$ (d) $\sim q \to \sim p$

(viii) The statement "Every integer greater than 2 is a sum of two prime numbers" is:

(a) theorem (b) conjecture (c) axiom (d) postulates

(ix) The statement "A straight line can be drawn between any two points" is :

(a) theorem (b) conjecture (c) axiom (d) logic

(x) The statement "The sum of the interior angle of a triangle is 180°" is:

(a) converse (b) theorem (c) axiom (d) conditional

2. Write the converse, inverse and contrapositive of the following conditionals:

(i) $\sim p \to q$ (ii) $q \to p$ (iii) $\sim p \to \sim q$ (iv) $\sim q \to \sim p$

Solution

Conditional	**Converse**	**Inverse**	**Contra Positive**
~p → q	q → ~p	p → ~q	~q → p
q → p	p → q	~q → ~p	~p → ~q
~p → ~q	~q → ~p	p → q	q → p
~q → ~p	~p → ~q	q → p	p → q

3. Write the truth table of the following

(i) $\sim(p \vee q) \vee (\sim q)$ (ii) $\sim(\sim q \vee \sim p)$ (iii) $(p \vee q) \leftrightarrow (p \wedge q)$

Solution

3(i) $\sim(\mathbf{p} \vee \mathbf{q}) \vee (\sim\mathbf{q})$

p	q	~q	p ∨ q	~(p ∨ q)	~(p ∨ q) ∨ (~q)
T	T	F	T	F	F
T	F	T	T	F	T
F	T	F	T	F	F
F	F	T	F	T	T

3(ii) $\sim(\sim \mathbf{q} \vee \sim \mathbf{p})$

p	q	~p	~q	(~q ∨ ~p)	~(~q ∨ ~p)
T	T	F	F	F	T
T	F	*F*	T	T	*F*
F	T	*T*	F	T	F
F	F	T	T	T	F

3(iii) $(\mathbf{p} \vee \mathbf{q}) \leftrightarrow (\mathbf{p} \wedge \mathbf{q})$

p	q	(p ∨ q)	(p ∧ q)	(p ∨ q) ↔ (p ∧ q)
T	T	T	T	T
T	F	T	F	*F*
F	T	T	F	F
F	F	F	F	T

4. Differentiate between a mathematical statement and its proof. Given two examples.

Solution

A **"mathematical statement"** is simply a declarative sentence that may be either true or false but not both, while a **"mathematical proof"** is a logical argument that demonstrates the truth of a mathematical statement using established axioms and theorems, effectively showing why a statement is true; essentially, a statement is the claim itself, and a proof is the process of verifying that claim is true.

Examples of Mathematical Statement:

$\frac{22}{7} \notin Q'$; $Q \subseteq R$ these are true statements

$3 + 4 = 8$; $Z \subseteq W$ these are false statements

Examples of Mathematical Proof:

If x is an odd integer, then x^2 is also an odd integer

The sum of two odd numbers is an even number

5. What is the difference between an axiom and a theorem? Give examples of each.

Solution: Theorem:

A **theorem** is a mathematical statement that has been proved true based on previously known facts.

Example of Theorems:

Theorem: The sum of the interior angles of a quadrilateral is 360 degrees.

The Fundamental Theorem of Arithmetic: Every integer greater than 1 can be uniquely expressed as a product of prime numbers up to the order of the factors.

Fermat's Last Theorem: There are no three positive integers a, b, c, which satisfy the equation $a^n + b^n = c^n$, where $n \in N$ and $n > 2$

Axiom:

An **axiom** is a mathematical statement that we believe to be true without any evidence or requiring any proof.

Example Axioms:

Axiom: Through a given point, infinitely many lines can pass.

Euclid Axioms: A straight line can be drawn between any two points.

Peano Axioms: Every natural number has a successor, which is also a natural number.

Axiom of Extensionality: Two sets are equal if they have the same elements.

Axiom of Power Set: Any set has a set of all its subsets.

6. What is the importance of logical reasoning in mathematical proofs? Give an example to illustrate your point.

Solution

Logic is a systematic method of reasoning that enable you to interpret the meaning of statement, examine the truth of statements, and deduce new information from existing facts. Logic play a key role in problem solving and decision making. We generally use logic in our daily life and certainly while engaging in mathematics. For example, we often draw general conclusions from a limited number of observations or experiences. A person gets penicillin injection once or twice and experiences reaction soon afterwards. He generalizes that he is allergic to penicillin.

7. Indicate whether it is an axiom, conjecture or theorem and explain your reasoning.
   (i) There is exactly one straight line through any two points.
   (ii) Every even number greater than 2 can be written as the sum of two prime numbers."
   (iii) The sum of the angles in a triangle is 180°.

Solution

7(i) There is exactly one straight line through any two points.

This statement is a **Euclidean Axiom**. And it is believe to be true without any evidence or requiring any proof.

7(ii) Every even number greater than 2 can be written as the sum of two prime numbers.

This statement is a **Conjecture**, specifically known as **Goldbach's Conjecture**. And it has not been formally proven or disproven.

7(iii) The sum of angles in a triangle is 180°.

This statement is a **Theorem**. And it has been formally proven using established axioms and definitions of geometry.

8. Formulate simple deductive proofs for each of the following algebraic expressions, prove that the L.H.S is equal to the R.H.S:
   (i) prove that $(x-4)^2+9=x^2-8x+25$
   (ii) prove that $(x+1)^2-(x-1)^2=4x$
   (iii) prove that $(x+5)^2-(x-5)^2=20x$

Solution

Deductive Proof: L.H.S. = ……… (solve)

Conclusion: the L.H.S. is exactly same as the R.H.S.

8(i) $L.H.S. = (x-4)^2+9 = x^2-8x+16+9 = x^2-8x+25 = R.H.S.$

8(ii) $L.H.S. = (x+1)^2-(x-1)^2 = (x^2+2x+1)-(x^2-2x+1)$

$= x^2+2x+1-x^2+2x-1 = 4x = R.H.S.$

8(iii) $L.H.S. = (x+5)^2-(x-5)^2 = (x^2+10x+25)-(x^2-10x+25)$

$= x^2+10x+25-x^2+10x-25 = 20x = R.H.S.$

9. Prove the following by justifying each step:

(i) $\dfrac{4+16x}{4}=1+4x$ (ii) $\dfrac{6x^2+18x}{3x^2-27}=\dfrac{2x}{x-3}$

(iii) $\dfrac{x^2+7x+10}{x^2-3x-10}=\dfrac{x+5}{x-5}$

Solution

(i) $\boldsymbol{\frac{4+16x}{4}=1+4x}$

$\frac{4+16x}{4}=\frac{1}{4}\times(4+16x)$ $\because \frac{a}{b}=\frac{1}{b}\times a$

$=\frac{1}{4}\times(4\times 1+4\times 4x)$ $\because$ multiplicative identity

$=\frac{1}{4}\times 4\times(1+4x)$ $\because$ distributive Law

$=\left(\frac{1}{4}\times 4\right)\times(1+4x)$ $\because$ associative Law

$=1\times(1+4x)$ $\because$ multiplicative inverse

$=1+4x$ $\because$ multiplicative identity

(ii) $\boldsymbol{\frac{6x^2+18x}{3x^2-27}=\frac{2x}{x-3}}$

$\frac{6x^2+18x}{3x^2-27}=\frac{6x(x+3)}{3(x^2-9)}$ $\because$ left distributive property

$=\frac{2x(x+3)}{(x-3)(x+3)}$ $\because$ Factorization

$=\frac{2x}{x-3}$ $\because$ cancellation property

(iii) $\boldsymbol{\frac{x^2+7x+10}{x^2-3x-10}=\frac{x+5}{x-5}}$

$\frac{x^2+7x+10}{x^2-3x-10}=\frac{(x+2)(x+5)}{(x+2)(x-5)}$ $\because$ Factorization

$=\frac{x+5}{x-5}$ $\because$ cancellation property

10. Suppose x is an integer. Then x is odd if and only if $9x + 4$ is odd.

Solution

If x is odd, then 9x + 4 is odd

Let x be an odd integer. Then, x = 2k + 1 for some integer k.

9x + 4 = 9(2k + 1) + 4= 18k + 9 + 4= 18k + 13= 2(9k + 6) + 1

Since 9k + 6 is an integer, 2(9k + 6) + 1 is odd. Therefore, 9x + 4 is odd.

If 9x + 4 is odd, then x is odd

Let 9x + 4 be odd. Then, 9x + 4 = 2m + 1 for some integer m.

9x = 2m - 3

x = (2m - 3)/9

Since 2m - 3 is odd, (2m - 3)/9 is either an integer or a non-integer.

If x is a non-integer, then 9x + 4 is not an integer, which contradicts the assumption. Therefore, x must be an integer.

x = (2m - 3)/9 = 2((m - 1)/9) - 1/3 (since m is odd, m - 1 is even)

Since (m - 1)/9 is an integer, x is odd.

Therefore, if x is an integer, then x is odd if and only if 9x + 4 is odd.

11. Suppose x is an integer. If x is odd, then $7x + 5$ is even.

Solution

Let x be an odd integer. Then, x = 2k + 1 for some integer k.

7x + 5 = 7(2k + 1) + 5 = 14k + 7 + 5 = 14k + 12 = 2(7k + 6)

Since 7k + 6 is an integer, 2(7k + 6) is even.

Therefore, 7x + 5 is even.

12. Prove the following statements

(a) If x is an odd integer, then show that it $x^2 - 4x + 6$ is odd.

(b) If x is an even integer then show that $x^2 + 2x + 4$ is even.

Solution

12(a) Let x be an odd integer. Then, $x = 2k + 1$ for some integer k.

$x^2 - 4x + 6 = (2k + 1)^2 - 4(2k + 1) + 6$

$= 4k^2 + 4k + 1 - 8k - 4 + 6 = 4k^2 - 4k + 3 = 4k(k - 1) + 3$ **odd integer**

Therefore, $x^2 - 4x + 6$ is odd.

12(b) Let x be an even integer. Then, $x = 2k$ for some integer k.

$x^2 + 2x + 4 = (2k)^2 + 2(2k) + 4 = 4k^2 + 4k + 4 = 4(k^2 + k + 1)$

Since $k^2 + k + 1$ is an integer, $4(k^2 + k + 1)$ is even. Therefore, $x^2 + 2x + 4$ is even.

13. Prove that for any two non-empty sets A and B, $(A \cap B)' = A' \cup B'$.

Solution

Let $x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B) \Rightarrow x \notin A$ or $x \notin B \Rightarrow x \in A'$ or $x \in B' \Rightarrow x \in A' \cup B'$

$\Rightarrow (A \cap B)' \subseteq A' \cup B'$…………….. (i)

Again, let $y \in A' \cup B'$

$\Rightarrow y \in A'$ or $y \in B' \Rightarrow y \notin A$ or $y \notin B \Rightarrow y \notin (A \cap B) \Rightarrow y \in (A \cap B)'$

$\Rightarrow A' \cup B' \subseteq (A \cap B)'$…………….. (ii)

From (i) and (ii) we have

$(A \cap B)' = A' \cup B'$

14. If x and y are positive real numbers and $x^2 < y^2$ then $x < y$.

Solution

Given: $x^2 < y^2$

$\sqrt{x^2} < \sqrt{y^2}$

$x < y$

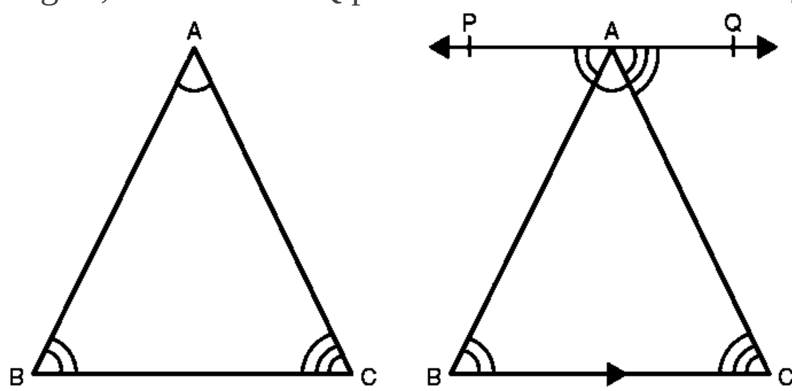
Therefore, if $x^2 < y^2$, then $x < y$.

Note: This proof relies on the fact that the square root function is monotonically increasing for positive real numbers.

15. The sum of the interior angles of a triangle is 180°

Solution

Consider a ΔABC, as shown in the figure below. To prove the above property of triangles, draw a line PQ parallel to the side BC of the given triangle.



Since PQ is a straight line, it can be concluded that:

$\angle PAB + \angle BAC + \angle QAC = 180°$ ………(1)

Since PQ||BC and AB, AC are transversals,

Therefore, $\angle QAC = \angle ACB$ (a pair of alternate angle)

Also, $\angle PAB = \angle CBA$ (a pair of alternate angle)

Substituting the value of $\angle QAC$ and $\angle PAB$ in equation (1),

$\angle ACB + \angle BAC + \angle CBA = 180°$

Thus, the sum of the interior angles of a triangle is 180°.

16. If a, b and c are non-zero real numbers, prove that:

(a) $\frac{a}{b}=\frac{c}{d}\Leftrightarrow ad=bc$ (b) $\frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}$ (c) $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$

Solution

(a) $\boldsymbol{\frac{a}{b}=\frac{c}{d}\Leftrightarrow ad=bc}$

$\frac{a}{b}=\frac{c}{d}$

$\frac{a}{b}\times bd=\frac{c}{d}\times bd$ ∵ left multiplication

$\frac{b}{b}\times ad=\frac{d}{d}\times bc$ ∵ associative property

$1\times ad=1\times bc$ ∵ identity

$ad=bc$

(b) $\boldsymbol{\frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}}$

$\frac{a}{b}.\frac{c}{d}$

$=a\times\frac{1}{b}.c\times\frac{1}{d}$

$=a\times\left(\frac{1}{b}.c\right)\times\frac{1}{d}$ ∵ associative property

$=a\times\left(c.\frac{1}{b}\right)\times\frac{1}{d}$ ∵ commutative property

$=ac\times\frac{1}{bd}$ ∵ associative property

$=\frac{ac}{bd}$ ∵ multiplication property

(c) $\boldsymbol{\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}}$

$\frac{a}{b}+\frac{c}{b}$

$=a\times\frac{1}{b}+c\times\frac{1}{b}$ ∵ $\frac{a}{b}=a\times\frac{1}{b}$

$=(a+c)\times\frac{1}{b}$ ∵ distributive property

$=\frac{a+c}{b}$ ∵ multiplication property