MATHEMATICS – 9
PTB - SYLLABUS

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Unit 1 Real Numbers

EXERCISE 1.1

1. Identify each of the following as a rational or irrational number:

(i) 2.353535 (ii) $0.\overline{6}$ (iii) $2.236067...$ (iv) $\sqrt{7}$

(v) e (vi) π (vii) $5+\sqrt{11}$ (viii) $\sqrt{3}+\sqrt{13}$

(ix) $\frac{15}{4}$ (x) $(2-\sqrt{2})(2+\sqrt{2})$

Solution

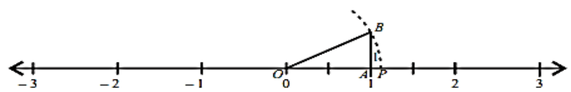
(i) Rational	(ii) Rational	(iii) Irrational	(iv) Irrational	(v) Irrational
(vi) Irrational	(vii) Irrational	(viii) Irrational	(ix) Rational	(x) rational

2. Represent the following numbers on number line:

(i) $\sqrt{2}$ (ii) $\sqrt{3}$ (iii) $4\frac{1}{3}$

(iv) $-2\frac{1}{7}$ (v) $\frac{5}{8}$ (vi) $2\frac{3}{4}$

Solution

(i)


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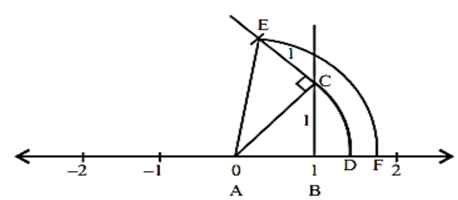

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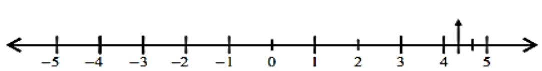

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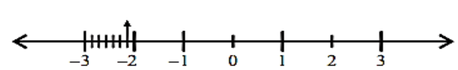

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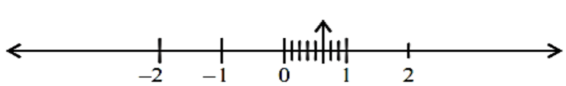
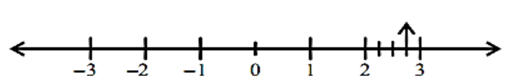


(vi)


3. Express the following as a rational number $\frac{p}{q}$ where p and q are integers and $q \neq 0$:

(i) $0.\overline{4}$ (ii) $0.\overline{37}$ (iii) $0.\overline{21}$

$\boldsymbol{x = 0.\overline{4}}$ $x = 0.4444\ldots$ $10x = 10(0.4444\ldots)$ $10x = 4.4444\ldots$ $10x - x = (4.4444\ldots) - (0.4444\ldots)$ $9x = 4 \Rightarrow \boldsymbol{x = \frac{4}{9}}$	$\boldsymbol{x = 0.\overline{37}}$ $x = 0.3737\ldots$ $100x = 100(0.3737\ldots)$ $100x = 37.3737\ldots$ $100x - x = (37.3737\ldots) - (0.3737\ldots)$ $99x = 37 \Rightarrow \boldsymbol{x = \frac{37}{99}}$
$\boldsymbol{x = 0.\overline{21}}$ $x = 0.2121\ldots$ $100x = 100(0.2121\ldots)$ $100x = 21.2121\ldots$ $100x - x = (21.2121\ldots) - (0.2121\ldots)$ $99x = 21 \Rightarrow \boldsymbol{x = \frac{21}{99}}$	

4. Name the property used in the following:

(i) $(a+4)+b = a+(4+b)$ (ii) $\sqrt{2}+\sqrt{3} = \sqrt{3}+\sqrt{2}$

(iii) $x - x = 0$ (iv) $a(b+c) = ab+ac$

(v) $16+0 = 16$ (vi) $100 \times 1 = 100$

(vii) $4 \times (5 \times 8) = (4 \times 5) \times 8$ (viii) $ab = ba$

Solution

(i) Associative property over addition
(ii) Commutative property over addition
(iii) Additive inverse
(iv) Left distributive property **Of multiplication over +**
(v) Additive identity
(vi) Multiplicative identity
(vii) Associative property under multiplication
(viii) Commutative property under multiplication

5. Name the property used in the following:

(i) $-3 < -1 \Rightarrow 0 < 2$

(ii) If $a < b$ then $\frac{1}{a} > \frac{1}{b}$

(iii) If $a < b$ then $a + c < b + c$

(iv) If $ac < bc$ and $c > 0$ then $a < b$

(v) If $ac < bc$ and $c < 0$ then $a > b$

(vi) Either $a > b$ or $a = b$ or $a < b$

Solution

(i) Additive property
(ii) Reciprocal property
(iii) Additive property
(iv) Multiplicative property
(v) Multiplicative property
(vi) Trichotomy property

6. Insert two rational numbers between:

(i) $\frac{1}{3}$ and $\frac{1}{4}$ (ii) 3 and 4 (iii) $\frac{3}{5}$ and $\frac{4}{5}$

Solution

i. $q_1 = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right) = \frac{1}{2}\left(\frac{7}{12}\right) = \frac{7}{24}$ and $q_2 = \frac{1}{2}\left(\frac{7}{24} + \frac{1}{4}\right) = \frac{1}{2}\left(\frac{13}{24}\right) = \frac{13}{48}$

Hence required rational are $\mathbf{\frac{7}{24}, \frac{13}{48}}$

ii. $q_1 = \frac{1}{2}(3 + 4) = \frac{7}{2}$ and $q_2 = \frac{1}{2}\left(\frac{7}{2} + 4\right) = \frac{1}{2}\left(\frac{15}{2}\right) = \frac{15}{4}$

Hence required rational are $\mathbf{\frac{7}{2}, \frac{15}{4}}$

iii. $q_1 = \frac{1}{2}\left(\frac{3}{5} + \frac{4}{5}\right) = \frac{1}{2}\left(\frac{7}{5}\right) = \frac{7}{10}$ and $q_2 = \frac{1}{2}\left(\frac{7}{10} + \frac{4}{5}\right) = \frac{1}{2}\left(\frac{15}{10}\right) = \frac{3}{4}$

Hence required rational are $\mathbf{\frac{7}{10}, \frac{3}{4}}$

EXERCISE 1.2

1. Rationalize the denominator of following:

(i) $\frac{13}{4+\sqrt{3}}$ (ii) $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{3}}$ (iii) $\frac{\sqrt{2}-1}{\sqrt{5}}$

(iv) $\frac{6-4\sqrt{2}}{6+4\sqrt{2}}$ (v) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ (vi) $\frac{4\sqrt{3}}{\sqrt{7}+\sqrt{5}}$

Solution

i. $\frac{\mathbf{13}}{\mathbf{4+\sqrt{3}}} = \frac{13}{4+\sqrt{3}} \times \frac{4-\sqrt{3}}{4-\sqrt{3}} = \frac{13(4-\sqrt{3})}{(4)^2-(\sqrt{3})^2} = \frac{13(4-\sqrt{3})}{16-3} = \frac{13(4-\sqrt{3})}{13} = \mathbf{4-\sqrt{3}}$

ii. $\frac{\mathbf{\sqrt{2}+\sqrt{5}}}{\mathbf{\sqrt{3}}} = \frac{\sqrt{2}+\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{(\sqrt{2}+\sqrt{5})\sqrt{3}}{\sqrt{3}.\sqrt{3}} = \frac{\sqrt{2}.\sqrt{3}+\sqrt{5}.\sqrt{3}}{3} = \frac{\mathbf{\sqrt{6}+\sqrt{15}}}{\mathbf{3}}$

iii. $\frac{\mathbf{\sqrt{2}-1}}{\mathbf{\sqrt{5}}} = \frac{\sqrt{2}-1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{(\sqrt{2}-1)\sqrt{5}}{\sqrt{5}.\sqrt{5}} = \frac{\sqrt{2}.\sqrt{5}-1.\sqrt{5}}{5} = \frac{\mathbf{\sqrt{10}-\sqrt{5}}}{\mathbf{5}}$

iv. $\frac{\mathbf{6-4\sqrt{2}}}{\mathbf{6+4\sqrt{2}}} = \frac{6-4\sqrt{2}}{6+4\sqrt{2}} \times \frac{6-4\sqrt{2}}{6-4\sqrt{2}} = \frac{(6-4\sqrt{2})^2}{(6)^2-(4\sqrt{2})^2} = \frac{(6)^2+(4\sqrt{2})^2-2(6)(4\sqrt{2})}{36-16(2)}$

$= \frac{36+32-48\sqrt{2}}{36-32} = \frac{68-48\sqrt{2}}{4} = \mathbf{17-12\sqrt{2}}$

v. $\frac{\mathbf{\sqrt{3}-\sqrt{2}}}{\mathbf{\sqrt{3}+\sqrt{2}}} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{(\sqrt{3}-\sqrt{2})^2}{(\sqrt{3})^2-(\sqrt{2})^2} = \frac{(\sqrt{3})^2+(\sqrt{2})^2-2(\sqrt{3})(\sqrt{2})}{3-2}$

$= \frac{3+2-2\sqrt{6}}{1} = \mathbf{5-2\sqrt{6}}$

vi. $\frac{\mathbf{4\sqrt{3}}}{\mathbf{\sqrt{7}+\sqrt{5}}} = \frac{4\sqrt{3}}{\sqrt{7}+\sqrt{5}} \times \frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}-\sqrt{5}} = \frac{4\sqrt{3}(\sqrt{7}-\sqrt{5})}{(\sqrt{7})^2-(\sqrt{5})^2} = \frac{4\sqrt{3}(\sqrt{7}-\sqrt{5})}{7-5}$

$= \frac{4\sqrt{3}(\sqrt{7}-\sqrt{5})}{2} = \mathbf{2\sqrt{3}}(\sqrt{7}-\sqrt{5})$

2. Simplify the following:

(i) $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$ (ii) $\left(\frac{3}{4}\right)^{-2} \div \left(\frac{4}{9}\right)^{3} \times \frac{16}{27}$ (iii) $(0.027)^{-\frac{1}{3}}$

(iv) $\sqrt[7]{\frac{x^{14} \times y^{21} \times z^{35}}{y^{14} z^{7}}}$ (v) $\frac{5\cdot(25)^{n+1} - 25\cdot(5)^{2n}}{5\cdot(5)^{2n+3} - (25)^{n+1}}$

(vi) $\frac{(16)^{x+1} + 20(4^{2x})}{2^{x-3} \times 8^{x+2}}$ (vii) $(64)^{-\frac{2}{3}} \div (9)^{-\frac{3}{2}}$

(viii) $\frac{3^{n} \times 9^{n+1}}{3^{n-1} \times 9^{n-1}}$ (ix) $\frac{5^{n+3} - 6.5^{n+1}}{9 \times 5^{n} - 4 \times 5^{n}}$

Solution

i. $\left(\frac{81}{16}\right)^{-\frac{3}{4}} = \left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2^4}{3^4}\right)^{\frac{3}{4}} = \frac{2^{4\times\frac{3}{4}}}{3^{4\times\frac{3}{4}}} = \frac{2^3}{3^3} = \mathbf{\frac{8}{27}}$

ii. $\left(\frac{3}{4}\right)^{-2} \div \left(\frac{4}{9}\right)^{3} \times \frac{16}{27} = \left(\frac{4}{3}\right)^{2} \div \left(\frac{4}{9}\right)^{3} \times \frac{16}{27} = \frac{4^2}{3^2} \times \frac{9^3}{4^3} \times \frac{16}{27} = \frac{16\times729\times16}{9\times64\times27} = \mathbf{12}$

iii. $(0.027)^{-\frac{1}{3}} = \left(\frac{27}{1000}\right)^{-\frac{1}{3}} = \left(\frac{1000}{27}\right)^{\frac{1}{3}} = \left(\frac{10^3}{3^3}\right)^{\frac{1}{3}} = \frac{10^{3\times\frac{1}{3}}}{3^{3\times\frac{1}{3}}} = \mathbf{\frac{10}{3}}$

iv. $\sqrt[7]{\frac{x^{14}\times y^{21}\times z^{35}}{y^{14}\times z^{7}}} = \left(\frac{x^{14}\times y^{21}\times z^{35}}{y^{14}\times z^{7}}\right)^{\frac{1}{7}} = (x^{14} \times y^{7} \times z^{28})^{\frac{1}{7}}$

$= x^{14\times\frac{1}{7}} \times y^{7\times\frac{1}{7}} \times z^{28\times\frac{1}{7}} = \boldsymbol{x^2yz^4}$

v. $\frac{5.(25)^{n+1}-25.(5)^{2n}}{5.(5)^{2n+3}-(25)^{n+1}} = \frac{5.(5^2)^{n+1}-5^2.(5)^{2n}}{5.(5)^{2n+3}-(5^2)^{n+1}} = \frac{5.5^{2n+2}-5^2.5^{2n}}{5.5^{2n+3}-5^{2n+2}} = \frac{5^{2n+3}-5^{2n+2}}{5^{2n+4}-5^{2n+2}}$

$= \frac{5^{2n+2}(5-1)}{5^{2n+2}(5^2-1)} = \frac{5-1}{25-1} = \frac{4}{24} = \mathbf{\frac{1}{6}}$

vi. $\frac{(16)^{x+1}+20(4^{2x})}{2^{x-3}\times8^{x+2}} = \frac{(2^4)^{x+1}+20.(2^2)^{2x}}{2^{x-3}\times(2^3)^{x+2}} = \frac{2^{4x+4}+20.2^{4x}}{2^{x-3}\times2^{3x+6}} = \frac{2^{4x+4}+20.2^{4x}}{2^{4x+3}}$

$= \frac{2^{4x}(2^4+20)}{2^{4x}.2^3} = \frac{(16+20)}{2^3} = \frac{36}{8} = \mathbf{\frac{9}{2}}$

vii. $(\mathbf{64})^{-\frac{2}{3}} \div (\mathbf{9})^{-\frac{3}{2}} = \frac{(64)^{-\frac{2}{3}}}{(9)^{-\frac{3}{2}}} = \frac{(9)^{\frac{3}{2}}}{(64)^{\frac{2}{3}}} = \frac{(3^2)^{\frac{3}{2}}}{(4^3)^{\frac{2}{3}}} = \frac{3^3}{4^2} = \mathbf{\frac{27}{16}}$

viii. $\frac{\mathbf{3^n \times 9^{n+1}}}{\mathbf{3^{n-1} \times 9^{n-1}}} = \frac{3^n \times (3^2)^{n+1}}{3^{n-1} \times (3^2)^{n-1}} = \frac{3^n \times 3^{2n+2}}{3^{n-1} \times 3^{2n-2}} = \frac{3^{3n+2}}{3^{3n-3}} = 3^{3n+2-3n+3} = 3^5 = \mathbf{243}$

ix. $\frac{\mathbf{5^{n+3} - 6.5^{n+1}}}{\mathbf{9 \times 5^n - 4 \times 5^n}}$ =???

$= \frac{5^n(5^3 - 6.5^1)}{5^n(9 - 2^2)}$

$= \frac{125 - 30}{9 - 4} = \frac{95}{5} = \mathbf{19}$

3. If $x = 3 + \sqrt{8}$ then find the value of:

(i) $x + \frac{1}{x}$ (ii) $x - \frac{1}{x}$ (iii) $x^2 + \frac{1}{x^2}$

(iv) $x^2 - \frac{1}{x^2}$ (v) $x^4 + \frac{1}{x^4}$ (vi) $\left(x - \frac{1}{x}\right)^2$

Solution

$x = 3 + \sqrt{8} \Rightarrow \frac{1}{x} = \frac{1}{3+\sqrt{8}} = \frac{1}{3+\sqrt{8}} \times \frac{3-\sqrt{8}}{3-\sqrt{8}} = \frac{3-\sqrt{8}}{(3)^2 - (\sqrt{8})^2} = \frac{3-\sqrt{8}}{9-8} = 3 - \sqrt{8}$

Hence $\boldsymbol{x} = \mathbf{3} + \sqrt{\mathbf{8}}$ and $\frac{\mathbf{1}}{\boldsymbol{x}} = \mathbf{3} - \sqrt{\mathbf{8}}$

i. $x + \frac{1}{x} = (3 + \sqrt{8}) + (3 - \sqrt{8}) = \mathbf{6}$

ii. $x - \frac{1}{x} = (3 + \sqrt{8}) - (3 - \sqrt{8}) = \mathbf{2\sqrt{8}}$

iii. $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = (6)^2 - 2 = 36 - 2 = \mathbf{34}$

iv. $x^2 - \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right) = (6)(2\sqrt{8}) = \mathbf{12\sqrt{8}}$

v. $x^4 + \frac{1}{x^4} = \left(x^2 + \frac{1}{x^2}\right)^2 - 2 = (34)^2 - 2 = 1156 - 2 = \mathbf{1154}$

vi. $\left(x - \frac{1}{x}\right)^2 = (2\sqrt{8})^2 = 4 \times 8 = \mathbf{32}$

4. Find the rational numbers p and q such that $\frac{8-3\sqrt{2}}{4+3\sqrt{2}} = p + q\sqrt{2}$

Solution

$\frac{8-3\sqrt{2}}{4+3\sqrt{2}} = p + q\sqrt{2}$

$\frac{8-3\sqrt{2}}{4+3\sqrt{2}} \times \frac{4-3\sqrt{2}}{4-3\sqrt{2}} = p + q\sqrt{2}$

$\frac{32-24\sqrt{2}-12\sqrt{2}+18}{(4)^2-(3\sqrt{2})^2} = p + q\sqrt{2}$

$\frac{50-36\sqrt{2}}{16-18} = p + q\sqrt{2}$

$\frac{50-36\sqrt{2}}{-2} = p + q\sqrt{2}$

$-25 + 18\sqrt{2} = p + q\sqrt{2}$

Hence $\boldsymbol{p = -25}$ and $\boldsymbol{q = 18}$

5. Simplify the following:

(i) $\frac{(25)^{\frac{3}{2}} \times (243)^{\frac{3}{5}}}{(16)^{\frac{5}{4}} \times (8)^{\frac{4}{3}}}$ (ii) $\frac{54 \times \sqrt[3]{(27)^{2x}}}{9^{x+1} + 216(3^{2x-1})}$

(iii) $\sqrt{\frac{(216)^{\frac{2}{3}} \times (25)^{\frac{1}{2}}}{(0.04)^{\frac{-3}{2}}}}$ (iv) $\left(a^{\frac{1}{3}} + b^{\frac{2}{3}}\right) \times \left(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}\right)$

Solution

i. $\dfrac{(\mathbf{25})^{\frac{3}{2}}\times(\mathbf{243})^{\frac{3}{5}}}{(\mathbf{16})^{\frac{5}{4}}\times(\mathbf{8})^{\frac{4}{3}}}=\dfrac{(5^2)^{\frac{3}{2}}\times(3^5)^{\frac{3}{5}}}{(2^4)^{\frac{5}{4}}\times(2^3)^{\frac{4}{3}}}=\dfrac{5^3\times3^3}{2^5\times2^4}=\dfrac{5^3\times3^3}{2^9}=\dfrac{125\times27}{512}=\dfrac{\mathbf{3375}}{\mathbf{512}}$

ii. $\dfrac{\mathbf{54}\times\sqrt[3]{(\mathbf{27})^{2x}}}{\mathbf{9}^{x+1}+\mathbf{216}(\mathbf{3}^{2x-1})}=\dfrac{54\times(27)^{\frac{2x}{3}}}{9^{x+1}+216(3^{2x-1})}=\dfrac{54\times(3^3)^{\frac{2x}{3}}}{(3^2)^{x+1}+216(3^{2x-1})}=\dfrac{54\times3^{2x}}{3^{2x+2}+216(3^{2x-1})}$

$=\dfrac{54\times3^{2x}}{3^{2x}\left(3^2+216(3^{-1})\right)}=\dfrac{54}{\left(3^2+\frac{216}{3}\right)}=\dfrac{54}{9+72}=\dfrac{54}{81}=\dfrac{\mathbf{2}}{\mathbf{3}}$

iii. $\sqrt{\dfrac{(\mathbf{216})^{\frac{2}{3}}\times(\mathbf{25})^{\frac{1}{2}}}{(\mathbf{0.04})^{-\frac{3}{2}}}}=\left(\dfrac{(6^3)^{\frac{2}{3}}\times(5^2)^{\frac{1}{2}}}{\left(\frac{4}{100}\right)^{-\frac{3}{2}}}\right)^{\frac{1}{2}}=\left(\dfrac{6^2\times5}{\left(\frac{100}{4}\right)^{\frac{3}{2}}}\right)^{\frac{1}{2}}=\left(\dfrac{6^2\times5}{(25)^{\frac{3}{2}}}\right)^{\frac{1}{2}}=\left(\dfrac{6^2\times5}{(5^2)^{\frac{3}{2}}}\right)^{\frac{1}{2}}$

$=\left(\dfrac{6^2\times5}{5^3}\right)^{\frac{1}{2}}=\left(\dfrac{6^2}{5^2}\right)^{\frac{1}{2}}=\dfrac{\mathbf{6}}{\mathbf{5}}$

iv. $\left(\boldsymbol{a}^{\frac{1}{3}}+\boldsymbol{b}^{\frac{2}{3}}\right)\times\left(\boldsymbol{a}^{\frac{2}{3}}-\boldsymbol{a}^{\frac{1}{3}}\boldsymbol{b}^{\frac{2}{3}}+\boldsymbol{b}^{\frac{4}{3}}\right)$

$=\left(a^{\frac{1}{3}}+b^{\frac{2}{3}}\right)\times\left(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}\right)$

$=\left(a^{\frac{1}{3}}a^{\frac{2}{3}}-a^{\frac{1}{3}}a^{\frac{1}{3}}b^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{4}{3}}+b^{\frac{2}{3}}a^{\frac{2}{3}}-b^{\frac{2}{3}}a^{\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{2}{3}}b^{\frac{4}{3}}\right)$

$=\left(a^{\frac{1}{3}+\frac{2}{3}}-a^{\frac{1}{3}+\frac{1}{3}}b^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}+\frac{2}{3}}+b^{\frac{2}{3}+\frac{4}{3}}\right)$

$=\left(a^{\frac{3}{3}}-\cancel{a^{\frac{2}{3}}b^{\frac{2}{3}}}+\cancel{a^{\frac{1}{3}}b^{\frac{4}{3}}}+\cancel{a^{\frac{2}{3}}b^{\frac{2}{3}}}-\cancel{a^{\frac{1}{3}}b^{\frac{4}{3}}}+b^{\frac{6}{3}}\right)$

$=\boldsymbol{a}+\boldsymbol{b}^{\mathbf{2}}$

EXERCISE 1.3

1. The sum of three consecutive integers is forty-two, find the three integers.

Solution

Consider three consecutive integers are x , $(x+1)$ and $(x+2)$

$(x)+(x+1)+(x+2)=42$

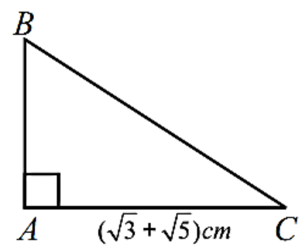
$3x+3=42$

$3x=39$

$x=13$

Hence the three consecutive integers are **13**, **14**, and **15**.

2. The diagram shows right angled ΔABC in which the length of $\overline{AC}$ is $(\sqrt{3}+\sqrt{5})$cm. The area of ΔABC is $(1+\sqrt{15})$ cm^2. Find the length $\overline{AB}$ in the form $(a\sqrt{3}+b\sqrt{5})$ cm, where a and b are integers.



Solution

Length of $\overline{AC}=\left(\sqrt{3}+\sqrt{5}\right)$ cm

Area of $\Delta ABC=\left(1+\sqrt{15}\right)$ cm^2

Area of $\Delta ABC=\frac{1}{2}\times$ base $\times$ height

$\left(1+\sqrt{15}\right)=\frac{1}{2}\times\left(\sqrt{3}+\sqrt{5}\right)\times\overline{AB}$

$\left(2+2\sqrt{15}\right)=\left(\sqrt{3}+\sqrt{5}\right)\times\overline{AB}$

$\overline{AB}=\frac{2+2\sqrt{15}}{\sqrt{3}+\sqrt{5}}=\frac{2+2\sqrt{15}}{\sqrt{3}+\sqrt{5}}\times\frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}}=\frac{2\sqrt{3}-2\sqrt{5}+2\sqrt{45}-2\sqrt{75}}{\left(\sqrt{3}\right)^2-\left(\sqrt{5}\right)^2}$

$\overline{AB}=\frac{2\sqrt{3}-2\sqrt{5}+6\sqrt{5}-10\sqrt{3}}{3-5}=\frac{-8\sqrt{3}+4\sqrt{5}}{-2}=\mathbf{\left(4\sqrt{3}-2\sqrt{5}\right)}$

3. A rectangle has sides of length $2+\sqrt{18}$ m and $\left(5-\frac{4}{\sqrt{2}}\right)$m. Express the area of the rectangle in the form $a+b\sqrt{2}$, where a and b are integers.

Solution

Area $=L\times W=\left(2+\sqrt{18}\right)\times\left(5-\frac{4}{\sqrt{2}}\right)=10-\frac{8}{\sqrt{2}}+5\sqrt{18}-\sqrt{18}\left(\frac{4}{\sqrt{2}}\right)$

Area $=10-\frac{4\times 2}{\sqrt{2}}+5\sqrt{9\times 2}-4\sqrt{\frac{18}{2}}=10-4\sqrt{2}+5\times 3\sqrt{2}-4\sqrt{9}$

Area $=10-4\sqrt{2}+15\sqrt{2}-12=\mathbf{\left(11\sqrt{2}-2\right)}$ m^2

4. Find two numbers whose sum is 68 and difference is 22.

Solution

Let x equal the first number and y equal the second number. Then

According to condition: $x + y = 68$ and $x - y = 22$

$x + y = 68$ $x - y = 22$ adding both $\boldsymbol{x = 45}$	$x + y = 68$ $-x \mp y = 22$ subtracting both $\boldsymbol{y = 23}$

5. The weather in Lahore was unusually warm during the summer of 2024. The TV news reported temperature as high as $48°C$. By using the formula, $(°F = \frac{9}{5}°C + 32)$ find the temperature as Fahrenheit scale.

Solution

°F = 9/5**°C** + 32

°F = 9/5× 48**°C** + 32 = **118.4°F**

6. The sum of the ages of the father and son is 72 years. Six years ago, the father's age was 2 times the age of the son. What was son's age six years ago?

Solution

Son's current age = x year

Father's current age = 72 − x year

Six years ago, Son's age = x − 6 year

Six years ago, Father's age = (72 – x) – 6 = 66 – x year

Six years ago, according to condition: $66 - x = 2(x - 6)$

Simplifying we get: $x = 26$

Six years ago, Son's age = 26 − 6 = **20 year**

7. Mirha bought a toy for Rs. 1500 and sold for Rs. 1520. What was her profit percentage?

Solution

$\text{CP} = \text{Rs.}\,1500$

$\text{SP} = \text{Rs.}\,1520$

$\text{Profit} = \text{SP} - \text{CP} = 1520 - 1500 = \text{Rs.}\,20$

$\text{Profit Percentage} = \frac{\text{Profit}}{\text{CP}} \times 100\%$

$\text{Profit Percentage} = \frac{20}{1500} \times 100\%$

$\text{Profit Percentage} = 1.33\%$

8. The annual income of Tayyab is Rs. 9,60,000, while the exempted amount is Rs. 1,30,000. How much tax would he have to pay at the rate of 0.75%?

Solution

Taxable Income = Total Income $-$ Exempted Amount
Taxable Income = $\text{Rs.}\,960000 - \text{Rs.}\,130000$
Taxable Income = $\text{Rs.}\,830000$
Tax Rate = $0.75\% = 0.0075$
Tax Amount = Taxable Income $\times$ Tax Rate
Tax Amount = $\text{Rs.}\,830000 \times 0.0075$
Tax Amount = $\textbf{Rs.}\,\mathbf{6225}$

9. Find the compound markup on Rs. 3,75,000 for one year at the rate of 14% compounded annually.

Solution

Principal Amount (P) = $\text{Rs.}\,375000$
Rate of Interest (R) = $14\% = 0.14$
Time (T) = 1 year
Compound Interest (CI) = $\text{P} \times (1 + \text{R})^{\text{T}} - \text{P}$
Compound Interest (CI) = $\text{Rs.}\,375000 \times (1 + 0.14)^{1} - \text{Rs.}\,375000$
Compound Interest (CI) = $\textbf{Rs.}\,\mathbf{52500}$

REVIEW EXERCISE 1

1. Four options are given against each statement. Encircle the correct option.

(i) $\sqrt{7}$ is:

(a) integer (b) rational number
(c)✓ irrational number (d) natural number

(ii) π and e are:

(a) natural numbers (b) integers
(c) rational numbers (d)✓ irrational numbers

(iii) If n is not a perfect square, then $\sqrt{n}$ is:

(a) rational number (b) natural number
(c) integer (d)✓ irrational number

(iv) $\sqrt{3}+\sqrt{5}$ is:

(a) whole number (b) integer
(c) rational number (d)✓ irrational number

(v) For all $x \in R$, $x = x$ is called:

(a)✓ reflexive property (b) transitive number
(c) symmetric property (d) trichotomy property

(vi) Let $a, b, c \in R$, then $a > b$ and $b > c \Rightarrow a > c$ is called __________ property.

(a) trichotomy (b)✓ transitive
(c) additive (d) multiplicative

(vii) $2^x \times 8^x = 64$ then $x =$

(a)✓ $\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $\frac{5}{6}$ (d) $\frac{2}{3}$

(viii) Let $a, b \in R$, then $a = b$ and $b = a$ is called __________ property.

(a) reflexive (b)✓ symmetric
(c) transitive (d) additive

(ix) $\sqrt{75}+\sqrt{27}=$

(a) $\sqrt{102}$ (b) $9\sqrt{3}$ (c) $5\sqrt{3}$ (d) ✓ $8\sqrt{3}$

(x) The product of $(3+\sqrt{5})(3-\sqrt{5})$ is:

(a) prime number (b) odd number

(c) irrational number (d) ✓ rational number

2. If $a=\frac{3}{2}$, $b=\frac{5}{3}$ and $c=\frac{7}{5}$, then verify that

(i) $a(b+c)=ab+ac$ (ii) $(a+b)c=ac+bc$

Solution

i. $\boldsymbol{a(b+c)=ab+ac}$

$\mathrm{L.H.S}=\mathrm{a(b+c)}=\frac{3}{2}\left(\frac{5}{3}+\frac{7}{5}\right)=\frac{3}{2}\left(\frac{25+21}{15}\right)=\frac{3}{2}\left(\frac{46}{15}\right)=\frac{138}{30}=\frac{23}{5}$

$\mathrm{R.H.S}=ab+ac=\frac{3}{2}\left(\frac{5}{3}\right)+\frac{3}{2}\left(\frac{7}{5}\right)=\frac{15}{6}+\frac{21}{10}=\frac{5}{2}+\frac{21}{10}=\frac{46}{10}=\frac{23}{5}$

Hence $a(b+c)=ab+ac$

ii. $\boldsymbol{(a+b)c=ac+bc}$

$\mathrm{L.H.S}=(a+b)c=\left(\frac{3}{2}+\frac{5}{3}\right)\frac{7}{5}=\left(\frac{9+10}{6}\right)\frac{7}{5}=\left(\frac{19}{6}\right)\frac{7}{5}=\frac{133}{30}$

$\mathrm{R.H.S}=ac+bc=\left(\frac{3}{2}\right)\frac{7}{5}+\left(\frac{5}{3}\right)\frac{7}{5}=\frac{21}{10}+\frac{35}{15}=\frac{21}{10}+\frac{7}{3}=\frac{133}{30}$

Hence $(a+b)c=ac+bc$

3. If $a=\frac{4}{3}$, $b=\frac{5}{2}$, $c=\frac{7}{4}$, then verify the associative property of real numbers w.r.t addition and multiplication.

Solution

We have to verify

$(a+b)+c=a+(b+c)$ and $(a\times b)\times c=a\times(b\times c)$

i. $\boldsymbol{(a+b)+c=a+(b+c)}$

$\mathrm{L.H.S}=(a+b)+c=\left(\frac{4}{3}+\frac{5}{2}\right)+\frac{7}{4}=\left(\frac{8+15}{6}\right)+\frac{7}{4}=\frac{23}{6}+\frac{7}{4}=\frac{67}{12}$

$\mathrm{R.H.S}=a+(b+c)=\frac{4}{3}+\left(\frac{5}{2}+\frac{7}{4}\right)=\frac{4}{3}+\left(\frac{10+7}{4}\right)=\frac{4}{3}+\frac{17}{4}=\frac{67}{12}$

Hence $(a+b)+c=a+(b+c)$

ii. $\boldsymbol{(a\times b)\times c=a\times(b\times c)}$

$\mathrm{L.H.S}=(a\times b)\times c=\left(\frac{4}{3}\times\frac{5}{2}\right)\times\frac{7}{4}=\frac{20}{6}\times\frac{7}{4}=\frac{10}{3}\times\frac{7}{4}=\frac{70}{12}=\frac{35}{6}$

$\mathrm{R.H.S}=a\times(b\times c)=\frac{4}{3}\times\left(\frac{5}{2}\times\frac{7}{4}\right)=\frac{4}{3}\times\frac{35}{8}=\frac{140}{24}=\frac{35}{6}$

Hence $(a\times b)\times c=a\times(b\times c)$

4. Is 0 a rational number? Explain.

Solution

Yes, zero is a rational number. A rational number is defined as a number that can be expressed as the ratio of two integers, i.e., $\frac{a}{b}$, where a and b are integers and b is non-zero. Zero can be expressed as a ratio of two integers, such as: 0 = 0/1

In this case, both 0 and 1 are integers, and 1 is non-zero. Therefore, zero meets the definition of a rational number.

5. State trichotomy property of real numbers.

Solution

For any two real numbers a and b, exactly one of the following is true:

1. $a < b$ 2. $a = b$ 3. $a > b$

6. Find two rational numbers between 4 and 5.

Solution

$q_1 = \frac{1}{2}(4+5) = \frac{9}{2}$ and $q_2 = \frac{1}{2}\left(\frac{9}{2}+5\right) = \frac{1}{2}\left(\frac{19}{2}\right) = \frac{19}{4}$

Hence required rational are $\mathbf{\frac{9}{2}, \frac{19}{4}}$

7. Simplify the following:

(i) $\sqrt[5]{\frac{x^{15}y^{35}}{z^{20}}}$ (ii) $\sqrt[3]{(27)^{2x}}$ (iii) $\frac{6(3)^{n+2}}{3^{n+1}-3^n}$

Solution

i. $\sqrt[5]{\frac{x^{15}y^{35}}{z^{20}}} = \left(\frac{x^{15}y^{35}}{z^{20}}\right)^{\frac{1}{5}} = \frac{x^{15\times\frac{1}{5}}y^{35\times\frac{1}{5}}}{z^{20\times\frac{1}{5}}} = \frac{x^3y^7}{z^4}$

ii. $\sqrt[3]{(27)^{2x}} = (27)^{\frac{2x}{3}} = (3^3)^{\frac{2x}{3}} = \mathbf{3^{2x}}$

iii. $\frac{6(3)^{n+2}}{(3)^{n+1}-3^n} = \frac{3^n(6\times3^2)}{3^n(3-1)} = \frac{6\times9}{2} = \mathbf{27}$

8. The sum of three consecutive odd integers is 51. Find the three integers.

Solution

Let the three consecutive odd integers be x, x+2, and x+4.

$x + (x+2) + (x+4) = 51$

$3x + 6 = 51$

$3x = 45$

$x = 15$

Now that we know x, we can find the other two integers:

$x+2 = 17$

$x+4 = 19$

So, the three consecutive integers are **15**, **17**, and **19**.

9. Abdullah picked up 96 balls and placed them into two buckets. One bucket has twenty-eight more balls than the other bucket. How many balls were in each bucket?

Solution

Let's say the number of balls in the smaller bucket is x. Since the other bucket has 28 more balls, the number of balls in the larger bucket is x + 28.

We know that the total number of balls is 96, so we can set up the equation:

$x + (x + 28) = 96$

$2x + 28 = 96$

$2x = 68$

$x = 34$

So, the smaller bucket has 34 balls.

The larger bucket has 34 + 28 = 62 balls.

Therefore, the two buckets have **34** and **62** balls, respectively.

10. Salma invested Rs. 3,50,000 in a bank, which paid simple profit at the rate of $7\frac{1}{4}\%$ per annum. After 2 years, the rate was increased to 8% per annum. Find the amount she had at the end of 7 years.

Solution

Initial Investment = Rs. 3,50,000

Rate of interest for the first 2 years = $7\frac{1}{4}\%$ = 7.25% per annum

Interest for the first 2 years = (3,50,000 × 7.25% × 2) = Rs. 50,750

Rate of interest for the next 5 years = 8% per annum

Interest for the next 5 years = (3,50,000 × 8% × 5) = Rs. 1,40,000

Amount after 7 years = 3,50,000 + 50,750+ 1,40,000 = Rs. 5,40,750

Therefore, Salma had **Rs. 5,40,750** at the end of 7 years.

Unit 2 Logarithms

EXERCISE 2.1

1. Express the following numbers in scientific notation:

(i) 2000000 (ii) 48900 (iii) 0.0042

(iv) 0.0000009 (v) 73×10^3 (vi) 0.65×10^2

Solution

(i) 2×10^6 (ii) 4.89×10^4 (iii) 4.2×10^{-3} (iv) 9×10^{-7} (v) 7.3×10^4

(vi) 6.5×10^1

2. Express the following numbers in ordinary notation:

(i) 8.04×10^2 (ii) 3×10^5 (iii) 1.5×10^{-2}

(iv) 1.77×10^7 (v) 5.5×10^{-6} (vi) 4×10^{-5}

Solution

(i) 804 (ii) 300000 (iii) 0.015 (iv) 17700000 (v) 0.0000055 (vi) 0.00004

3. The speed of light is approximately 3×10^8 metres per second. Express it in standard form.

4. The circumference of the Earth at the equator is about 40075000 metres. Express this number in scientific notation.

5. The diameter of Mars is 6.7779×10^3 km. Express this number in standard form.

6. The diameter of Earth is about 1.2756×10^4 km. Express this number in standard form.

Solution

3. 300,000,000 m/sec **4.** 4.0075×10^7 m

5. 6779 km **6.** 12756 km

EXERCISE 2.2

1. Express each of the following in logarithmic form:

(i) $10^3 = 1000$ (ii) $2^8 = 256$ (iii) $3^{-3} = \frac{1}{27}$

(iv) $20^2 = 400$ (v) $16^{-\frac{1}{4}} = \frac{1}{2}$ (vi) $11^2 = 121$

(vii) $p = q^r$ (viii) $(32)^{\frac{-1}{5}} = \frac{1}{2}$

Solution: $\log_b(x) = y \Leftrightarrow b^y = x \quad ; b > 0, x > 0, b \neq 1$

(i) $\log_{10} 1000 = 3$ (ii) $\log_2 256 = 8$ (iii) $\log_3 \frac{1}{27} = -3$ (iv) $\log_{20} 400 = 2$

(v) $\log_{16} \frac{1}{2} = -\frac{1}{4}$ (vi) $\log_{11} 121 = 2$ (vii) $\log_q p = r$ (viii) $\log_{32} \frac{1}{2} = -\frac{1}{5}$

2. Express each of the following in exponential form:

(i) $\log_5 125 = 3$ (ii) $\log_2 16 = 4$ (iii) $\log_{23} 1 = 0$

(iv) $\log_5 5 = 1$ (v) $\log_2 \frac{1}{8} = -3$ (vi) $\frac{1}{2} = \log_9 3$

(vii) $5 = \log_{10} 100000$ (viii) $\log_4 \frac{1}{16} = -2$

Solution: $\log_b(x) = y \Leftrightarrow b^y = x \quad ; b > 0, x > 0, b \neq 1$

(i) $5^3 = 125$ (ii) $2^4 = 16$ (iii) $23^0 = 1$ (iv) $5^1 = 5$

(v) $2^{-3} = \frac{1}{8}$ (vi) $9^{\frac{1}{2}} = 3$ (vii) $10^5 = 100000$ (viii) $4^{-2} = \frac{1}{16}$

3. Find the value of x in each of the following:

(i)	$\log_x 64 = 3$	(ii)	$\log_5 1 = x$	(iii)	$\log_x 8 = 1$
(iv)	$\log_{10} x = -3$	(v)	$\log_4 x = \frac{3}{2}$	(vi)	$\log_2 1024 = x$

Solution: $\mathbf{\log_b(x) = y \Leftrightarrow b^y = x \quad ; b > 0, x > 0, b \neq 1}$

i. $\log_x 64 = 3 \Rightarrow x^3 = 64 \Rightarrow x^3 = 4^3 \Rightarrow \mathbf{x = 4}$

ii. $\log_5 1 = x \Rightarrow 5^x = 1 \Rightarrow 5^x = 5^0 \Rightarrow \mathbf{x = 0}$

iii. $\log_x 8 = 1 \Rightarrow x^1 = 8 \Rightarrow \mathbf{x = 8}$

iv. $\log_{10} x = -3 \Rightarrow 10^{-3} = x \Rightarrow x = \frac{1}{10^3} \Rightarrow \mathbf{x = \frac{1}{1000}}$

v. $\log_4 x = \frac{3}{2} \Rightarrow 4^{\frac{3}{2}} = x \Rightarrow x = (2^2)^{\frac{3}{2}} \Rightarrow x = 2^3 \Rightarrow \mathbf{x = 8}$

vi. $\log_2 1024 = x \Rightarrow 2^x = 1024 \Rightarrow 2^x = 2^{10} \Rightarrow \mathbf{x = 10}$

EXERCISE 2.3

1. Find characteristic of the following numbers:

(i)	5287	(ii)	59.28	(iii)	0.0567
(iv)	234.7	(v)	0.000049	(vi)	145000

Solution

(i) 3 (ii) 1 (iii) −2 (iv) 2 (v) −5 (vi) 5

2. Find logarithm of the following numbers:

(i)	43	(ii)	579	(iii)	1.982
(iv)	0.0876	(v)	0.047	(vi)	0.000354

Solution

i. $\log 43 = 1.6335$ Characteristic $= 1$, Mantissa $= 0.6335$

ii. $\log 579 = 2.7627$ Characteristic $= 2$, Mantissa $= 0.7627$

iii. $\log 19.82 = 1.2971$ Characteristic $= 1$, Mantissa $= 0.2971$

iv. $\log 0.0876 = -2 + 0.9425 = -1.0575$ Characteristic $= -2$, Mantissa $= 0.9425$

v. $\log 0.047 = -2 + 0.6721 = -1.3279$ Characteristic $= -2$, Mantissa $= 0.6721$

vi. $\log 0.000354 = -4 + 0.5490 = -3.4518$ Characteristic $= -4$, Mantissa $= 0.5490$

3. If log 3.177 = 0.5019, then find:

(i)	log 3177	(ii)	log 31.77	(iii)	log 0.03177

Solution

i. $\log 3177 = 3.5019$ Characteristic $= 3$, Mantissa $= 0.5019$

ii. $\log 31.77 = 1.5019$ Characteristic $= 1$, Mantissa $= 0.5019$

iii. $\log 0.03177 = -2 + 0.5019 = -1.4981$ Characteristic $= -2$, Mantissa $= 0.5019$

4. Find the value of x.

(i) $\log x = 0.0065$ (ii) $\log x = 1.192$ (iii) $\log x = -3.434$

(iv) $\log x = -1.5726$ (v) $\log x = 4.3561$ (vi) $\log x = -2.0184$

Solution

i. $\log x = 0.0065 \Rightarrow x = \text{antilog}(0.0065) \Rightarrow x = 1.015$

ii. $\log x = 1.192 \Rightarrow x = \text{antilog}(1.192) \Rightarrow x = 15.56$

iii. $\log x = -3.434 \Rightarrow \log x = -4 + 4 - 3.434 \Rightarrow x = \text{antilog}(\bar{4}.566)$

$\Rightarrow x = 0.0003681$

iv. $\log x = -1.5726 \Rightarrow \log x = -2 + 2 - 1.5726 \Rightarrow x = \text{antilog}(\bar{2}.4274)$

$\Rightarrow x = 0.02675$

v. $\log x = 4.3561 \Rightarrow x = \text{antilog}(4.3561) \Rightarrow x = 22700$

vi. $\log x = -2.0184 \Rightarrow \log x = -3 + 3 - 2.0184 \Rightarrow x = \text{antilog}(\bar{3}.9816)$

$\Rightarrow x = 0.009585$

EXERCISE 2.4

1. Without using calculator, evaluate the following:

(i) $\log_2 18 - \log_2 9$ (ii) $\log_2 64 + \log_2 2$ (iii) $\frac{1}{3}\log_3 8 - \log_3 18$

(iv) $2\log 2 + \log 25$ (v) $\frac{1}{3}\log_4 64 + 2\log_5 25$ (vi) $\log_3 12 + \log_3 0.25$

Solution

i. $\log_2 18 - \log_2 9 = \log_2(2 \times 9) - \log_2 9 = \log_2 2 + \log_2 9 - \log_2 9$

$= \log_2 2 = \mathbf{1}$

ii. $\log_2 64 + \log_2 2 = \log_2(2 \times 2 \times 2 \times 2 \times 2 \times 2) + \log_2 2$

$= \log_2(2^6) + \log_2 2 = 6\log_2 2 + \log_2 2 = 7\log_2 2 = 7(1) = \mathbf{7}$

iii. $\frac{1}{3}\log_3 8 - \log_3 18 = \frac{1}{3}\log_3(2 \times 2 \times 2) - \log_3(2 \times 3 \times 3)$

$= \frac{1}{3}\log_3(2^3) - \log_3(2 \times 3^2) = \frac{3}{3}\log_3 2 - \log_3 2 - 2\log_3 3$

$= \log_3 2 - \log_3 2 - 2\log_3 3 = -2(1) = \mathbf{-2}$

iv. $2\log 2 + \log 25 = 2\log 2 + \log(5^2) = 2\log 2 + 2\log 5 = 2(\log 2 + \log 5)$

$= 2\log(2 \times 5) = 2\log 10 = 2(1) = \mathbf{2}$

v. $\frac{1}{3}\log_4 64 + 2\log_5 25 = \frac{1}{3}\log_4(4^3) + 2\log_5(5^2) = \frac{3}{3}\log_4 4 + 2 \times 2\log_5 5$

$= \log_4 4 + 4\log_5 5 = (1) + 4(1) = 1 + 4 = \mathbf{5}$

vi. $\log_3 12 + \log_3 0.25 = \log_3 12 + \log_3 \frac{25}{100} = \log_3 12 + \log_3 \frac{1}{4} = \log_3 \frac{12}{4}$

$= \log_3 3 = \mathbf{1}$

2. Write the following as a single logarithm:

(i) $\frac{1}{2}\log 25 + 2\log 3$ (ii) $\log 9 - \log\frac{1}{3}$

(iii) $\log_5 b^2 . \log_a 5^3$ (iv) $2\log_3 x + \log_3 y$

(v) $4\log_5 x - \log_5 y + \log_5 z$ (vi) $2\ln a + 3\ln b - 4\ln c$

Solution

i. $\frac{1}{2}\log 25 + 2\log 3 = \frac{1}{2}\log(5^2) + \log(3^2) = \log 5 + \log 9 = \log(5\times 9) = \mathbf{\log 45}$

ii. $\log 9 - \log\frac{1}{3} = \log\left(\frac{9}{\frac{1}{3}}\right) = \log(9\times 3) = \mathbf{\log 27}$

iii. $\log_5 b^2 . \log_a 5^3 = 2\log_5 b \times 3\log_a 5 = 2\frac{\log_a b}{\log_a 5}\times 3\frac{\log_a 5}{\log_a a} = 6\frac{\log_a b}{(1)} = \mathbf{6\log_a b}$

vi. $2\log_3 x + \log_3 y = \log_3(x^2) + \log_3 y = \mathbf{\log_3 x^2 y}$

vi. $2\ln a + 3\ln b - 4\ln c = \ln a^2 + \ln b^3 - \ln c^4 = \mathbf{\ln\frac{a^2b^3}{c^4}}$

3. Expand the following using laws of logarithms:

(i) $\log\left(\frac{11}{5}\right)$ (ii) $\log_5\sqrt{8a^6}$ (iii) $\ln\left(\frac{a^2b}{c}\right)$

(iv) $\log\left(\frac{xy}{z}\right)^{\frac{1}{9}}$ (v) $\ln\sqrt[3]{16x^3}$ (vi) $\log_2\left(\frac{1-a}{b}\right)^5$

Solution

i. $\log\left(\frac{11}{5}\right) = \mathbf{\log 11 - \log 5}$

ii. $\log_5\sqrt{8a^6} = \log_5(2^3\times a^6)^{\frac{1}{2}} = \log_5\left(2^{\frac{3}{2}}\times a^3\right) = \mathbf{\frac{3}{2}\log_5 2 + 3\log_5 a}$

iii. $\ln\left(\frac{a^2b}{c}\right) = \ln a^2 + \ln b - \ln c = \mathbf{2\ln a + \ln b - \ln c}$

iv. $\ln\left(\frac{xy}{z}\right)^{\frac{1}{9}} = \frac{1}{9}\ln\left(\frac{xy}{z}\right) = \frac{1}{9}[\mathbf{\ln x + \ln y - \ln z}]$

v. $\ln\sqrt[3]{16x^3} = \ln(2^4\times x^3)^{\frac{1}{3}} = \ln\left(2^{\frac{4}{3}}\times x\right) = \mathbf{\frac{4}{3}\ln 2 + \ln x}$

vi. $\log_2\left(\frac{1-a}{b}\right)^5 = \mathbf{5}\log_2\left(\frac{1-a}{b}\right) = \mathbf{5[\log_2(1-a) - \log_2 b]}$

4. Find the value of x in the following equations:

(i) $\log 2+\log x=1$ (ii) $\log_2 x+\log_2 8=5$

(iii) $(81)^x=(243)^{x+2}$ (iv) $\left(\frac{1}{27}\right)^{x-6}=27$

(v) $\log(5x-10)=2$ (vi) $\log_2(x+1)-\log_2(x-4)=2$

Solution

i. $\log 2+\log x=1 \Rightarrow \log 2x=\log 10 \Rightarrow 2x=10 \Rightarrow \mathbf{x=5}$

ii. $\log_2 x+\log_2 8=5 \Rightarrow \log_2 x+\log_2 8=5\log_2 2 \Rightarrow \log_2 8x=\log_2 2^5 \Rightarrow 8x=32 \Rightarrow \mathbf{x=4}$

iii. $(81)^x=(243)^{x+2} \Rightarrow (3^4)^x=(3^5)^{x+2} \Rightarrow 3^{4x}=3^{5x+10} \Rightarrow 5x+10=4x \Rightarrow \mathbf{x=-10}$

iv. $\left(\frac{1}{27}\right)^{x-6}=27 \Rightarrow (3^{-3})^{x-6}=3^3 \Rightarrow 3^{-3x+18}=3^3 \Rightarrow -3x+18=3 \Rightarrow \mathbf{x=5}$

v. $\log(5x-10)=2 \Rightarrow \log(5x-10)=2\log 10 \Rightarrow \log(5x-10)=\log 10^2$

$\Rightarrow 5x-10=100 \Rightarrow 5x=110 \Rightarrow \mathbf{x=22}$

vi. $\log_2(x+1)-\log_2(x-4)=2 \Rightarrow \log_2\left(\frac{x+1}{x-4}\right)=2\log_2 2$

$\Rightarrow \log_2\left(\frac{x+1}{x-4}\right)=\log_2 2^2 \Rightarrow \frac{x+1}{x-4}=4 \Rightarrow x+1=4x-16$

$\Rightarrow 3x=17 \Rightarrow x=\frac{17}{3} \Rightarrow \mathbf{x=5\frac{2}{3}}$

5. Find the values of the following with the help of logarithm table:

(i) $\frac{3.68\times 4.21}{5.234}$ (ii) $4.67\times 2.11\times 2.397$

(iii) $\frac{(20.46)^2\times(2.4122)}{754.3}$ (iv) $\frac{\sqrt[3]{9.364}\times 21.64}{3.21}$

5(i). $\mathbf{\log\left(\frac{3.68\times 4.21}{5.234}\right)}$ =???

Solution

Let $x=\frac{3.68\times 4.21}{5.234}$

$\log x=\log\left(\frac{3.68\times 4.21}{5.234}\right)$ taking logarithm on both sides

$\log x=\log(3.68)+\log(4.21)-\log(5.234)$

$\log x=0.5658+0.6243-0.7188$

$\log x=0.4713$

$x=\text{antilog}(0.4713)$

$\Rightarrow \mathbf{\log\left(\frac{3.68\times 4.21}{5.234}\right)=2.960}$

5(ii). $\log(4.67 \times 2.11 \times 2.397) = ???$

Solution

Let $x = 4.67 \times 2.11 \times 2.397$

$\log x = \log(4.67 \times 2.11 \times 2.397)$ taking logarithm on both sides

$\log x = \log(4.67) + \log(2.11) + \log(2.397)$

$\log x = 0.6693 + 0.3243 + 0.3797$

$\log x = 1.3733$

$x = \text{antilog}(1.3733)$

$\Rightarrow \mathbf{\log(4.67 \times 2.11 \times 2.397) = 23.62}$

5(iii). $\log\left[\frac{(20.46)^2 \times (2.4122)}{754.3}\right] = ???$

Solution

Let $x = \frac{(20.46)^2 \times (2.4122)}{754.3}$

$\log x = \log\left[\frac{(20.46)^2 \times (2.4122)}{754.3}\right]$ taking logarithm on both sides

$\log x = 2\log(20.46) + \log(2.4122) - \log(754.3)$

$\log x = 2(1.3109) + 0.3824 - 2.8776$

$\log x = 0.1266$

$x = \text{antilog}(0.1266)$

$\Rightarrow \mathbf{\log\left[\frac{(20.46)^2 \times (2.4122)}{754.3}\right] = 1.339}$

5(iv). $\log\left[\frac{\sqrt[3]{9.364} \times (21.64)}{3.21}\right] = ???$

Solution

Let $x = \frac{\sqrt[3]{9.364} \times (21.64)}{3.21}$

$\log x = \log\left[\frac{\sqrt[3]{9.364} \times (21.64)}{3.21}\right]$ taking logarithm on both sides

$\log x = \frac{1}{3}\log(9.364) + \log(21.64) - \log(3.21)$

$\log x = \frac{1}{3}(0.9715) + 1.3353 - 0.5065$

$\log x = 1.1526$

$x = \text{antilog}(1.1526)$

$\Rightarrow \mathbf{\log\left[\frac{\sqrt[3]{9.364} \times (21.64)}{3.21}\right] = 14.21}$

6. The formula to measure the magnitude of earthquakes is given by $M=\log_{10}\left(\frac{A}{A_o}\right)$. If amplitude ($A$) is 10,000 and reference amplitude (A_o) is 10. What is the magnitude of the earthquake?

Solution

$$\mathbf{M = \log_{10}\left[\frac{A}{A_0}\right] = \log_{10}\left[\frac{10000}{10}\right]} = ???$$

$$M = \log_{10}\left[\frac{10000}{10}\right] \Rightarrow M = \log_{10}[1000] \Rightarrow M = \log_{10}[10^3] \Rightarrow \mathrm{M} = 3\log_{10}(10)$$

$$\Rightarrow \mathbf{M = \log_{10}\left[\frac{A}{A_0}\right] = \log_{10}\left[\frac{10000}{10}\right] = 3}\ \ \textbf{rector scale}$$

7. Abdullah invested Rs. 100,000 in a saving scheme and gains interest at the rate of 5% per annum so that the total value of this investment after t years is Rs y. This is modelled by an equation $y = 100{,}000\,(1.05)^t$, $t \geq 0$. Find after how many years the investment will be double.

Solution

Initial investment = Rs. 100000

Interest rate = 5% per annum

Total value after t years = y

The equation modeling this situation is:

$$\mathrm{y} = 100000 \times (1.05)^{\mathrm{t}}$$

We want to find years when the investment will be double, i.e., y = 2,00,000

$$2{,}00{,}000 = 1{,}00{,}000 \times (1.05)^{\mathrm{t}}$$

$$2 = (1.05)^t \Rightarrow \log 2 = \log(1.05)^{\mathrm{t}} \Rightarrow \log 2 = \mathrm{t} \times \log(1.05)$$

$$\Rightarrow \mathrm{t} = \frac{\log 2}{\log(1.05)} \Rightarrow t = \frac{0.3010}{0.0212} \Rightarrow \mathbf{t \approx 14.21\ years}$$

8. Huria is hiking up a mountain where the temperature (T) decreases by 3% (or a factor of 0.97) for every 100 metres gained in altitude. The initial temperature (T_i) at sea level is 20°C. Using the formula $T = T_i \times 0.97^{\frac{h}{100}}$, calculate the temperature at an altitude (h) of 500 metres.

Solution

First method	**Second method**
$\mathrm{T} = \mathrm{T_i} \times (0.97)^{\frac{h}{100}}$	$\mathrm{T} = \mathrm{T_i} \times (0.97)^{\frac{h}{100}}$
$\mathrm{T} = 20 \times (0.97)^{\frac{500}{100}}$	$\mathrm{T} = 20 \times (0.97)^{\frac{500}{100}}$
$\mathrm{T} = 20 \times (0.97)^5$	$\log\mathrm{T} = \log(20 \times (0.97)^5)$
$\mathrm{T} = 20 \times 0.859$	$\log\mathrm{T} = \log 20 + 5\log(0.97) = 1.3011 + 5(-0.0133)$
$\mathbf{T \approx 17.17°C}$	$\log\mathrm{T} = 1.2346 \Rightarrow \mathrm{T} = \text{antilog}(1.2346)$
	$\mathbf{T \approx 17.17°C}$

REVIEW EXERCISE 2

1. Four options are given against each statement. Encircle the correct option.

(i) The standard form of 5.2×10^6 is:

(a) 52,000 (b) 520,000 (c) ✓ 5,200,000 (d) 52,000,000

(ii) Scientific notation of 0.00034 is:

(a) 3.4×10^3 (b) ✓ 3.4×10^{-4} (c) 3.4×10^4 (d) 3.4×10^{-3}

(iii) The base of common logarithm is:

(a) 2 (b) ✓ 10 (c) 5 (d) e

(iv) $\log_2 2^3 =$ __________.

(a) 1 (b) 2 (c) 5 (d) ✓ 3

(v) $\log 100 =$ __________.

(a) ✓ 2 (b) 3 (c) 10 (d) 1

(vi) If $\log 2 = 0.3010$, then $\log 200$ is:

(a) 1.3010 (b) 0.6010 (c) ✓ 2.3010 (d) 2.6010

(vii) $\log (0) =$ __________.

(a) positive (b) negative (c) zero (d) ✓ undefined

(viii) $\log 10{,}000 =$

(a) 2 (b) 3 (c) ✓ 4 (d) 5

(ix) $\log 5 + \log 3 =$ __________.

(a) $\log 0$ (b) $\log 2$ (c) $\log\left(\frac{5}{3}\right)$ (d) ✓ $\log 15$

(x) $3^4 = 81$ in logarithmic form is:

(a) $\log_3 4 = 81$ (b) $\log_4 3 = 81$

(c) ✓ $\log_3 81 = 4$ (d) $\log_4 81 = 3$

2. Express the following numbers in scientific notation:

(i) 0.000567 (ii) 734 (iii) 0.33×10^3

Solution

(i) 5.67×10^{-4} (ii) 7.34×10^2 (iii) 3.3×10^2

3. Express the following numbers in ordinary notation:

(i) 2.6×10^3 (ii) 8.794×10^{-4} (iii) 6×10^{-6}

Solution

(i) 2600 (ii) 0.0008794 (iii) 0.000006

4. Express each of the following in logarithmic form:

(i) $3^7 = 2187$ (ii) $a^b = c$ (iii) $(12)^2 = 144$

Solution

(i) $\log_3 2187 = 7$ (ii) $\log_a c = b$ (iii) $\log_{12} 144 = 2$

5. Express each of the following in exponential form:

(i) $\log_4 8 = x$ (ii) $\log_9 729 = 3$ (iii) $\log_4 1024 = 5$

Solution

(i) $4^x = 8$ (ii) $9^3 = 729$ (iii) $4^5 = 1024$

6. Find value of x in the following:

(i) $\log_9 x = 0.5$ (ii) $\left(\frac{1}{9}\right)^{3x} = 27$ (iii) $\left(\frac{1}{32}\right)^{2x} = 64$

Solution

i. $\log_9 x = 0.5 \Rightarrow x = 9^{0.5} \Rightarrow x = (3^2)^{\frac{1}{2}} \Rightarrow \mathbf{x = 3}$

ii. $\left(\frac{1}{9}\right)^{3x} = 27 \Rightarrow \left(\frac{1}{3^2}\right)^{3x} = 3^3 \Rightarrow (3^{-2})^{3x} = 3^3 \Rightarrow 3^{-6x} = 3^3$

$\Rightarrow -6x = 3 \Rightarrow x = -\frac{3}{6} \Rightarrow \mathbf{x = -\frac{1}{2}}$

iii. $\left(\frac{1}{32}\right)^{2x} = 64 \Rightarrow \left(\frac{1}{2^5}\right)^{2x} = 2^6 \Rightarrow (2^{-5})^{2x} = 2^6 \Rightarrow 2^{-10x} = 2^6$

$\Rightarrow -10x = 6 \Rightarrow x = -\frac{6}{10} \Rightarrow \mathbf{x = -\frac{3}{5}}$

7. Write the following as a single logarithm:

(i) $7\log x - 3\log y^2$ (ii) $3\log 4 - \log 32$

(iii) $\frac{1}{3}(\log_5 8 + \log_5 27) - \log_5 3$

Solution

i. $7\text{log}x - 3\text{log}y^2 = \text{log}x^7 - \text{log}y^6 = \mathbf{log\frac{x^7}{y^6}}$

ii. $3\text{log}4 - \text{log}32 = \text{log}4^3 - \text{log}32 = \text{log}\frac{4^3}{32} = \text{log}\frac{64}{32} = \mathbf{log2}$

iii. $\frac{1}{3}(\log_5 8 + \log_5 27) - \log_5 3 = \frac{1}{3}[\log_5(8 \times 27)] - \log_5 3$

$= \frac{1}{3}[\log_5(216)] - \log_5 3 = \log_5(6^3)^{\frac{1}{3}} - \log_5 3$

$= \log_5 6 - \log_5 3 = \log_5\frac{6}{3} = \mathbf{log_5 2}$

8. Expand the following using laws of logarithms:

(i) $\log(x y z^6)$ (ii) $\log_3 \sqrt[6]{m^5 n^3}$ (iii) $\log\sqrt{8x^3}$

Solution

i. $\text{log}(xyz^6) = \text{log}x + \text{log}y + \text{log}z^6 = \mathbf{logx + logy + 6logz}$

ii. $\log_3\sqrt[6]{m^5n^3} = \log_3(m^5n^3)^{\frac{1}{6}} = \frac{1}{6}[\log_3 m^5 + \log_3 n^3] = \mathbf{\frac{1}{6}[5log_3 m + 3log_3 n]}$

iii. $\text{log}\sqrt{8x^3} = \text{log}(8x^3)^{\frac{1}{2}} = \text{log}(2^3x^3)^{\frac{1}{2}} = \text{log}(2\text{x})^{\frac{3}{2}} = \mathbf{\frac{3}{2}[log2 + logx]}$

9. Find the values of the following with the help of logarithm table:

(i) $\sqrt[3]{68.24}$ (ii) 319.8×3.543 (iii) $\frac{36.12 \times 750.9}{113.2 \times 9.98}$

9(i). $\mathbf{log[\sqrt[3]{68.24}]}$ =???

Solution

Let $x = \sqrt[3]{68.24} = (68.24)^{\frac{1}{3}}$

$\text{log}x = \text{log}(68.24)^{\frac{1}{3}}$ taking logarithm on both sides

$\text{log}x = \frac{1}{3}\text{log}(68.24) = \frac{1}{3}(1.8340)$

$\text{log}x = 0.6113$

$x = \text{antilog}(0.6113)$

$\Rightarrow \mathbf{log[\sqrt[3]{68.24}] = 4.086}$

9(ii). $\log(\mathbf{319.8 \times 3.543})$ =???

Solution

Let $x = 319.8 \times 3.543$

$\log x = \log(319.8 \times 3.543)$ taking logarithm on both sides

$\log x = \log(319.8) + \log(3.543)$

$\log x = 2.5049 + 0.5494$

$\log x = 3.0543$

$x = \text{antilog}(3.0543)$

$\Rightarrow \mathbf{\log(319.8 \times 3.543) = 1133}$

9(iii). $\log\left(\frac{\mathbf{36.12 \times 750.9}}{\mathbf{113.2 \times 9.98}}\right)$ =???

Solution

Let $x = \frac{36.12 \times 750.9}{113.2 \times 9.98}$

$\log x = \log\left(\frac{36.12 \times 750.9}{113.2 \times 9.98}\right)$ taking logarithm on both sides

$\log x = \log(36.12) + \log(750.9) - \log(113.2) - \log(9.98)$

$\log x = 1.5578 + 2.8756 - 2.0539 - 0.9991$

$\log x = 1.3804$

$x = \text{antilog}(1.3804)$

$\Rightarrow \mathbf{\log\left(\frac{36.12 \times 750.9}{113.2 \times 9.98}\right) = 24.01}$

10. In the year 2016, the population of a city was 22 millions and was growing at a rate of 2.5% per year. The function $p(t) = 22(1.025)^t$ gives the population in millions, *t* years after 2016. Use the model to determine in which year the population will reach 35 millions. Round the answer to the nearest year.

Solution

$P(t) = 22 \times (1.025)^t$

$35 = 22 \times (1.025)^t$ when P(t) = 35

$1.591 = (1.025)^t$ dividing by 22

$\log 1.591 = t \times \log 1.025$ taking logarithm on both sides

$0.2014 = t \times 0.0107 \Rightarrow t = \frac{0.2014}{0.0107}$

$t = 18.81 \approx 19$ years

Since t represents years after 2016, add 19 to 2016:

Year ≈ 2016 + 19 ≈ 2035

Unit 3 Sets and Functions

EXERCISE 3.1

1. Write the following sets in set builder notation:
   - (i) $\{1, 4, 9, 16, 25, 36, \ldots, 484\}$
   - (ii) $\{2, 4, 8, 16, \ldots, 256\}$
   - (iii) $\{0, \pm 1, \pm 2, \ldots, \pm 1000\}$
   - (iv) $\{6, 12, 18, \ldots, 120\}$
   - (v) $\{100, 102, 104, \ldots, 400\}$
   - (vi) $\{1, 3, 9, 27, 81, \ldots\}$
   - (vii) $\{1, 2, 4, 5, 10, 20, 25, 50, 100\}$
   - (viii) $\{5, 10, 15, \ldots, 100\}$
   - (ix) The set of all integers between – 100 and 1000

Solution

(i) $\{x|x = n^2, n \in N \wedge 1 \leq n \leq 22\}$ **(ii)** $\{x|x = 2^n, n \in N \wedge 1 \leq n \leq 8\}$
(iii) $\{x|x \in Z \wedge -1000 \leq x \leq 1000\}$ **(iv)** $\{x|x = 6n, n \in N \wedge 1 \leq n \leq 20\}$
(v) $\{x|x = 100 + 2n, n \in W \wedge 0 \leq n \leq 150\}$ **(vi)** $\{x|x = 3^n, n \in W\}$
(vii) $\{x|x \text{ is a divisor of } 100\}$ **(viii)** $\{x|x = 5n, n \in N \wedge 1 \leq n \leq 20\}$
(ix) $\{x|x \in Z \wedge -100 < x < 1000\}$

2. Write each of the following sets in tabular forms:
   - (i) $\{x|x \text{ is a multiple of } 3 \wedge x \leq 36\}$
   - (ii) $\{x|x \in R \wedge 2x + 1 = 0\}$
   - (iii) $\{x|x \in P \wedge x < 12\}$
   - (iv) $\{x|x \text{ is a divisor of } 128\}$
   - (v) $\{x|x = 2^n, n \in N \wedge n < 8\}$
   - (vi) $\{x|x \in N \wedge x + 4 = 0\}$
   - (vii) $\{x|x \in N \wedge x = x\}$
   - (viii) $\{x|x \in Z \wedge 3x + 1 = 0\}$

Solution

(i) $\{3, 6, 9, \ldots, 36\}$ (ii) $\left\{-\frac{1}{2}\right\}$

(iii) $\{2, 3, 5, 7, 11\}$ (iv) $\{1, 2, 4, 8, 16, 32, 64, 128\}$ (v) $\{2,4,8,16,32,64,128\}$
(vi) $\{\ \}$ (vii) $\{1, 2, 3, 4, 5, \ldots\}$ (viii) $\{\ \}$

3. Write two proper subsets of each of the following sets:

(i) $\{a, b, c\}$ (ii) $\{0, 1\}$ (iii) N (iv) Z

(v) Q (vi) R (vii) $\{x \mid x \in Q \wedge 0 < x \leq 2\}$

Solution

i. The Proper subsets of {a, b, c} are {a}, {b}.
ii. The Proper subsets of {0,1} are { 0 }, { 1 }.
iii. The Proper subsets of N = {1,2,3,… } are {1}, {2}.
iv. The Proper subsets of Z = {... , −3, −2, −1,0,1,2,3, ... } are {1}, {2}.
v. The Proper subsets of Q are {1}, {2}.
vi. The Proper subsets of R are {1}, {2}.
vii. The Proper subsets of $\{x|x \in Q \wedge 0 < x \leq 2\}$ are {1}, {2}.

4. Is there any set which has no proper subset? If so, name that set.

Solution

Yes, { } or φ

5. What is the difference between $\{a, b\}$ and $\{\{a,b\}\}$?

Solution

{a,b} is a set containing two elements a and b while {{a,b}} is a set containing one element {a,b}.

6. What is the number of elements of the power set of each of the following sets?

(i) { } (ii) {0, 1} (iii) {1, 2, 3, 4, 5, 6, 7}

(iv) {0, 1, 2, 3, 4, 5, 6, 7} (v) $\{a, \{b, c\}\}$

(vi) $\{\{a, b\}, \{b, c\}, \{d, e\}\}$

Solution

(i) 1 (ii) 4 (iii) 128 (iv) 256 (v) 4 (vi) 8

7. Write down the power set of each of the following sets:

(i) {9, 11} (ii) $\{+, -, \times, \div\}$ (iii) $\{\phi\}$ (iv) $\{a, \{b,c\}\}$

Solution

i. The Power set of {9,11} is { φ, {9}, {11},{9,11}}.
ii. The Power set of $\{+, -, \times, \div\}$ is
$\{\varphi, \{+\}, \{-\}, \{\times\}, \{\div\}, \{+,-\}, \{+\ \times\}, \{+,\div\}, \{-,\times\}, \{-,\div\}, \{\times,\div\}, \{+,-,\times\},$
$\{+,-,\div\}, \{+,\times,\div\}, \{-,\times,\div\}, \{+,-,\times,\div\}\}.$
iii. The Power set of {φ} is { φ, {φ}}.
iv. The Power set of $\{a, \{b, c\}\}$ is $\{\varphi, \{a\}. \{\{b, c\}\}, \{a, \{b, c\}\}\}$.

Exercise 3.2

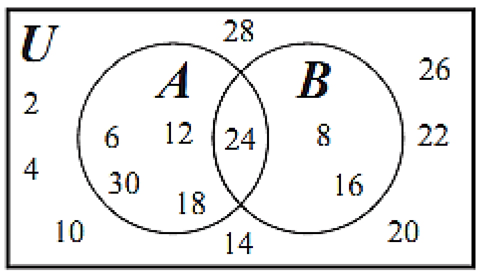
1. Consider the universal set $U = \{x : x \text{ is multiple of 2 and } 0 < x \leq 30\}$, $A = \{x : x \text{ is a multiple of } 6\}$ and $B = \{x : x \text{ is a multiple of } 8\}$

   (i) List all elements of sets A and B in tabular form

   (ii) Find $A \cap B$ (iii) Draw a Venn diagram

Solution

(i) $A = \{6, 12, 18, 24, 30\}$, $B = \{8, 16, 24\}$ (ii) $A \cap B = \{24\}$



2. Let, $U = \{x : x \text{ is an integer and } 0 < x \leq 150\}$,

   $G = \{x : x = 2^m \text{ for integer } m\}$ and

   $H = \{x : x \text{ is a square}\}$

   (i) List all elements of sets G and H in tabular form

   (ii) Find $G \cup H$ (iii) Find $G \cap H$

Solution

(i) $G = \{1, 2, 4, 8, 16, 32, 64, 128\}$,
$H = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144\}$

(ii) $G \cup H = \{1, 2, 4, 8, 9, 16, 25, 32, 36, 49, 64, 81, 100, 121, 128, 144\}$

(iii) $G \cap H = \{1, 4, 16, 64\}$

3. Consider the sets $P = \{x : x \text{ is a prime number and } 0 < x \leq 20\}$ and $Q = \{x : x \text{ is a divisor of 210 and } 0 < x \leq 20\}$

   (i) Find $P \cap Q$ (ii) Find $P \cup Q$

Solution

(i) $P \cap Q = \{2,3,5,7,9,11,13,17,19\} \cap \{1, 2, 3, 5, 6, 7, 10, 14, 15\} = \{2,3,5,7\}$

(ii) $P \cup Q = \{2,3,5,7,9,11,13,17,19\} \cup \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$

$P \cup Q = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19\}$

4. Verify the commutative properties of union and intersection for the following pairs of sets:

(i) $A = \{1, 2, 3, 4, 5\},\ B = \{4, 6, 8, 10\}$ (ii) N, Z

(iii) $A = \{ x \mid x \in R \wedge x \geq 0\},\ B = R.$

Solution

4.(i) $\boldsymbol{A \cup B = B \cup A}$ **also** $\boldsymbol{A \cap B = B \cap A}$

$A \cup B = \{1,2,3,4,5\} \cup \{4,6,8,10\} = \{1,2,3,4,5,6,8,10\}$

$B \cup A = \{4,6,8,10\} \cup \{1,2,3,4,5\} = \{1,2,3,4,5,6,8,10\}$

Hence $A \cup B = B \cup A$

$A \cap B = \{1,2,3,4,5\} \cap \{4,6,8,10\} = \{4\}$

$B \cap A = \{4,6,8,10\} \cap \{1,2,3,4,5\} = \{4\}$

Hence $A \cap B = B \cap A$

4.(ii) $\boldsymbol{N \cup Z = Z \cup N}$ **also** $\boldsymbol{N \cap Z = Z \cap N}$

$N \cup Z = \{1,2,3, \dots\} \cup \{0, \pm1, \pm2, \pm3, \dots\} = \{0, \pm1, \pm2, \pm3, \dots\}$

$Z \cup N = \{0, \pm1, \pm2, \pm3, \dots\} \cup \{1,2,3, \dots\} = \{0, \pm1, \pm2, \pm3, \dots\}$

Hence $N \cup Z = Z \cup N$

$N \cap Z = \{1,2,3, \dots\} \cap \{0, \pm1, \pm2, \pm3, \dots\} = \{1,2,3, \dots\}$

$Z \cap N = \{0, \pm1, \pm2, \pm3, \dots\} \cap \{1,2,3, \dots\} = \{1,2,3, \dots\}$

Hence $N \cap Z = Z \cap N$

4.(iii) $\boldsymbol{A \cup B = B \cup A}$ **also** $\boldsymbol{A \cap B = B \cap A}$

$A \cup B = \{0,1,2,3,4,5, \dots\} \cup R = R$

$B \cup A = R \cup \{0,1,2,3,4,5, \dots\} = R$

Hence $A \cup B = B \cup A$

$A \cap B = \{0,1,2,3,4,5, \dots\} \cap R = \{0,1,2,3,4,5, \dots\}$

$B \cap A = R \cap \{0,1,2,3,4,5, \dots\} = \{0,1,2,3,4,5, \dots\}$

Hence $A \cap B = B \cap A$

5. Let $U = \{a, b, c, d, e, f, g, h, i, j\}$

$A = \{a, b, c, d, g, h\},\quad B = \{c, d, e, f, j\},$

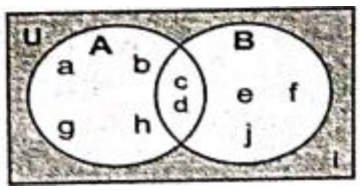
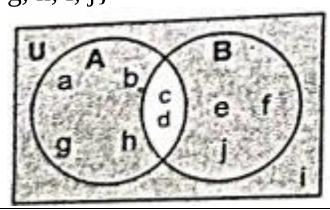
Verify De Morgan's Laws for these sets. Draw Venn diagram

Solution

We have to verify

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

$(\mathbf{A} \cup \mathbf{B})' = \mathbf{A}' \cap \mathbf{B}'$	$(\mathbf{A} \cap \mathbf{B})' = \mathbf{A}' \cup \mathbf{B}'$
A = {a, b, c, d, g, h} B = {c, d, e, f, j} U = {a, b, c, d, e, f, g, h, i, j} A′ = U - A = {a, b, c, d, e, f, g, h, i, j} - {a, b, c, d, g, h} = {e, f, i, j} B′ = U - B = {a, b, c, d, e, f, g, h, i, j} - {c, d, e, f, j} = {a, b, g, h, i} A ∪ B = {a, b, c, d, e, f, g, h, j} (A ∪ B)′ = U - (A ∪ B) = {a, b, c, d, e, f, g, h, i, j} - {a, b, c, d, e, f, g, h, j} = {i} A′ ∩ B′= {e, f, i, j} ∩ {a, b, g, h, i} = {i}  	A = {a, b, c, d, g, h} B = {c, d, e, f, j} U = {a, b, c, d, e, f, g, h, i, j} A′ = U - A = {a, b, c, d, e, f, g, h, i, j} - {a, b, c, d, g, h} = {e, f, i, j} B′ = U - B = {a, b, c, d, e, f, g, h, i, j} - {c, d, e, f, j} = {a, b, g, h, i} A ∩ B = {c, d} (A ∩ B)′ = U - (A ∩ B) = {a, b, c, d, e, f, g, h, i, j} - {c, d} = {a, b, e, f, g, h, i, j} A′ ∪ B′= {e, f, i, j} ∪ {a, b, g, h, i} = {a, b, e, f, g, h, i, j} 

6. If $U = \{1, 2, 3, \ldots, 20\}$ and $A = \{1, 3, 5, \ldots, 19\}$, verify the following:

(i) $A \cup A' = U$ (ii) $A \cap U = A$ (iii) $A \cap A' = \phi$

Solution

$U = \{1,2,3, \ldots ,20\}$ and $A = \{1,3,5, \ldots ,19\}$

$A' = U - A = \{1,2,3, \ldots ,20\} - \{1,3,5, \ldots ,19\} = \{2,4,6, \ldots ,20\}$

(i) $A \cup A' = \{1,3,5, \ldots ,19\} \cup \{2,4,6, \ldots ,20\} = \{1,2,3, \ldots ,20\} = U$

(ii) $A \cap U = \{1,3,5, \ldots ,19\} \cap \{1,2,3, \ldots ,20\} = \{1,3,5, \ldots ,19\} = A$

(iii) $A \cap A' = \{1,3,5, \ldots ,19\} \cap \{2,4,6, \ldots ,20\} = \varphi$

7. In a class of 55 students, 34 like to play cricket and 30 like to play hockey. Also each student likes to play at least one of the two games. How many students like to play both games?

Solution

n (C) = 34 ; n (H) = 30 ; n (U) = 55 ; n (C ∪ H) = 55

n(C ∪ H) = n(C) + n(H) – n(C ∩ H)

$55 = 34 + 30 - n(C \cap H) \Rightarrow 55 = 64 - n(C \cap H)$

$\Rightarrow n(C \cap H) = 64 - 55$

$\Rightarrow \mathbf{n(C \cap H) = 9}.$

8. In a group of 500 employees, 250 can speak Urdu, 150 can speak English, 50 can speak Punjabi, 40 can speak Urdu and English, 30 can speak both English and Punjabi, and 10 can speak Urdu and Punjabi. How many can speak all three languages?

Solution

$n(U \cup E \cup P) = 500$; $n(U) = 250$; $n(E) = 150$; $n(P) = 50$

$n(U \cap E) = 40$; $n(E \cap P) = 30$; $n(U \cap P) = 10$

$n(U \cap E \cap P) = ???$

$n(U \cup E \cup P) = n(U) + n(E) + n(P) – n(U \cap E) – n(E \cap P) – n(U \cap P) + n(U \cap E \cap P)$

$500 = 250 + 150 + 50 - 40 - 30 - 10 + n(U \cap E \cap P)$

$500 = 450 - 80 + n(U \cap E \cap P)$

$500 = 370 + n(U \cap E \cap P)$

$\mathbf{n(U \cap E \cap P) = 130}$

9. In sports events, 19 people wear blue shirts, 15 wear green shirts, 3 wear blue and green shirts, 4 wear a cap and blue shirts, and 2 wear a cap and green shirts. The total number of people with either a blue or green shirt or cap is 34. How many people are wearing caps?

Solution

$n(B) = 19$; $n(G) = 15$; $n(C) = ?$; $n(B \cap G) = 3$; $n(B \cap C) = 4$

$n(G \cap C) = 2$; $n(B \cup G \cup C) = 34$; $n(B \cap G \cap C) = 0$

$n(B \cup G \cup C) = n(B) + n(G) + n(C) – n(B \cap G) – n(B \cap C) – n(G \cap C) + n(B \cap G \cap C)$

$34 = 19 + 15 + n(C) - 3 - 4 - 2 + 0$

$34 = 34 – 9 + n(C)$

$\mathbf{n(C) = 9}$

10. In a training session,17 participants have laptops, 11 have tablets, 9 have laptops and tablets, 6 have laptops and books, and 4 have both tablets and books. Eight participants have all three items. The total number of participants with laptops, tablets, or books is 35. How many participants have books?

Solution

$n(L) = 17$; $n(T) = 11$; $n(L \cap T) = 9$; $n(L \cap B) = 6$; $n(T \cap B) = 4$

$n(L \cap T \cap B) = 8$; $n(L \cup T \cup B) = 35$

$n(L \cup T \cup B) = n(L) + n(T) + n(B) - n(L \cap T) - n(L \cap B) - n(T \cap B) + n(L \cap T \cap B)$

$35 = 17 + 11 + n(B) - 9 - 6 - 4 + 8$

$35 = 17 + n(B)$

$\mathbf{n(B) = 18}$

11. A shopping mall has150 employees labelled 1 to 150, representing the Universal set U. The employees fall into the following categories:

- Set A: 40 employees with a salary range of 30k-45k, labelled from 50 to 89.
- Set B: 50 employees with a salary range of 50k-80k, labelled from 101 to 150.
- Set C: 60 employees with a salary range of 100k-150k, labelled from 1 to 49 and 90 to 100.

(a) Find $(A' \cup B') \cap C$ (a) Find $n\{A \cap (B^c \cap C^C)\}$

Solution

$U = \{1,2,3,\dots,150\}$; $n(U) = 150$

$A = \{50,51,52,\dots,89\}$; $n(A) = 40$

$B = \{101,102,\dots,150\}$; $n(B) = 50$

$C = \{1,2,3,\dots,49,90,91,\dots,100\}$; $n(C) = 60$

$A' = U - A = \{1,2,3,\dots,150\} - \{50,51,52,\dots,89\} = \{1,2,3,\dots,49,90,91,\dots,150\}$
$B' = U - B = \{1,2,3,\dots,150\} - \{101,102,\dots,150\} = \{1,2,3,\dots,100\}$
$C' = U - C = \{1,2,3,\dots,150\} - \{1,2,3,\dots,49,90,91,\dots,100\} = \{50,51,52,\dots,89\}$

(i) $(\mathbf{A}' \cup \mathbf{B}') \cap \mathbf{C} = ???$
$A' \cup B' = \{1,2,3,\dots,49,90,91,\dots,150\} \cup \{1,2,3,\dots,100\} = \{1,2,3,\dots,150\}$
$(A' \cup B') \cap C = \{1,2,3,\dots,150\} \cap \{1,2,3,\dots,49,90,91,\dots,100\}$
$(A' \cup B') \cap C = \{1,2,3,\dots,49,90,91,\dots,100\}$

(ii) $\boldsymbol{n}\{\mathbf{A} \cap (\mathbf{B}' \cap \boldsymbol{C}')\} = ??$

$B' \cap C' = \{1,2,3, \dots ,100\} \cap \{50,51,52, \dots ,89\} = \{50,51,52, \dots ,89\}$

$A \cap (B' \cap C') = \{50,51,52, \dots ,89\} \cap \{50,51,52, \dots ,89\} = \{50,51,52, \dots ,89\}$

$\boldsymbol{n}\{\mathbf{A} \cap (\mathbf{B}' \cap \boldsymbol{C}')\} = \mathbf{40}$

12. In a secondary school with 125 students participate in at least one of the following sports: cricket, football, or hockey.
    - 60 students play cricket.
    - 70 students play football.
    - 40 students play hockey.
    - 25 students play both cricket and football.
    - 15 students play both football and hockey.
    - 10 students play both cricket and hockey.
        - (a) How many students play all three sports?
        - (b) Draw a Venn diagram showing the distribution of sports participation in all the games.

Solution

$n(C \cup F \cup H) = 125$; $n(C) = 60$; $n(F) = 70$; $n(H) = 40$

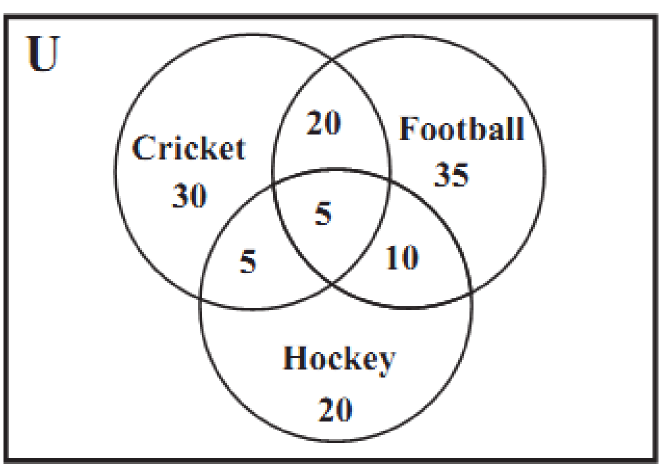
$n(C \cap F) = 25$; $n(F \cap H) = 15$; $n(C \cap H) = 10$; $n(C \cap F \cap H) = ???$

$n(C \cup F \cup H) = n(C) + n(F) + n(H) - n(C \cap F) - n(F \cap H) - n(C \cap H) + n(C \cap F \cap H)$

$125 = 60 + 70 + 40 - 25 - 15 - 10 + n(C \cap F \cap H)$

$n(C \cap F \cap H) = 125 - 60 - 70 - 40 + 25 + 15 + 10$

$\mathbf{n(C \cap F \cap H) = 5}$



13. A survey was conducted in which 130 people were asked about their favourite foods. The survey results showed the following information:

- 40 people said they liked nihari
- 65 people said they liked biryani
- 50 people said they liked korma
- 20 people said they liked nihari and biryani
- 35 people said they liked biryani and korma
- 27 people said they liked nihari and korma
- 12 people said they liked all three foods nihari, biryani, and korma
  - (a) At least how many people like nihari, biryani or korma?
  - (b) How many people did not like nihari, biryani, or korma?
  - (c) How many people like only one of the following foods: nihari, biryani, or korma?
  - (d) Draw a Venn diagram.

Solution

$n(N \cup B \cup K) = ???$; $n(N) = 40$; $n(B) = 65$; $n(K) = 50$

$n(N \cap B) = 20$; $n(B \cap K) = 35$; $n(N \cap K) = 27$; $n(N \cap B \cap K) = 12$

a) At least how many people like Nihari, Biryani or Korma:

$n(N \cup B \cup K) = n(N) + n(B) + n(K) - n(N \cap B) - n(B \cap K) - n(N \cap K) + n(N \cap B \cap K)$

$n(N \cup B \cup K) = 40 + 65 + 50 - 20 - 35 - 27 + 12$

$n(N \cup B \cup K) = \mathbf{85}$

b) How many people did not like Nihari, Biryani or Korma:

Total people = 130

People who like nihari, biryani, or korma = 85

People who did not like nihari, biryani, or korma = 130 – 85 = **45**

c) How many people like only one of the Nihari, Biryani or Korma:

People who like only nihari = $n(N) - n(N \cap B) - n(N \cap K) + n(N \cap B \cap K)$

$= 40 - 20 - 27 + 12 = 5$

People who like only biryani = n(B) – n(N ∩ B) – n(B ∩ K) + n(N ∩ B ∩ K)

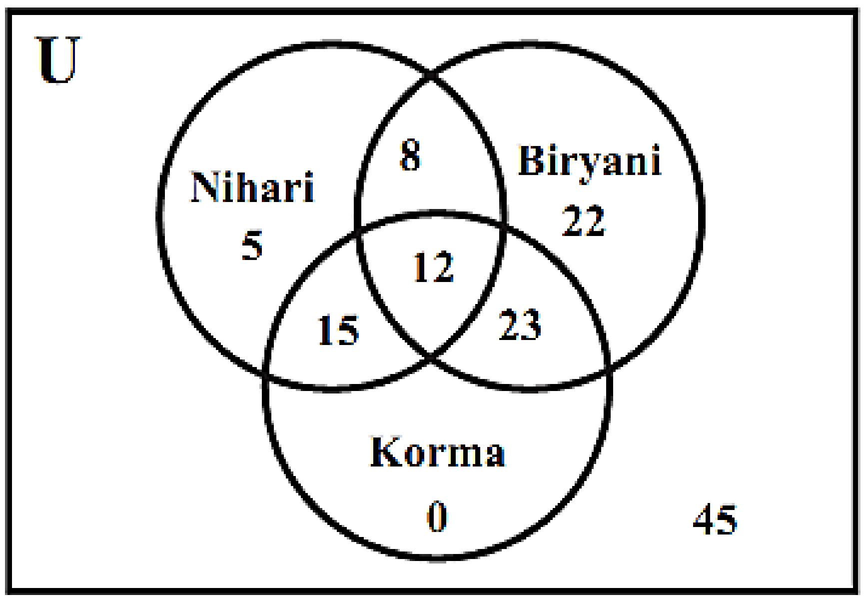
= 65 - 20 - 35 + 12 = 22

People who like only korma = n(K) – n(N ∩ K) – n(B ∩ K) + n(N ∩ B ∩ K)

= 50 - 27 - 35 + 12 = 0

Total people who like only one food = 5 + 22 + 0 = **27**

(d)



EXERCISE 3.3

1. For $A = \{1, 2, 3, 4\}$, find the following relations in A. State the domain and range of each relation.

(i) $\{(x, y) \mid y = x\}$ (ii) $\{(x, y) \mid y + x = 5\}$

(iii) $\{(x, y) \mid x + y < 5\}$ (iv) $\{(x, y) \mid x + y > 5\}$

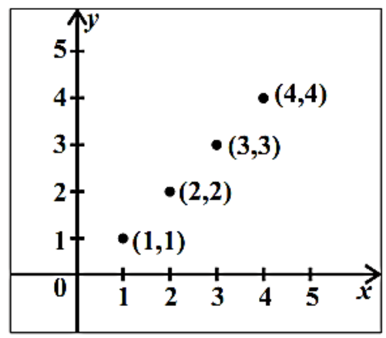
Solution

$A = \{1,2,3,4\}$

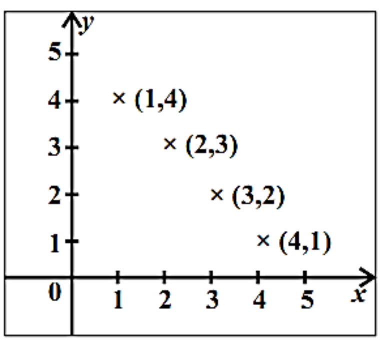
$A \times A = \{1,2,3,4\} \times \{1,2,3,4\} =$

$\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

(i) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
Domain of (i) = $\{1, 2, 3, 4\}$
Range of (i) = $\{1, 2, 3, 4\}$

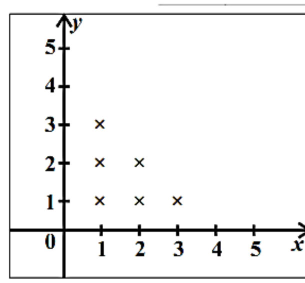


(ii)

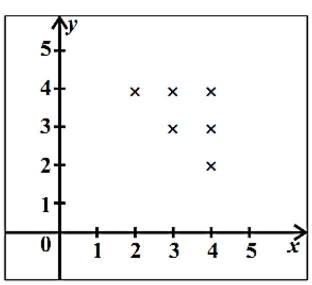


$\{(1, 4), (2, 3), (3, 2), (4, 1)\}$
Domain of (ii) = $\{1, 2, 3, 4\}$
Range of (ii) = $\{1, 2, 3, 4\}$

(iii) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$
Domain of (iii) = $\{1, 2, 3\}$
Range of (iii) = $\{1, 2, 3\}$



(iv)



$\{(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$
Domain of (iv) = $\{2, 3, 4\}$
Range of (iv) = $\{2, 3, 4\}$

2. Which of the following diagrams represent functions and of which type?

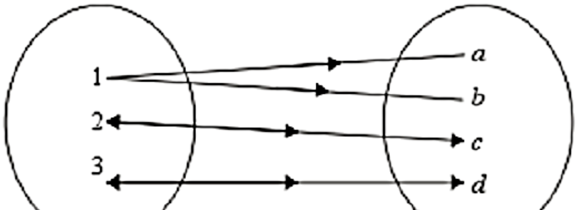


Fig (1)

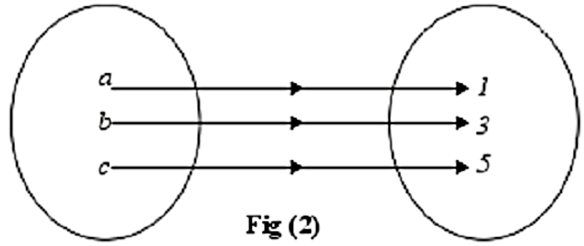


Fig (2)

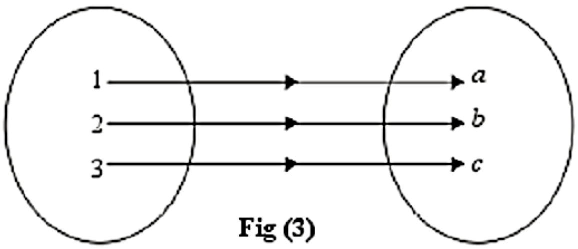


Fig (3)

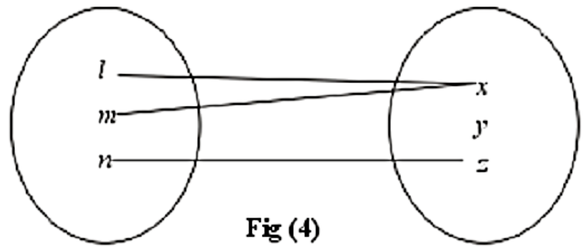


Fig (4)

Solution

Fig (1) does not represent a function. Fig (2) represents a function, which is a bijective function.
Fig (3) represents a function, which is a bijective function.
Fig (4) represents a function, which is an into function.

3. If $g(x) = 3x + 2$ and $h(x) = x^2 + 1$, then find:

(i) $g(0)$ (ii) $g(-3)$ (iii) $g\left(\frac{2}{3}\right)$

(iv) $h(1)$ (v) $h(-4)$ (vi) $h\left(-\frac{1}{2}\right)$

Solution

i. $g(x) = 3x + 2 \Rightarrow g(0) = 3(0) + 2 \Rightarrow g(0) = 2$

ii. $g(x) = 3x + 2 \Rightarrow g(-3) = 3(-3) + 2 \Rightarrow g(-3) = -9 + 2 \Rightarrow g(-3) = -7$

iii. $g(x) = 3x + 2 \Rightarrow g\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right) + 2 \Rightarrow g\left(\frac{2}{3}\right) = 2 + 2 \Rightarrow g\left(\frac{2}{3}\right) = 4$

iv. $h(x) = x^2 + 1 \Rightarrow h(1) = (1)^2 + 1 \Rightarrow h(1) = 1 + 1 \Rightarrow h(1) = 2$

v. $h(x) = x^2 + 1 \Rightarrow h(-4) = (-4)^2 + 1 \Rightarrow h(-4) = 16 + 1 \Rightarrow h(-4) = 17$

vi. $h(x) = x^2 + 1 \Rightarrow h\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + 1 \Rightarrow h\left(-\frac{1}{2}\right) = \frac{1}{4} + 1 \Rightarrow h\left(-\frac{1}{2}\right) = \frac{5}{4}$

4. Given that $f(x) = ax + b + 1$, where a and b are constant numbers. If $f(3) = 8$ and $f(6) = 14$, then find the values of a and b.

Solution

$f(x) = ax + b + 1$

$f(3) = a(3) + b + 1$	$f(6) = a(6) + b + 1$
$8 = 3a + b + 1$	$14 = 6a + b + 1$
$8 - 1 = 3a + b$	$14 - 1 = 6a + b$
$\mathbf{3a + b = 7}$ **……(i)**	$\mathbf{6a + b = 13}$ **……(ii)**

2(i) - (ii)	**(ii) - (i)**
$6a + 2b = 14$	$6a + b = 13$
$-6a \pm b = -13$	$-3a \pm b = -7$
$\mathbf{b = 1}$	$\mathbf{a = 2}$

5. Given that $g(x) = ax + b + 5$, where a and b are constant numbers. If $g(-1) = 0$ and $g(2) = 10$, find the values of a and b.

Solution

$g(x) = ax + b + 5$

$g(-1) = a(-1) + b + 5$	$g(2) = a(2) + b + 5$
$0 = -a + b + 5$	$10 = 2a + b + 5$
$0 - 5 = -a + b$	$10 - 5 = 2a + b$
$\mathbf{-a + b = -5}$ **……(i)**	$\mathbf{2a + b = 5}$ **……(ii)**

(i) - (ii)	**2(i) + (ii)**
$-a + b = -5$	$-2a + 2b = -10$
$-2a \pm b = -5$	$2a + b = 5$
$\mathbf{a = \frac{10}{3}}$	$\mathbf{b = -\frac{5}{3}}$

6. Consider the function defined by $f(x) = 5x + 2$. If $f(x) = 32$, find the x value.

Solution

$f(x) = 5x + 2$

$f(x) = 32$

$5x + 2 = 32$

$5x = 32 - 2$

$x = \frac{30}{5} = 6$

7. Consider the function $f(x) = cx^2 + d$, where c and d are constant numbers. If $f(1) = 6$ and $f(-2) = 10$, then find the values of c and d.

Solution

$f(x) = cx^2 + d$

$f(1) = c(1)^2 + d$	$f(-2) = c(-2)^2 + d$
$\mathbf{c + d = 6}$ **……(i)**	$\mathbf{4c + d = 10}$ **……(ii)**

(i) - (ii)	**4(i) - (ii)**
$c + d = 6$	$4c + 4d = 24$
$-4c \pm d = -10$	$-4c \pm d = -10$
$\mathbf{c = \frac{4}{3}}$	$\mathbf{d = \frac{14}{3}}$

REVIEW EXERCISE 3

1. Four options are given against each statement. Encircle the correct option.

(i) The set builder form of the set $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\right\}$ is:

(a) $\left\{x \mid x = \frac{1}{n}, n \in W\right\}$ (b) $\left\{x \mid x = \frac{1}{2n+1}, n \in W\right\}$

(c) $\left\{x \mid x = \frac{1}{n+1}, n \in W\right\}$ (d) $\{x \mid x = 2n+1, n \in W\}$

(ii) If $A = \{\ \}$, then $P(A)$ is:

(a) $\{\ \}$ (b) $\{1\}$ (c) $\{\{\ \}\}$ (d) ϕ

(iii) If $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $U - (A \cap B)$ is:

(a) $\{1, 2, 4, 5\}$ (b) $\{2, 3\}$ (c) $\{1, 3, 4, 5\}$ (d) $\{1, 2, 3\}$

(iv) If A and B are overlapping sets, then $n(A - B)$ is equal to

(a) $n(A)$ (b) $n(B)$ (c) $A \cap B$ (d) $n(A) - n(A \cap B)$

(v) If $A \subseteq B$ and $B - A \neq \phi$, then $n(B - A)$ is equal to

(a) 0 (b) $n(B)$ (c) $n(A)$ (d) $n(B) - n(A)$

(vi) If $n(A \cup B) = 50$, $n(A) = 30$ and $n(B) = 35$, then $n(A \cap B) =$:

(a) 23 (b) 15 (c) 9 (d) 40

(vii) If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$, then cartesian product of A and B contains exactly ________ elements.

(a) 13 (b) 12 (c) 10 (d) 6

(viii) If $f(x) = x^2 - 3x + 2$, then the value of $f(a + 1)$ is equal to:

(a) $a + 1$ (b) $a^2 + 1$ (c) $a^2 + 2a + 1$ (d) $a^2 - a$

(ix) Given that $f(x) = 3x+1$, if $f(x)=28$, then the value of x is:

(a) 9 (b) 27 (c) 3 (d) 18

(x) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$ two non-empty sets and $f: A \rightarrow B$ be a function defined as $f = \{(1, a), (2, b), (3, b)\}$, then which of the following statement is true?

(a) f is injective (b) f is surjective (c) f is bijective (d) f is into only

2. Write each of the following sets in tabular forms:

(i) $\{x|x=2n, n\in N\}$ (ii) $\{x|x=2m+1, m\in N\}$

(iii) $\{x|x=11n, n\in W \wedge n<11\}$ (iv) $\{x|x\in E \wedge 4<x<6\}$

(v) $\{x|x\in O \wedge 5<x<7\}$ (vi) $\{x|x\in Q \wedge x^2=2\}$

(vii) $\{x|x\in Q \wedge x=-x\}$ (viii) $\{x|x\in R \wedge x\notin Q'\}$

Solution

(i) $\{2, 4, 6, 8, 10, \ldots\}$ (ii) $\{3, 5, 7, 9, 11, \ldots\}$

(iii) $\{0, 11, 22, 33, 44, 55, 66, 77, 88, 99, 110\}$

(iv) ϕ (v) ϕ (vi) ϕ

(vii) $\{0\}$ (viii) Q

3. Let $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A=\{2, 4, 6, 8, 10\}$, $B=\{1, 2, 3, 4, 5\}$ and $C=\{1, 3, 5, 7, 9\}$

List the members of each of the following sets:

(i) A' (ii) B' (iii) $A\cup B$ (iv) $A-B$

(v) $A\cap C$ (vi) $A'\cup C'$ (vii) $A'\cup C$ (viii) U'

Solution

i. $A' = U - A = \{1,2,3,4,5,6,7,8,9,10\} - \{2,4,6,8,10\} = \{1,3,5,7,9\}$

ii. $B' = U - B = \{1,2,3,4,5,6,7,8,9,10\} - \{1,2,3,4,5\} = \{6,7,8,9,10\}$

iii. $A \cup B = \{2,4,6,8,10\} \cup \{1,2,3,4,5\} = \{1,2,3,4,5,6,7,8,9,10\}$

iv. $A - B = \{2,4,6,8,10\} - \{1,2,3,4,5\} = \{6,8,10\}$

v. $A \cap C = \{2,4,6,8,10\} \cap \{1,3,5,7,9\} = \varphi$

vi. $A' \cup C' = \{1,3,5,7,9\} \cup \{2,4,6,8,10\} = \{1,2,3,4,5,6,7,8,9,10\}$

vii. $A' \cup C = \{1,3,5,7,9\} \cup \{1,3,5,7,9\} = \{1,3,5,7,9\}$

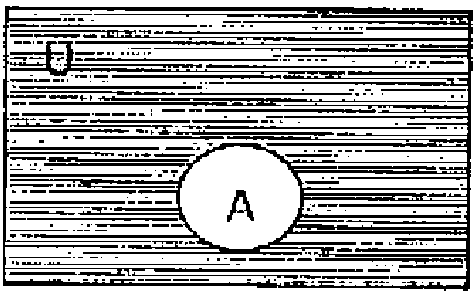
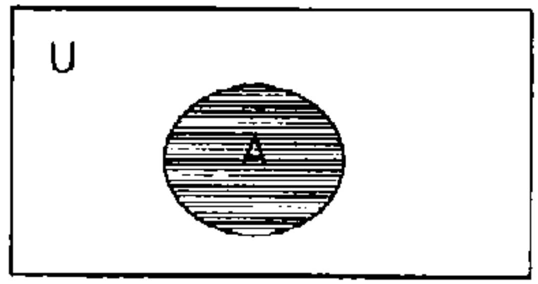
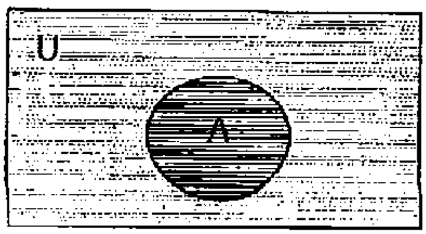
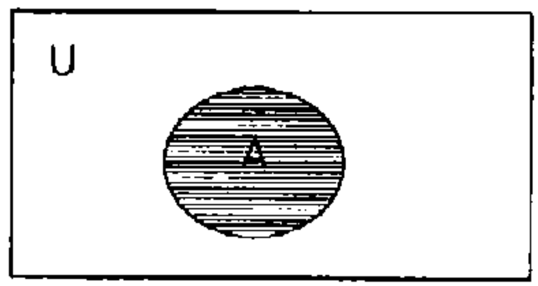
viii. $U' = U - U = \{1,2,3,4,5,6,7,8,9,10\} - \{1,2,3,4,5,6,7,8,9,10\} = \varphi$

4. Using the Venn diagrams, if necessary, find the single sets equal to the following:

(i) A' (ii) $A \cap U$ (iii) $A \cup U$

(iv) $A \cup \phi$ (v) $\phi \cap \phi$

Solution

(i) $\boldsymbol{A' = U - A}$ 	**(ii)** $\boldsymbol{A \cap U = A}$ 
(iii) $\boldsymbol{A \cup U = U}$ 	**(iv)** $\boldsymbol{A \cup \varphi = A}$ 
(v) $\boldsymbol{\varphi \cap \varphi = \varphi}$ It has no Venn diagram.	

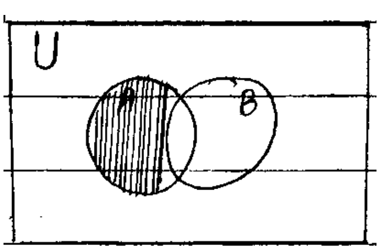
5. Use Venn diagrams to verify the following:

(i) $A - B = A \cup B'$ (ii) $(A-B)' \cap B = B$

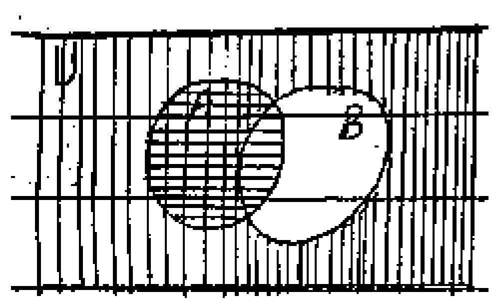
Solution

(i) $\boldsymbol{A - B = A \cup B'}$

Case – I: If A and B are Overlapping

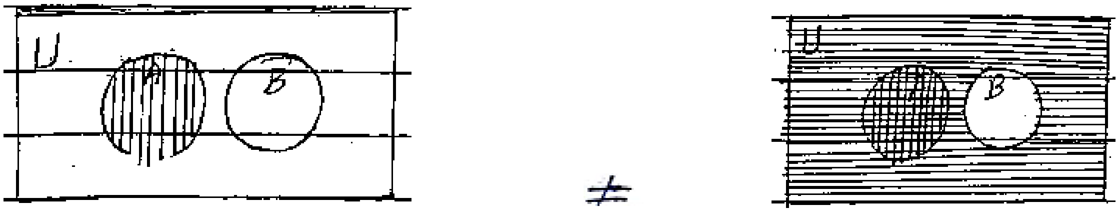


$\neq$



$$\boldsymbol{A - B \neq A \cup B'}$$

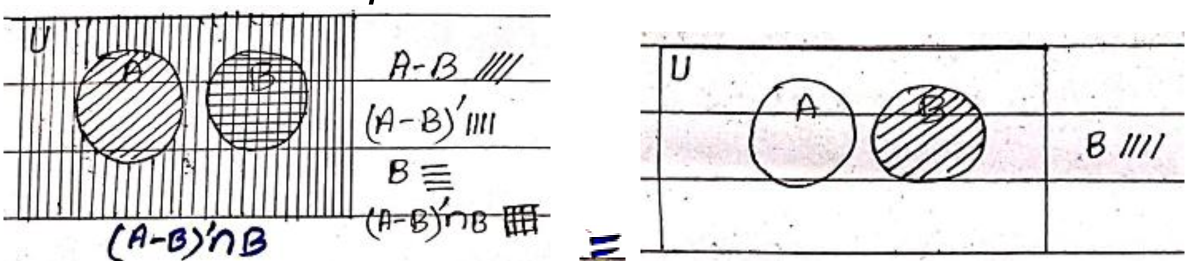
Case – II: If A and B are Disjoint



$$\boldsymbol{A-B \neq A \cup B'}$$

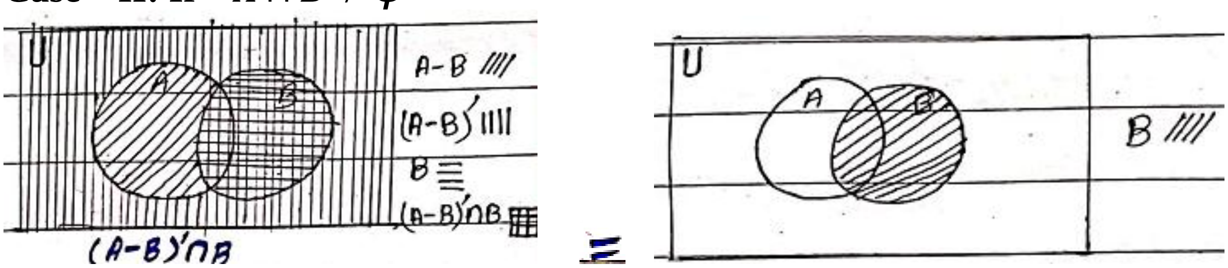
(ii) $\boldsymbol{(A-B)' \cap B = B}$

Case – I: If $\boldsymbol{A \cap B = \varphi}$



$$\boldsymbol{(A-B)' \cap B = B}$$

Case – II: If $\boldsymbol{A \cap B \neq \varphi}$



$$\boldsymbol{(A-B)' \cap B = B}$$

6. Verify the properties for the sets A, B and C given below:

(i) Associativity of Union (ii) Associativity of intersection.

(iii) Distributivity of Union over intersection.

(iv) Distributivity of intersection over union.

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$

(b) $A = \phi$, $B = \{0\}$, $C = \{0, 1, 2\}$

(c) $A = N, B = Z, C = Q$

Solution

(a) $A = \{1,2,3,4\}\,; B = \{3,4,5,6,7,8\}\,; C = \{5,6,7,9,10\}$

i. Associativity of Union: $\mathbf{A \cup (B \cup C) = (A \cup B) \cup C}$

$L.H.S = A \cup (B \cup C) = \{1,2,3,4\} \cup [\{3,4,5,6,7,8\} \cup \{5,6,7,9,10\}]$

$A \cup (B \cup C) = \{1,2,3,4\} \cup \{3,4,5,6,7,8,9,10\} = \{1,2,3,4,5,6,7,8,9,10\}$

$R.H.S = (A \cup B) \cup C = [\{1,2,3,4\} \cup \{3,4,5,6,7,8\}] \cup \{5,6,7,9,10\}$

$(A \cup B) \cup C = \{1,2,3,4,5,6,7,8\} \cup \{5,6,7,9,10\} = \{1,2,3,4,5,6,7,8,9,10\}$

Hence $A \cup (B \cup C) = (A \cup B) \cup C$

ii. Associativity of Intersection: $\mathbf{A \cap (B \cap C) = (A \cap B) \cap C}$

$L.H.S = A \cap (B \cap C) = \{1,2,3,4\} \cap [\{3,4,5,6,7,8\} \cap \{5,6,7,9,10\}]$

$A \cap (B \cap C) = \{1,2,3,4\} \cap \{5,6,7\} = \{\ \ \}$

$R.H.S = (A \cap B) \cap C = [\{1,2,3,4\} \cap \{3,4,5,6,7,8\}] \cap \{5,6,7,9,10\}$

$(A \cap B) \cap C = \{3,4\} \cap \{5,6,7,9,10\} = \{\ \ \}$

Hence $A \cap (B \cap C) = (A \cap B) \cap C$

iii. Distributivity of Union over Intersection: $\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

$L.H.S = A \cup (B \cap C) = \{1,2,3,4\} \cup [\{3,4,5,6,7,8\} \cap \{5,6,7,9,10\}]$

$A \cup (B \cap C) = \{1,2,3,4\} \cup \{5,6,7\} = \{1,2,3,4,5,6,7\}$

$R.H.S = (A \cup B) \cap (A \cup C) = [\{1,2,3,4\} \cup \{3,4,5,6,7,8\}] \cap [\{1,2,3,4\} \cup \{5,6,7,9,10\}]$

$(A \cup B) \cap (A \cup C) = \{1,2,3,4,5,6,7,8\} \cap \{1,2,3,4,5,6,7,9,10\} = \{1,2,3,4,5,6,7\}$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv. Distributivity of Intersection over Union: $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

$L.H.S = A \cap (B \cup C) = \{1,2,3,4\} \cap [\{3,4,5,6,7,8\} \cup \{5,6,7,9,10\}]$

$A \cap (B \cup C) = \{1,2,3,4\} \cap \{3,4,5,6,7,8,9,10\} = \{3,4\}$

$R.H.S = (A \cap B) \cup (A \cap C) = [\{1,2,3,4\} \cap \{3,4,5,6,7,8\}] \cup [\{1,2,3,4\} \cap \{5,6,7,9,10\}]$

$(A \cap B) \cup (A \cap C) = \{3,4\} \cup \{\ \ \} = \{3,4\}$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $A = \varphi\,; B = \{0\}\,; C = \{0,1,2\}$

i. Associativity of Union: $\mathbf{A \cup (B \cup C) = (A \cup B) \cup C}$

$\text{L.H.S} = A \cup (B \cup C) = \varphi \cup [\{0\} \cup \{0,1,2\}] = \varphi \cup \{0,1,2\} = \{0,1,2\}$

$\text{R.H.S} = (A \cup B) \cup C = [\varphi \cup \{0\}] \cup \{0,1,2\} = \{0\} \cup \{0,1,2\} = \{0,1,2\}$

Hence $A \cup (B \cup C) = (A \cup B) \cup C$

ii. Associativity of Intersection: $\mathbf{A \cap (B \cap C) = (A \cap B) \cap C}$

$\text{L.H.S} = A \cap (B \cap C) = \varphi \cap [\{0\} \cap \{0,1,2\}] = \varphi \cap \{0\} = \varphi$

$\text{R.H.S} = (A \cap B) \cap C = [\varphi \cap \{0\}] \cap \{0,1,2\} = \varphi \cap \{0,1,2\} = \varphi$

Hence $A \cap (B \cap C) = (A \cap B) \cap C$

iii. Distributivity of Union over Intersection: $\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

$\text{L.H.S} = A \cup (B \cap C) = \varphi \cup [\{0\} \cap \{0,1,2\}] = \varphi \cup \{0\} = \{0\}$

$\text{R.H.S} = (A \cup B) \cap (A \cup C) = [\varphi \cup \{0\}] \cap [\varphi \cup \{0,1,2\}] = \{0\} \cap \{0,1,2\} = \{0\}$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv. Distributivity of Intersection over Union: $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

$\text{L.H.S} = A \cap (B \cup C) = \varphi \cap [\{0\} \cup \{0,1,2\}] = \varphi \cap \{0,1,2\} = \varphi$

$\text{R.H.S} = (A \cap B) \cup (A \cap C) = [\varphi \cap \{0\}] \cup [\varphi \cap \{0,1,2\}] = \varphi \cup \varphi = \varphi$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(c) $A = N = \{1,2,3,\dots\}\,; B = Z = \{0, \pm1, \pm2, \pm3, \dots\}\,; C = Q\,;\ \mathbf{N \le Z \le Q}$

i. Associativity of Union: $\mathbf{A \cup (B \cup C) = (A \cup B) \cup C}$

$\text{L.H.S} = A \cup (B \cup C) = N \cup [Z \cup Q] = N \cup Q = Q$

$\text{R.H.S} = (A \cup B) \cup C = [N \cup Z] \cup Q = Z \cup Q = Q$

Hence $A \cup (B \cup C) = (A \cup B) \cup C$

ii. Associativity of Intersection: $\mathbf{A \cap (B \cap C) = (A \cap B) \cap C}$

$\text{L.H.S} = A \cap (B \cap C) = N \cap [Z \cap Q] = N \cap Z = N$

$\text{R.H.S} = (A \cap B) \cap C = [N \cap Z] \cap Q = N \cap Q = N$

Hence $A \cap (B \cap C) = (A \cap B) \cap C$

iii. Distributivity of Union over Intersection: $\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

$\text{L.H.S} = A \cup (B \cap C) = N \cup [Z \cap Q] = N \cup Z = Z$

$\text{R.H.S} = (A \cup B) \cap (A \cup C) = [N \cup Z] \cap [N \cup Q] = Z \cap Q = Z$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv. Distributivity of Intersection over Union: $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

$\text{L.H.S} = A \cap (B \cup C) = N \cap [Z \cup Q] = N \cap Q = N$

$\text{R.H.S} = (A \cap B) \cup (A \cap C) = [N \cap Z] \cup [N \cap Q] = N \cup N = N$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

7. Verify De Morgan's Laws for the following sets:
$U = \{1, 2, 3, \ldots, 20\}$, $A = \{2, 4, 6, \ldots, 20\}$ and $B = \{1, 3, 5, \ldots, 19\}$.

Solution

$(A \cup B)' = A' \cap B'$	$(A \cap B)' = A' \cup B'$
U = {1, 2, 3, ..., 20}	U = {1, 2, 3, ..., 20}
A = {2, 4, 6, ..., 20}	A = {2, 4, 6, ..., 20}
B = {1, 3, 5, ..., 19}	B = {1, 3, 5, ..., 19}
A′ = U - A	A′ = U - A
= {1, 2, 3, ..., 20} - {2, 4, 6, ..., 20}	= {1, 2, 3, ..., 20} - {2, 4, 6, ..., 20}
= {1, 3, 5, ..., 19}	= {1, 3, 5, ..., 19}
B′ = U - B	B′ = U - B
= {1, 2, 3, ..., 20} - {1, 3, 5, ..., 19}	= {1, 2, 3, ..., 20} - {1, 3, 5, ..., 19}
= {2, 4, 6, ..., 20}	= {2, 4, 6, ..., 20}
A ∪ B = {1, 2, 3, ..., 20}	A ∩ B = { }
(A ∪ B)′ = U - (A ∪ B)	(A ∩ B)′ = U - (A ∩ B)
= {1, 2, 3, ..., 20} - {1, 2, 3, ..., 20}	= {1, 2, 3, ..., 20} - { }
= { }	= {1, 2, 3, ..., 20}
A′ ∩ B′= {1, 3, 5, ..., 19} ∩ {2, 4, 6, ..., 20}	A′ ∪ B′= {1, 3, 5, ..., 19} ∪ {2, 4, 6, ..., 20}
= { }	= {1, 2, 3, ..., 20}
Hence (A ∪ B)′ = A′ ∩ B′	Hence (A ∩ B)′ = A′ ∪ B′

8. Consider the set $P = \{x \mid x = 5m, m \in N\}$ and $Q = \{x \mid x = 2m, m \in N\}$. Find $P \cap Q$

Solution

P = {x | x = 5m, m ∈ N} = {5, 10, 15, 20, 25, ...}

Q = {x | x = 2m, m ∈ N} = {2, 4, 6, 8, 10, 12, ...}

P ∩ Q = {5, 10, 15, 20, 25, ...} ∩ {2, 4, 6, 8, 10, 12, ...}

P ∩ Q = {10, 20, 30, 40, 50, ...} = {x | x = 10m, m ∈ N}

9. From suitable properties of union and intersection, deduce the following results:
(i) $A \cap (A \cup B) = A \cup (A \cap B)$ (ii) $A \cup (A \cap B) = A \cap (A \cup B)$

Solution

(i) L.H.S. = A ∩ (A ∪ B) = (A ∩ A) ∪ (A ∩ B) = A ∪ (A ∩ B) = **R.H.S.**

(i) L.H.S. = A ∪ (A ∩ B) = (A ∪ A) ∩ (A ∪ B) = A ∩ (A ∪ B) = **R.H.S.**

10. If $g(x) = 7x - 2$ and $s(x) = 8x^2 - 3$ find:

(i) $g(0)$ (ii) $g(-1)$ (iii) $g\left(-\frac{5}{3}\right)$ (iv) $s(1)$ (v) $s(-9)$ (vi) $s\left(\frac{7}{2}\right)$

Solution

i. $g(x) = 7x - 2 \Rightarrow g(0) = 7(0) - 2 \Rightarrow g(0) = -2$

ii. $g(x) = 7x - 2 \Rightarrow g(-1) = 7(-1) - 2 \Rightarrow g(-1) = -7 - 2 \Rightarrow g(-1) = -9$

iii. $g(x) = 7x - 2 \Rightarrow g\left(-\frac{5}{3}\right) = 7\left(-\frac{5}{3}\right) - 2 \Rightarrow g\left(-\frac{5}{3}\right) = -\frac{35}{3} - 2 \Rightarrow g\left(-\frac{5}{3}\right) = -\frac{41}{3}$

iv. $s(x) = 8x^2 - 3 \Rightarrow s(1) = 8(1)^2 - 3 \Rightarrow s(1) = 8 - 3 \Rightarrow s(1) = 5$

v. $s(x) = 8x^2 - 3 \Rightarrow s(-9) = 8(-9)^2 - 3 \Rightarrow s(-9) = 648 - 3 \Rightarrow s(-9) = 645$

vi. $s(x) = 8x^2 - 3 \Rightarrow s\left(\frac{7}{2}\right) = 8\left(\frac{7}{2}\right)^2 - 3 \Rightarrow s\left(\frac{7}{2}\right) = 98 - 3 \Rightarrow s\left(\frac{7}{2}\right) = 95$

11. Given that $f(x) = ax + b$, where a and b are constant numbers. If $f(-2) = 3$ and $f(4) = 10$, then find the values of a and b.

Solution

$f(x) = ax + b$

$f(-2) = a(-2) + b$	$f(4) = a(4) + b$
$\mathbf{-2a + b = 3}$ **……(i)**	$\mathbf{4a + b = 10}$ **……(ii)**

(i) - (ii)	**2(i) + (ii)**
$-2a + b = 3$	$-4a + 2b = 6$
$-4a \pm b = -10$	$4a + b = 10$
$\mathbf{a = \frac{7}{6}}$	$\mathbf{b = \frac{16}{3}}$

12. Consider the function defined by $k(x) = 7x - 5$. If $k(x) = 100$, find the value of x.

Solution

$k(x) = 7x - 5$

Using $k(x) = 100$

$7x - 5 = 100$

$7x = 100 + 5$

$x = \frac{105}{7}$

13. Consider the function $g(x) = mx^2+n$, where m and n are constant numbers. If

$g(4) = 20$ and $g(0) = 5$, find the values of m and n.

Solution

$g(x) = mx^2 + n$

$g(0) = m(0)^2 + n$

$\mathbf{n = 5}$

Now $g(4) = m(4)^2 + n$

$\mathbf{16m + n = 20}$

Using $n = 5$

$16m + 5 = 20$

$\mathbf{m = \frac{15}{16}}$

14. A shopping mall has 100 products from various categories labeled 1 to 100, representing the universal set U. The products are categorized as follows:

- Set A: Electronics, consisting of 30 products labeled from 1 to 30.
- Set B: Clothing comprises 25 products labeled from 31 to 55.
- Set C: Beauty Products, comprising 25 products labeled from 76 to 100.

Write each set in tabular form, and find the union of all three sets.

Solution

U = {1, 2, 3, ... , 100}

A = {1, 2, 3, ... , 30}

B = {31, 32, 33, ... , 55}

C = {76, 77, 78, ... , 100}

A ∪ B ∪ C = {1, 2, 3, ..., 30} ∪ {31, 32, ..., 55} ∪ {76, 77, ..., 100}

A ∪ B ∪ C = {1, 2, 3, ..., 30, 31, 32, ..., 55, 76, 77, ..., 100}

15. Out of the 180 students who appeared in the annual examination, 120 passed the math test, 90 passed the science test, and 60 passed both the math and science tests.

(a) How many passed either the math or science test?
(b) How many did not pass either of the two tests?
(c) How many passed the science test but not the math test?
(d) How many failed the science test?

Solution

Total students = 180

Passed Math = 120, Passed Science = 90, Passed both Math and Science = 60

1. How many passed either the Math or Science test?

Passed either Math or Science = Passed Math + Passed Science - Passed both

= 120 + 90 – 60 = 150

2. How many did not pass either of the two tests?

Failed both Math and Science = Total students - Passed either Math or Science

= 180 – 150 = 30

3. How many passed the Science test but not the Math test?

Passed Science but not Math = Passed Science - Passed both

= 90 – 60 = 30

4. How many failed the Science test?

Failed Science = Total students - Passed Science

= 180 – 90 = 90

16. In a software house of a city with 300 software developers, a survey was conducted to determine which programming languages are liked more. The survey revealed the following statistics:

- 150 developers like Python.
- 130 developers like Java.
- 120 developers like PHP.
- 70 developers like both Python and Java.
- 60 developers like both Python and PHP.
- 50 developers like both Java and PHP.
- 40 developers like all three languages: Python, Java and PHP.
  - (a) How many developers use at least one of these languages?
  - (b) How many developers use only one of these languages?
  - (c) How many developers do not use any of these languages?
  - (d) How many developers use only PHP?

Solution

Total developers = 300

$n(P) = 150$, $n(J) = 130$, $n(H) = 120$, $n(P \cap J) = 70$, $n(P \cap H) = 60$, $n(J \cap H) = 50$

$n(P \cap J \cap H) = 40$

1. How many developers use at least one of these languages?

$n(P \cup J \cup H) = n(P) + n(J) + n(H) - n(P \cap J) - n(P \cap H) - n(J \cap H) + n(P \cap J \cap H)$

$= 150 + 130 + 120 - 70 - 60 - 50 + 40 =$ **260**

2. How many developers use only one of these languages?

Developers who use only P $= n(P) - n(P \cap J) - n(P \cap H) + n(P \cap J \cap H)$

$= 150 - 70 - 60 + 40 = 60$

Developers who use only J = n(J) – n(P ∩ J) – n(J ∩ H) + n(P ∩ J ∩ H)

= 130 - 70 - 50 + 40 = 50

Developers who use only H = n(H) – n(P ∩ H) – n(J ∩ H) + n(P ∩ J ∩ H)

= 120 - 60 - 50 + 40 = 50

Total developers who use only one language = 60 + 50 + 50 = **160**

3. How many developers do not use any of these languages?

Developers who do not use any language = Total developers - Developers who use at least one language

= 300 – 260 = **40**

4. How many developers use only PHP?

Developers who use only PHP = n(H) = **50**

Unit 4 Factorization and Algebraic Manipulation

EXERCISE 4.1

1. Factorize by identifying common factors.

(i) $6x + 12$ (ii) $15y^2 + 20y$ (iii) $-12x^2 - 3x$

(iv) $4a^2b + 8ab^2$ (v) $xy - 3x^2 + 2x$ (vi) $3a^2b - 9ab^2 + 15ab$

Solution:

(i) $6(x + 2)$ (ii) $5y(3y + 4)$ (iii) $-3x(4x + 1)$

(iv) $4ab(a + 2b)$ (v) $x(y - 3x + 2)$

(vi) $3ab(a - 3b + 5)$

2. Factorize and represent pictorially:

(i) $5x + 15$ (ii) $x^2 + 4x + 3$ (iii) $x^2 + 6x + 8$

(iv) $x^2 + 4x + 4$

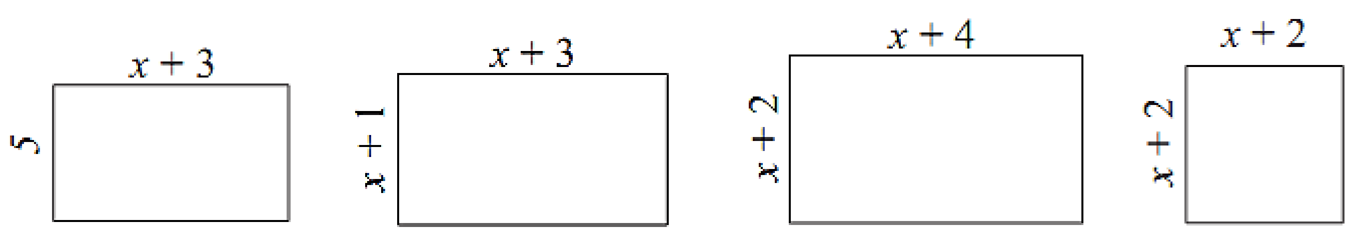
Solution

(i) $5(x+3)$

(ii) $x^2 + 4x + 3 = x^2 + 3x + x + 3 = x(x + 3) + 1(x + 3) = (x + 1)(x + 3)$

(iii) $x^2 + 6x + 8 = x^2 + 4x + 2x + 8 = x(x + 4) + 2(x + 4) = (x + 2)(x + 4)$

(iv) $x^2 + 4x + 4 = x^2 + 2x + 2x + 4 = x(x + 2) + 2(x + 2) = (x + 2)(x + 2) = (x + 2)^2$



3. Factorize:

(i) $x^2 + x - 12$ (ii) $x^2 + 7x + 10$ (iii) $x^2 - 6x + 8$

(iv) $x^2 - x - 56$ (v) $x^2 - 10x - 24$ (vi) $y^2 + 4y - 12$

(vii) $y^2 + 13y + 36$ (viii) $x^2 - x - 2$

Solution

(i) $x^2 + x - 12 = x^2 + 4x - 3x - 12 = x(x+4) - 3(x+4) = (x+4)(x-3)$

(ii) $x^2 + 7x + 10 = x^2 + 5x + 2x + 10 = x(x+5) + 2(x+5) = (x+5)(x+2)$

(iii) $x^2 - 6x + 8 = x^2 - 4x - 2x + 8 = x(x-4) - 2(x-4) = (x-4)(x-2)$

(iv) $x^2 - x - 56 = x^2 - 8x + 7x - 56 = x(x-8) + 7(x-8) = (x-8)(x+7)$

(v) $x^2 - 10x - 24 = x^2 - 12x + 2x - 24 = x(x-12) + 2(x-12) = (x-12)(x+2)$

(vi) $y^2 + 4y - 12 = y^2 + 6y - 2y - 12 = y(y+6) - 2(y+6) = (y+6)(y-2)$

(vii) $y^2 + 13y + 36 = y^2 + 9y + 4y + 36 = y(y+9) + 4(y+9) = (y+9)(y+4)$

(viii) $x^2 - x - 2 = x^2 - 2x + x - 2 = x(x-2) + 1(x-2) = (x-2)(x+1)$

4. Factorize:

(i) $2x^2 + 7x + 3$ (ii) $2x^2 + 11x + 15$ (iii) $4x^2 + 13x + 3$

(iv) $3x^2 + 5x + 2$ (v) $3y^2 - 11y + 6$ (vi) $2y^2 - 5y + 2$

(vii) $4z^2 - 11z + 6$ (viii) $6 + 7x - 3x^2$

Solution

(i) $2x^2 + 7x + 3 = 2x^2 + 6x + x + 3 = 2x(x+3) + 1(x+3) = (2x+1)(x+3)$

(ii) $2x^2 + 11x + 15 = 2x^2 + 6x + 5x + 15 = 2x(x+3) + 5(x+3) = (2x+5)(x+3)$

(iii) $4x^2 + 13x + 3 = 4x^2 + 12x + x + 3 = 4x(x+3) + 1(x+3) = (4x+1)(x+3)$

(iv) $3x^2 + 5x + 2 = 3x^2 + 3x + 2x + 2 = 3x(x+1) + 2(x+1) = (3x+2)(x+1)$

(v) $3y^2 - 11y + 6 = 3y^2 - 9y - 2y + 6 = 3y(y-3) - 2(y-3) = (3y-2)(y-3)$

(vi) $2y^2 - 5y + 2 = 2y^2 - 4y - y + 2 = 2y(y-2) - 1(y-2) = (2y-1)(y-2)$

(vii) $4z^2 - 11z + 6 = 4z^2 - 8z - 3z + 6 = 4z(z-2) - 3(z-2) = (4z-3)(z-2)$

(viii) $6 + 7x - 3x^2 = -3x^2 + 7x + 6 = -3x^2 + 9x - 2x + 6$

$= 3x(-x+3) + 2(-x+3) = (3x+2)(3-x)$

EXERCISE 4.2

1. Factorize each of the following expressions:

(i) $4x^4 + 81y^4$ (ii) $a^4 + 64b^4$ (iii) $x^4 + 4x^2 + 16$

(iv) $x^4 - 14x^2 + 1$ (v) $x^4 - 30x^2y^2 + 9y^4$ (vi) $x^4 - 7x^2y^2 + y^4$

Solution

1.(i) $4x^4 + 81y^4$

$= (2x^2)^2 + (9y^2)^2 = (2x^2)^2 + (9y^2)^2 + 2(2x^2)(9y^2) - 2(2x^2)(9y^2)$

$= (2x^2 + 9y^2)^2 - 36x^2y^2 = (2x^2 + 9y^2)^2 - (6xy)^2$

$= (2x^2 + 9y^2 - 6xy)(2x^2 + 9y^2 + 6xy)$

1.(ii) $a^4 + 64b^4$

$= (a^2)^2 + (8b^2)^2 = (a^2)^2 + (8b^2)^2 + 2(a^2)(8b^2) - 2(a^2)(8b^2)$

$= (a^2 + 8b^2)^2 - 16\,a^2b^2 = (a^2 + 8b^2)^2 - (4ab)^2$

$= (a^2 + 8b^2 + 4ab)(a^2 + 8b^2 - 4ab)$

1.(iii) $x^4 + 4x^2 + 16$

$= x^4 + 8x^2 - 4x^2 + 16 = x^4 + 8x^2 + 16 - 4x^2$

$= (x^2 + 4)^2 - (2x)^2$

$= (x^2 + 4 - 2x)(x^2 + 4 + 2x)$

1.(iv) $x^4 - 14x^2 + 1$

$= x^4 - 16x^2 + 2x^2 + 1 = x^4 + 2x^2 + 1 - 16x^2$

$= (x^2 + 1)^2 - (4x)^2$

$= (x^2 + 1 - 4x)(x^2 + 1 + 4x)$

1.(v) $\boldsymbol{x^4 - 30x^2y^2 + 9y^4}$

$= x^4 - 36x^2y^2 + 6x^2y^2 + 9y^4 = x^4 + 6x^2y^2 + 9y^4 - 36x^2y^2$

$= (x^2 + 3y^2)^2 - (6xy)^2$

$= (x^2 - 6xy + 3y^2)(x^2 + 6xy + 3y^2)$

1.(vi) $\boldsymbol{x^4 - 7x^2y^2 + y^4}$

$= x^4 - 9x^2y^2 + 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - 9x^2y^2$

$= (x^2 + y^2)^2 - (3xy)^2$

$= (x^2 - 3xy + y^2)(x^2 + 3xy + y^2)$

2. Factorize each of the following expressions:

(i) $(x+1)(x+2)(x+3)(x+4)+1$ (ii) $(x+2)(x-7)(x-4)(x-1)+17$

(iii) $(2x^2+7x+3)(2x^2+7x+5)+1$ (iv) $(3x^2+5x+3)(3x^2+5x+5)-3$

(v) $(x+1)(x+2)(x+3)(x+6)-3x^2$ (vi) $(x+1)(x-1)(x+2)(x-2)-16x^2$

Solution

2.(i) $\boldsymbol{(x+1)(x+2)(x+3)(x+4)+1}$

$= (x+1)(x+4)(x+2)(x+3)+1$

$= (x^2+5x+4)(x^2+5x+6)+1$

$= (y+4)(y+6)+1 = y^2+10y+24+1 = y^2+10y+25$

$= (y+5)^2 = (x^2+5x+5)^2$

2.(ii) $\boldsymbol{(x+2)(x-7)(x-4)(x-1)+17}$

$= (x+2)(x-7)(x-4)(x-1)+17$

$= (x^2-5x-14)(x^2-5x+4)+17$

$= (y-14)(y+4)+17 = y^2-10y-56+17 = y^2-10y-39$

$= y^2-13y+3y-39 = y(y-13)+3(y-13)$

$= (y-13)(y+3) = (x^2-5x-13)(x^2-5x+3)$

2.(iii) $(2x^2 + 7x + 3)(2x^2 + 7x + 5) + 1$

$= (2x^2 + 7x + 3)(2x^2 + 7x + 5) + 1$

$= (y + 3)(y + 5) + 1 = y^2 + 8y + 15 + 1 = y^2 + 8y + 16$

$= (y + 4)^2 = (2x^2 + 7x + 4)^2$

2.(iv) $(3x^2 + 5x + 3)(3x^2 + 5x + 5) - 3$

$= (3x^2 + 5x + 3)(3x^2 + 5x + 5) - 3$

$= (y + 3)(y + 5) - 3 = y^2 + 8y + 15 - 3 = y^2 + 8y + 12$

$= y^2 + 6y + 2y + 12 = y(y + 6) + 2(y + 6)$

$= (y + 6)(y + 2) = (3x^2 + 5x + 6)(3x^2 + 5x + 2)$

2.(v) $(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$

$= (x + 1)(x + 6)(x + 2)(x + 3) - 3x^2$

$= (x^2 + 7x + 6)(x^2 + 5x + 6) - 3x^2$

$= (x^2 + 6 + 7x)(x^2 + 6 + 5x) - 3x^2$

$= (y + 7x)(y + 5x) - 3x^2 = y^2 + 12xy + 35x^2 - 3x^2$

$= y^2 + 12xy + 32x^2 = y^2 + 8xy + 4xy + 32x^2 = y(y + 8x) + 4x(y + 8x)$

$= (y + 8x)(y + 4x) = (x^2 + 8x + 6)(x^2 + 4x + 6)$

2.(vi) $(x + 1)(x - 1)(x + 2)(x - 2) - 16x^2$

$= (x + 1)(x + 2)(x - 1)(x - 2) - 16x^2$

$= (x^2 + 3x + 2)(x^2 - 3x + 2) - 16x^2$

$= (x^2 + 2 + 3x)(x^2 + 2 - 3x) - 16x^2$

$= (y + 3x)(y - 3x) - 16x^2 = y^2 - 9x^2 - 16x^2 = y^2 - 25x^2$

$= (y - 5x)(y + 5x) = (x^2 - 5x + 2)(x^2 + 5x + 2)$

3. Factorize:

(i) $8x^3 + 12x^2 + 6x + 1$ (ii) $27a^3 + 108a^2b + 144ab^2 + 64b^3$

(iii) $x^3 + 18x^2y + 108xy^2 + 216y^3$ (iv) $8x^3 - 125y^3 + 150xy^2 - 60x^2y$

Solution

Remember!

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

3.(i) $\mathbf{8x^3 + 12x^2 + 6x + 1}$

$= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3$

$= (2x + 1)^3$

3.(ii) $\mathbf{27a^3 + 108a^2b + 144ab^2 + 64b^3}$

$= (3a)^3 + 3(3a)^2(4b) + 3(3a)(4b)^2 + (4b)^3$

$= (3a + 4b)^3$

3.(iii) $\mathbf{x^3 + 18x^2y + 108xy^2 + 216y^3}$

$= (x)^3 + 3(x)^2(6y) + 3(x)(6y)^2 + (6y)^3$

$= (x + 6y)^3$

3.(iv) $\mathbf{8x^3 - 125y^3 + 150xy^2 - 60x^2y}$

$= (2x)^3 + (-5y)^3 + 3(2x)(-5y)^2 + 3(2x)^2(-5y)$

$= (2x - 5y)^3$

4. Factorize:

(i) $125a^3 - 1$ (ii) $64x^3 + 125$ (iii) $x^6 - 27$

(iv) $1000a^3 + 1$ (v) $343x^3 + 216$ (vi) $27 - 512y^3$

Solution

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

4.(i) $\mathbf{125a^3 - 1}$

$= (5a)^3 - (1)^3$

$= (5a - 1)[(5a)^2 + (5a)(1) + (1)^2]$

$= (5a - 1)(25a^2 + 5a + 1)$

4.(ii) $\mathbf{64x^3 + 125}$

$= (4x)^3 + (5)^3$

$= (4x + 5)[(4x)^2 - (4x)(5) + (5)^2]$

$= (4x + 5)(16x^2 - 20x + 25)$

4.(iii) $\mathbf{x^6 - 27}$

$= (x^2)^3 - (3)^3$

$= (x^2 - 3)[(x^2)^2 + (x^2)(3) + (3)^2]$

$= (x^2 - 3)(x^4 + 3x^2 + 9)$

4.(iv) $\mathbf{1000a^3 + 1}$

$= (10a)^3 + (1)^3$

$= (10a + 1)[(10a)^2 - (10a)(1) + (1)^2]$

$= (10a + 1)(100a^2 - 10a + 1)$

4.(v) $\mathbf{343x^3 + 216}$

$= (7x)^3 + (6)^3$

$= (7x + 6)[(7x)^2 - (7x)(6) + (6)^2]$

$= (7x + 6)(49x^2 - 42x + 36)$

4.(vi) $\mathbf{27 - 512y^3}$

$= (3)^3 - (8y)^3$

$= (3 - 8y)[(3)^2 + (3)(8y) + (8y)^2]$

$= (3 - 8y)(9 + 24y + 64y^2)$

EXERCISE 4.3

1. Find HCF by factorization method.

(i) $21x^2y, 35xy^2$ (ii) $4x^2 - 9y^2, 2x^2 - 3xy$

(iii) $x^3 - 1, x^2 + x + 1$ (iv) $a^3 + 2a^2 - 3a, 2a^3 + 5a^2 - 3a$

(v) $t^2 - 3t - 4, t^2 + 5t + 4, t^2 - 1$ (vi) $x^2 + 15x + 56, x^2 + 5x - 24, x^2 + 8x$

Solution

1.(i) $21x^2y, 35xy^2$

$21x^2y = 3 \times 7 \times x \times x \times y$

$35xy^2 = 5 \times 7 \times x \times y \times y$

$HCF = 7 \times x \times y = 7xy$

1.(ii) $4x^2 - 9y^2, 2x^2 - 3xy$

$4x^2 - 9y^2 = (2x)^2 - (3y)^2 = (2x - 3y)(2x + 3y)$

$2x^2 - 3xy = x(2x - 3y)$

$HCF = 2x - 3y$

1.(iii) $x^3 - 1, x^2 + x + 1$

$x^3 - 1 = (x - 1)(x^2 + x + 1)$

$x^2 + x + 1 = x^2 + x + 1$

$HCF = x^2 + x + 1$

1.(iv) $a^3 + 2a^2 - 3a, 2a^3 + 5a^2 - 3a$

$a^3 + 2a^2 - 3a = a(a^2 + 2a - 3) = a(a^2 + 3a - a - 3) = a(a + 3)(a - 1)$

$2a^3 + 5a^2 - 3a = a(2a^2 + 5a - 3) = a(2a^2 + 6a - a - 3) = a(a + 3)(2a - 1)$

$HCF = a(a + 3)$

1.(v) $\boldsymbol{t^2 - 3t - 4, t^2 + 5t + 4, t^2 - 1}$

$\mathrm{t^2 - 3t - 4 = t^2 - 4t + t - 4 = t(t - 4) + 1(t - 4) = (t - 4)(t + 1)}$

$\mathrm{t^2 + 5t + 4 = t^2 + 4t + t + 4 = t(t + 4) + 1(t + 4) = (t + 4)(t + 1)}$

$\mathrm{t^2 - 1 = (t - 1)(t + 1)}$

$\mathrm{HCF = t + 1}$

1.(vi) $\boldsymbol{x^2 + 15x + 56, x^2 + 5x - 24, x^2 + 8x}$

$\mathrm{x^2 + 15x + 56 = x^2 + 8x + 7x + 56 = x(x + 8) + 7(x + 8) = (x + 8)(x + 7)}$

$\mathrm{x^2 + 5x - 24 = x^2 + 8x - 3x - 24 = x(x + 8) - 3(x + 8) = (x + 8)(x - 3)}$

$\mathrm{x^2 + 8x = x(x + 8)}$

$\mathrm{HCF = x + 8}$

2. Find HCF of the following expressions by using division method:

(i) $27x^3 + 9x^2 - 3x - 10,\ 3x - 2$ (ii) $x^3 - 9x^2 + 23x - 15,\ x^2 - 4x + 3$

(iii) $2x^3 + 2x^2 + 2x + 2,\ 6x^3 + 12x^2 + 6x + 12$

(iv) $2x^3 - 4x^2 + 16x, x^3 - 4x,\ 3x^2 + 6x$

Solution

2.(i) $\boldsymbol{27x^3 + 9x^2 - 3x - 10, 3x - 2}$

Quotient: $9x^2 + 9x + 5$

$3x - 2 \overline{)\, 27x^3 + 9x^2 - 3x - 10}$

$\pm 27x^3 \mp 18x^2$

$27x^2 - 3x - 10$

$\pm 27x^2 \mp 18x$

$15x - 10$

$\pm 15x \mp 10$

0

$\mathrm{HCF = 3x - 2}$

2.(ii) $\boldsymbol{x^3 - 9x^2 + 23x - 15, x^2 - 4x + 3}$

$$
\begin{array}{r|l}
 & x-5 \\
\hline
x^2-4x+3 & x^3-9x^2+23x-15 \\
 & \underline{\pm x^3 \mp 4x^2 \pm 3x} \\
 & \quad -5x^2+20x-15 \\
 & \quad \underline{\mp 5x^2 \pm 20x \mp 15} \\
 & \qquad 0
\end{array}
$$

HCF $= x^2 - 4x + 3$

2.(iii) $\boldsymbol{2x^3 + 2x^2 + 2x + 2, 6x^3 + 12x^2 + 6x + 12}$

$$
\begin{array}{r|l}
 & 3 \\
\hline
2x^3+2x^2+2x+2 & 6x^3+12x^2+6x+12 \\
 & \underline{\pm 6x^3 \pm 6x^2 \pm 6x \pm 6} \\
 & \qquad 6x^2 \qquad +6
\end{array}
$$

$$
\begin{array}{r|l}
 & \frac{x}{3} \\
\hline
6x^2+6 & 2x^3+2x^2+2x+2 \\
 & \underline{\pm 2x^3 \qquad \pm 2x} \\
 & \qquad 2x^2 \qquad +2
\end{array}
$$

and

$$
\begin{array}{r|l}
 & 3 \\
\hline
2x^2+2 & 6x^2+6 \\
 & \underline{\pm 6x^2 \pm 6} \\
 & \quad 0
\end{array}
$$

HCF $= 2x^2 + 2 = 2(x^2 + 1)$

2.(iv) $2x^3 - 4x^2 - 16x, x^3 - 4x, 3x^2 + 6x$

Let $3x^2 + 6x = 3(x^2 + 2x)$

3 is not common in both given polynomials, so ignoring it, we consider only $x^2 + 2x$

$$
\begin{array}{r|l}
 & x - 2 \\
x^2 + 2x & x^3 - 4x \\
 & \pm x^3 \pm 2x^2 \\
\hline
 & -2x^2 - 4x \\
 & \mp 2x^2 \mp 4x \\
\hline
 & 0
\end{array}
$$

Now,

$$
\begin{array}{r|l}
 & 2x - 8 \\
x^2 + 2x & 2x^3 - 4x^2 - 16x \\
 & \pm 2x^3 \pm 4x^2 \\
\hline
 & -8x^2 - 16x \\
 & \mp 8x^2 \mp 16x \\
\hline
 & 0
\end{array}
$$

HCF $= x^2 + 2x = x(x + 2)$

3. Find LCM of the following expressions by using prime factorization method.

(i) $2a^2b,\ 4ab^2,\ 6ab$ (ii) $x^2+x,\ x^3+x^2$

(iii) $a^2-4a+4,\ a^2-2a$ (iv) $x^4-16,\ x^3-4x$

(v) $16-4x^2,\ x^2+x-6,\ 4-x^2$

Solution

3.(i) $2a^2b, 4ab^2, 6ab$

$2a^2b = 2\times a\times a\times b$

$4ab^2 = 2\times 2\times a\times b\times b$

$6ab = 2\times 3\times a\times b$

$\text{Common Factors} = 2\times a\times b = 2ab$

$\text{non} - \text{Common Factors} = 2\times 3\times a\times b = 6ab$

$\text{LCM} = \text{CF}\times\text{NCF} = 2ab\times 6ab = 12a^2b^2$

3.(ii) x^2+x, x^3+x^2

$x^2+x = x(x+1)$

$x^3+x^2 = x^2(x+1) = x\times x\times (x+1)$

$\text{Common Factors} = x(x+1)$

$\text{non} - \text{Common Factors} = x$

$\text{LCM} = \text{CF}\times\text{NCF} = x(x+1)\times x = x^2(x+1)$

3.(iii) a^2-4a+4, a^2-2a

$a^2-4a+4 = (a-2)^2 = (a-2)(a-2)$

$a^2-2a = a(a-2)$

$\text{Common Factors} = (a-2)$

$\text{non} - \text{Common Factors} = a(a-2)$

$\text{LCM} = \text{CF}\times\text{NCF} = (a-2)\times a(a-2) = a(a-2)^2$

3.(iv) $x^4 - 16, x^3 - 4x$

$x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$

$x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2)$

$\text{Common Factors} = (x - 2)(x + 2)$

$\text{non} - \text{ Common Factors} = x(x^2 + 4)$

$\text{LCM} = \text{CF} \times \text{NCF} = (x - 2)(x + 2) \times x(x^2 + 4) = x(x^4 - 16)$

3.(v) $16 - 4x^2, x^2 + x - 6, 4 - x^2$

$16 - 4x^2 = 4(4 - x^2) = 4(2 - x)(2 + x)$

$x^2 + x - 6 = x^2 + 3x - 2x - 6 = (x + 3)(x - 2) = -(x + 3)(2 - x)$

$4 - x^2 = (2 - x)(2 + x)$

$\text{Common Factors} = (2 - x)(2 + x)$

$\text{non} - \text{ Common Factors} = -4(x + 3)$

$\text{LCM} = \text{CF} \times \text{NCF} = (2 - x)(2 + x) \times -4(x + 3) = 4(x^2 - 4)(x + 3)$

4. The HCF of two polynomials is $y - 7$ and their LCM is $y^3 - 10y^2 + 11y + 70$. If one of the polynomials is $y^2 - 5y - 14$, find the other.

Solution

$\text{HCF} = y - 7$

$\text{LCM} = y^3 - 10y^2 + 11y + 70$

$p(y) = y^2 - 5y - 14$

$q(y) = ?$

Using formula: $p(y) \times q(y) = \text{HCF} \times \text{LCM}$

$(y^2 - 5y - 14) \times q(y) = (y - 7) \times (y^3 - 10y^2 + 11y + 70)$

$$q(y) = \frac{(y-7)\times(y^3-10y^2+11y+70)}{(y^2-5y-14)}$$

$$q(y) = \frac{(y-7)\times(y^3-10y^2+11y+70)}{(y-7)(y+2)}$$

$$q(y) = \frac{y^3-10y^2+11y+70}{y+2}$$

$$y+2 \;\Big)\; y^3 - 10y^2 + 11y + 70 \;\Big(\; y^2 - 12y + 35$$

$$\pm y^3 \pm 2y^2$$

$$-12y^2 + 11y + 70$$

$$\mp 12y^2 \mp 24y$$

$$35y + 70$$

$$\pm 35y \pm 70$$

$$0$$

$$\mathbf{q(y) = \frac{y^3 - 10y^2 + 11y + 70}{y+2} = y^2 - 12y + 35}$$

5. The LCM and HCF of two polynomial $p(x)$ and $q(x)$ are $36x^3(x + a)(x^3 - a^3)$ and $x^2(x - a)$ respectively. If $p(x) = 4x^2(x^2 - a^2)$, find $q(x)$.

Solution

$\text{HCF} = x^2(x - a)$

$\text{LCM} = 36x^3(x + a)(x^3 - a^3)$

$p(x) = 4x^2(x^2 - a^2)$

$q(x) = ?$

Using formula: $p(x) \times q(x) = \text{HCF} \times \text{LCM}$

$4x^2(x^2 - a^2) \times q(x) = x^2(x - a) \times 36x^3(x + a)(x^3 - a^3)$

$q(x) = \frac{x^2(x-a) \times 36x^3(x+a)(x^3-a^3)}{4x^2(x^2-a^2)}$

$q(x) = 9x^3(x^3 - a^3)$

6. The HCF and LCM of two polynomials is $(x + a)$ and $12x^2(x + a)(x^2 - a^2)$ respectively. Find the product of the two polynomials.

Solution

$\text{HCF} = (x + a)$

$\text{LCM} = 12x^2(x + a)(x^2 - a^2)$

$p(x) \times q(x) = ?$

Using formula: $p(x) \times q(x) = \text{HCF} \times \text{LCM}$

$p(x) \times q(x) = (x + a) \times 12x^2(x + a)(x^2 - a^2)$

$p(x) \times q(x) = 12x^2(x + a)^2(x^2 - a^2) = 12x^2(x + a)^3(x - a)$

EXERCISE 4.4

1. Find the square root of the following polynomials by factorization method:

(i) $x^2 - 8x + 16$ (ii) $9x^2 + 12x + 4$

(iii) $36a^2 + 84a + 49$ (iv) $64y^2 - 32y + 4$

(v) $200t^2 - 120t + 18$ (vi) $40x^2 + 120x + 90$

Solution

1.(i) $\sqrt{x^2 - 8x + 16}$ =???

$$x^2 - 8x + 16 = (x)^2 - 2(x)(4) + (4)^2 = (x - 4)^2$$

$$\sqrt{x^2 - 8x + 16} = \sqrt{(x - 4)^2}$$

$$\sqrt{x^2 - 8x + 16} = \pm(x - 4)$$

1.(ii) $\sqrt{9x^2 + 12x + 4}$ =???

$$9x^2 + 12x + 4 = (3x)^2 + 2(3x)(2) + (2)^2 = (3x + 2)^2$$

$$\sqrt{9x^2 + 12x + 4} = \sqrt{(3x + 2)^2}$$

$$\sqrt{9x^2 + 12x + 4} = \pm(3x + 2)$$

1.(iii) $\sqrt{36a^2 + 84a + 49}$ =???

$$36a^2 + 84a + 49 = (6a)^2 + 2(6a)(7) + (7)^2 = (6a + 7)^2$$

$$\sqrt{36a^2 + 84a + 49} = \sqrt{(6a + 7)^2}$$

$$\sqrt{36a^2 + 84a + 49} = \pm(6a + 7)$$

1.(iv) $\sqrt{64y^2 - 32y + 4}$ =???

$$64y^2 - 32y + 4 = (8y)^2 - 2(8y)(2) + (2)^2 = (8y - 2)^2$$

$$\sqrt{64y^2 - 32y + 4} = \sqrt{(8y - 2)^2}$$

$$\sqrt{64y^2 - 32y + 4} = \pm(8y - 2)$$

1.(v) $\sqrt{\mathbf{200t^2 - 120t + 18}}$ =???

$200t^2 - 120t + 18 = 2[100t^2 - 60t + 9]$

$200t^2 - 120t + 18 = 2[(10t)^2 - 2(10t)(3) + (3)^2] = 2(10t - 3)^2$

$\sqrt{64y^2 - 32y + 18} = \sqrt{2(10t - 3)^2}$

$\sqrt{64y^2 - 32y + 18} = \pm\sqrt{2}(10t - 3)$

1.(vi) $\sqrt{\mathbf{40x^2 + 120x + 90}}$ =???

$40x^2 + 120x + 90 = 10(4x^2 + 12x + 9) = 10[(2x)^2 + 2(2x)(3) + (3)^2]$

$40x^2 + 120x + 90 = 10(2x + 3)^2$

$\sqrt{40x^2 + 120x + 90} = \sqrt{10(2x + 3)^2}$

$\sqrt{40x^2 + 120x + 90} = \pm\sqrt{10}(2x + 3)$

2. Find the square root of the following polynomials by division method:

(i) $4x^4 - 28x^3 + 37x^2 + 42x + 9$

(ii) $121x^4 - 198x^3 - 183x^2 + 216x + 144$

(iii) $x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4$

(iv) $4x^4 - 12x^3 + 37x^2 - 42x + 49$

Solution

1.(i) $\sqrt{\mathbf{4x^4 - 28x^3 + 37x^2 + 42x + 9}}$ =???

	$2x^2 - 7x - 3$
$2x^2$	$4x^4 - 28x^3 + 37x^2 + 42x + 9$ $\pm 4x^4$
$4x^2 - 7x$	$-28x^3 + 37x^2$ $\mp 28x^3 \pm 49x^2$
$4x^2 - 14x - 3$	$-12x^2 + 42x + 9$ $\mp 12x^2 \pm 42x \pm 9$
	0

$\sqrt{4x^4 - 28x^3 + 37x^2 + 42x + 9} = \pm(2x^2 - 7x - 3)$

1.(ii) $\sqrt{\mathbf{121x^4 - 198x^3 - 183x^2 + 216x + 144}} = ???$

	$11x^2 - 9x - 12$
$11x^2$	$121x^4 - 198x^3 - 183x^2 + 216x + 144$ $\pm 121x^4$
$22x^2 - 9x$	$-198x^3 - 183x^2$ $\mp 198x^3 \pm 81x^2$
$22x^2 - 18x - 12$	$-264x^2 + 216x + 144$ $\mp 264x^2 \pm 216x \pm 144$
	0

$$\sqrt{121x^4 - 198x^3 - 183x^2 + 216x + 144} = \pm(11x^2 - 9x - 12)$$

1.(iii) $\sqrt{\mathbf{x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4}} = ???$

	$x^2 - 5xy + y^2$
x^2	$x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4$ $\pm x^4$
$2x^2 - 5xy$	$-10x^3y + 27x^2y^2$ $\mp 10x^3y \pm 25x^2y^2$
$2x^2 - 10xy + y^2$	$2x^2y^2 - 10xy^3 + y^4$ $\pm 2x^2y^2 \mp 10xy^3 \pm y^4$
	0

$$\sqrt{x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4} = \pm(x^2 - 5xy + y^2)$$

1.(iv) $\sqrt{\mathbf{4x^4 - 12x^3 + 37x^2 - 42x + 49}} = ???$

	$2x^2 - 3x + 7$
$2x^2$	$4x^4 - 12x^3 + 37x^2 - 42x + 49$ $\pm 4x^4$
$4x^2 - 3x$	$-12x^3 + 37x^2$ $\mp 12x^3 \pm 9x^2$
$4x^2 - 6x + 7$	$28x^2 - 42x + 49$ $\pm 28x^2 \mp 42x \pm 49$
	0

$$\sqrt{4x^4 - 12x^3 + 37x^2 - 42x + 49} = \pm(2x^2 - 3x + 7)$$

3. An investor's return $R(x)$ in rupees after investing x thousand rupees is given by quadratic expression:

$$R(x) = -x^2 + 6x - 8$$

Factorize the expression and find the investment levels that result in zero return.

Solution

$R(x) = -x^2 + 6x - 8 = -x^2 + 4x + 2x - 8 = -x(x - 4) + 2(x - 4)$

$R(x) = (-x + 2)(x - 4)$

For zero return $R(x) = 0$ we have $(-x + 2)(x - 4) = 0$

$-x + 2 = 0$ or $x - 4 = 0$

$x = 2$ or $x = 4$

Investment levels that result in zero return will be $x = 2$ and $x = 4$

4. A company's profit $P(x)$ in rupees from selling x units of a product is modeled by the cubic expression:

$$P(x) = x^3 - 15x^2 + 75x - 125$$

Find the break-even point(s), where the profit is zero.

Solution

$P(x) = x^3 - 15x^2 + 75x - 125$

$P(x) = (x)^3 - 3(x)^2(5) + 3(x)(5)^2 - (5)^3 = (x - 5)^3$

Since profit is zero, using $P(x) = 0$ we have $(x - 5)^3 = 0$

After taking cube root on both sides we have $x = 5$

5. The potential energy $V(x)$ in an electric field varies as a cubic function of distance x, given by:

$$V(x) = 2x^3 - 6x^2 + 4x$$

Determine where the potential energy is zero.

Solution

$V(x) = 2x^3 - 6x^2 + 4x$

$V(x) = 2x(x^2 - 3x + 2) = 2x(x - 2)(x - 1)$

For zero potential energy, using $V(x) = 0$ we have $2x(x - 1)(x - 2) = 0$

Then $x = 0, x = 1, x = 2$

6. In structural engineering, the deflection $Y(x)$ of a beam is given by:

$$Y(x) = 2x^2 - 8x + 6$$

This equation gives the vertical deflection at any point x along the beam. Find the points of zero deflection.

Solution

$$Y(x) = 2x^2 - 8x^2 + 6$$

$$Y(x) = 2(x^2 - 4x + 3)$$

$$Y(x) = 2(x^2 - 3x - x + 3)$$

$$Y(x) = 2(x - 1)(x - 3)$$

For zero potential deflection, using $Y(x) = 0$ we have $2(x - 1)(x - 3) = 0$

$2 \neq 0$ then $x - 1 = 0$ or $x - 3 = 0$

Then $x = 1, x = 3$

REVIEW EXERCISE 4

1. Four options are given against each statement. Encircle the correct option.

   i. The factorization of $12x + 36$ is:

   (a) ✓ $12(x + 3)$ (b) $12(3x)$ (c) $12(3x + 1)$ (d) $x(12 + 36x)$

   ii. The factors of $4x^2 - 12y + 9$ are:

   (a) $(2x + 3)^2$ (b) ✓ $(2x - 3)^2$
   (c) $(2x - 3)(2x + 3)$ (d) $(2 + 3x)(2 - 3x)^2$

   iii. The HCF of a^3b^3 and ab^2 is:

   (a) a^3b^3 (b) ✓ ab^2 (c) a^4b^5 (d) a^2b

   iv. The LCM of $16x^2$, $4x$ and $30xy$ is:

   (a) $480x^3y$ (b) $240xy$ (c) ✓ $240x^2y$ (d) $120x^4y$

   v. Product of LCM and HCF = __________ of two polynomials.

   (a) sum (b) difference (c) ✓ product (d) quotient

   vi. The square root of $x^2 - 6x + 9$ is:

   (a) ✓ $\pm(x - 3)$ (b) $\pm(x + 3)$ (c) $x - 3$ (d) $x + 3$

   vii. The LCM of $(a - b)^2$ and $(a - b)^4$ is:

   (a) $(a - b)^2$ (b) $(a - b)^3$ (c) ✓ $(a - b)^4$ (d) $(a - b)^6$

   viii. Factorization of $x^3 + 3x^2 + 3x + 1$ is:

   (a) ✓ $(x + 1)^3$ (b) $(x - 1)^3$
   (c) $(x + 1)(x^2 + x + 1)$ (d) $(x - 1)(x^2 - x + 1)$

   ix. Cubic polynomial has degree:

   (a) 1 (b) 2 (c) ✓ 3 (d) 4

   x. One of the factors of $x^3 - 27$ is:

   (a) ✓ $x - 3$ (b) $x + 3$ (c) $x^2 - 3x + 9$ (d) Both *a* and *c*

2. Factorize the following expressions:

(i) $4x^3 + 18x^2 - 12x$ (ii) $x^3 + 64y^3$

(iii) $x^3y^3 - 8$ (iv) $-x^2 - 23x - 60$

(v) $2x^2 + 7x + 3$ (vi) $x^4 + 64$

(vii) $x^4 + 2x^2 + 9$ (viii) $(x + 3)(x + 4)(x + 5)(x + 6) - 360$

(ix) $(x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

Solution

2.(i) $\mathbf{4x^3 + 18x^2 - 12x}$

$= 2x(2x^2 + 9x - 6)$

2.(ii) $\mathbf{x^3 + 64y^3}$

$= (x)^3 + (4y)^3$

$= (x + 4y)[(x)^2 - (x)(4y) + (4y)^2]$

$= (x + 4y)(x^2 - 4xy + 16y^2)$

2.(iii) $\mathbf{x^3y^3 - 8}$

$= (xy)^3 - (2)^3$

$= (xy - 2)[(xy)^2 + (xy)(2) + (2)^2]$

$= (xy - 2)(x^2y^2 + 2xy + 4)$

2.(iv) $\mathbf{-x^2 - 23x - 60}$

$= -(x^2 + 23x + 60) = -[x^2 + 20x + 3x + 60] = -[x(x + 20) + 3(x + 20)]$

$= -(x + 3)(x + 20)$

2.(v) $\mathbf{2x^2 + 7x + 3}$

$= 2x^2 + 6x + x + 3 = 2x(x + 3) + 1(x + 3)$

$= (2x + 1)(x + 3)$

2.(vi) $\boldsymbol{x^4 + 64}$

$= (x^2)^2 + (8)^2 + 2(x^2)(8) - 2(x^2)(8) = (x^2 + 8)^2 - 16x^2$

$= (x^2 + 8)^2 - (4x)^2 = (x^2 + 8 - 4x)(x^2 + 8 + 4x)$

$= (x^2 - 4x + 8)(x^2 + 4x + 8)$

2.(vii) $\boldsymbol{x^4 + 2x^2 + 9}$

$= (x^2)^2 + 2x^2 + (3)^2$

$= (x^2)^2 + 2(x^2)(3) + (3)^2 + 2x^2 - 2(x^2)(3)$

$= (x^2 + 3)^2 - 4x^2$

$= (x^2 + 3)^2 - (2x)^2 = (x^2 + 3 - 2x)(x^2 + 3 + 2x)$

$= (x^2 - 2x + 3)(x^2 + 2x + 3)$

2.(viii) $\boldsymbol{(x + 3)(x + 4)(x + 5)(x + 6) - 360}$

$= (x + 3)(x + 6)(x + 4)(x + 5) - 360$

$= (x^2 + 9x + 18)(x^2 + 9x + 20) - 360$

$= (y + 18)(y + 20) - 360 = y^2 + 38y + 360 - 360 = y^2 + 38y$

$= y(y + 38) = (x^2 + 9x)(x^2 + 9x + 38)$

$= x(x + 9)(x^2 + 9x + 38)$

2.(ix) $\boldsymbol{(x^2 + 6x + 3)(x^2 + 6x - 9) + 36}$

$= (x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

$= (y + 3)(y - 9) + 36 = y^2 - 6y - 27 + 36 = y^2 - 6y + 9$

$= (y)^2 - 2(y)(3) + (3)^2 = (y - 3)^2$

$= (x^2 + 6x - 3)^2$

3. Find LCM and HCF by prime factorization method:

(i) $4x^3 + 12x^2,\ 8x^2 + 16x$ (ii) $x^3 + 3x^2 - 4x,\ x^2 - x - 6$

(iii) $x^2 + 8x + 16, x^2 - 16$ (iv) $x^3 - 9x,\ x^2 - 4x + 3$

Solution

3.(i) $4x^3 + 12x^2, 8x^2 + 16x$

$4x^3 + 12x^2 = 4x^2(x+3) = 4 \times x \times x \times (x+3)$

$8x^2 + 16x = 8x(x+2) = 2 \times 4 \times x \times (x+2)$

$\text{Common Factors} = 4x$

$\text{Un} - \text{ Common Factors} = x \times (x+3) \times 2 \times (x+2) = 2x(x+3)(x+2)$

$\text{HCF} = 4x$

$\text{LCM} = \text{CF} \times \text{UCF} = 4x \times 2x(x+3)(x+2) = 8x^2(x+2)(x+3)$

3.(ii) $x^3 + 3x^2 - 4x, x^2 - x - 6$

$x^3 + 3x^2 - 4x = x(x^2 + 3x - 4) = x(x^2 + 4x - x - 4) = x(x-1)(x+4)$

$x^2 - x - 6 = x^2 - 3x + 2x - 6 = (x-3)(x+2)$

$\text{Common Factors} = 1$

$\text{Un} - \text{ Common Factors} = x(x-1)(x+4)(x-3)(x+2)$

$\text{HCF} = 1$

$\text{LCM} = \text{CF} \times \text{UCF} = x(x-1)(x+2)(x-3)(x+4)$ **wrong answer in book**

3.(iii) $x^2 + 8x + 16, x^2 - 16$

$x^2 + 8x + 16 = (x)^2 + 2(x)(4) + (4)^2 = (x+4)^2 = (x+4)(x+4)$

$x^2 - 16 = (x)^2 - (4)^2 = (x-4)(x+4)$

$\text{Common Factors} = (x+4)$

$\text{Un} - \text{ Common Factors} = (x-4)(x+4)$

$\text{HCF} = (x+4)$

$\text{LCM} = \text{CF} \times \text{UCF} = (x+4) \times (x-4)(x+4) = (x-4)(x+4)^2$

3.(iv) $x^3 - 9x, x^2 - 4x + 3$

$x^3 - 9x = x(x^2 - 9) = x(x-3)(x+3)$

$x^2 - 4x + 3 = x^2 - 3x - x + 3 = (x-3)(x-1)$

$\text{Common Factors} = (x-3)$

$\text{Un} - \text{ Common Factors} = x(x+3)(x-1)$

$\text{HCF} = (x-3)$

$\text{LCM} = \text{CF} \times \text{UCF} = (x-3) \times x(x+3)(x-1)$

$\text{LCM} = x(x-1)(x^2-9)$ **wrong answer in book**

4. Find square root by factorization and division method of the expression $16x^4 + 8x^2 + 1$.

Solution

Factorization:

$16x^4 + 8x^2 + 1 = (4x^2)^2 + 2(4x^2)(1) + (1)^2$

$16x^4 + 8x^2 + 1 = (4x^2+1)^2$

$\sqrt{16x^4 + 8x^2 + 1} = \sqrt{(4x^2+1)^2}$

$\sqrt{16x^4 + 8x^2 + 1} = \pm(4x^2+1)$

Division Method:

	$4x^2 + 1$
	$16x^4 + 8x^2 + 1$
$4x^2$	$16x^4$
	$8x^2 + 1$
$8x^2 + 1$	$\pm 8x^2 \pm 1$
	0

$\sqrt{16x^4 + 8x^2 + 1} = \pm(4x^2+1)$

5. Huria is analyzing the total cost of her loan, modeled by the expression $C(x) = x^2 - 8x + 15$, where x represents the number of years. What is the optimal repayment period for Huria's loan?

Solution

$C(x) = x^2 - 8x + 15 = x^2 - 5x - 3x + 15 = (x-5)(x-3)$

For optimal repayment, using $C(x) = 0$ we have $(x-5)(x-3) = 0$

We have $x = 3, x = 5$. That is 3 years or 5 years.

Unit 5 Linear Equations and Inequalities

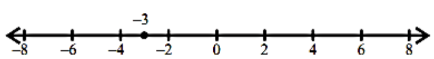
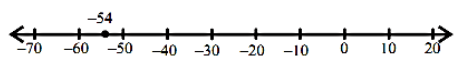
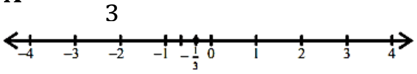
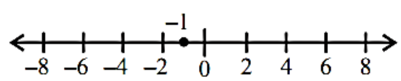
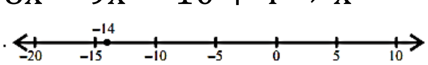
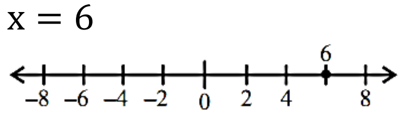
EXERCISE 5.1

1. Solve and represent the solution on a real line.

(i) $12x + 30 = -6$ (ii) $\frac{x}{3} + 6 = -12$ (iii) $\frac{x}{2} - \frac{3x}{4} = \frac{1}{12}$

(iv) $2 = 7(2x + 4) + 12x$ (v) $\frac{2x-1}{3} - \frac{3}{4}\frac{x}{} = \frac{5}{6}$ (vi) $\frac{-5x}{10} = 9 - \frac{10}{5}x$

Solution

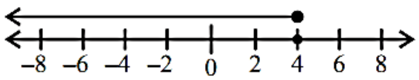
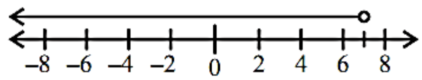
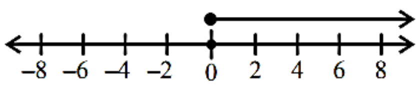
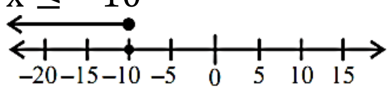
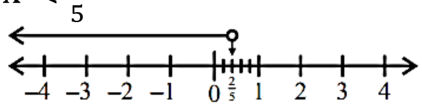
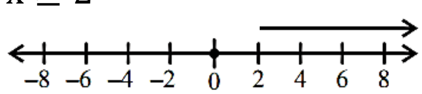
(i) $12x + 30 = -6$ $12x = -6 - 30$ $12x = -36$ $x = -\frac{36}{12}$ $x = -3$ 	**(ii)** $\frac{x}{3} + 6 = -12$ $\frac{x}{3} = -12 - 6$ $\frac{x}{3} = -18$ $x = -18 \times 3 \Rightarrow x = -54$ 
(iii) $\frac{x}{2} - \frac{3x}{4} = \frac{1}{12}$ $12 \times \left(\frac{x}{2}\right) - 12 \times \left(\frac{3x}{4}\right) = 12 \times \left(\frac{1}{12}\right)$ $6x - 9x = 1 \Rightarrow -3x = 1$ $x = -\frac{1}{3}$ 	**(iv)** $2 = 7(2x + 4) + 12x$ $2 = 14x + 28 + 12x$ $2 - 28 = 14x + 12x$ $-26 = 26x \Rightarrow x = -\frac{26}{26}$ $x = -1$ 
(v) $\frac{2x-1}{3} - \frac{3x}{4} = \frac{5}{6}$ $12 \times \left(\frac{2x-1}{3}\right) - 12 \times \left(\frac{3x}{4}\right) = 12 \times \left(\frac{5}{6}\right)$ $4(2x - 1) - 9x = 10$ $8x - 4 - 9x = 10$ $8x - 9x = 10 + 4 \Rightarrow x = -14$ 	**(vi)** $-\frac{5x}{10} = 9 - \frac{10}{5}x$ $10 \times \left(-\frac{5x}{10}\right) = 10 \times (9) - 10 \times \left(\frac{10}{5}x\right)$ $-5x = 90 - 20x$ $-5x + 20x = 90 \Rightarrow 15x = 90$ $x = 6$ 

2. Solve each inequality and represent the solution on a real line.

(i) $x-6\le -2$ (ii) $-9>-16+x$ (iii) $3+2x\ge 3$

(iv) $6(x+10)\le 0$ (v) $\frac{5}{3}x-\frac{3}{4}<\frac{-1}{12}$ (vi) $\frac{1}{4}x-\frac{1}{2}\le -1+\frac{1}{2}x$

Solution

(i) $x-6\le -2$ $x\le -2+6$ $x\le 4$ 	**(ii)** $-9>-16+x$ $-9+16>x$ $7>x$ or $x<7$ 
(iii) $3+2x\ge 3$ $2x\ge 3-3$ $2x\ge 0$ $x\ge 0$ 	**(iv)** $6(x+10)\le 0$ $6x+60\le 0$ $6x\le -60$ $x\le -\frac{60}{6}$ $x\le -10$ 
(v) $\frac{5}{3}x-\frac{3}{4}<-\frac{1}{12}$ $12\times\left(\frac{5}{3}x\right)-12\times\left(\frac{3}{4}\right)<12\times\left(-\frac{1}{12}\right)$ $4(5x)-9<-1$ $20x<-1+9$ $20x<8$ $x<\frac{8}{20}$ $x<\frac{2}{5}$ 	**(vi)** $\frac{1}{4}x-\frac{1}{2}\le -1+\frac{1}{2}x$ $4\times\left(\frac{1}{4}x\right)-4\times\left(\frac{1}{2}\right)\le 4\times(-1)+4\times\left(\frac{1}{2}x\right)$ $x-2\le -4+2x$ $-2+4\le 2x-x$ $2\le x$ $x\ge 2$ 

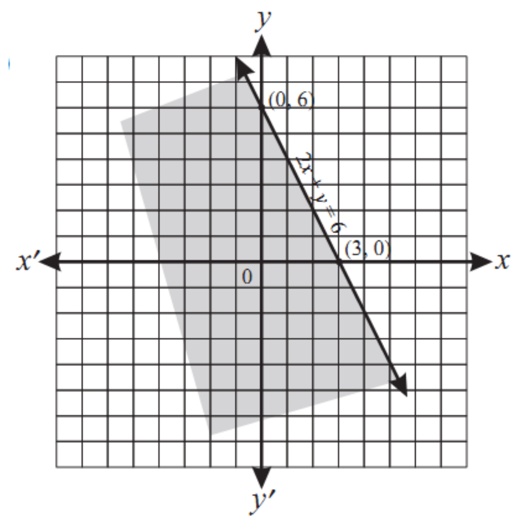
3. Shade the solution region for the following linear inequalities in xy-plane:

(i) $2x + y \leq 6$ (ii) $3x + 7y \geq 21$ (iii) $3x - 2y \geq 6$

(iv) $5x - 4y \leq 20$ (v) $2x + 1 \geq 0$ (vi) $3y - 4 \leq 0$

Solution

3 (i) $2x + y \leq 6$



Associated equations: $2x + y = 6$

To find Points:

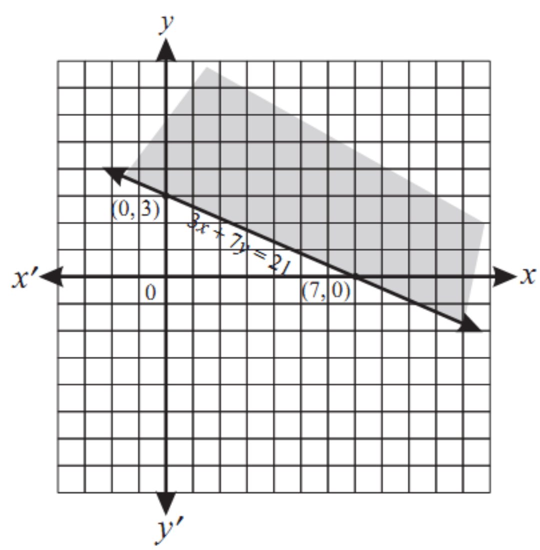
Put $x = 0$, $y = 6$ then point is (0,6)

Put $y = 0$, $x = 3$ then point is (3,0)

To check Region put $(0, 0)$ in given eq.

$0 < 6$ true, graph towards the origin

3 (ii) $3x + 7y \geq 21$



Associated equations: $3x + 7y = 21$

To find Points

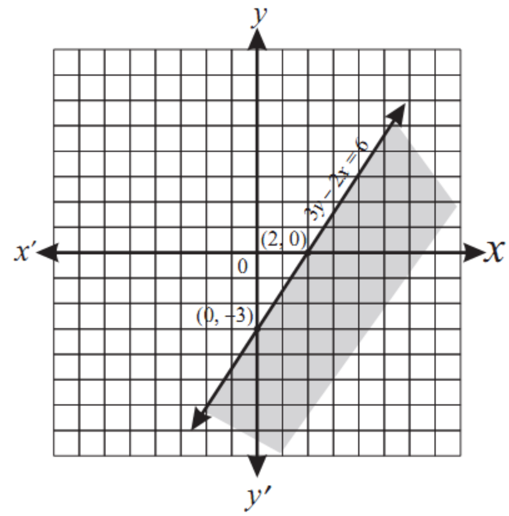
Put $x = 0$, $y = 3$ then point is (0,3)

Put $y = 0$, $x = 7$ then point is (7,0)

To check Region put $(0, 0)$ in given eq.

$0 > 21$ false, graph away from origin

3 (iii) $3x - 2y \geq 6$



Associated equations: $3x - 2y = 6$

To find Points:

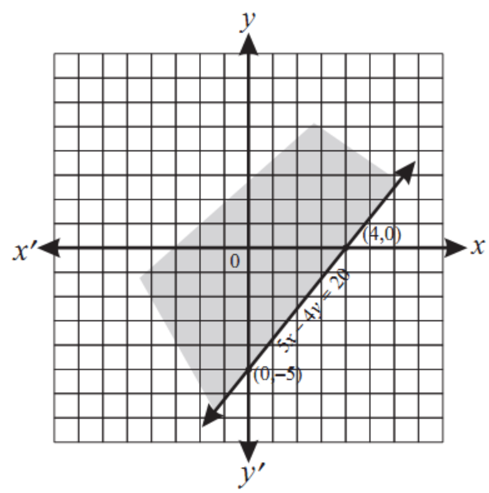
Put $x = 0$, $y = -3$ then point is $(0, -3)$

Put $y = 0$, $x = 2$ then point is $(2,0)$

To check Region put $(0, 0)$ in given eq.

$0 > 6$ false, graph away from origin

3 (iv) $5x - 4y \leq 20$



Associated equations: $5x - 4y = 20$

To find Points

Put $x = 0$, $y = -5$ then point is $(0, -5)$

Put $y = 0$, $x = 4$ then point is $(4,0)$

To check Region put $(0, 0)$ in given eq.

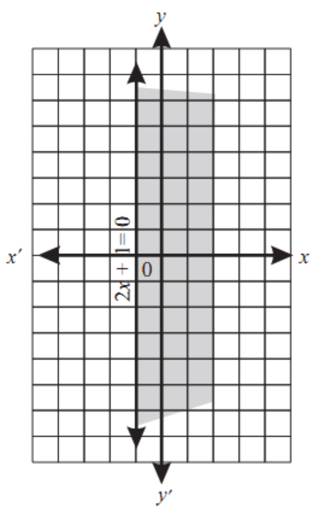
$0 < 20$ true, graph towards the origin

3 (v) $2x + 1 \geq 0$

Associated equations: $2x + 1 = 0$

Point: $x = -\frac{1}{2}$

Region: $1 > 0$ true, graph towards the origin

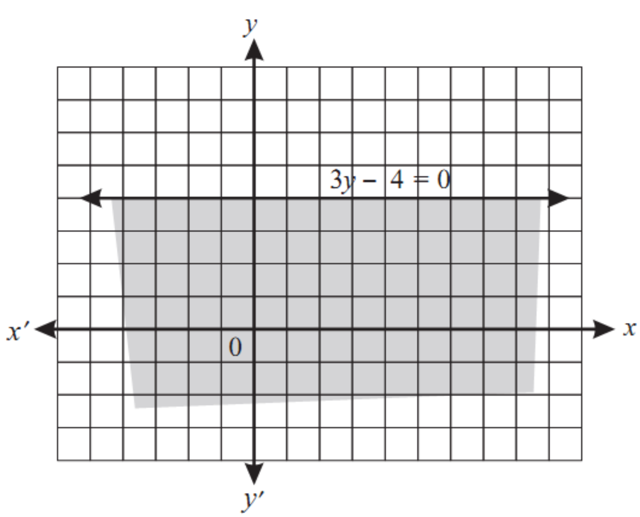


3 (vi) $3y - 4 \leq 0$

Associated equations: $3y - 4 = 0$

Point: $y = \frac{4}{3}$

Region: $0 < 4$ true, graph towards the origin



4. Indicate the solution region of the following linear inequalities by shading:

(i) $2x-3y\le 6$, $2x+3y\le 12$ (ii) $x+y\ge 5$, $-y+x\le 1$ (iii) $3x+7y\ge 21$, $x-y\le 2$

(iv) $4x-3y\le 12$, $x\ge -\frac{3}{2}$ (v) $3x+7y\ge 21$, $y\le 4$ (vi) $5x+7y\le 35$, $x-2y\le 2$

Solution

4 (i)

$\mathbf{2x - 3y \le 6}$ ………(i)

$\mathbf{2x + 3y \le 12}$ ………(ii)

Associated equations

$2x - 3y = 6$ ………(iii)

$2x + 3y = 12$ ………(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = -2$ then point is $(0, -2)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 3$ then point is $(3,0)$

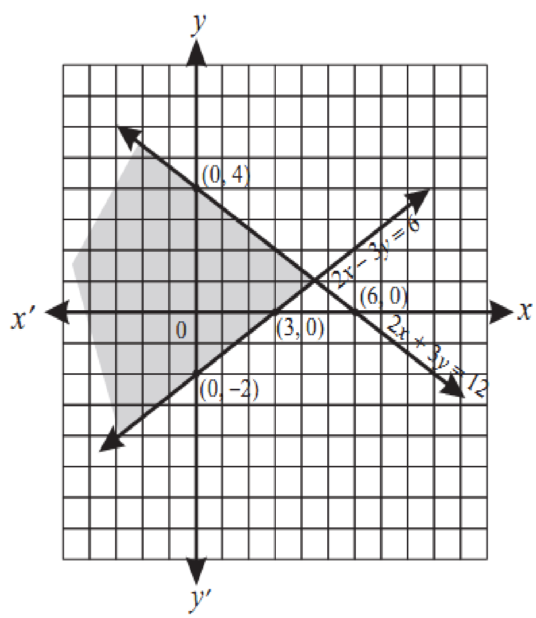
(iv) $\Rightarrow$ Put $x = 0$, $y = 4$ then point is $(0,4)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 6$ then point is $(6,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) $\Rightarrow 0 < 6$ true, graph towards the origin

(ii) $\Rightarrow 0 < 12$ true, graph towards the origin



4 (ii)

$\mathbf{x + y \geq 5}$ ………(**i**)

$\mathbf{-y + x \leq 1}$ ………(**ii**)

Associated equations

$x + y = 5$ ………(iii)

$x - y = 1$ ………(iv)

To find Points

(iii) ⇒ Put $x = 0$, $y = 5$ then point is (0,5)

(iii) ⇒ Put $y = 0$, $x = 5$ then point is (5,0)

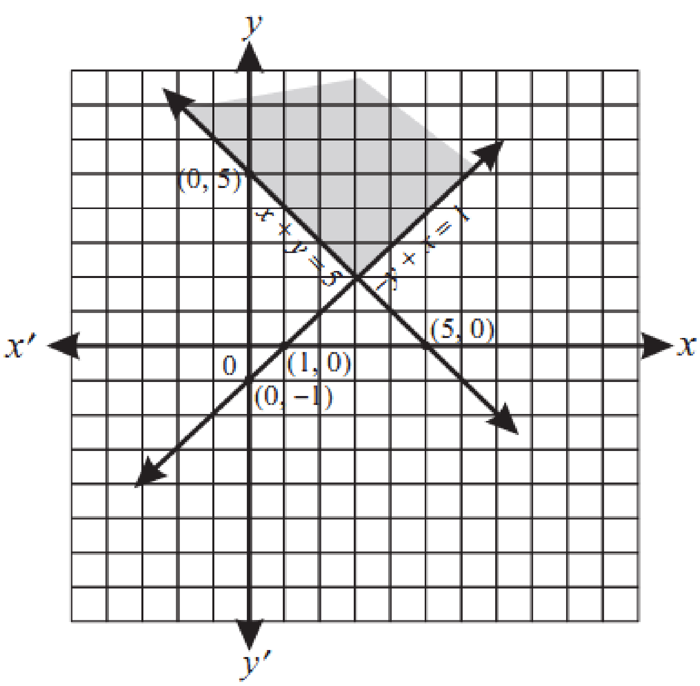
(iv) ⇒ Put $x = 0$, $y = -1$ then point is $(0, -1)$

(iv) ⇒ Put $y = 0$, $x = 1$ then point is (1,0)

To check Region put $(\mathbf{0}, \mathbf{0})$ in (i) and (ii)

(i) ⇒ $0 > 5$ false, graph away from origin

(ii) ⇒ $0 < 1$ true, graph towards the origin



4 (iii)

$\mathbf{3x + 7y \geq 21}$ **.........(i)**

$\mathbf{x - y \leq 2}$ **.........(ii)**

Associated equations

$3x + 7y = 21$ ....(iii)

$x - y = 2$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 7$ then point is $(7,0)$

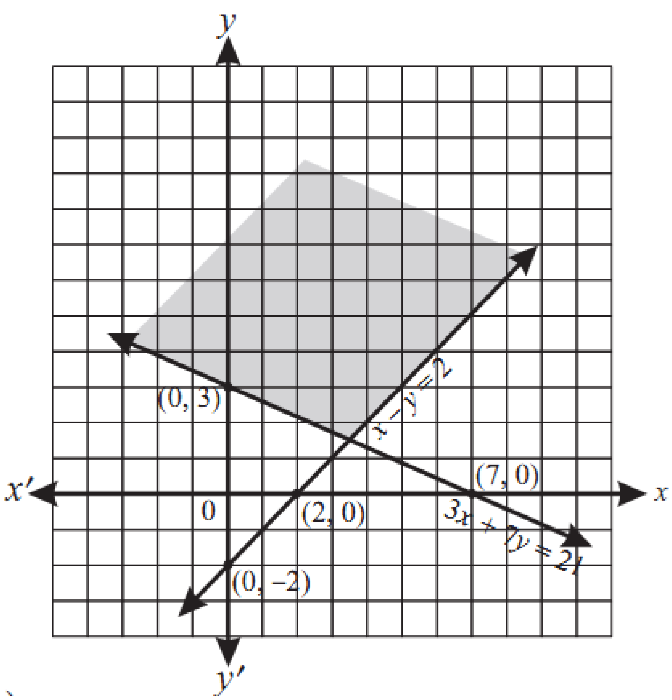
(iv) $\Rightarrow$ Put $x = 0$, $y = -2$ then point is $(0,-2)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

To check Region put $(\mathbf{0,0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 > 21$ false, graph away from origin

(ii) $\Rightarrow 0 < 2$ true, graph towards the origin



4 (iv)

$\mathbf{4x - 3y \leq 12}$ ………(i)

$\mathbf{x \geq -\frac{3}{2}}$ ………(ii)

Associated equations

$4x - 3y = 12$ ………(iii)

$x = -\frac{3}{2}$ ………(iv)

To find Points

(iii) ⇒ Put $x = 0$, $y = -4$ then point is $(0, -4)$

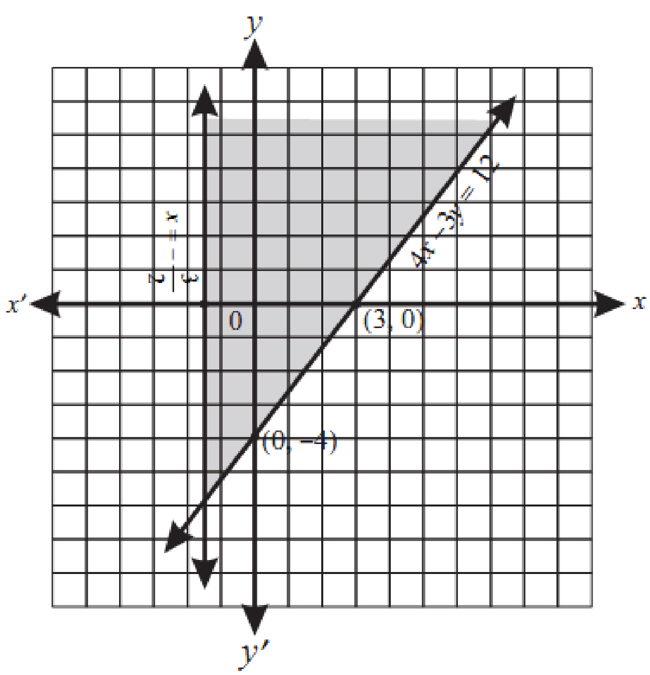
(iii) ⇒ Put $y = 0$, $x = 3$ then point is $(3,0)$

(iv) ⇒ we have $y = 0$, $x = -\frac{3}{2}$ then point is $\left(-\frac{3}{2}, 0\right)$

To check Region put $(\mathbf{0}, \mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) ⇒ $0 < 12$ true, graph towards the origin

(ii) ⇒ $0 > -\frac{3}{2}$ true, graph towards the origin



4 (v)

$\mathbf{3x + 7y \geq 12}$ **………(i)**

$\mathbf{y \leq 4}$ **………(ii)**

Associated equations

$3x + 7y = 12$ ………(iii)

$y = 4$ ………(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

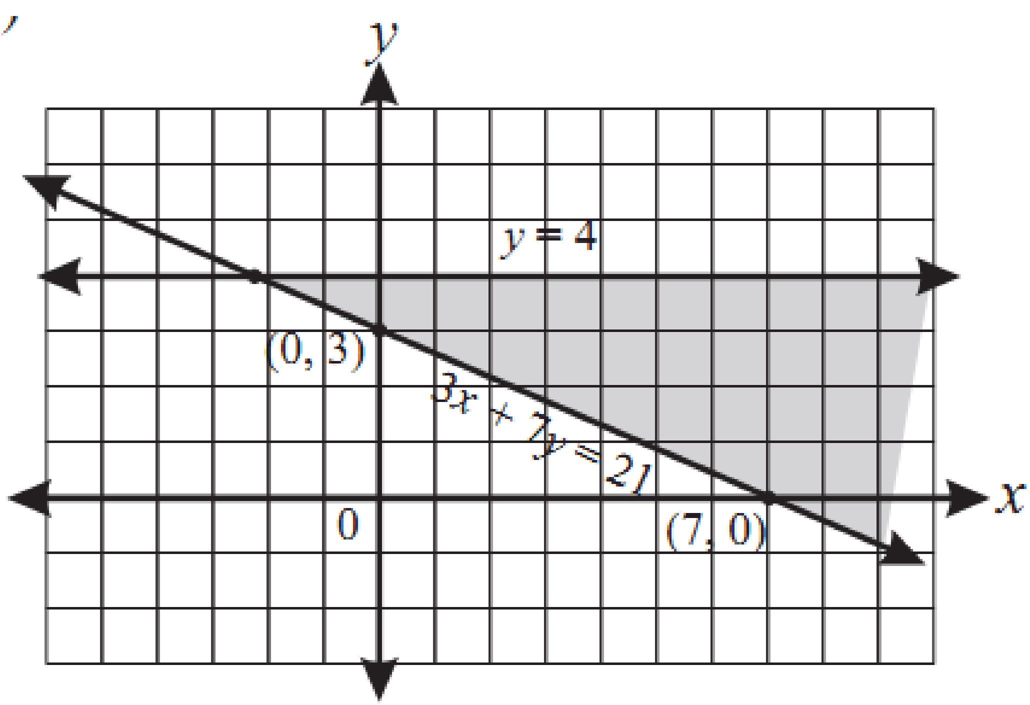
(iii) $\Rightarrow$ Put $y = 0$, $x = 7$ then point is $(7,0)$

(iv) $\Rightarrow$ we have $x = 0$, $y = 4$ then point is $(0,4)$

To check Region put $(\mathbf{0,0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 > 12$ false, graph away from origin

(ii) $\Rightarrow 0 < 4$ true, graph towards the origin



4 (vi)

$\mathbf{5x + 7y \leq 35}$ ………(i)

$\mathbf{x - 2y \leq 2}$ ………(ii)

Associated equations

$5x + 7y = 35$ ………(iii)

$x - 2y = 2$ ………(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 5$ then point is $(0,5)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 7$ then point is $(7,0)$

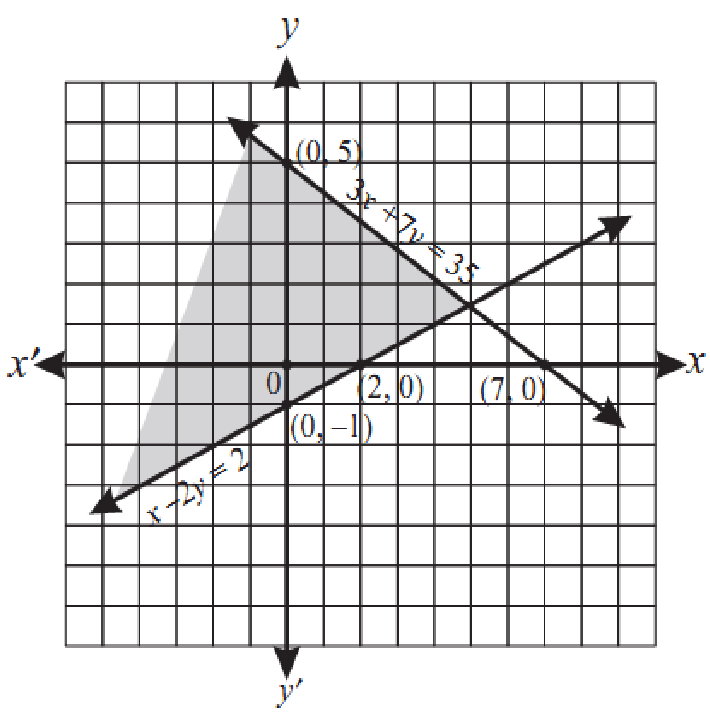
(iv) $\Rightarrow$ Put $x = 0$, $y = -1$ then point is $(0, -1)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

To check Region put $(\mathbf{0}, \mathbf{0})$ in (i) and (ii)

(i) $\Rightarrow 0 < 35$ true, graph towards the origin

(ii) $\Rightarrow 0 < 2$ true, graph towards the origin



EXERCISE 5.2

1. Maximize $f(x, y) = 2x + 5y$; subject to the constraints

$$2y - x \leq 8 \quad ; \quad x - y \leq 4 \quad ; \quad x \geq 0; \;\; y \geq 0$$

Solution

$-x + 2y \leq 8$ ………(i)

$x - y \leq 4$ ………(ii)

Associated equations

$-x + 2y = 8$ ………(iii)

$x - y = 4$ ………(iv)

To find Points

(iii) ⇒ Put $x = 0$, $y = 4$ then point is $(0,4)$

(iii) ⇒ Put $y = 0$, $x = -8$ then point is $(-8,0)$

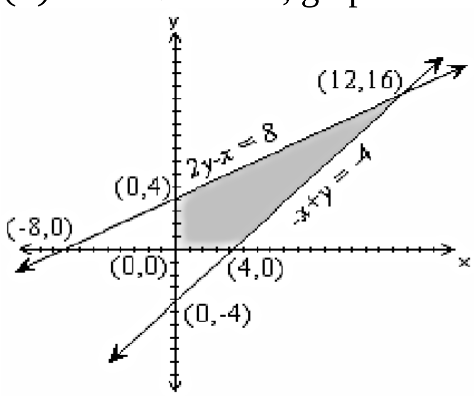
(iv) ⇒ Put $x = 0$, $y = -4$ then point is $(0,-4)$

(iv) ⇒ Put $y = 0$, $x = 4$ then point is $(4,0)$

To check Region put $(\mathbf{0},\mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) ⇒ $0 < 8$ true, graph towards the origin

(ii) ⇒ $0 < 4$ true, graph towards the origin



Solve $(\mathbf{iii}) + (\mathbf{iv})$

$(-x + 2y) + (x - y) = 8 + 4$ we have $y = 12$

Put $y = 12$ in (iii) we have $x = 16$ and $D(16,12)$

Corner Points of Feasible Region: $A(0,0), B(4,0), C(0,4), D(16,12)$

At A: $z = f(0,0) = 2(0) + 5(0) = 0$

At B: $z = f(4,0) = 2(4) + 5(0) = 8$

At C: $z = f(0,4) = 2(0) + 5(4) = 20$

At D: $z = f(16,12) = 2(16) + 5(12) = 92$

So $z = 2x + 5y$ is maximum at $(16,12)$

2. Maximize $f(x, y) = x + 3y$; subject to the constraints

$$2x + 5y \le 30 \quad ; \quad 5x + 4y \le 20 \quad ; \quad x \ge 0 \; ; y \ge 0$$

Solution

$2x + 5y \le 30$ ………(i)

$5x + 4y \le 20$ ………(ii)

Associated equations

$2x + 5y = 30$ ………(iii)

$5x + 4y = 20$ ………(iv)

To find Points

(iii) ⇒ Put $x = 0$, $y = 6$ then point is $(0,6)$

(iii) ⇒ Put $y = 0$, $x = 15$ then point is $(15,0)$

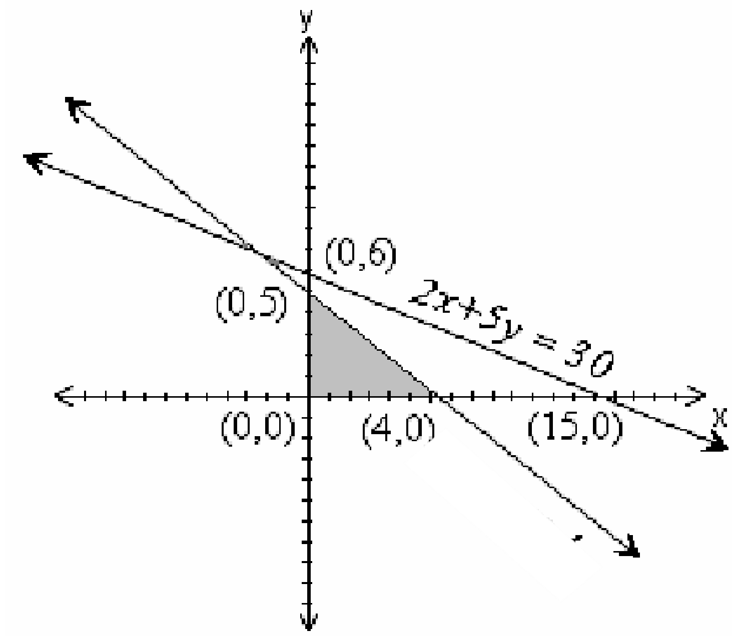
(iv) ⇒ Put $x = 0$, $y = 5$ then point is $(0,5)$

(iv) ⇒ Put $y = 0$, $x = 4$ then point is $(4,0)$

To check Region put $(\mathbf{0},\mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) ⇒ $0 < 30$ true, graph towards the origin

(ii) ⇒ $0 < 20$ true, graph towards the origin



Corner Points of Feasible Region: $\text{A}(0,0), \text{B}(4,0), C(0,5)$

At A: $z = f(0,0) = (0) + 3(0) = 0$

At B: $z = f(4,0) = (4) + 3(0) = 4$

At C: $z = f(0,5) = (0) + 3(5) = 15$

So $z = x + 3y$ is maximum at $(0,5)$

3. Maximize $z = 2x + 3y$; subject to the constraints:

$$2x + y \leq 4 \quad ; \quad 4x - y \leq 2 \quad ; \quad x \geq 0: \; y \geq 0$$

Solution

$2x + y \leq 4$ ....(i)

$4x - y \leq 2$ ....(ii)

Associated equations

$2x + y = 4$ ....(iii)

$4x - y = 2$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 4$ then point is $(0,4)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

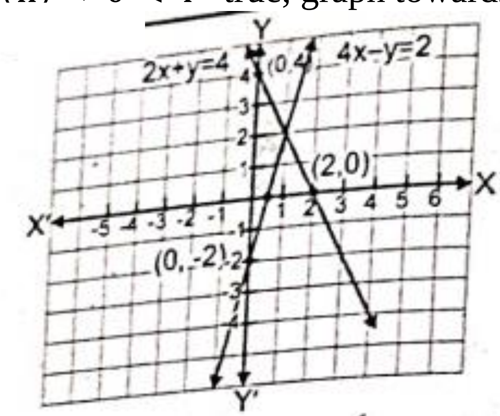
(iv) $\Rightarrow$ Put $x = 0$, $y = -2$ then point is $(0,-2)$

(iv) $\Rightarrow$ Put $y = 0$, $x = \frac{1}{2}$ then point is $\left(\frac{1}{2}, 0\right)$

To check Region put $(\mathbf{0}, \mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 < 4$ true, graph towards the origin

(ii) $\Rightarrow 0 < 4$ true, graph towards the origin



Solve $(\mathbf{iii}) + (\mathbf{iv})$

$(2x + y) + (4x - y) = 4 + 2$ we have $x = 1$

Put $x = 1$ in (iii) we have $y = 2$ and $(1,2)$

Corner Points: $(0,0), \left(\frac{1}{2}, 0\right), (0,4), (1,2)$

At A: $z = f(0,0) = 2(0) + 3(0) = 0$

At B: $z = f\left(\frac{1}{2}, 0\right) = 2\left(\frac{1}{2}\right) + 3(0) = 1$

At C: $z = f(0,4) = 2(0) + 3(4) = 12$

At P: $z = f(1,2) = 2(1) + 3(2) = 7$

So $z = 2x + 3y$ is maximum at $(0,4)$

4. Minimize $z = 2x + y$; subject to the constraints:

$$x + y \geq 3 \quad ; \quad 7x + 5y \leq 35 \quad ; \quad x \geq 0; \ \ y \geq 0$$

Solution

$x + y \geq 3$ ....(i)

$7x + 5y \leq 35$ ....(ii)

Associated equations

$x + y = 3$ ....(iii)

$7x + 5y = 35$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 3$ then point is $(3,0)$

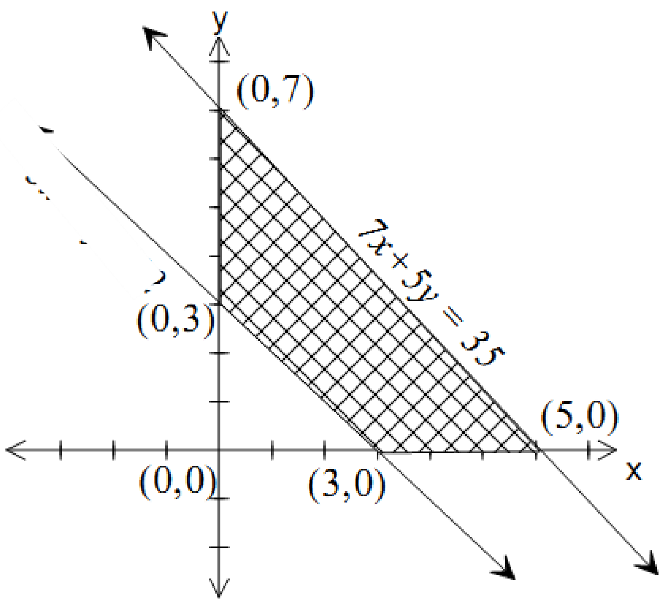
(iv) $\Rightarrow$ Put $x = 0$, $y = 7$ then point is $(0,7)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 5$ then point is $(5,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) $\Rightarrow 0 > 3$ false, graph away from the origin

(ii) $\Rightarrow 0 < 35$ true, graph towards the origin



Corner Points: $A(3,0), B(0,3), C(5,0), P(0,7)$

At A: $z = f(3,0) = 2(3) + (0) = 6$

At B: $z = f(0,3) = 2(0) + (3) = 3$

At C: $z = f(5,0) = 2(5) + (0) = 10$

At P: $z = f(0,7) = 2(0) + (7) = 7$

So $z = 2x + y$ is minimum at $(0,3)$

5. Maximize the function defined as; $f(x, y) = 2x + 3y$ subject to the constraints:

$$2x + y \leq 10 \quad ; \quad x + 2y \leq 14 \quad ; \quad x \geq 0; \; y \geq 0$$

Solution

$2x + y \leq 10$ ....(i)

$x + 2y \leq 14$ ....(ii)

Associated equations

$2x + y = 10$ ....(iii)

$x + 2y = 14$ ....(iv)

To find Points

(iii) ⇒ Put $x = 0$, $y = 10$ then point is $(0,10)$

(iii) ⇒ Put $y = 0$, $x = 5$ then point is $(5,0)$

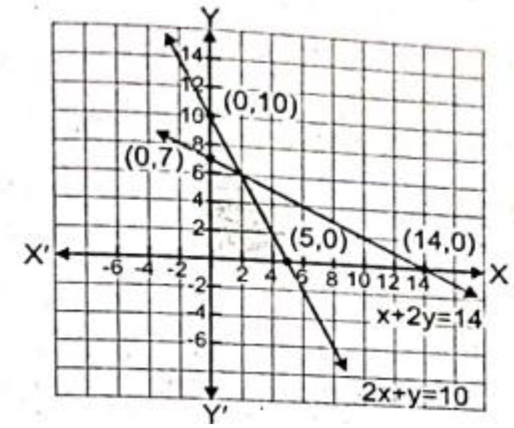
(iv) ⇒ Put $x = 0$, $y = 7$ then point is $(0,7)$

(iv) ⇒ Put $y = 0$, $x = 14$ then point is $(14,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) ⇒ $0 < 10$ true, graph towards the origin

(ii) ⇒ $0 < 14$ true, graph towards the origin



Solve 2(iii) − (iv)

$(4x + 2y) - (x + 2y) = 20 - 14$ we have $x = 2$

Put $x = 2$ in (iii) we have $y = 6$ and $C(2,6)$

Corner Points: $A(0,0), B(5,0), C(2,6), D(0,7)$

At A: $z = f(0,0) = 2(0) + 3(0) = 0$

At B: $z = f(5,0) = 2(5) + 3(0) = 10$

At C: $z = f(2,6) = 2(2) + 3(6) = 22$

At D: $z = f(0,7) = 2(0) + 3(7) = 21$

So $z = 2x + 3y$ is maximum at $(2,6)$

6. Find minimum and maximum values of $z = 3x + y$; subject to the constraints:

$$3x + 5y \geq 15 \quad ; \quad x + 3y \leq 9 \quad ; \quad x \geq 0; \quad y \geq 0$$

Solution

$3x + 5y \geq 15$ ....(i)

$x + 3y \leq 9$ ....(ii)

Associated equations

$3x + 5y = 15$ ....(iii)

$x + 3y = 9$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 5$ then point is $(5,0)$

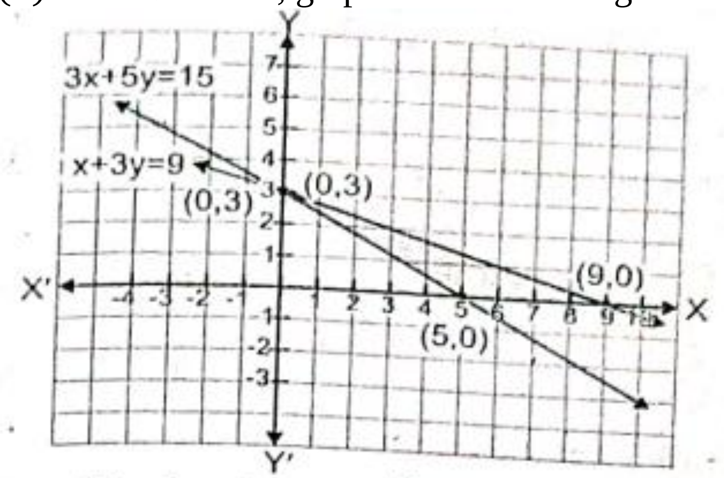
(iv) $\Rightarrow$ Put $x = 0$, $y = 2$ then point is $(0,2)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 9$ then point is $(9,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) $\Rightarrow 0 > 15$ false, graph away from the origin

(ii) $\Rightarrow 0 < 9$ true, graph towards the origin



Corner Points: $(0,3), (5,0), (9,0)$

At A: $z = f(0,3) = 3(0) + 3 = 3$

At B: $z = f(5,0) = 3(5) + 0 = 15$

At C: $z = f(9,0) = 3(9) + 0 = 27$

So $z = 3x + y$ is minimum at $(0,3)$ and maximum at $(9,0)$

REVIEW EXERCISE 5

1. Four options are given against each statement. Encircle the correct one.

i. In the following, linear equation is:

(a) $5x > 7$ (b) $4x - 2 < 1$
✓(c) $2x + 1 = 1$ (d) $4 = 1 + 3$

ii. Solution of $5x - 10 = 10$ is:

(a) 0 (b) 50
✓(c) 4 (d) – 4

iii. If $7x + 4 < 6x + 6$, then x belongs to the interval

(a) $(2, \infty)$ (b) $[2,\infty)$
✓(c) $(-\infty, 2)$ (d) $(-\infty, 2]$

iv. A vertical line divides the plane into

(a) left half plane (b) right half plane
(c) full plane ✓(d) two half planes

v. The linear equation formed out of the linear inequality is called

(a) cubic equation ✓(b) associated equation
(c) quadratic equal (d) feasible region

vi. $3x + 4 < 0$ is:

(a) equation ✓(b) inequality
(c) not inequality (d) identity

vii. Corner point is also called:

(a) code ✓(b) vertex
(c) curve (d) region

viii. (0,0) is solution of inequality:

(a) $4x+5y>8$ (b) $3x+y>6$

✓(c) $-2x+3y<0$ (d) $x+y>4$

ix. The solution region restricted to the first quadrant is called:

(a) objective region ✓(b) feasible region

(c) solution region (d) constraints region

x. A function that is to be maximized or minimized is called:

(a) solution function ✓(b) objective function

(c) feasible function (d) none of these

2. Solve and represent their solutions on real line.

(i) $\dfrac{x+5}{3}=1-x$ (ii) $\dfrac{2x+1}{3}+\dfrac{1}{2}=1-\dfrac{x-1}{3}$

(iii) $3x+7<16$ (iv) $5(x-3)\geq 26x-(10x+4)$

Solution

(i) $\frac{x+5}{3}=1-x$

$x+5=3-3x$

$x+3x=3-5$

$4x=-2$

$x=-\frac{2}{4}$

$x=-\frac{1}{2}$

(ii) $\frac{2x+1}{3}+\frac{1}{2}=1-\frac{x-1}{3}$

$6\times\left(\frac{2x+1}{3}\right)+6\times\left(\frac{1}{2}\right)=6\times(1)-6\times\left(\frac{x-1}{3}\right)$

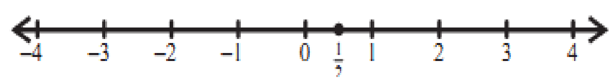
$2(2x+1)+3=6-2(x-1)$

$4x+2+3=6-2x+2$

$4x+2x=6+2-2-3$

$6x=3$

$x=\frac{1}{2}$



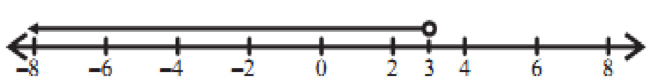
(iii) $3x+7<16$

$3x<16-7$

$3x<9$

$x<\frac{9}{3}$

$x<3$



(iv) $5(x-3)\geq 26x-(10x+4)$

$5x-15\geq 26x-10x-4$

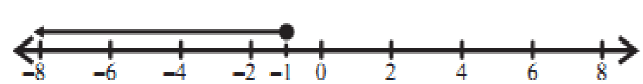
$5x-15\geq 16x-4$

$5x-16x\geq -4+15$

$-11x\geq 11$

$x\leq -\frac{11}{11}$

$x\leq -1$



3. Find the solution region of the following linear equalities:

(i) $3x - 4y \leq 12$; $3x + 2y \geq 3$

(ii) $2x + y \leq 4$; $x + 2y \leq 6$

Solution

3 (i)

$3x - 4y \leq 12$ ………(i)

$3x + 2y \geq 3$ ………(ii)

Associated equations

$3x - 4y = 12$ ………(iii)

$3x + 2y = 3$ ………(iv)

To find Points

(iii) ⇒ Put $x = 0$, $y = -3$ then point is $(0, -3)$

(iii) ⇒ Put $y = 0$, $x = 4$ then point is $(4,0)$

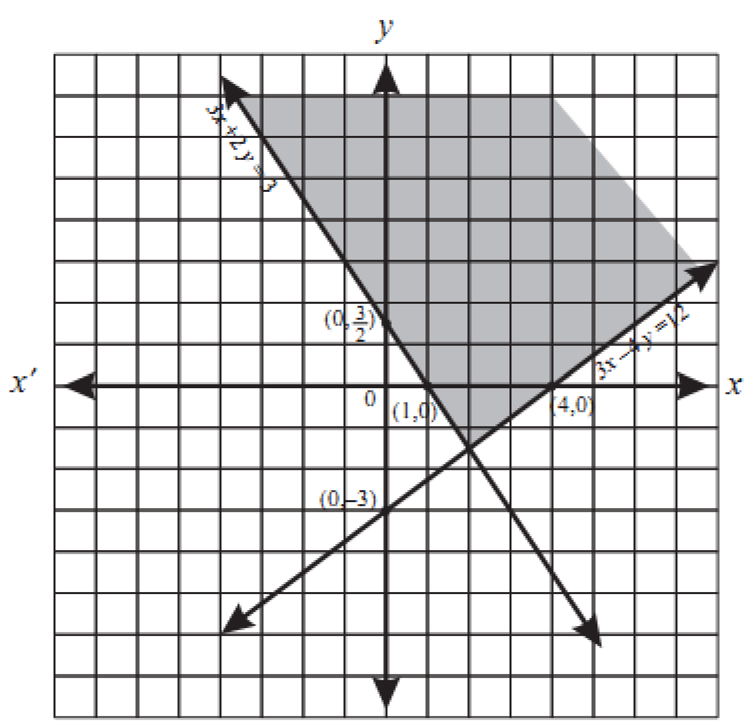
(iv) ⇒ Put $x = 0$, $y = \frac{3}{2}$ then point is $\left(0, \frac{3}{2}\right)$

(iv) ⇒ Put $y = 0$, $x = 1$ then point is $(1,0)$

To check Region put $(\mathbf{0}, \mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) ⇒ $0 < 12$ true, graph towards the origin

(ii) ⇒ $0 > 3$ false, graph away from the origin



3 (ii)

$2x + y \leq 4$ ....(i)

$x + 2y \leq 6$ ....(ii)

Associated equations

$2x + y = 4$ ....(iii)

$x + 2y = 6$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 4$ then point is $(0,4)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

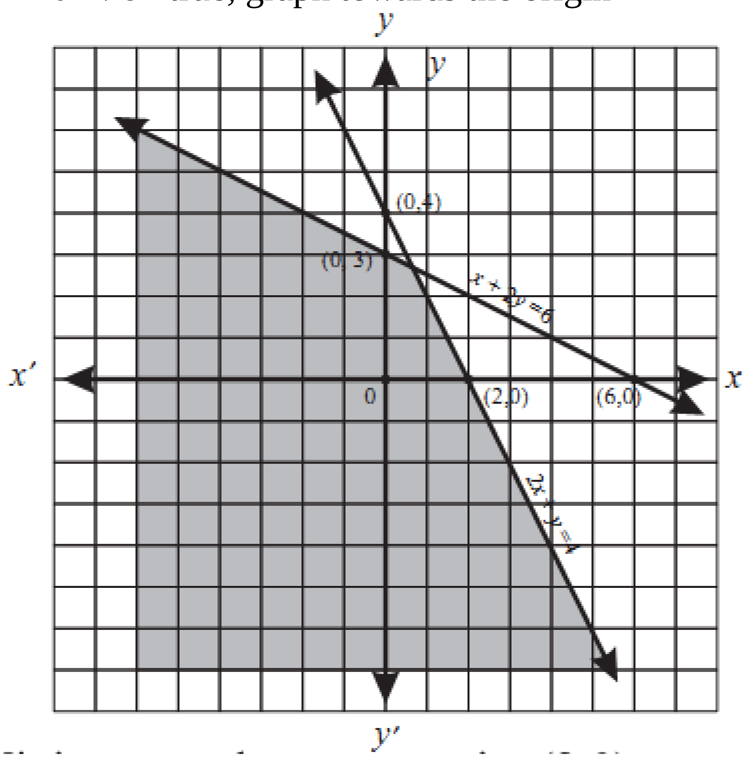
(iv) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 6$ then point is $(6,0)$

To check Region put $(\mathbf{0}, \mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 < 4$ true, graph towards the origin

(ii) $\Rightarrow 0 < 6$ true, graph towards the origin



4. Find the maximum value of $g(x,y) = x + 4y$ subject to constraints $x + y \leq 4$, $x \geq 0$ and $y \geq 0$.

Solution

$x + y \leq 4$

Associated equations

$x + y = 4$

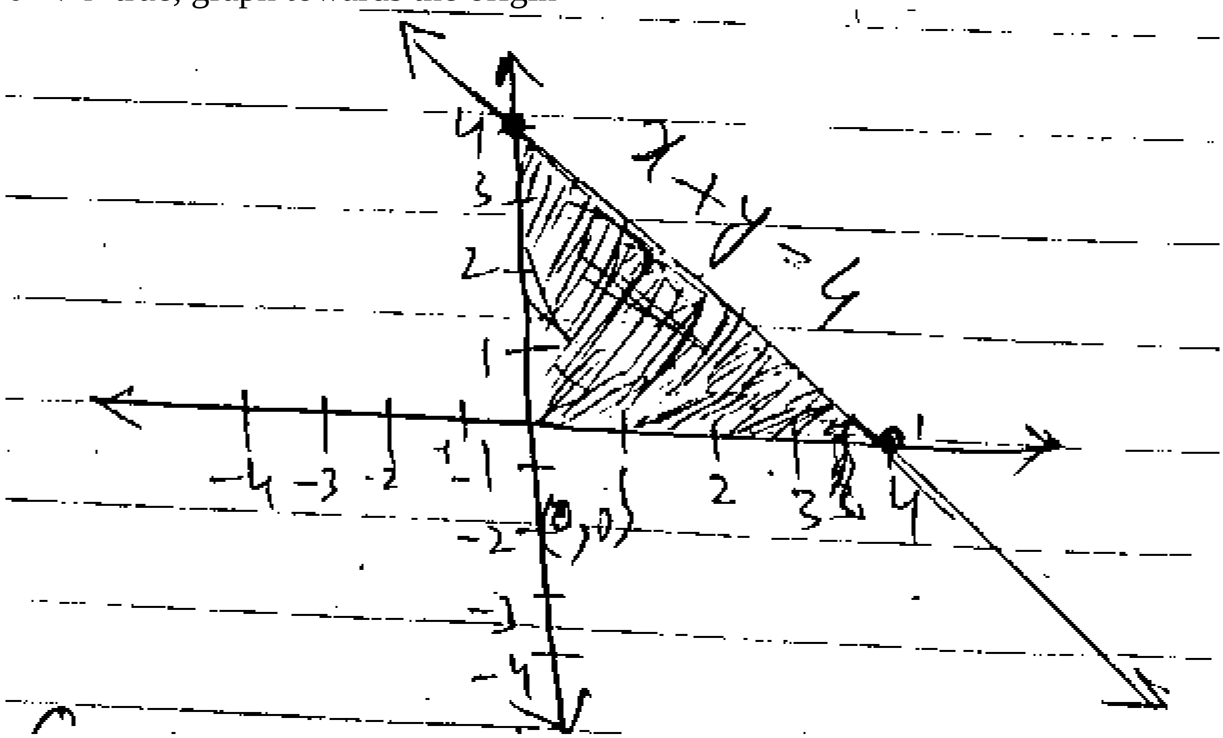
To find Points

$\Rightarrow$ Put $x = 0$, $y = 4$ then point is $(0,4)$

$\Rightarrow$ Put $y = 0$, $x = 4$ then point is $(4,0)$

To check Region put $(0,0)$ in (i) and (ii)

$0 < 4$ true, graph towards the origin



Corner Points: $A(0,0), B(0,4), C(4,0)$

At A: $z = g(0,0) = (0) + 4(0) = 0$

At B: $z = g(0,4) = (0) + 4(4) = 16$

At C: $z = g(4,0) = (4) + 0(0) = 4$

So $z = x + 4y$ is maximum at $(0,4)$

5. Find the minimum value of $f(x,y) = 3x + 5y$ subject to constraints

$x + 3y \geq 3, \quad x + y \geq 2, \quad x \geq 0, \quad y \geq 0.$

Solution

$x + 3y \geq 3$ ....(i)

$x + y \geq 2$ ....(ii)

Associated equations

$x + 3y = 3$ ....(iii)

$x + y = 2$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 1$ then point is $(0,1)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 3$ then point is $(3,0)$

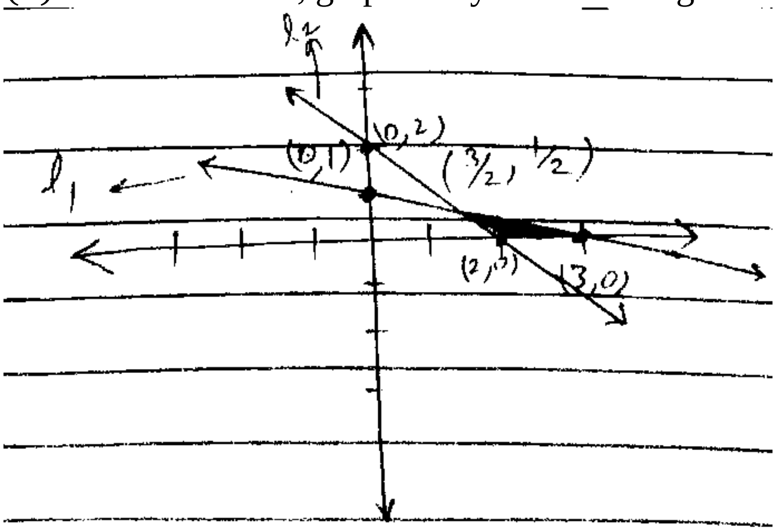
(iv) $\Rightarrow$ Put $x = 0$, $y = 2$ then point is $(0,2)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) $\Rightarrow 0 > 3$ false, graph away from the origin

(ii) $\Rightarrow 0 > 2$ false, graph away from the origin



Solve (iii) − (iv)

$(x + 3y) - (x + y) = 3 - 2$ we have $y = \frac{1}{2}$

Put $y = \frac{1}{2}$ in (iii) we have $x = \frac{3}{2}$ and $P\left(\frac{3}{2}, \frac{1}{2}\right)$

Corner Points: $A(2,0), B(3,0), P\left(\frac{3}{2}, \frac{1}{2}\right)$

At A: $z = f(2,0) = 3(2) + 5(0) = 6$

At B: $z = f(3,0) = 3(3) + 5(0) = 9$

At P: $z = f\left(\frac{3}{2}, \frac{1}{2}\right) = 3\left(\frac{3}{2}\right) + 5\left(\frac{1}{2}\right) = 8$

So $z = 3x + 5y$ is minimum at $(2,0)$ and maximum at $(3,0)$

Unit 6 Trigonometry

EXERCISE 6.1

1. Find in which quadrant the following angles lie. Write a co-terminal angle for each:

(i) 65° (ii) 135° (iii) – 40° (iv) 210° (v) –150°

Solution

(i) 1^{st} (ii) 2^{nd} (iii) 4^{th} (iv) 3^{rd} (v) 3^{rd}

2. Convert the following into degrees, minutes, and seconds:

(i) 123.456° (ii) 58.7891° (iii) 90.5678°

Solution

2(i): $\mathbf{123.456°}$

123

$0.456 \times 60 = 27.36$

$0.36 \times 60 = 21.6$

$123.456° \approx 123°\ 27'\ 22''$

2(ii): $\mathbf{58.7891°}$

58

$0.7891 \times 60 = 47.346$

$0.346 \times 60 = 20.76$

$58.7891° \approx 58°\ 47'\ 21''$

2(iii): $\mathbf{90.5678°}$

90

$0.5678 \times 60 = 34.068$

$0.068 \times 60 = 4.08$

$90.5678° \approx 90°\ 34'4''$

3. Convert the following into decimal degrees:

(i) 65° 32' 15" (ii) 42° 18' 45" (iii) 78° 45' 36"

Solution

3(i): $\mathbf{65°32'15''}$

$65°32'15'' = 65 + \frac{32}{60} + \frac{15}{60\times60} = 65 + 0.5333 + 0.0042 = 65.5375°$

3(ii): $\mathbf{42°18'45''}$

$42°18'45'' = 42 + \frac{18}{60} + \frac{45}{60\times60} = 42 + 0.3 + 0.0125 = 42.3125°$

3(iii): $\mathbf{78°45'36''}$

$78°45'36'' = 78 + \frac{45}{60} + \frac{36}{60\times60} = 78 + 0.75 + 0.01 = 78.76°$

4. Convert the following into radians:

(i) 36° (ii) 22.5° (iii) 67.5°

Solution

4(i): $\mathbf{36°} = 36 \times \frac{\pi}{180} = \frac{\pi}{5}\text{rad}$

4(ii): $\mathbf{22.5°} = 22.5 \times \frac{\pi}{180} = \frac{\pi}{8}\text{rad}$

4(iii): $\mathbf{67.5°} = 67.5 \times \frac{\pi}{180} = \frac{3\pi}{8}\text{rad}$

5. Convert the following into degrees:

(i) $\frac{\pi}{16}$ rad (ii) $\frac{11\pi}{5}$ rad (iii) $\frac{7\pi}{6}$ rad

Solution

5(i): $\mathbf{\frac{\pi}{16}\text{rad}} = \frac{\pi}{16} \times \frac{180°}{\pi} = 11.25°$

5(ii): $\mathbf{\frac{11\pi}{5}\text{rad}} = \frac{11\pi}{5} \times \frac{180°}{\pi} = 396°$

5(iii): $\mathbf{\frac{7\pi}{6}\text{rad}} = \frac{7\pi}{6} \times \frac{180°}{\pi} = 210°$

6. Find the arc length and area of a sector if:

(i) $r = 6$ cm and central angle $\theta = \frac{\pi}{3}$ radians.

(ii) $r = \frac{4.8}{\pi}$ cm and central angle $\theta = \frac{5\pi}{6}$ radians.

Solution

6(i): $\boldsymbol{l} = \text{r}\theta = 6 \times \frac{\pi}{3} = 6.28\text{cm}$

$\boldsymbol{A} = \frac{1}{2}\text{r}^2\theta = \frac{1}{2} \times (6)^2 \times \frac{\pi}{3} = 18.84\text{cm}^2$

6(ii): $\boldsymbol{l} = \text{r}\theta = \frac{4.8}{\pi} \times \frac{5\pi}{6} = 4\text{cm}$

$\boldsymbol{A} = \frac{1}{2}\text{r}^2\theta = \frac{1}{2} \times \left(\frac{4.8}{\pi}\right)^2 \times \frac{5\pi}{6} = 3.06\text{cm}^2$

7. If the central angle of a sector is 60° and the radius of the circle is 12 cm, find the area of the sector and the percentage of the total area of the circle it represents.

Solution

$\theta = 60° = 60 \times \frac{\pi}{180} = \frac{\pi}{3}\text{rad}$

Area of the sector $= \frac{1}{2}r^2\theta = \frac{1}{2} \times (12)^2 \times \frac{\pi}{3} = 75.4\text{cm}^2$

Total area of the circle $= \pi r^2 = 3.14159 \times (12)^2 = 452.389\text{cm}^2$

Percentage $= \frac{\text{Area of the sector}}{\text{Total area of the circle}} \times 100\%$

Percentage $= \frac{75.4\text{cm}^2}{452.389\text{cm}^2} \times 100\% = 16.67\%$

8. Find the percentage of the area of sector subtending an angle $\frac{\pi}{8}$ radians.

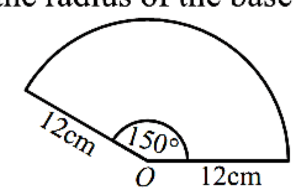
Solution

Percentage $= \frac{\text{Area of the sector}}{\text{Total area of the circle}} \times 100\%$

Percentage $= \frac{\theta}{2\pi} \times 100\% = \frac{\frac{\pi}{8}}{2\pi} \times 100\% = \frac{1}{16} \times 100\% = 6.25\%$

9. A circular sector of radius $r = 12$ cm has an angle of 150°. This sector is cut out and then bent to form a cone. What is the slant height and the radius of the base of this cone?

Hint: Arc length of sector = circumference of cone.



Solution

Radius of the sector $= r = 12\text{cm}$

Angle of the sector $= \theta = 150° = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}\text{rad}$

Arc Length $= l = r\theta = 12 \times \frac{5\pi}{6} = 10\pi\text{cm}$

Now

Circumference of base of the cone $= 2\pi r'$

$10\pi = 2\pi r'$

radius of base $= r' = 5\text{cm}$

slant height $= l = r = 12\text{cm}$

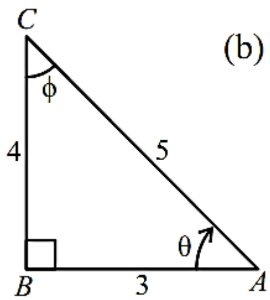
EXERCISE 6.2

1. For each of the following right-angled triangles, find the trigonometric ratios:

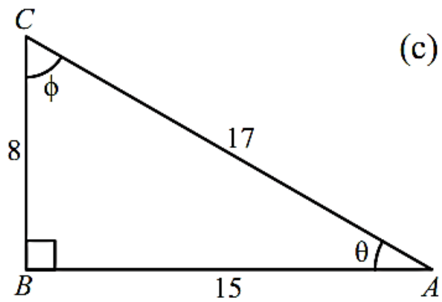
(i) $\sin\theta$ (ii) $\cos\theta$ (iii) $\tan\theta$ (iv) $\sec\theta$ (v) $\text{cosec}\,\theta$

(vi) $\cot\phi$ (vii) $\tan\phi$ (viii) $\text{cosec}\,\phi$ (ix) $\sec\phi$ (x) $\cos\phi$

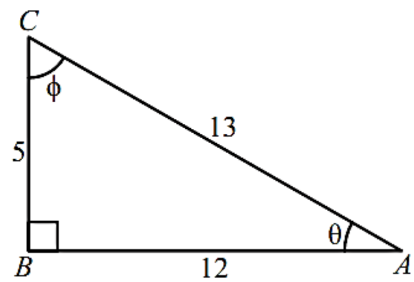
(a)



(b)



(c)



Solution

(a) (i) $\frac{4}{5}$ (ii) $\frac{3}{5}$ (iii) $\frac{4}{3}$ (iv) $\frac{5}{3}$ (v) $\frac{5}{4}$ (vi) $\frac{4}{3}$ (vii) $\frac{3}{4}$ (viii) $\frac{5}{3}$ (ix) $\frac{5}{4}$ (x) $\frac{4}{5}$

(b) (i) $\frac{8}{17}$ (ii) $\frac{15}{17}$ (iii) $\frac{8}{15}$ (iv) $\frac{17}{15}$ (v) $\frac{17}{8}$ (vi) $\frac{8}{15}$ (vii) $\frac{15}{8}$ (viii) $\frac{17}{15}$ (ix) $\frac{17}{8}$ (x) $\frac{8}{17}$

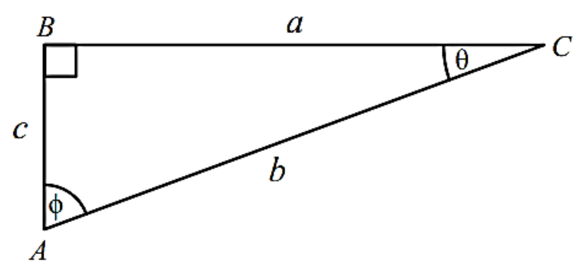
(c) (i) $\frac{5}{13}$ (ii) $\frac{12}{13}$ (iii) $\frac{5}{12}$ (iv) $\frac{13}{5}$ (v) $\frac{13}{12}$ (vi) $\frac{5}{12}$ (vii) $\frac{12}{5}$ (viii) $\frac{13}{12}$ (ix) $\frac{13}{5}$ (x) $\frac{5}{13}$

2. For the following right-angled triangle ABC find the trigonometric ratios for which $m\angle A = \phi$ and $m\angle C = \theta$

(i) $\sin\theta$ (ii) $\cos\theta$

(iii) $\tan\theta$ (iv) $\sin\phi$

(v) $\cos\phi$ (vi) $\tan\phi$



Solution

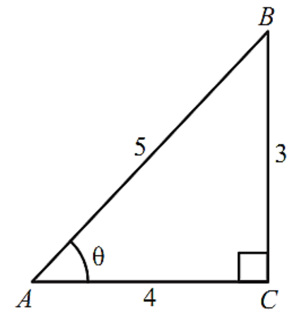
(i) $\frac{c}{b}$ (ii) $\frac{a}{b}$ (iii) $\frac{c}{a}$ (iv) $\frac{a}{b}$ (v) $\frac{c}{b}$ (vi) $\frac{a}{c}$

3. Considering the adjoining triangle ABC, verify that:

(i) $\sin\theta \ \operatorname{cosec}\theta = 1$

(ii) $\cos\theta \ \sec\theta = 1$

(iii) $\tan\theta \ \cot\theta = 1$



Solution

3.(i) $sin\theta cosec\theta = \frac{3}{5} \times \frac{5}{3} = 1$

3.(ii) $cos\theta sec\theta = \frac{4}{5} \times \frac{5}{4} = 1$

3.(iii) $tan\theta cot\theta = \frac{3}{4} \times \frac{4}{3} = 1$

4. Fill in the blanks.

(i) $\sin 30^\circ = \sin(90^\circ - 60^\circ) =$ $cos60^\circ$

(ii) $\cos 30^\circ = \cos(90^\circ - 60^\circ) =$ $sin60^\circ$

(iii) $\tan 30^\circ = \tan(90^\circ - 60^\circ) =$ $cot60^\circ$

(iv) $\tan 60^\circ = \tan(90^\circ - 30^\circ) =$ $cot30^\circ$

(v) $\sin 60^\circ = \sin(90^\circ - 30^\circ) =$ $cos30^\circ$

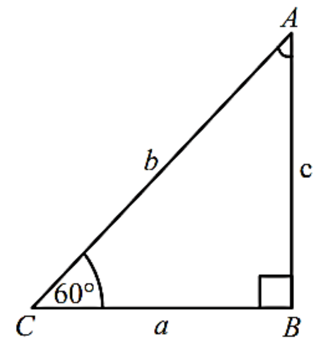
(vi) $\cos 60^\circ = \cos(90^\circ - 30^\circ) =$ $sin30^\circ$

(vii) $\sin 45^\circ = \sin(90^\circ - 45^\circ) =$ $cos45^\circ$

(viii) $\tan 45^\circ = \tan(90^\circ - 45^\circ) =$ $cot45^\circ$

(ix) $\cos 45^\circ = \cos(90^\circ - 45^\circ) =$ $sin45^\circ$

5. In a right angled triangle ABC, $m\angle B = 90°$ and C is an acute angle of $60°$. Alsc $\sin m\angle A = \frac{a}{b}$, then find the following trigonometric ratios:



(i) $\frac{m\overline{BC}}{m\overline{AB}}$ (ii) $\cos 60°$

(iii) $\tan 60^{\circ}$ (iv) $\text{cosec}\ \frac{\pi}{3}$

(v) $\cot 60°$ (vi) $\sin 30°$

(vii) $\cos 30°$ (viii) $\tan \frac{\pi}{6}$

(ix) $\sec 30°$ (x) $\cot 30°$

Solution

(i) $\frac{a}{c}$ (ii) $\frac{a}{b}$ (iii) $\frac{c}{a}$ (iv) $\frac{b}{c}$ (v) $\frac{a}{c}$

(vi) $\frac{a}{b}$ (vii) $\frac{c}{b}$ (viii) $\frac{a}{c}$ (ix) $\frac{b}{c}$ (x) $\frac{c}{a}$

EXERCISE 6.3

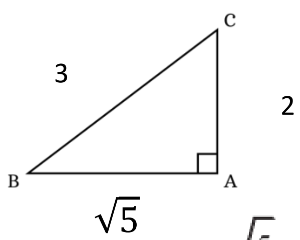
1. If θ lies in first quadrant, find the remaining trigonometric ratios of θ.

(i) $\sin\theta = \frac{2}{3}$ (ii) $\cos\theta = \frac{3}{4}$ (iii) $\tan\theta = \frac{1}{2}$

(iv) $\sec\theta = 3$ (v) $\cot\theta = \sqrt{\frac{3}{2}}$

Solution

1.(i) $\boldsymbol{sin\theta = \frac{2}{3}}$



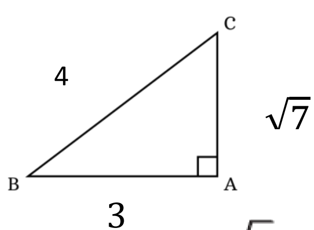
By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow 3^2 = 2^2 + B^2$$

$$\Rightarrow B^2 = 9 - 4 = 5 \Rightarrow B = \sqrt{5}$$

(i) $\cos\theta = \frac{\sqrt{5}}{3}$, $\tan\theta = \frac{2}{\sqrt{5}}$, $\text{cosec}\,\theta = \frac{3}{2}$, $\sec\theta = \frac{3}{\sqrt{5}}$, $\cot\theta = \frac{\sqrt{5}}{2}$

1.(ii) $\boldsymbol{cos\theta = \frac{3}{4}}$



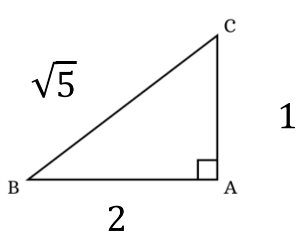
By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow 4^2 = P^2 + 3^2$$

$$\Rightarrow P^2 = 16 - 9 = 7 \Rightarrow P = \sqrt{7}$$

(ii) $\sin\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = \frac{\sqrt{7}}{3}$, $\text{cosec}\,\theta = \frac{4}{\sqrt{7}}$, $\sec\theta = \frac{4}{3}$, $\cot\theta = \frac{3}{\sqrt{7}}$

1.(iii) $\boldsymbol{tan\theta = \frac{1}{2}}$



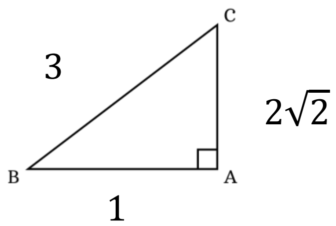
By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow H^2 = 1^2 + 2^2$$

$$\Rightarrow H^2 = 1 + 4 = 5 \Rightarrow H = \sqrt{5}$$

(iii) $\sin\theta = \frac{1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$, $\text{cosec}\,\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = 2$

1.(iv) $\boldsymbol{sec\theta = 3 = \frac{3}{1}}$



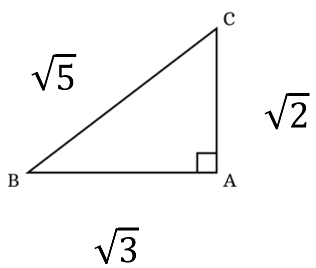
By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow 3^2 = P^2 + 1^2$$

$$\Rightarrow P^2 = 9 - 1 = 8 \Rightarrow P = 2\sqrt{2}$$

(iv) $\sin\theta = \dfrac{2\sqrt{2}}{3}$, $\cos\theta = \dfrac{1}{3}$, $\tan\theta = 2\sqrt{2}$, $\operatorname{cosec}\theta = \dfrac{3}{2\sqrt{2}}$, $\cot\theta = \dfrac{1}{2\sqrt{2}}$

1.(v) $\boldsymbol{cot\theta = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}}}$



By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow H^2 = \left(\sqrt{2}\right)^2 + \left(\sqrt{3}\right)^2$$

$$\Rightarrow H^2 = 2 + 3 = 5 \Rightarrow H = \sqrt{5}$$

(v) $\sin\theta = \sqrt{\dfrac{2}{5}}$, $\cos\theta = \sqrt{\dfrac{3}{5}}$, $\tan\theta = \sqrt{\dfrac{2}{3}}$, $\operatorname{cosec}\theta = \sqrt{\dfrac{5}{2}}$, $\sec\theta = \sqrt{\dfrac{5}{3}}$

Prove the Following Trigonometric Identities

2. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Solution

$(\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$

$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

3. $\dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

Solution

$\frac{\cos\theta}{\sin\theta} = \cot\theta = \frac{1}{\tan\theta}$

4. $$\frac{\sin\theta}{\text{cosec }\theta}+\frac{\cos\theta}{\sec\theta}=1$$

Solution

$$\frac{\sin\theta}{\text{cosec}\theta}+\frac{\cos\theta}{\sec\theta}=\sin\theta\times\frac{1}{\text{cosec}\theta}+\cos\theta\times\frac{1}{\sec\theta}$$
$$\frac{\sin\theta}{\text{cosec}\theta}+\frac{\cos\theta}{\sec\theta}=\sin\theta\times\sin\theta+\cos\theta\times\cos\theta=\sin^2\theta+\cos^2\theta=1$$

5. $\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$

Solution

$$\cos^2\theta-\sin^2\theta=\cos^2\theta-(1-\cos^2\theta)=\cos^2\theta-1+\cos^2\theta$$
$$\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$$

6. $\cos^2\theta-\sin^2\theta=1-2\sin^2\theta$

Solution

$$\cos^2\theta-\sin^2\theta=(1-\sin^2\theta)-\sin^2\theta=1-\sin^2\theta-\sin^2\theta$$
$$\cos^2\theta-\sin^2\theta=1-2\sin^2\theta$$

7. $$\frac{1-\sin\theta}{\cos\theta}=\frac{\cos\theta}{1+\sin\theta}$$

Solution

$$\frac{1-\sin\theta}{\cos\theta}=\frac{(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)}=\frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}=\frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}=\frac{\cos\theta}{1+\sin\theta}$$

8. $$(\sec\theta-\tan\theta)^2=\frac{1-\sin\theta}{1+\sin\theta}$$

Solution

$$(\sec\theta-\tan\theta)^2=\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)^2=\left(\frac{1-\sin\theta}{\cos\theta}\right)^2=\frac{(1-\sin\theta)^2}{\cos^2\theta}=\frac{(1-\sin\theta)(1-\sin\theta)}{1-\sin^2\theta}$$
$$(\sec\theta-\tan\theta)^2=\frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}=\frac{1-\sin\theta}{1+\sin\theta}$$

9. $(\tan\theta+\cot\theta)^2=\sec^2\theta\,\text{cosec}^2\theta$

Solution

$$(\tan\theta+\cot\theta)^2=\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)^2=\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)^2=\left(\frac{1}{\cos\theta\sin\theta}\right)^2$$
$$(\tan\theta+\cot\theta)^2=\frac{1}{\cos^2\theta}\times\frac{1}{\sin^2\theta}=\sec^2\theta\,\text{cosec}^2\theta$$

$$10. \quad \frac{\tan\ \theta+\sec\ \theta-1}{\tan\ \theta-\sec\ \theta+1}=\tan\theta+\sec\theta$$

Solution

$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$

$= \frac{\tan\theta+\sec\theta-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$

$= \frac{\tan\theta+\sec\theta-(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)}{\tan\theta-\sec\theta+1}$

$= \frac{(\tan\theta+\sec\theta)[1-(\sec\theta-\tan\theta)]}{\tan\theta-\sec\theta+1}$

$= \frac{(\tan\theta+\sec\theta)[1-\sec\theta+\tan\theta]}{1-\sec\theta+\tan\theta}$

$= \tan\theta+\sec\theta$

$$11. \quad \sin^3\theta-\cos^3\theta=(\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)$$

Solution

$\sin^3\theta-\cos^3\theta$

$= (\sin\theta-\cos\theta)(\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta)$

$= (\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)$

$$12. \quad \sin^6\theta-\cos^6\theta=(\sin^2\theta-\cos^2\theta)(1-\sin^2\theta\cos^2\theta)$$

Solution

$\sin^6\theta-\cos^6\theta$

$= (\sin^2\theta)^3-(\cos^2\theta)^3$

$= (\sin^2\theta-\cos^2\theta)[(\sin^2\theta)^2+(\cos^2\theta)^2+\sin^2\theta\cos^2\theta]$

$= (\sin^2\theta-\cos^2\theta)[(\sin^2\theta)^2+(\cos^2\theta)^2+2\sin^2\theta\cos^2\theta-\sin^2\theta\cos^2\theta]$

$= (\sin^2\theta-\cos^2\theta)[(\sin^2\theta+\cos^2\theta)^2-\sin^2\theta\cos^2\theta]$

$= (\sin^2\theta-\cos^2\theta)(1-\sin^2\theta\cos^2\theta)$

EXERCISE 6.4

θ	0	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$	$90° = \frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

1. Find the value of the following trigonometric ratios without using the calculator.

(i) sim 30° (ii) cos 30° (iii) $\tan\frac{\pi}{6}$ (iv) tan 60°

(v) sec 60° (vi) $\cos\frac{\pi}{3}$ (vii) cot 60° (viii) sin 60°

(ix) sec 30° (x) cosec 30° (xi) sin 45° (xii) $\cos\frac{\pi}{4}$

Solution

(i) $\frac{1}{2}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{\sqrt{3}}{3}$ (iv) $\sqrt{3}$

(v) 2 (vi) $\frac{1}{2}$ (vii) $\frac{\sqrt{3}}{3}$ (viii) $\frac{\sqrt{3}}{2}$

(ix) $\frac{2\sqrt{3}}{3}$ (x) 2 (xi) $\frac{\sqrt{2}}{2}$ (xii) $\frac{\sqrt{2}}{2}$

2. Evaluate:

(i) $2\sin 60° \cos 60°$ (ii) $2\cos\dfrac{\pi}{6}\sin\dfrac{\pi}{6}$

(iii) $2\sin 45° + 2\cos 45°$ (iv) $\sin 60°\cos 30° + \cos 60°\sin 30°$

(v) $\cos 60°\cos 30° - \sin 60°\sin 30°$ (vi) $\sin 60°\cos 30° - \cos 60°\sin 30°$

(vii) $\cos 60°\cos 30° + \sin 60°\sin 30°$ (viii) $\tan\dfrac{\pi}{6}\cot\dfrac{\pi}{6} + 1$

Solution

2(i): $2\sin60°\cos60° = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$

2(ii): $2\cos\frac{\pi}{6}\sin\frac{\pi}{6} = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

2(iii): $2\sin45° + 2\cos45° = 2 \times \frac{1}{\sqrt{2}} + 2 \times \frac{1}{\sqrt{2}} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

2(iv): $\sin60°\cos30° + \cos60°\sin30° = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$

2(v): $\cos60°\cos30° - \sin60°\sin30° = \frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$

2(vi): $\sin60°\cos30° - \cos60°\sin30° = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

2(vii): $\cos60°\cos30° + \sin60°\sin30° = \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

2(viii): $\tan\frac{\pi}{6}\cot\frac{\pi}{6} + 1 = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{1} + 1 = 1 + 1 = 2$

3. If $\sin\dfrac{\pi}{4}$ and $\cos\dfrac{\pi}{4}$ equal to $\dfrac{1}{\sqrt{2}}$ each, then find the value of the followings:

(i) $2\sin 45° - 2\cos 45°$ (ii) $3\cos 45° + 4\sin 45°$

(iii) $5\cos 45° - 3\sin 45°$

Solution

3(i): $2\sin45° - 2\cos45° = 2 \times \frac{1}{\sqrt{2}} - 2 \times \frac{1}{\sqrt{2}} = \sqrt{2} - \sqrt{2} = 0$

3(ii): $3\cos45° + 4\sin45° = 3 \times \frac{1}{\sqrt{2}} + 4 \times \frac{1}{\sqrt{2}} = \frac{7}{\sqrt{2}}$

3(iii): $5\cos45° - 3\sin45° = 5 \times \frac{1}{\sqrt{2}} - 3 \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

EXERCISE 6.5

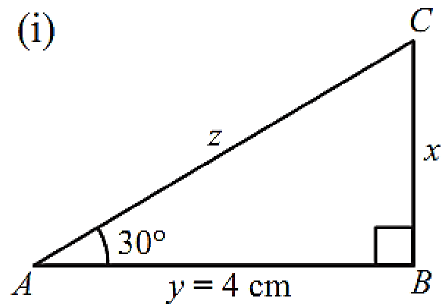
1. Find the values of x, y and z from the following right angled triangles.

1(i) $m\angle A = 30°, y = 4cm$

Solution

$m\angle C = m\angle B - m\angle A = 90° - 30°$

$m\angle C = 60°$



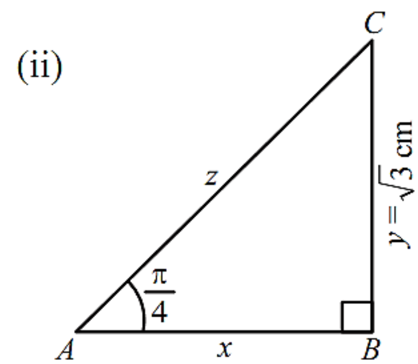
$\frac{x}{y} = \tan 30°$ $\frac{x}{4} = \frac{1}{\sqrt{3}}$ $x = \frac{4}{\sqrt{3}}$	$\frac{y}{z} = \cos 30°$ $\frac{4}{z} = \frac{\sqrt{3}}{2}$ $z = 4 \times \frac{2}{\sqrt{3}} = \frac{8}{\sqrt{3}}$

1(ii) $m\angle A = 45°, y = \sqrt{3}cm$

Solution

$m\angle C = m\angle B - m\angle A = 90° - 45°$

$m\angle C = 45°$



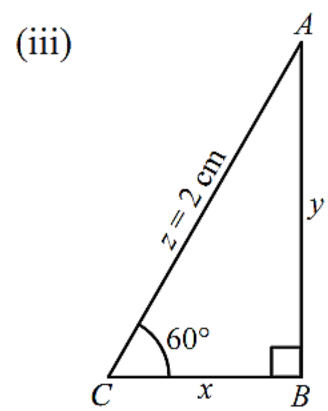
$\frac{y}{x} = \tan 45°$ $\frac{\sqrt{3}}{x} = 1$ $x = \sqrt{3}$	$\frac{y}{z} = \sin 45°$ $\frac{\sqrt{3}}{z} = \frac{1}{\sqrt{2}}$ $z = \sqrt{3} \times \sqrt{2} = \sqrt{6}$

1(iii) $m\angle C = 60°, z = 2cm$

Solution

$m\angle A = m\angle B - m\angle C = 90° - 60°$

$m\angle A = 30°$



$\frac{x}{z} = \cos 60°$ $\frac{x}{2} = \frac{1}{2}$ $x = \frac{2}{2}$ $x = 1$	$\frac{y}{z} = \sin 60°$ $\frac{y}{2} = \frac{\sqrt{3}}{2}$ $y = \frac{2 \times \sqrt{3}}{2}$ $y = \sqrt{3}$

1(iv) $m\angle A = 45°, y = 4cm$

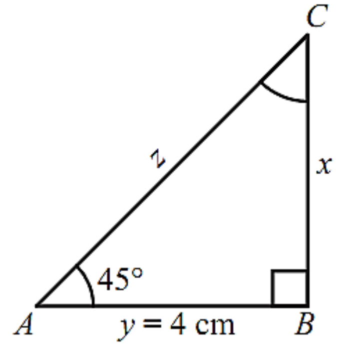
Solution

$m\angle C = m\angle B - m\angle A = 90° - 45°$

$m\angle C = 45°$

$\frac{x}{y} = \tan 45°$	$\frac{y}{z} = \cos 45°$
$\frac{x}{4} = 1$	$\frac{4}{z} = \frac{1}{\sqrt{2}}$
$x = 4$	$z = 4\sqrt{2}$

(iv)

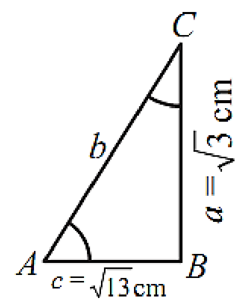


2. Find the unknown side and angles of the following triangles.

2(i)

By Pythagoras Formula

(i)



$$b^2 = a^2 + c^2 \Rightarrow b^2 = \left(\sqrt{3}\right)^2 + \left(\sqrt{13}\right)^2$$

$$\Rightarrow b^2 = 3 + 13 = 16 \Rightarrow b = 4$$

$$\sin A = \frac{a}{b} = \frac{\sqrt{3}}{4} = 0.4330$$

$$A = \sin^{-1}(0.4330) = 25.64°$$

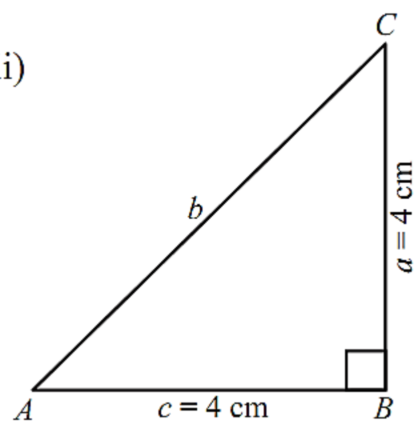
$m\angle C = m\angle B - m\angle A = 90° - 25.64°$

$m\angle C = 64.36°$

2(ii)

By Pythagoras Formula

(ii)



$$b^2 = a^2 + c^2 \Rightarrow b^2 = (4)^2 + (4)^2$$

$$\Rightarrow b^2 = 16 + 16 = 32 \Rightarrow b = 4\sqrt{2}$$

$$\cos A = \frac{c}{b} = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}} = 0.7071$$

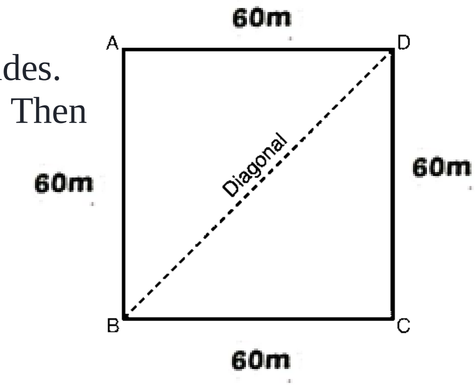
$$A = \cos^{-1}(0.7071) = 45°$$

$m\angle C = m\angle B - m\angle A = 90° - 45°$

$m\angle C = 45°$

3. Each side of a square field is 60 m long. Find the lengths of the diagonals of the field.

Solution



A square's diagonal forms a right-angled triangle with two sides.
If 'a' and 'b' are the sides of the square and 'c' is the diagonal. Then
Using Pythagorean Theorem:

$c^2 = a^2 + b^2$

In this case, $a = b = 60$m.

Therefore, $c^2 = 60^2 + 60^2$

$c^2 = 3600 + 3600 = 7200$

$c = \sqrt{7200} = \sqrt{3600 \times 2} = 60\sqrt{2}$m

Solve the following triangles when $m\angle B = 90°$:

4. $m\angle C = 60°,\ c = 3\sqrt{3}$ cm

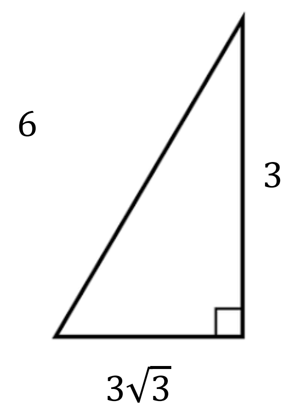
Solution

$m\angle C = 60°, c = 3\sqrt{3}\text{cm}$

$m\angle A = m\angle B - m\angle C = 90° - 60°$

$m\angle A = 30°$

$\frac{c}{b} = \sin 60°$ $\frac{3\sqrt{3}}{b} = \frac{\sqrt{3}}{2}$ $b = \frac{2\times 3\sqrt{3}}{\sqrt{3}}$ $b = 6\text{cm}$	$\frac{a}{b} = \sin 30°$ $\frac{a}{6} = \frac{1}{2}$ $a = \frac{6}{2}$ $a = 3\text{cm}$



Solve the following triangles when $m\angle B = 90°$:

5. $m\angle C = 45°,\ a = 8$ cm

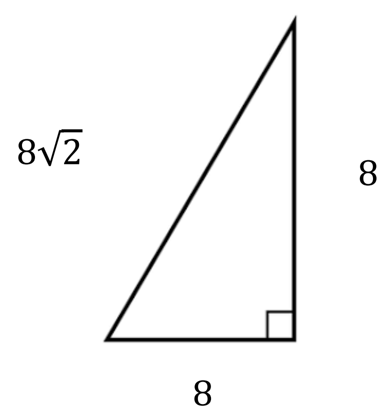
Solution

$m\angle C = 45°, a = 8\text{cm}$

$m\angle A = m\angle B - m\angle C = 90° - 45°$

$m\angle A = 45°$

$\frac{a}{b} = \sin 45°$ $\frac{8}{b} = \frac{1}{\sqrt{2}}$ $b = 8\sqrt{2}\text{cm}$	$\frac{c}{b} = \cos 45°$ $\frac{c}{8\sqrt{2}} = \frac{1}{\sqrt{2}}$ $c = \frac{8\sqrt{2}}{\sqrt{2}}$ $c = 8\text{cm}$



Solve the following triangles when $m\angle B = 90°$:

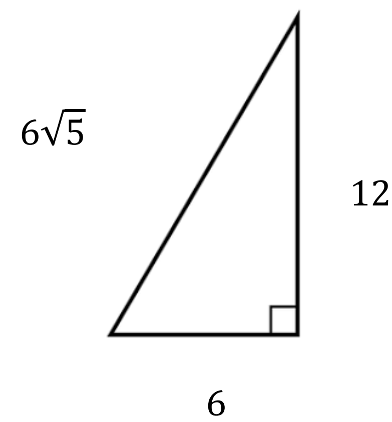
6. $a = 12$ cm, $c = 6$ cm

Solution

By Pythagoras Formula

$b^2 = a^2 + c^2 \Rightarrow b^2 = (12)^2 + (6)^2$

$\Rightarrow b^2 = 144 + 36 = 180 \Rightarrow b = 6\sqrt{5}$



$\text{sinA} = \frac{\text{a}}{\text{b}} = \frac{12}{6\sqrt{5}} = 0.8944$

$\text{A} = \sin^{-1}(0.8944) = 63.4°$

$\text{m}\angle\text{C} = \text{m}\angle\text{B} - \text{m}\angle\text{A} = 90° - 63.4°$

$\text{m}\angle\text{C} = 26.6°$

Solve the following triangles when $m\angle B = 90°$:

7. $m\angle A = 60°$, $c = 4$ cm

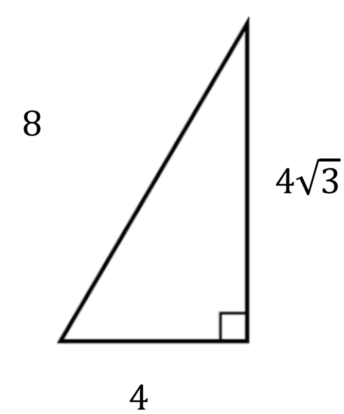
Solution

$\text{m}\angle\text{A} = 60°, \text{c} = 4\text{cm}$

$\text{m}\angle\text{C} = \text{m}\angle\text{B} - \text{m}\angle\text{C} = 90° - 60°$

$\text{m}\angle\text{C} = 30°$

$\frac{\text{c}}{\text{b}} = \cos 60°$ $\frac{4}{\text{b}} = \frac{1}{2}$ $\text{b} = 4 \times 2$ $\text{b} = 8\text{cm}$	$\frac{\text{a}}{\text{b}} = \sin 60°$ $\frac{\text{a}}{8} = \frac{\sqrt{3}}{2}$ $\text{a} = \frac{8\sqrt{3}}{2}$ $\text{a} = 4\sqrt{3}\text{cm}$



Solve the following triangles when $m\angle B = 90^\circ$:

8. $m\angle A = 30^\circ, c = 4cm$

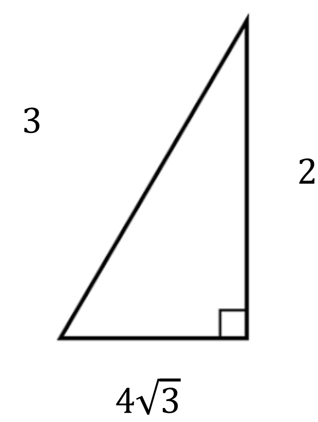
Solution

$m\angle A = 30^\circ, c = 4cm$

$m\angle C = m\angle B - m\angle C = 90^\circ - 30^\circ$

$m\angle C = 60^\circ$

$\frac{c}{b} = \cos 60^\circ$ $\frac{4}{b} = \frac{1}{2}$ $b = 4 \times 2$ $b = 8cm$	$\frac{a}{b} = \sin 60^\circ$ $\frac{a}{8} = \frac{\sqrt{3}}{2}$ $a = \frac{8\sqrt{3}}{2}$ $a = 4\sqrt{3}cm$



Solve the following triangles when $m\angle B = 90^\circ$:

9. $b = 10$ cm, $a = 6$ cm

Solution

By Pythagoras Formula

$b^2 = c^2 + a^2 \Rightarrow (10)^2 = c^2 + (6)^2$

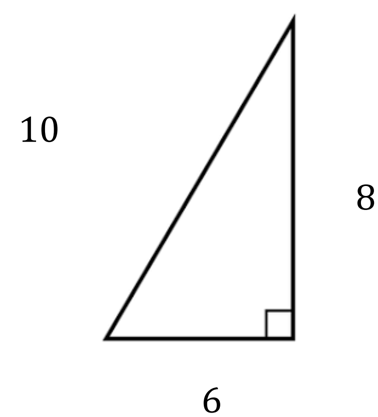
$\Rightarrow c^2 = 100 - 36 = 64 \Rightarrow c = 8$

$\sin C = \frac{c}{b} = \frac{8}{10} = 0.8$

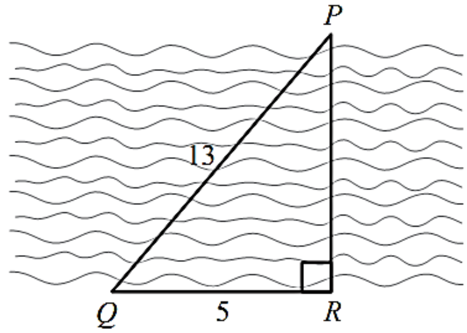
$C = \sin^{-1}(0.8) = 53.1^\circ$

$m\angle A = m\angle B - m\angle C = 90^\circ - 53.1^\circ$

$m\angle A = 36.9^\circ$



10. Let Q and R be the two points on the same bank of a canal. The point P is placed on the other bank straight to point R. Find the width of the canal and the angle PQR.



Solution

By Pythagoras Formula

$|PQ|^2 = |PR|^2 + |QR|^2$

$(13)^2 = |PR|^2 + (5)^2$

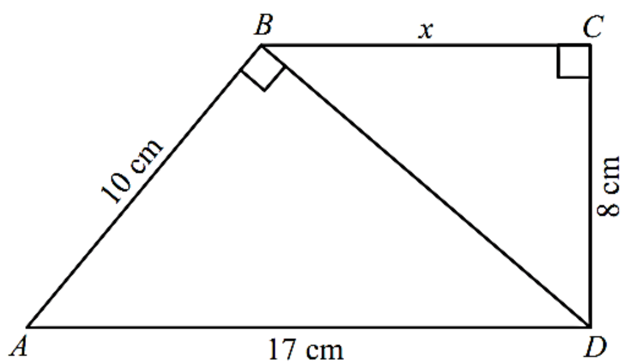
$|PR|^2 = 169 - 25 = 144$

$|PR| = 12km$

$\tan(\angle \mathrm{PQR}) = \frac{\mathrm{PR}}{\mathrm{QR}} = \frac{12}{5} = 2.4$

$\angle \mathrm{PQR} = \tan^{-1}(2.4) = 67.38°$

11. Calculate the length x in the adjoining figure.



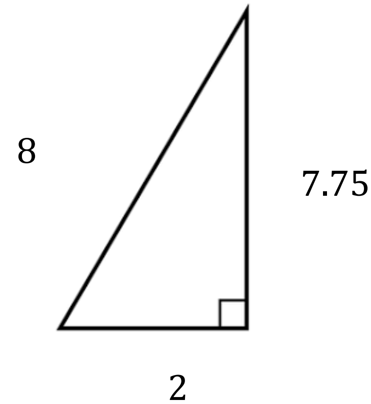
Solution

Applying Pythagoras Formula For ΔABD	Again applying Pythagoras Formula ForΔBCD
$\lvert AD\rvert^2 = \lvert BD\rvert^2 + \lvert AB\rvert^2$ $(17)^2 = \lvert BD\rvert^2 + (10)^2$ $\lvert BD\rvert^2 = 289 - 100 = 189$ $\lvert BD\rvert = 3\sqrt{21}$	$\lvert BD\rvert^2 = \lvert BC\rvert^2 + \lvert CD\rvert^2$ $\left(3\sqrt{21}\right)^2 = x^2 + (8)^2$ $x^2 = 189 - 64 = 125$ $x = 5\sqrt{5}$

12. If the ladder is placed along the wall such that the foot of the ladder is 2 m away from the wall. If the length of the ladder is 8 m, find the height of the wall.

Solution

By Pythagoras Formula



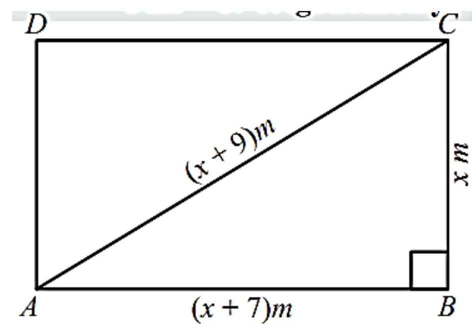
$8^2 = H^2 + 2^2$

$64 = H^2 + 4$

$H^2 = 64 - 4 = 60$

$H = 7.75m$

13. The diagonal of a rectangular field $ABCD$ is $(x + 9)m$ and the sides are $(x + 7)$m and x m. Find the value of x.



Solution

By Pythagoras Formula

$(x + 9)^2 = (x + 7)^2 + x^2$

$x^2 + 18x + 81 = x^2 + 14x + 49 + x^2$

$x^2 + 18x + 81 = 2x^2 + 14x + 49$

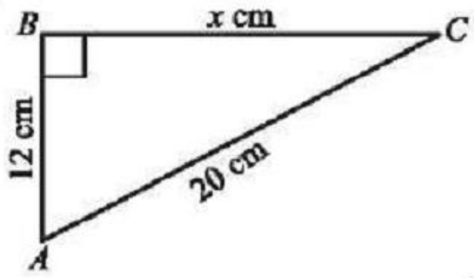
$x^2 - 4x - 32 = 0$

$(x - 8)(x + 4) = 0$

$x = 8$ or $x = -4$

Since x cannot be negative, therefore $x = 8$

14. Calculate the value of ‘x’ in each case.



Solution

By Pythagoras Formula

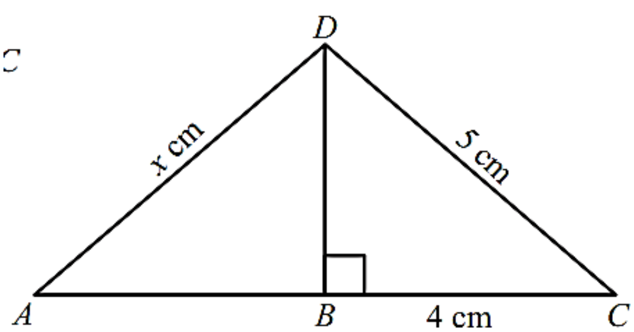
$|AC|^2 = |BC|^2 + |AB|^2$

$(20)^2 = x^2 + (12)^2$

$x^2 = 400 - 144 = 256$

$x = 16cm$

14. Calculate the value of ‘x’ in each case.



$m\overline{AB} = m\overline{BC}$

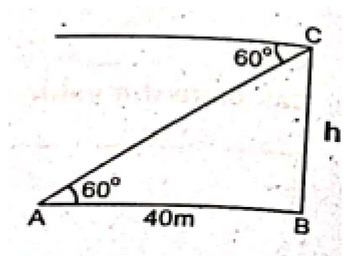
Solution

Applying Pythagoras Formula For ΔDBC $\|DC\|^2 = \|DB\|^2 + \|BC\|^2$ $(5)^2 = \|DB\|^2 + (4)^2$ $\|DB\|^2 = 25 - 16 = 9$ $\|DB\| = 3cm$	Again applying Pythagoras Formula For ΔDBA $\|AD\|^2 = \|DB\|^2 + \|AB\|^2$ $x^2 = (3)^2 + (4)^2$ $x^2 = 9 + 16 = 25$ $x = 5\ cm$

EXERCISE 6.6

1. The angle of elevation of the top of a flag post from a point on the ground level 40 m away from the flag post is 60°. Find the height of the post.

Solution



$\tan(60°) = \frac{h}{40}$

$h = 40 \times \tan(60°)$

$h = 40 \times \sqrt{3}$

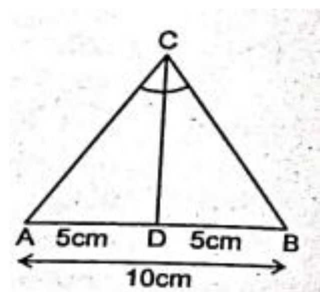
$h \approx 40 \times 1.732050807$

$h \approx 69.28$ meters

So, the height of the flag post is approximately 69.28 meters.

2. An isosceles triangle has a vertical angle of 120° and a base 10 cm long. Find the length of its altitude.

Solution



$\tan 60° = \frac{5}{h}$

$h = \frac{5}{\tan 60°}$

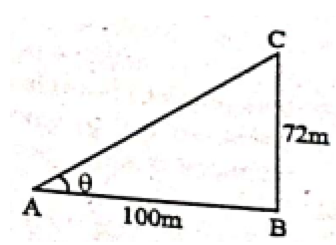
$h = \frac{5}{\sqrt{3}}$

$h \approx 2.89$ cm

So, the length of the altitude is approximately 2.89 cm.

3. A tree is 72 m high. Find the angle of elevation of its top from a point 100 m away on the ground level.

Solution



$\tan(\theta) = \frac{h}{d}$

$\tan(\theta) = \frac{72}{100}$

$\theta = \arctan\left(\frac{72}{100}\right)$

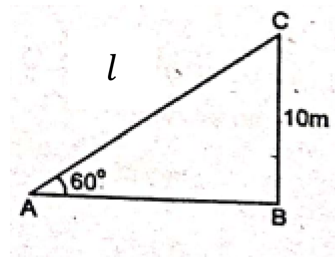
$\theta \approx 35.99°$

So, the angle of elevation of the top of the tree is approximately 35.99°.

4. A ladder makes an angle of 60° with the ground and reaches a height of 10m along the wall. Find the length of the ladder.

Solution

$\sin(\theta) = \frac{h}{l}$



$\sin(60°) = \frac{10}{l}$

$\frac{\sqrt{3}}{2} = \frac{10}{l}$

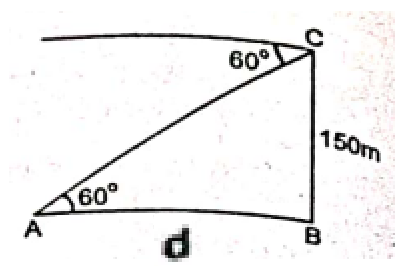
$l = \frac{20}{\sqrt{3}} \approx 11.55$ meters

So, the length of the ladder is approximately 11.55 meters.

5. A light house tower is 150 m high from the sea level. The angle of depression from the top of the tower to a ship is 60°. Find the distance between the ship and the tower.

Solution

$\tan(\theta) = \frac{h}{d}$



$\tan(60°) = \frac{150}{d}$

$d = \frac{150}{\tan(60°)}$

$d = \frac{150}{\sqrt{3}} \approx 86.60$ meters

So, the distance between the ship and the tower is approximately 86.60 meters.

6. Measure of an angle of elevation of the top of a pole is 15° from a point on the ground, in walking 100 m towards the pole the measure of angle is found to be 30°. Find the height of the pole.

Solution

Initial distance from the pole (d_1) = x + 100 meters

Initial angle of elevation (θ_1) = 15°

Final distance from the pole (d_2) = x meters (after walking 100 meters)

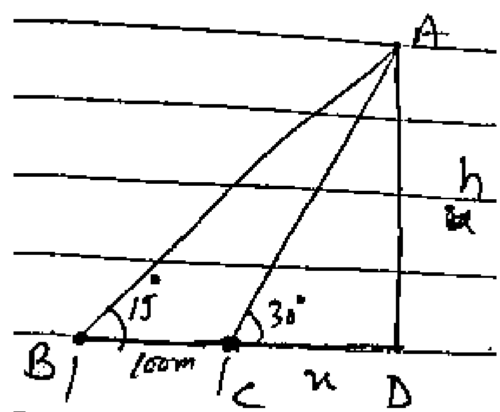
Final angle of elevation (θ_2) = 30°

We can use the tangent function to relate the angles, distances, and height (h) of the pole:

$$\tan(\theta_1) = \frac{h}{d_1}$$

$$\tan(15°) = \frac{h}{(x + 100)}$$

h = (x + 100) × tan(15°)



$$\tan(\theta_2) = \frac{h}{d_2}$$

$$\tan(30°) = \frac{h}{x}$$

h = x × tan(30°)

Equating the two expressions:

(x + 100) × tan(15°) = x × tan(30°)

(x + 100) × 0.2679 = x × 0.5773

0.2679x + 26.79 = 0.5773x

26.79 = 0.3094x

x ≈ 86.73 meters

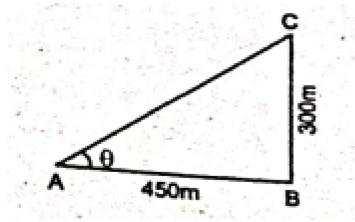
Now that we have x, we can find the height (h) of the pole:

h = x × tan(30°) = 86.73 × 0.5773 ≈ 50 meters

So, the height of the pole is approximately 50 meters.

7. Find the measure of an angle of elevation of the Sun, if a tower 300 m high casts a shadow 450 m long.

Solution



$\tan(\theta) = \frac{h}{s}$

$\tan(\theta) = \frac{300}{450}$

$\theta = \arctan\left(\frac{300}{450}\right)$

$\theta \approx 33.69°$

So, the measure of the angle of elevation of the sun is approximately 33.69°.

8. Measure of angle of elevation of the top of a cliff is 25°, on walking 100 metres towards the cliff, measure of angle of elevation of the top is 45°. Find the height of the cliff.

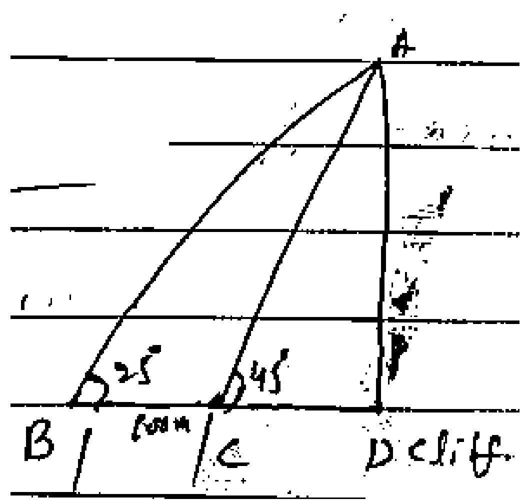
Solution

Initial distance from the cliff (d_1) = x + 100 meters

Initial angle of elevation (θ_1) = 25°

Final distance from the cliff (d_2) = x meters

Final angle of elevation (θ_2) = 45°



$\tan(\theta_1) = \frac{h}{d_1}$

$\tan(25°) = \frac{h}{(x + 100)}$

$h = (x + 100) \times \tan(25°)$

$\tan(\theta_2) = \frac{h}{d_2}$

$\tan(45°) = \frac{h}{x}$

$h = x \times \tan(45°)$

Equating the two expressions:

$(x + 100) \times \tan(25°) = x \times \tan(45°)$

$(x + 100) \times 0.4663 = x \times 1$

$0.4663x + 46.63 = x$

$46.63 = 0.5337x$

$x \approx 87.32$ meters

Now that we have x, we can find the height (h) of the cliff:

$h = x \times \tan(45°) = 87.32 \times 1 \approx 87.32$ meters

So, the height of the cliff is approximately 87.32 meters.

9. From the top of a hill 300 m high, the measure of the angle of depression of a point on the nearer shore of the river is 70° and measure of the angle of depression of a point, directly across the river is 50°. Find the width of the river How far is the river from the foot of the hill?

Solution

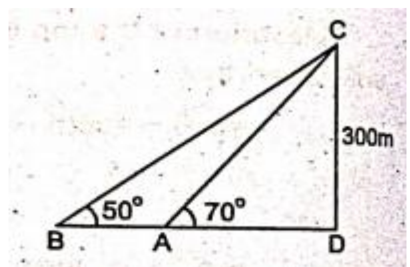
Distance to the nearer shore:

tan(70°) = 300 / x

x = 300 / tan(70°) ≈ 300 / 2.748 ≈ 109.2 meters

Distance to the point across the river:

tan(50°) = 300 / (x + w)



where w is the width of the river.

1.192 = 300 / (x + w)

x + w ≈ 300 / 1.192

x + w ≈ 251.7 meters

109.2 + w ≈ 251.7

w ≈ 142.5 meters

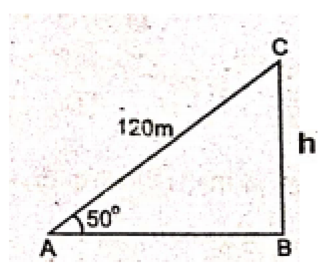
So, the width of the river is approximately 142.5 meters.

The distance from the foot of the hill to the river is approximately 109.2 meters.

10. A kite has 120 m of string attached to it when at an angle of elevation of 50°. How far is it above the hand holding it? (Assume that the string is stretched.

Solution

$\sin(\theta) = \frac{h}{l}$



$\sin(50°) = \frac{h}{120}$

h = 120 × sin(50°)

h ≈ 120 × 0.766 ≈ 91.92 meters

So, the kite is approximately 91.92 meters above the hand holding it.

REVIEW EXERCISE 6

1. Four options are given against each statement. Encircle the correct one.

(i) The value of $\tan^{-1} 2$ in radians is:

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) 1.11π (d) ✓ 1.11

(ii) In a right triangle, the hypotenuse is 13 units and one of the angles is $\theta = 30°$. The length of the opposite side is:

(a) 6.5 units (b) ✓ 7.5 units (c) 6 units (d) 5 units

(iii) A person standing 50 m away from a building sees the top of the building at an angle of elevation of 45°. Height of the building is:

(a) ✓ 50 m (b) 25 m (c) 35 m (d) 70 m

(iv) $\sec^2\theta - \tan^2\theta =$ __________.

(a) $\sin^2\theta$ (b) ✓ 1 (c) $\cos^2\theta$ (d) $\cot^2\theta$

(v) If $\sin\theta = \frac{3}{5}$ and θ is an acute angle, $\cos^2\theta =$ __________.

(a) $\frac{7}{25}$ (b) $\frac{24}{25}$ (c) ✓ $\frac{16}{25}$ (d) $\frac{4}{25}$

(vi) $\frac{5\pi}{24}$ rad = __________ degrees.

(a) 30° (b) ✓ 37.5° (c) 45° (d) 52.5°

(vii) $292.5° =$ __________ rad.

(a) $\frac{17\pi}{6}$ (b) $\frac{17\pi}{4}$ (c) 1.6π (d) ✓ 1.625π

(viii) Which of the following is a valid identity?

(a) ✓ $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$ (b) $\cos\left(\frac{\pi}{2}-\theta\right)=\cos\theta$

(c) $\cos\left(\frac{\pi}{2}-\theta\right)=\sec\theta$ (d) $\cos\left(\frac{\pi}{2}-\theta\right)=\text{cosec}\,\theta$

ix. $\sin 60° =$ ________.

(a) 1 (b) $\frac{1}{2}$ (c) $\sqrt{(3)^2}$ (d) ✓ $\frac{\sqrt{3}}{2}$

x. $\cos^2 100\pi + \sin^2 100\pi =$ _________.

(a) ✓ 1 (b) 2 (c) 3 (d) 4

2. Convert the given angles from:

(a) degrees to radians giving answer in terms of π.

(i) 255° (ii) 75° 45′ (iii) 142.5°

Solution

2(i): $\mathbf{255°} = 255\times\frac{\pi}{180} = \frac{17\pi}{2}\text{rad}$

2(ii): $\mathbf{75°45'} = \left(75+\frac{45}{60}\right)° = 75.75°\times\frac{\pi}{180} = \frac{101\pi}{240}\text{rad}$

2(iii): $\mathbf{142.5°} = 142.5\times\frac{\pi}{180} = \frac{19\pi}{24}\text{rad}$

2. Convert the given angles from:

(b) radians to degrees giving answer in degrees and minutes.

(i) $\frac{17\pi}{24}$ (ii) $\frac{7\pi}{12}$ (iii) $\frac{11\pi}{16}$

Solution

2(i): $\frac{\mathbf{17\pi}}{\mathbf{24}}\textbf{rad} = \frac{17\pi}{24}\times\frac{180°}{\pi} = 127.5° = 127°30'$

2(ii): $\frac{\mathbf{7\pi}}{\mathbf{12}}\textbf{rad} = \frac{7\pi}{12}\times\frac{180°}{\pi} = 105°$

2(iii): $\frac{\mathbf{11\pi}}{\mathbf{16}}\textbf{rad} = \frac{11\pi}{16}\times\frac{180°}{\pi} = 123°45'$

3. Prove the following trigonometric identities:

(i) $$\frac{\sin\theta}{1-\cos\theta}=\frac{1+\cos\theta}{\sin\theta}$$

Solution

$$\frac{\sin\theta}{1-\cos\theta}=\frac{\sin\theta}{1-\cos\theta}\times\frac{1+\cos\theta}{1+\cos\theta}=\frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}=\frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}=\frac{1+\cos\theta}{\sin\theta}$$

3. Prove the following trigonometric identities:

(ii) $$\sin\theta\,(\operatorname{cosec}\theta-\sin\theta)=\frac{1}{\sec^2\theta}$$

Solution

$$\sin\theta(\operatorname{cosec}\theta-\sin\theta)=\sin\theta\times\frac{1}{\sin\theta}-\sin^2\theta=1-\sin^2\theta=\cos^2\theta=\frac{1}{\sec^2\theta}$$

3. Prove the following trigonometric identities:

(iii) $$\frac{\operatorname{cosec}\theta-\sec\theta}{\operatorname{cosec}\theta+\sec\theta}=\frac{1-\tan\theta}{1+\tan\theta}$$

Solution

$$\frac{\operatorname{cosec}\theta-\sec\theta}{\operatorname{cosec}\theta+\sec\theta}=\frac{\frac{1}{\sin\theta}-\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}+\frac{1}{\cos\theta}}=\frac{\frac{\cos\theta-\sin\theta}{\sin\theta\cos\theta}}{\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}}=\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\frac{\cos\theta\left(1-\frac{\sin\theta}{\cos\theta}\right)}{\cos\theta\left(1+\frac{\sin\theta}{\cos\theta}\right)}=\frac{1-\tan\theta}{1+\tan\theta}$$

3. Prove the following trigonometric identities:

(iv) $$\tan\theta+\cot\theta=\frac{1}{\sin\theta\,\cos\theta}$$

Solution

$$\tan\theta+\cot\theta=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}=\frac{1}{\sin\theta\cos\theta}$$

3. Prove the following trigonometric identities:

(v) $$\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}+\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\frac{2}{1-2\sin^2\theta}$$

Solution

$$\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}+\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$$
$$=\frac{(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$
$$=\frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta+\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$$
$$=\frac{1+1}{1-\sin^2\theta-\sin^2\theta}=\frac{2}{1-2\sin^2\theta}$$

3. Prove the following trigonometric identities:

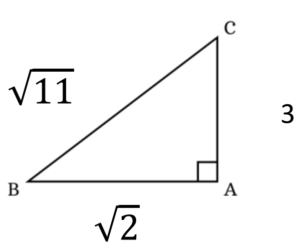
(vi) $$\frac{1+\cos\theta}{1-\cos\theta}=(\operatorname{cosec}\theta+\cot\theta)^2$$

Solution

$$(\operatorname{cosec}\theta+\cot\theta)^2$$
$$=\left(\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\right)^2=\frac{(1+\cos\theta)^2}{\sin^2\theta}=\frac{(1+\cos\theta)^2}{1-\cos^2\theta}=\frac{(1+\cos\theta)(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}=\frac{1+\cos\theta}{1-\cos\theta}$$

4. If $\tan\theta=\frac{3}{\sqrt{2}}$ then find the remaining trigonometric ratios when θ lies in first quadrant.

Solution



By Pythagoras Formula

$$H^2=P^2+B^2\Rightarrow H^2=3^2+\left(\sqrt{2}\right)^2$$
$$\Rightarrow H^2=9+2=11\Rightarrow H=\sqrt{11}$$

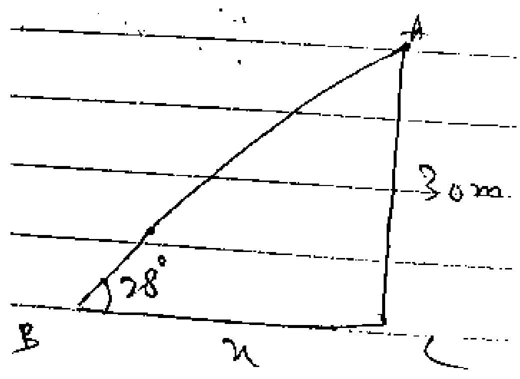
(i) $\sin\theta=\frac{3}{\sqrt{11}}$ (ii) $\cos\theta=\frac{\sqrt{2}}{\sqrt{11}}$ (iii) $\cot\theta=\frac{\sqrt{2}}{3}$

(iv) $\operatorname{cosec}\theta=\frac{\sqrt{11}}{3}$ (v) $\sec\theta=\frac{\sqrt{11}}{\sqrt{2}}$

5. From a point on the ground, the angle of elevation to the top of a 30 m high building is 28°. How far is the point from the base of the building?

Solution
Let's denote the distance from the point to the base of the building as x.

We know the angle of elevation (θ) is 28 degrees, and the height of the building (h) is 30 meters.



$$\tan(\theta) = \frac{h}{x}$$

$$\tan(28°) = \frac{30}{x}$$

$$x = \frac{30}{\tan(28°)}$$

$$x \approx \frac{30}{0.53170943}$$

x ≈ 56.42 meters

So, the point is approximately 56.42 meters away from the base of the building.

6. A ladder leaning against a wall forms an angle of 65° with the ground. If the ladder is 10 m long, how high does it reach on the wall?

Solution
Let's denote the height the ladder reaches on the wall as h.

We know the angle between the ladder and the ground (θ) is 65 degrees, and the length of the ladder (l) is 10 meters.

$$\sin(\theta) = \frac{h}{l}$$

$$\sin(65°) = \frac{h}{10}$$

$$h = 10 \times \sin(65°)$$

$$h \approx 10 \times 0.906307787$$

$h \approx 9.06$ meters

So, the ladder reaches approximately 9.06 meters high on the wall.

Unit 7 Coordinate Geometry

EXERCISE 7.1

1. Describe the location in the plane of the point $P(x, y)$, for which

(i) $x > 0$ (ii) $x > 0$ and $y > 0$ (iii) $x = 0$ (iv) $y = 0$

(v) $x > 0$ and $y \leq 0$ (vi) $y = 0, x = 0$ (vii) $x = y$

(viii) $x \geq 3$ (ix) $y > 0$ (x) x and y have opposite signs.

Solution

(i) Right half plane (ii) The 1st quadrant (iii) y-axis (iv) x-axis (v) 4th quadrant and negative y-axis (vi) Origin (vii) It is a line bisecting 1st and 3rd quadrant.

(vii) The set of points lying on and right side of the line $x = 3$.

(ix) The set of points lying above x-axis. (x) The set of points in 2nd and 4th quadrants.

2. Find the distance between the points:

(i) $A(6, 7), B(0, -2)$ (ii) $C(-5, -2), D(3, 2)$

(iii) $L(0, 3), M(-2, -4)$ (iv) $P(-8, -7), Q(0, 0)$

Solution

2 (i) $d = |\overline{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(0 - 6)^2 + (-2 - 7)^2}$

$= \sqrt{(-6)^2 + (-9)^2} = \sqrt{36 + 81} = \sqrt{117} = 3\sqrt{13}$

2 (ii) $d = |\overline{CD}| = \sqrt{\left(3 - (-5)\right)^2 + \left(2 - (-2)\right)^2}$

$= \sqrt{(8)^2 + (4)^2} = \sqrt{64 + 16} = \sqrt{80} = 4\sqrt{5}$

2 (iii) $d = |\overline{LM}| = \sqrt{(-2 - 0)^2 + \left(3 - (-4)\right)^2}$

$= \sqrt{(-2)^2 + (7)^2} = \sqrt{4 + 49} = \sqrt{53}$

2 (iv) $d = |\overline{PQ}| = \sqrt{\left(0 - (-8)\right)^2 + \left(0 - (-7)\right)^2}$

$= \sqrt{(8)^2 + (7)^2} = \sqrt{64 + 49} = \sqrt{113}$

3. Find in each of the following:

(i) The distance between the two given points

(ii) Midpoint of the line segment joining the two points:

(a) $A(3, 1)$, $B(-2, -4)$ (b) $A(-8, 3)$, $B(2, -1)$

(c) $A\left(-\sqrt{5}, -\frac{1}{3}\right)$, $B\left(-3\sqrt{5}, 5)\right)$

Solution

3(i) $\boldsymbol{A(3,1)}$, $\boldsymbol{B(-2,-4)}$

$$|AB| = \sqrt{(-2-3)^2 + (-4-1)^2} = \sqrt{(-5)^2 + (-5)^2}$$
$$= \sqrt{25+25} = \sqrt{50} = \sqrt{25\times 2} = 5\sqrt{2}$$

$$\text{Midpoint of } AB = \left(\frac{3-2}{2}, \frac{1-4}{2}\right) = \left(\frac{1}{2}, \frac{-3}{2}\right)$$

3(ii) $\boldsymbol{A(-8,3)}$, $\boldsymbol{B(2,-1)}$

$|\overline{AB}| = \sqrt{(2-(-8))^2 + (-1-3)^2}$

$= \sqrt{(10)^2 + (-4)^2} = \sqrt{100+16} = \sqrt{116} = 2\sqrt{29}$

mid point of AB $= \left(\frac{-8+2}{2}, \frac{3+(-1)}{2}\right) = \left(\frac{-6}{2}, \frac{2}{2}\right) = (-3,1)$

3(iii) $\boldsymbol{A(-\sqrt{5}, -\frac{1}{3})}$, $\boldsymbol{B(-3\sqrt{5}, 5)}$

$$|AB| = \sqrt{\left(-3\sqrt{5}+\sqrt{5}\right)^2 + \left(5+\frac{1}{3}\right)^2} = \sqrt{\left(2\sqrt{5}\right)^2 + \left(\frac{16}{3}\right)^2}$$
$$= \sqrt{20 + \frac{256}{9}} = \sqrt{\frac{436}{9}} = \sqrt{\frac{4\times 109}{9}} = \frac{2\sqrt{109}}{3}$$

$$\text{Midpoint of } AB = \left(\frac{-\sqrt{5}-3\sqrt{5}}{2}, \frac{-1/3+5}{2}\right) = \left(\frac{-4\sqrt{5}}{2}, \frac{14/3}{2}\right) = \left(-2\sqrt{5}, \frac{7}{3}\right)$$

4. Which of the following points are at a distance of 15 units from the origin?

(i) $(\sqrt{176}, 7)$ (ii) $(10, -10)$ (iii) $(1, 15)$

Solution

(i) Distance of $\left(\sqrt{176},7\right)$ from origin $=\sqrt{\left(\sqrt{176}-0\right)^2+\left(7-0\right)^2}$

$=\sqrt{(176)+(49)}$

$=\sqrt{(176)+(49)} \quad =\sqrt{225} \quad =15$

$\Rightarrow$ the point $\left(\sqrt{176},7\right)$ is at 15 unit away from origin.

(ii) Distance of $(10,-10)$ from origin $=\sqrt{(10-0)^2+(-10-0)^2}$

$=\sqrt{100+100} \quad =\sqrt{200}$

$=\sqrt{100\times 2} \quad =10\sqrt{2}\neq 15$

$\Rightarrow$ the point $(10,-10)$ is not at distance of 15 unit from origin.

(iii) Distance of (1,15) from origin $=\sqrt{(1-0)^2+(15-0)^2}$

$=\sqrt{1+225}=\sqrt{226}$

the point(1,15) from is not at distance of 15 unit from origin

5. Show that:

(i) the points $A(0, 2)$, $B(\sqrt{3}, 1)$ and $C(0, -2)$ are vertices of a right triangle.

Solution

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\Rightarrow AB = \sqrt{(\sqrt{3} - 0)^2 + (1 - 2)^2}$$

$$= \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4}$$

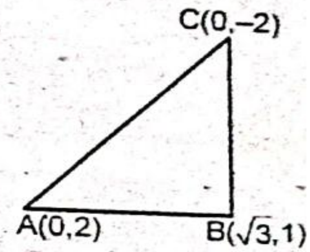
$$= 2$$

And $BC = \sqrt{(0 - \sqrt{3})^2 + (-2 - 1)^2}$

$$= \sqrt{(-\sqrt{3})^2 + (-3)^2} = \sqrt{12}$$

Also $AC = \sqrt{(0 - 0)^2 + (-2 - 2)^2}$

$$= \sqrt{0 + (-4)^2} = 4$$



By pythagoras theorem

$(AC)^2 = (AB)^2 + (BC)^2$

$(4)^2 = (2)^2 + (\sqrt{12})^2$

$16 = 4 + 12 = 16$

Therefore, the given points are vertices of a at right triangle.

5. Show that:

(ii) the points $A(3, 1)$, $B(-2, -3)$ and $C(2, 2)$ are vertices of an isosceles triangle.

Solution

Given: $A(3,1)$, $B(-2,-3)$ and $C(2,2)$

$$|AB| = \sqrt{(-2-3))^2 + (-3-1)^2} = \sqrt{(-5)^2 + (-4)^2} = \sqrt{25+16} = \sqrt{41}$$

$$|BC| = \sqrt{(2-(-2))^2 + (2-(-3))^2} = \sqrt{(4)^2 + (5)^2} = \sqrt{16+25} = \sqrt{41}$$

$$|CA| = \sqrt{(3-2)^2 + (1-2)^2} = \sqrt{(1)^2 + (-1)^2}$$

$$= \sqrt{1+1} = \sqrt{2}$$

$\because$ $|AB| = |BC|$ $\Rightarrow$ A, B & C are vertices of an isosceles triangle.

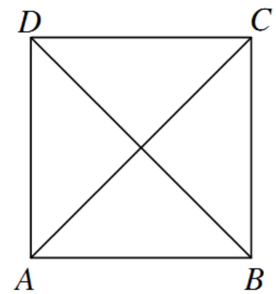
5. Show that:

(iii) the points $A(5, 2)$, $B(-2, 3)$, $C(-3, -4)$ and $D(4, -5)$ are vertices of a parallelogram.

Solution

Given: $A(5,2)$, $B(-2,3)$ & $C(-3,-4)$ and $D(4,-5)$

$$|AB| = \sqrt{(-2-5)^2 + (3-2)^2} = \sqrt{(-7)^2 + (1)^2}$$
$$= \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$$
$$|BC| = \sqrt{(-3+2)^2 + (-4-3)^2} = \sqrt{(-1)^2 + (-7)^2}$$
$$= \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$
$$|CD| = \sqrt{(4+3)^2 + (-5+4)^2} = \sqrt{(7)^2 + (-1)^2}$$
$$= \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$$
$$|DA| = \sqrt{(5-4)^2 + (2+5)^2} = \sqrt{(1)^2 + (7)^2}$$
$$= \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$



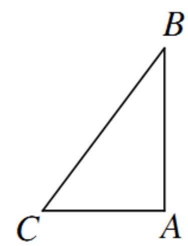
$\because$ $|AB| = |CD|$ and $|BC| = |DA|$ $\Rightarrow$ A, B, C and D are vertices of parallelogram.

6. Find h such that the points $A(\sqrt{3}, -1)$, $B(0, 2)$ and $C(h, -2)$ are vertices of a right triangle with right angle at the vertex A.

Solution

Since ABC is a right triangle therefore by Pythagoras theorem

$$|AB|^2 + |CA|^2 = |BC|^2$$
$$\Rightarrow \left[\left(0-\sqrt{3}\right)^2 + (2+1)^2\right] + \left[\left(\sqrt{3}-h\right)^2 + (-1+2)^2\right] = (h-0)^2 + (-2-2)^2$$
$$\Rightarrow [3+9] + \left[3 - 2\sqrt{3}h + h^2 + 1\right] = h^2 + 16$$
$$\Rightarrow 12 + 4 - 2\sqrt{3}\,h + h^2 = h^2 + 16$$
$$\Rightarrow -2\sqrt{3}\,h = h^2 + 16 - 12 - 4 - h^2 \quad \Rightarrow -2\sqrt{3}\,h = 0 \quad \Rightarrow \boxed{h = 0}.$$



7. Find h such that $A(-1, h)$, $B(3, 2)$ and $C(7, 3)$ are collinear.

Solution

Points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Since given points are collinear therefore

$$\begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix} = 0$$

$\Rightarrow -1(2-3) - h(3-7) + 1(9-14) = 0 \quad \Rightarrow -1(-1) - h(-4) + 1(-5) = 0$

$\Rightarrow 1 + 4h - 5 = 0 \quad \Rightarrow 4h - 4 = 0 \quad \Rightarrow 4h = 4 \quad \Rightarrow \boxed{h = 1}$

8. The points $A(-5, -2)$ and $B(5, -4)$ are ends of a diameter of a circle. Find the centre and radius of the circle.

Solution

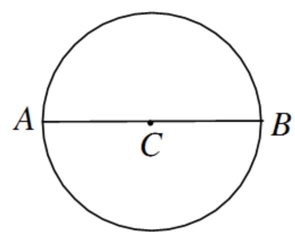
The centre of the circle is mid point of AB

i.e. centre 'C' $= \left(\dfrac{-5+5}{2}, \dfrac{-2-4}{2}\right) = \left(\dfrac{0}{2}, \dfrac{-6}{2}\right) = (0, -3)$

Now radius $= |AC|$

$= \sqrt{(0+5)^2 + (-3+2)^2}$

$= \sqrt{25+1} \quad = \sqrt{26}$



9. Find h such that the points $A(h, 1)$, $B(2, 7)$ and $C(-6, -7)$ are vertices of a right triangle with right angle at the vertex A.

Solution

According to the given condition

$|AB|^2 + |AC|^2 = |BC|^2$ ……(i)

$|\overline{AB}| = \sqrt{40 - 4h + h^2}$; $|\overline{BC}| = \sqrt{260}$; $|\overline{AC}| = \sqrt{h^2 + 12h + 100}$

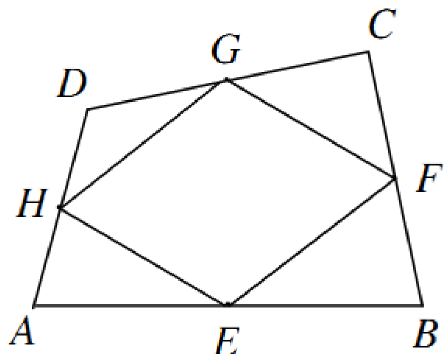
Putting in (i) we have $h^2 + 4h - 60 = 0$

$\Rightarrow (h+10)(h-6) = 0 \Rightarrow h + 10 = 0$; $h - 6 = 0$

$\Rightarrow h = -10$ or $h = 6$

10. A quadrilateral has the points $A(9, 3)$, $B(-7, 7)$, $C(-3, -7)$ and $D(5, -5)$ as its vertices. Find the midpoints of its sides. Show that the figure formed by joining the midpoints consecutively is a parallelogram.

Solution



Given: $A(9,3)$, $B(-7,7)$, $C(-3,-7)$ and $D(5,-5)$

Let E, F, G and H be the mid-points of sides of quadrilateral

Coordinate of $E = \left(\frac{9-7}{2},\frac{3+7}{2}\right) = \left(\frac{2}{2},\frac{10}{2}\right) = (1,5)$

Coordinate of $F = \left(\frac{-7-3}{2},\frac{7-7}{2}\right) = \left(\frac{-10}{2},\frac{0}{2}\right) = (-5,0)$

Coordinate of $G = \left(\frac{-3+5}{2},\frac{-7-5}{2}\right) = \left(\frac{2}{2},\frac{-12}{2}\right) = (1,-6)$

Coordinate of $H = \left(\frac{9+5}{2},\frac{3-5}{2}\right) = \left(\frac{14}{2},\frac{-2}{2}\right) = (7,-1)$

Now $|EF| = \sqrt{(-5-1)^2+(0-5)^2} = \sqrt{36+25} = \sqrt{61}$

$|FG| = \sqrt{(1+5)^2+(-6-0)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$

$|GH| = \sqrt{(7-1)^2+(-1+6)^2} = \sqrt{36+25} = \sqrt{61}$

$|HE| = \sqrt{(1-7)^2+(5+1)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$

Since $|EF| = |GH|$ and $|FG| = |HE|$

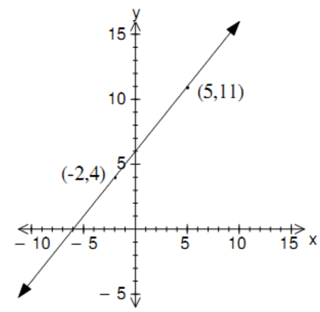
Therefore $EFGH$ is a parallelogram.

EXERCISE 7.2

1. Find the slope and inclination of the line joining the points:
   (i) (–2, 4) ; (5, 11) (ii) (3, –2) ; (2, 7) (iii) (4, 6) ; (4, 8)

Solution

(i) $(-2,4)$; $(5,11)$

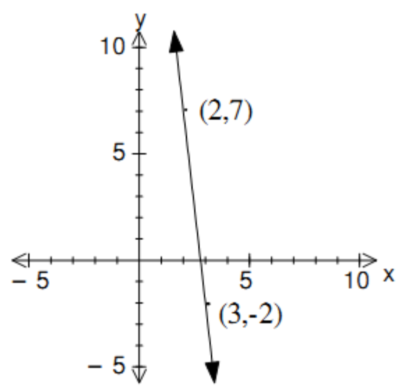


Slope $m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{11-4}{5+2}=\dfrac{7}{7}=1$

Since $\tan\alpha=m=1$

$\Rightarrow \alpha=\tan^{-1}(1)=45^\circ$

(ii) $(3,-2)$; $(2,7)$



Slope $m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{7+2}{2-3}=\dfrac{9}{-1}=-9$

Since $\tan\alpha=m=-9$

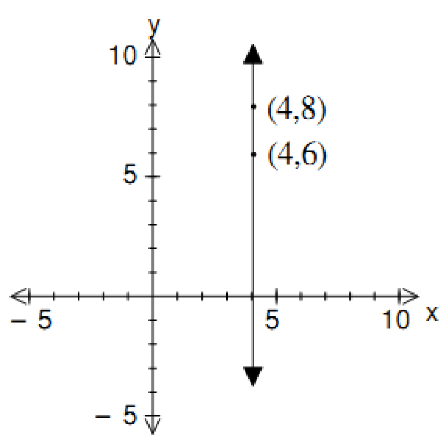
$\Rightarrow -\tan\alpha = 9 \Rightarrow \tan(180-\alpha) = 9$

$\Rightarrow 180-\alpha = \tan^{-1}(9)$

$\Rightarrow 180-\alpha = 83^\circ 40'$

$\Rightarrow \alpha = 180-83^\circ 40' = 96^\circ 20'$

(iii) $(4,6)$; $(4,8)$



Slope $m=\dfrac{y_2-y_1}{x_2-x_1}$

$=\dfrac{8-6}{4-4}=\dfrac{2}{0}=\infty$

Since $\tan\alpha=m=\infty$

$\Rightarrow \alpha=\tan^{-1}(\infty)$

$=90^\circ$

2. By means of slopes, show that the following points lie on the same line:

(i) $A(-1,-3)$; $B(1,5)$; $C(2,9)$ (ii) $P(4,-5)$; $Q(7,5)$; $R(10,15)$

(iii) $L(-4,6)$; $M(3,8)$; $N(10,10)$ (iv) $X(a,2b)$; $Y(c,a+b)$; $Z(2c-a,2a)$

Solution

(i) Let $A(-1,-3), B(1,5), C(2,9)$ be given points

Slope of AB $= \frac{5-(-3)}{1-(-1)} = \frac{5+3}{1+1} = \frac{8}{2} = 4$

Slope of BC $= \frac{9-5}{2-1} = \frac{4}{1} = 4$

Slope of AB = Slope of BC

Therefore A,B and C lie on the same line.

(ii) Let $P(4,-5), Q(7,5), R(10,15)$ be given points

Slope of PQ $= \frac{5-(-5)}{7-4} = \frac{5+5}{7-4} = \frac{10}{3}$

Slope of QR $= \frac{15-5}{10-7} = \frac{10}{3}$

Slope of PQ = Slope of QR

Therefore P,Q and R lie on the same line.

(iii) Let $L(-4,6), M(3,8), N(10,10)$ be given points

Slope of LM $= \frac{8-6}{3-(-4)} = \frac{8-6}{3+4} = \frac{2}{7}$

Slope of MN $= \frac{10-8}{10-3} = \frac{2}{7}$

Slope of LM = Slope of MN

Therefore P,Q and R lie on the same line.

(iv) Let $X(a,2b), Y(c,a+b), Z(2c-a,2a)$ be given points

Slope of XY $= \frac{(a+b)-2b}{c-a} = \frac{a-b}{c-a}$

Slope of YZ $= \frac{2a-(a+b)}{(2c-a)-c} = \frac{a-b}{c-a}$

Slope of XY = Slope of YZ

Therefore X,Y and Z lie on the same line.

3. Find k so that the line joining $A(7, 3)$; $B(k, -6)$ and the line joining $C(-4, 5)$; $D(-6, 4)$ are:

(i) parallel (ii) perpendicular.

Solution

Since $A(7,3)$, $B(k,-6)$, $C(-4,5)$ and $D(-6,4)$

Therefore slope of $AB = m_1 = \dfrac{-6-3}{k-7} = \dfrac{-9}{k-7}$

Slope of $CD = m_2 = \dfrac{4-5}{-6+4} = \dfrac{-1}{-2} = \dfrac{1}{2}$

(i) If AB and CD are parallel then $m_1 = m_2$

$$\Rightarrow \frac{-9}{k-7} = \frac{1}{2} \quad \Rightarrow -18 = k-7$$

$$\Rightarrow k = -18+7 \quad \Rightarrow \boxed{k = -11}$$

(ii) If AB and CD are perpendicular then $m_1 m_2 = -1$

$$\Rightarrow \left(\frac{-9}{k-7}\right)\left(\frac{1}{2}\right) = -1 \quad \Rightarrow -9 = -2(k-7)$$

$$\Rightarrow 9 = 2k-14 \quad \Rightarrow 2k = 9+14 = 23$$

$$\Rightarrow \boxed{k = \frac{23}{2}}$$

4. Using slopes, show that the triangle with its vertices $A(6, 1)$, $B(2, 7)$ and $C(-6, -7)$ is a right triangle.

Solution

Since $A(6,1), B(2,7)$ and $C(-6,-7)$ are vertices of triangle therefore

$$\text{Slope of } \overline{AB} = m_1 = \frac{7-1}{2-6} = \frac{6}{-4} = -\frac{3}{2}$$

$$\text{Slope of } \overline{BC} = m_2 = \frac{-7-7}{-6-2} = \frac{-12}{-8} = \frac{7}{4}$$

REMEMBER

The symbols

(*i*) || stands for 'parallel"

(*ii*) ∦ stands for "not parallel"

(*iii*) ⊥ stands for "perpendicular"

$$\text{Slope of } \overline{CA} = m_3 = \frac{1+7}{6+6} = \frac{8}{12} = \frac{2}{3}$$

$$\text{Since} \quad m_1 m_3 = \left(-\frac{3}{2}\right)\left(\frac{2}{3}\right) = -1$$

$\Rightarrow$ The triangle ABC is a right triangle with $m\angle A = 90^\circ$

5. Two pairs of points are given. Find whether the two lines determined by these points are:

(i) parallel (ii) perpendicular (iii) none.

(a) (1, –2), (2, 4) and (4, 1), (–8, 2)

(b) (–3, 4), (6, 2) and (4, 5), (–2, –7)

Solution

(a) Slope of line joining $(1,-2)$ and $(2,4) = m_1 = \frac{4+2}{2-1} = \frac{6}{1} = 6$

Slope of line joining $(4,1)$ and $(-8,2) = m_2 = \frac{2-1}{-8-4} = \frac{1}{-12}$

Since $m_1 \neq m_2$

Also $m_1 m_2 = 6 \cdot \frac{1}{-12} = -\frac{1}{2} \neq -1$

$\Rightarrow$ lines are neither parallel nor perpendicular.

(b) Slope of line joining $(-3,4)$ and $(6,2) = m_1 = \frac{2-4}{6-(-3)} = -\frac{2}{9}$

Slope of line joining $(4,5)$ and $(-2,-7) = m_2 = \frac{-7-5}{-2-4} = \frac{-12}{-6} = 2$

Since $m_1 \neq m_2$

Also $m_1 m_2 = \left(-\frac{2}{9}\right)(2) = -\frac{4}{9} \neq -1$

$\Rightarrow$ Lines are neither parallel nor perpendicular.

6. Find an equation of:

(a) the horizontal line through (7, –9) (b) the vertical line through (–5, 3)

(c) through A(–6, 5) having slope 7 (d) through (8, –3) having slope 0

(e) through (–8 , 5) having slope undefined

(f) through (–5, –3) and (9, –1)

(g) y-intercept: –7 and slope: –5

(h) x-intercept: –3 and y-intercept: 4

(i) x-intercept: –9 and slope: –4

Solution

(a) Since slope of horizontal line $= m = 0$

$\& \ (x_1, y_1) = (7, -9)$

therefore equation of line:

$$y - (-9) = 0(x - 7)$$

$$\Rightarrow \ x + 9 = 0$$

(b) Since slope of vertical line $m = \infty = \frac{1}{0}$

$\& \ (x_1, y_1) = (-5, 3)$

therefore required equation of line

$$y - 3 = \infty\left(x - (-5)\right)$$

$$\Rightarrow \ y - 3 = \frac{1}{0}(x + 5) \qquad \Rightarrow \ 0(y - 3) = 1(x + 5)$$

$$\Rightarrow \ x + 5 = 0$$

(c) $\because \quad (x_1, y_1) = (-6, 5)$

and slope of line $= m = 7$

so required equation

$$y - 5 = 7(x - (-6))$$

$$\Rightarrow \quad y - 5 = 7(x + 6) \qquad \Rightarrow \quad y - 5 = 7x + 42$$

$$\Rightarrow \quad 7x + 42 - y + 5 = 0 \qquad \Rightarrow \quad 7x - y + 47 = 0$$

(d) Slope $= m = 0$ and Point $= (x_1, y_1) = (8, -3)$

Equation of line is $y - y_1 = m(x - x_1)$

$$y - (-3) = 0(x - 8)$$

$$y + 3 = 0$$

(e) $\because \quad (x_1, y_1) = (-8, 5)$

and slope of line $= m = \infty$

So required equation

$$y - 5 = \infty(x - (-8))$$

$$\Rightarrow \quad y - 5 = \frac{1}{0}(x + 8) \qquad \Rightarrow \quad 0(y - 5) = 1(x + 8)$$

$$\Rightarrow \quad x + 8 = 0$$

(f) The line through $(-5, -3)$ and $(9, -1)$ is

$$y - (-3) = \frac{-1 - (-3)}{9 - (-5)}(x - (-5)) \qquad \Rightarrow \quad y + 3 = \frac{2}{14}(x + 5)$$

$$\Rightarrow \quad y + 3 = \frac{1}{7}(x + 5) \qquad \Rightarrow \quad 7y + 21 = x + 5$$

$$\Rightarrow \quad x + 5 - 7y - 21 = 0 \qquad \Rightarrow \quad x - 7y - 16 = 0$$

(g) $\because \quad y -$ intercept $= -7$

$\Rightarrow \quad (0, -7)$ lies on a required line

Also slope $= m = -5$

So required equation

$$y - (-7) = -5(x - 0)$$

$$\Rightarrow \quad y + 7 = -5x \qquad \Rightarrow \quad 5x + y + 7 = 0$$

(h) $x-\text{intercept} = a = -3$

$y-\text{intercept} = b = 4$

Using two-intercept form of equation line

$$\frac{x}{a}+\frac{y}{b}=1 \quad \Rightarrow \frac{x}{-3}+\frac{y}{4}=0$$

$$\Rightarrow 4x-3y=-12 \qquad \times\text{ing by } -12$$

$$\Rightarrow 4x-3y+12=0$$

(*i*) Slope $= m = -4$ and Point $= (x_1, y_1) = (-9,0)$

Equation of line is $y - 0 = -4\big(x-(-9)\big)$

$$y = -4(x+9)$$

$$4x + y + 36 = 0$$

7. Find an equation of the perpendicular bisector of the segment joining the points $A(3\ ,5)$ and $B(9,\ 8)$.

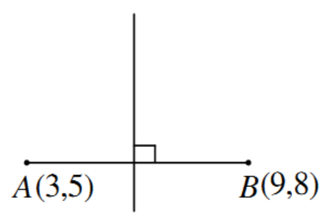
Solution

Given points $A(3,5)$ and $B(9,8)$

Midpoint of $\overline{AB} = \left(\frac{3+9}{2},\frac{5+8}{2}\right) = \left(\frac{12}{2},\frac{13}{2}\right) = \left(6,\frac{13}{2}\right)$

Slope of $\overline{AB} = m = \frac{8-5}{9-3} = \frac{3}{6} = \frac{1}{2}$

Slope of line $\perp$ to $\overline{AB} = -\frac{1}{m} = -\frac{1}{1/2} = --2$



Now equation of $\perp$ bisector having slope -2 through $\left(6,\frac{13}{2}\right)$

$$\Rightarrow y-\frac{13}{2}=-2(x-6)$$

$$\Rightarrow y-\frac{13}{2}=-2x+12 \qquad \Rightarrow y-\frac{13}{2}+2x-12=0$$

$$\Rightarrow 2x+y-\frac{37}{2}=0 \qquad \Rightarrow 4x+2y-37=0$$

8. Find an equation of the line through (–4, –6) and perpendicular to a line having slope $\frac{-3}{2}$.

Solution

Here $(x_1, y_1) = (-4, -6)$

Slope of given line $= m = \frac{-3}{2}$

$\because$ required line is $\perp$ to given line

$\therefore$ slope of required line $= -\frac{1}{m} = -\frac{1}{-3/2} = \frac{2}{3}$

Now equation of line having slope $\frac{2}{3}$ passing through $(-4, -6)$

$$y - (-6) = \frac{2}{3}(x - (-4))$$

$$\Rightarrow 3(y + 6) = 2(x + 4) \qquad \Rightarrow 3y + 18 = 2x + 8$$

$$\Rightarrow 2x + 8 - 3y - 18 = 0 \qquad \Rightarrow 2x - 3y - 10 = 0$$

9. Find an equation of the line through (11, –5) and parallel to a line with slope –24.

Solution

Here $(x_1, y_1) = (11, -5)$

Slope of given line $= m = -24$

$\because$ required line is || to given line

$\therefore$ slope of required line $= m = -24$

Now equation of line having slope -24 passing through $(11, -5)$

$$y - (-5) = -24(x - 11)$$

$$\Rightarrow y + 5 = -24x + 264 \qquad \Rightarrow 24x - 264 + y + 5 = 0$$

$$\Rightarrow 24x + y - 259 = 0$$

10. Convert each of the following equations into slope intercept form, two intercept form and normal form:

(a) $2x-4y+11=0$ (b) $4x+7y-2=0$ (c) $15y-8x+3=0$

Solution

(a) $2x-4y+11=0$

(i) - Slope-intercept form

$$\because\ 2x-4y+11=0$$

$$\Rightarrow 4y=2x+11 \quad \Rightarrow y=\frac{2x+11}{4}$$

$$\Rightarrow y=\frac{1}{2}x+\frac{11}{4}$$

is the intercept form of equation of line with $m=\frac{1}{2}$ and $c=\frac{11}{4}$

(ii) - Two-intercept form

$$\because\ 2x-4y+11=0 \quad \Rightarrow 2x-4y=-11$$

$$\Rightarrow \frac{2}{-11}x-\frac{4}{-11}y=1 \quad \Rightarrow \frac{x}{-11/2}+\frac{y}{11/4}=1$$

is the two-point form of equation of line with $a=-\frac{11}{2}$ and $b=\frac{11}{4}$.

(iii) - Normal form

$$\because\ 2x-4y+11=0 \quad \Rightarrow 2x-4y=-11$$

Dividing above equation by $\sqrt{(2)^2+(-4)^2}=\sqrt{20}=2\sqrt{5}$

$$\frac{2x}{2\sqrt{5}}-\frac{4y}{2\sqrt{5}}=\frac{-11}{2\sqrt{5}} \quad \Rightarrow \frac{x}{\sqrt{5}}-\frac{2y}{\sqrt{5}}=\frac{-11}{2\sqrt{5}}$$

$$\Rightarrow -\frac{x}{\sqrt{5}}+\frac{2y}{\sqrt{5}}=\frac{11}{2\sqrt{5}} \qquad \times\text{ing by } -1.$$

Suppose $\cos\alpha=-\frac{1}{\sqrt{5}}<0$ and $\sin\alpha=\frac{2}{\sqrt{5}}>0$

$\Rightarrow \alpha$ lies in 2nd quadrant and $\alpha=\cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)=116.57^\circ$

Hence the normal form is

$$x\cos\left(116.57^\circ\right)+y\sin\left(116.57^\circ\right)=\frac{11}{2\sqrt{5}}$$

And length of perpendicular from $(0,0)$ to line $=p=\frac{11}{2\sqrt{5}}$

(b) $4x + 7y - 2 = 0$

(i) - Slope-intercept form

$$\because \; 4x+7y-2=0$$

$$\Rightarrow \; 7y=-4x+2 \qquad \Rightarrow \; y=\frac{-4x+2}{7}$$

$$\Rightarrow \; y=-\frac{4}{7}x+\frac{2}{7}$$

is the intercept form of equation of line with $m=-\frac{4}{7}$ and $c=\frac{2}{7}$

(ii) - Two-intercept form

$$\because \; 4x+7y-2=0 \qquad \Rightarrow \; 4x+7y=2$$

$$\Rightarrow \; 2x+\frac{7}{2}y=1 \qquad \div\text{ing by } 2$$

$$\Rightarrow \; \frac{x}{1/2}+\frac{y}{2/7}=1$$

is the two-point form of equation of line with $a=\frac{1}{2}$ and $b=\frac{2}{7}$.

(iii) - Normal form

$$\because \; 4x+7y-2=0$$

$$\Rightarrow \; 4x+7y=2$$

Dividing above equation by $\sqrt{(4)^2+(7)^2}=\sqrt{16+49}=\sqrt{65}$

$$\Rightarrow \; \frac{4}{\sqrt{65}}x+\frac{7}{\sqrt{65}}y=\frac{2}{\sqrt{65}} \quad .$$

Suppose $\cos\alpha=\frac{4}{\sqrt{65}}>0$ and $\sin\alpha=\frac{7}{\sqrt{65}}>0$

$\Rightarrow$ α lies in first quadrant and $\alpha=\cos^{-1}\left(\frac{4}{\sqrt{65}}\right)=60.26^\circ$

Hence the normal form is

$$x\cos\left(60.26^\circ\right)+y\sin\left(60.26^\circ\right)=\frac{2}{\sqrt{65}}$$

And length of perpendicular from $(0,0)$ to line $= p=\frac{2}{\sqrt{65}}$

(c) $15y - 8x + 3 = 0$

(i) - Slope-intercept form

$$\because \quad 15y - 8x + 3 = 0$$

$$\Rightarrow \quad 15y = 8x - 3 \qquad \Rightarrow \quad y = \frac{8x-3}{15}$$

$$\Rightarrow \quad y = \frac{8}{15}x - \frac{3}{15} \qquad \Rightarrow \quad y = \frac{8}{15}x - \frac{1}{5}$$

is the intercept form of equation of line with $m = \frac{8}{15}$ and $c = -\frac{1}{5}$

(ii) - Two-intercept form

$$\because \quad 15y - 8x + 3 = 0 \qquad \Rightarrow \quad -8x + 15y = -3$$

$$\Rightarrow \quad \frac{8x}{3} - 5y = 1 \qquad \Rightarrow \quad \frac{x}{3/8} + \frac{y}{-1/5} = 1$$

is the two-point form of equation of line with $a = \frac{3}{8}$ and $b = -\frac{1}{5}$.

(iii) - Normal form

$$\because \; 15y - 8x + 3 = 0$$

$$\Rightarrow \; 8x - 15y = 3$$

Dividing above equation by $\sqrt{(8)^2 + (-15)^2} = \sqrt{64 + 225} = \sqrt{289} = 17$

$$\Rightarrow \quad \frac{8}{17}x - \frac{15}{17}y = \frac{3}{17} \quad .$$

Suppose $\cos\alpha = \frac{8}{17} > 0$ and $\sin\alpha = -\frac{15}{17} < 0$

$\Rightarrow$ α lies in 4^{th} quadrant and $\alpha = \cos^{-1}\left(\frac{8}{17}\right) = 298.07^\circ$

Hence the normal form is

$$x\cos\left(298.07^\circ\right) + y\sin\left(298.07^\circ\right) = \frac{3}{17}$$

And length of perpendicular from $(0,0)$ to line $= p = \frac{3}{17}$

> $\alpha = \cos^{-1}\left(\frac{8}{17}\right)$
> $= 61.93^\circ,\ 298.07^\circ$
> Taking value that lies in 4^{th} quadrant.

11. In each of the following check whether the two lines are

(i) parallel (ii) perpendicular (iii) neither parallel nor perpendicular

(a) $2x + y - 3 = 0$; $4x + 2y + 5 = 0$

(b) $3y = 2x + 5$; $3x + 2y - 8 = 0$

(c) $4y + 2x - 1 = 0$; $x - 2y - 7 = 0$

Solution

(a) Let $l_1 :\ 2x + y - 3 = 0$

$l_2 :\ 4x + 2y + 5 = 0$

$$\text{Slope of } l_1 = m_1 = -\frac{2}{1} = -2$$

$$\text{Slope of } l_2 = m_2 = -\frac{4}{2} = -2$$

Since $m_1 = m_2$ therefore l_1 and l_2 are parallel.

(b) Let $l_1 :\ 3y = 2x + 5 \ \Rightarrow\ 2x - 3y + 5 = 0$

$l_2 :\ 3x + 2y - 8 = 0$

$$\text{Slope of } l_1 = m_1 = -\frac{2}{-3} = \frac{2}{3}$$

$$\text{Slope of } l_2 = m_2 = -\frac{3}{2} =$$

Since $m_1 m_2 = \left(\frac{2}{3}\right)\left(-\frac{3}{2}\right) = -1 \ \Rightarrow\ l_1$ and l_2 are perpendicular.

(c) Let $l_1 :\ 4y + 2x - 1 = 0 \ \Rightarrow\ 2x + 4y - 1 = 0$

$l_2 :\ x - 2y - 7 = 0$

$$\text{Slope of } l_1 = m_1 = -\frac{2}{4} = -\frac{1}{2}$$

$$\text{Slope of } l_2 = m_2 = -\frac{1}{-2} = \frac{1}{2}$$

Since $m_1 \neq m_2$ and $m_1 m_2 = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{1}{4} \neq -1$

$\Rightarrow$ l_1 and l_2 are neither parallel nor perpendicular.

12. Find an equation of the line through (–4, 7) and parallel to the line $2x - 7y + 4 = 0$

Solution

Let $l: 2x - 7y + 4 = 0$

Slope of $l = m = -\frac{2}{-7} = \frac{2}{7}$

Since required line is parallel to l

> **REMEMBER**
> If $l: ax + by + c = 0$
> then slope of $l = -\frac{a}{b}$

Therefore slope of required line $= m = \frac{2}{7}$

Now equation of line having slope $\frac{2}{7}$ passing through $(-4, 7)$

$$y - 7 = \frac{2}{7}(x - (-4))$$

$$\Rightarrow 7(y - 7) = 2(x + 4)$$

$$\Rightarrow 7y - 49 = 2x + 8 \quad \Rightarrow 2x + 8 - 7y + 49 = 0$$

$$\Rightarrow 2x - 7y + 57 = 0$$

13. Find an equation of the line through (5, –8) and perpendicular to the join of $A(-15, -8)$, $B(10, 7)$.

Solution

Given: $A(-15, -18)$, $B(10, 7)$ and $(5, 8)$

Slope of $\overline{AB} = m = \frac{7 - (-18)}{10 - (-15)}$

$$= \frac{7 + 18}{10 + 15} = \frac{25}{25} = 1$$

Since required line is perpendicular to $\overline{AB}$

Therefore slope of required line $= -\frac{1}{m} = -\frac{1}{1} = -1$

Now equation of line having slope -1 through $(5, -8)$

$$y - (-8) = -1(x - 5)$$

$$\Rightarrow y + 8 = -x + 5$$

$$\Rightarrow x + y + 8 - 5 = 0 \quad \Rightarrow x + y + 3 = 0$$

Exercise 7.3

1. If the houses of two friends are represented by coordinates (2, 6) and (9, 12) on a grid. Find the straight line distance between their houses if the grid units represent kilometres?

Solution

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{distance} = \sqrt{(9-2)^2 + (12-6)^2} = \sqrt{(7)^2 + (6)^2} = \sqrt{49+36}$$

$$\text{distance} = \sqrt{85} \approx 9.22 \text{ km}$$

2. Consider a straight trail (represented by coordinate plane) that starts at point (5, 7) and ends at point (15, 3). What is the coordinate of the midpoint?

Solution

$$\text{Mid Point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\text{Mid Point} = \left(\frac{5+15}{2}, \frac{7+3}{2}\right) = \left(\frac{20}{2}, \frac{10}{2}\right)$$

$$\text{Mid Point} = (10,5)$$

3. An architect is designing a park with two buildings located at (10, 8) and (4, 3) on the grid. Calculate the straight-line distance between the buildings. Assume the coordinates are in meters.

Solution

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{distance} = \sqrt{(4-10)^2 + (3-8)^2} = \sqrt{(-6)^2 + (-5)^2} = \sqrt{36+25}$$

$$\text{distance} = \sqrt{61} \approx 7.81 \text{ m}$$

4. A delivery driver needs to calculate the distance between two delivery locations. One location is at (7, 2) and the other at (12, 10) on the city grid map, where each unit represents kilometres. What is the distance between the two locations?

Solution

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{distance} = \sqrt{(12 - 7)^2 + (10 - 2)^2} = \sqrt{(5)^2 + (8)^2} = \sqrt{25 + 64}$$

$$\text{distance} = \sqrt{89} \approx 9.43 \text{ km}$$

5. The start and end points of a race track are given by coordinates (3, 9) and (9, 13). What is the midpoint of the track.

Solution

$$\text{Mid Point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\text{Mid Point} = \left(\frac{3+9}{2}, \frac{9+13}{2}\right) = \left(\frac{12}{2}, \frac{22}{2}\right)$$

$$\text{Mid Point} = (6,11)$$

6. The coordinates of two points on a road are *A*(3, 4) and *B*(7, 10). Find the midpoint of the road.

Solution

$$\text{Mid Point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\text{Mid Point} = \left(\frac{3+7}{2}, \frac{4+10}{2}\right) = \left(\frac{10}{2}, \frac{14}{2}\right)$$

$$\text{Mid Point} = (5,7)$$

7. A ship is navigating from port *A* located at (12° N, 65° W) to port *B* at (20 ° N, 45 ° W). If the ship travels along the shortest path on the surface of the Earth, calculate the straight line distance between the points.

Solution

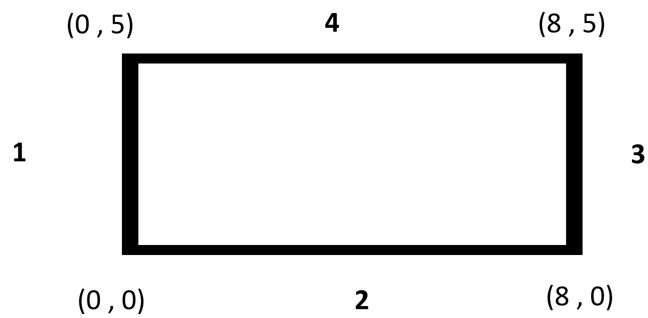
$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{distance} = \sqrt{(20 - 12)^2 + (45 - 65)^2} = \sqrt{(8)^2 + (-20)^2} = \sqrt{64 + 400}$$

$$\text{distance} = \sqrt{464} = \sqrt{16 \times 29} = 4\sqrt{29} \approx 21.5 \text{ unit}$$

8. Farah is fencing around a rectangular field with corners at (0,0), (0,5), (8, 5) and (8, 0). How much fencing material will she need to cover the entire perimeter of the field?

Solution



Length of side 1 $= 5 - 0 = 5$

Length of side 2 $= 8 - 0 = 8$

Length of side 3 $= 5 - 0 = 5$

Length of side 4 $= 8 - 0 = 8$

Perimeter = Side 1 + Side 2 + Side 3 + Side 4

Perimeter = 5 + 8 + 5 + 8 = 26 units

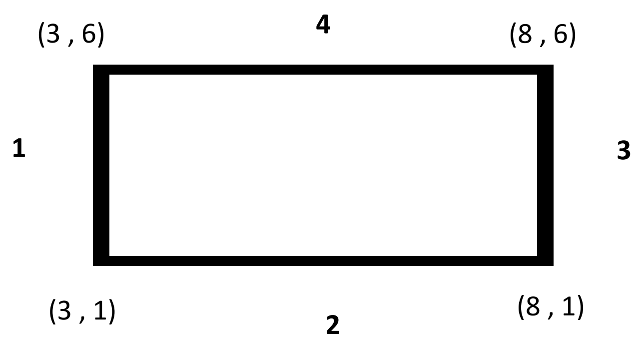
9. An airplane is flying from city *X* at (40° N, 100 ° W) to city *Y* at (50 ° N, 80° W). Use coordinate geometry, calculate the shortest distance between these two cities.

Solution

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\text{distance} = \sqrt{(50 - 40)^2 + (80 - 100)^2} = \sqrt{(-10)^2 + (-20)^2} = \sqrt{100 + 400}$$
$$\text{distance} = \sqrt{500} \approx 22.4 \text{ unit}$$

10. A land surveyor is marking out a rectangular plot of land with corners at (3, 1), (3, 6), (8, 6), and (8, 1). Calculate the perimeter.

Solution



Length of side 1 $= 6 - 1 = 5$

Length of side 2 $= 8 - 3 = 5$

Length of side 3 $= 6 - 1 = 5$

Length of side 4 $= 8 - 3 = 5$

Perimeter = Side 1 + Side 2 + Side 3 + Side 4

Perimeter = 5 + 5 + 5 + 5 = 20 units

11. A landscaper needs to install a fence around a rectangular garden. The garden has its corners at the coordinates: $A(0, 0)$, $B(5, 0)$, $C(5, 3)$, and $D(0, 3)$. How much fencing is required?

Solution

Solution

Length of side 1 = 5 − 0 = 5

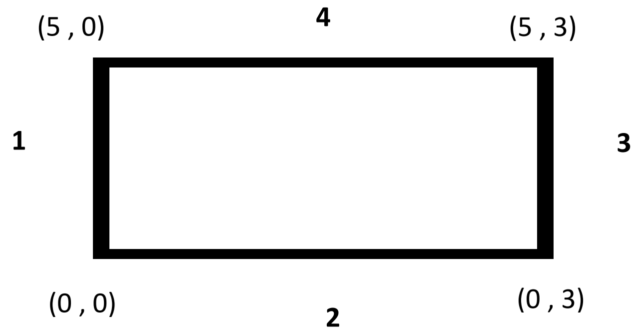
Length of side 2 = 3 − 0 = 3

Length of side 3 = 5 − 0 = 5

Length of side 4 = 3 − 0 = 3

Perimeter = Side 1 + Side 2 + Side 3 + Side 4

Perimeter = 5 + 3 + 5 + 3 = 16 units



REVIEW EXERCISE 7

1. Four options are given against each statement. Encircle the correct option.

(i) The equation of a straight line in the slope-intercept form is written as:
- (a) $y = m(x + c)$
- (b) $y - y_1 = m(x - x_1)$
- (c) ✓ $y = c + mx$
- (d) $ax + by + c = 0$

(ii) The gradients of two parallel lines are:
- (a) ✓ equal
- (b) zero
- (c) negative reciprocals of each other
- (d) always undefined

(iii) If the product of the gradients of two lines is −1, then the lines are:
- (a) Parallel
- (b) ✓ perpendicular
- (c) Collinear
- (d) coincident

(iv) Distance between two points $P(1, 2)$ and $Q(4, 6)$ is:
- (a) ✓ 5
- (b) 6
- (c) $\sqrt{13}$
- (d) 4

(v) The midpoint of a line segment with endpoints (–2, 4) and (6, –2) is:
- (a) (4, 2)
- (b) ✓ (2, 1)
- (c) (1, 1)
- (d) (0, 0)

(vi) A line passing through points (1, 2) and (4, 5) is:
- (a) ✓ $y = x + 1$
- (b) $y = 2x + 3$
- (c) $y = 3x - 2$
- (d) $y = x + 2$

(vii) The equation of a line in point-slope form is:

(a) $y = m(x + c)$ (b) ✓ $y - y_1 = m(x - x_1)$

(c) $y = c + mx$ (d) $ax + by + c = 0$

(viii) $2x + 3y - 6 = 0$ in the slope-intercept form is:

(a) ✓ $y = \frac{-2}{3}x + 2$ (b) $y = \frac{2}{3}x - 2$

(c) $y = \frac{2}{3}x + 1$ (d) $y = \frac{-2}{3}x - 2$

(ix) The equation of a line in symmetric form is:

(a) $\frac{x}{a} + \frac{y}{b} = 1$ (b) $\frac{x - x_1}{1} + \frac{y - y_1}{m} = \frac{z - z_1}{1}$

✓(c) $\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\sin\alpha} = r$ (d) $y - y_1 = m(x - x_1)$

(x) The equation of a line in normal form is:

(a) $y = mx + c$ (b) $\frac{x}{a} + \frac{y}{b} = 1$

(c) $\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\sin\alpha}$ (d) ✓ $x\cos\alpha + y\sin\alpha = p$

2. Find the distance between two points $A(2, 3)$ and $B(7, 8)$ on a coordinate plane.

Solution

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{distance} = \sqrt{(7-2)^2 + (8-3)^2} = \sqrt{(5)^2 + (5)^2} = \sqrt{25 + 25}$$

$$\text{distance} = \sqrt{50} = 5\sqrt{2}$$

3. Find the midpoint of the line segment joining the points (4, –2) and (–6, 3).

Solution

$$\text{Mid Point} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\text{Mid Point} = \left(\frac{-6+4}{2}, \frac{3-2}{2}\right) = \left(\frac{-2}{2}, \frac{1}{2}\right)$$

$$\text{Mid Point} = \left(-1, \frac{1}{2}\right)$$

4. Calculate the gradient (slope) of the line passing through the points (1, 2) and (4, 6).

Solution

$$\text{Gradient(Slope)} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{4-1} = \frac{4}{3}$$

5. Find the equation of the line in the form $y = mx + c$ that passes through the points (3, 7) and (5, 11).

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11-7}{5-3} = \frac{4}{2} = 2$$

Equation of line through (3,7):

$$y - y_1 = m(x - x_1)$$
$$y - 7 = 2(x - 3)$$
$$y - 7 = 2x - 6$$
$$y = 2x + 1$$

6. If two lines are parallel and one line has a gradient of $\frac{2}{3}$, what is the gradient of the other line?

Solution

Gradient (Slope) of one line $= m_1 = \frac{2}{3}$

For parallel lines $m_1 = m_2$

Gradient (Slope) of other line $= m_2 = \frac{2}{3}$

7. An airplane needs to fly from city *A* to coordinates (12, 5) to city *B* at coordinates (8, –4). Calculate the straight-line distance between these two cities.

Solution

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{distance} = \sqrt{(8-12)^2 + (-4-5)^2} = \sqrt{(-4)^2 + (-9)^2} = \sqrt{16 + 81}$$

$$\text{distance} = \sqrt{97} \approx 9.85 \text{ units}$$

8. In a landscaping project, the path starts at (2, 3) and ends at (10, 7). Find the midpoint.

Solution

$\text{Mid Point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$\text{Mid Point} = \left(\frac{10+2}{2}, \frac{3+7}{2}\right) = \left(\frac{12}{2}, \frac{10}{2}\right)$

$\text{Mid Point} = (6,5)$

9. A drone is flying from point (2, 3) to point (10, 15) on the grid. Calculate the gradient of the line along which the drone is flying and the total distance travelled.

Solution

$\text{Gradient(Slope)} = \frac{y_2-y_1}{x_2-x_1} = \frac{15-3}{10-2} = \frac{12}{8} = \frac{3}{2}$

$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\text{distance} = \sqrt{(10-2)^2 + (15-3)^2} = \sqrt{(8)^2 + (12)^2} = \sqrt{64 + 144}$

$\text{distance} = \sqrt{208} = \sqrt{16 \times 13} = 4\sqrt{13} \approx 14.4 \text{ units}$

10. For a line with a gradient of –3 and a y-intercept of 2, write the equation of the line in:

(a) Slope-intercept form
(b) Point-slope form using the point (1, 2)
(c) Two-point form using the points (1, 2) and (4, –7)
(d) Intercepts form
(e) Symmetric form
(f) Normal form

Solution

(a) Slope-Intercept Form

$y = mx + c$, where $m = -3$ and $c = 2$

$y = -3x + 2$

(b) Point-Slope Form with $m = -3, P(1, 2)$

$y - y_1 = m(x - x_1)$
$y - 2 = -3(x - 1)$

(c) Two-Point Form Using points (1, 2) and (4, −7)

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$
$\frac{y-2}{-7-2} = \frac{x-1}{4-1}$

(d) Intercept Form

y = −3x + 2

y + 3x = 2

$\frac{y}{2} + \frac{x}{\frac{2}{3}} = 1$

(e) Symmetric Form

y = −3x + 2

y + 3x = 2

Dividing $\sqrt{(3)^2 + (1)^2} = \sqrt{10}$ on both sides

$\frac{y}{\sqrt{10}} + \frac{3x}{\sqrt{10}} = \frac{2}{\sqrt{10}}$

(f) Normal Form

Slope = $tan\alpha = -3$

$\alpha = \tan^{-1}(-3) = -71.56°$

x cos α + y sin α = p where p = $\frac{2}{\sqrt{10}}$

x cos(−71.56°) + ysin(−71.56°) = $\frac{2}{\sqrt{10}}$

Unit 8 Logic

EXERCISE 8

1. Four options are given against each statement. Encircle the correct option.

(i) Which of the following expressions is often related to inductive reasoning?

✓(a) based on repeated experiments (b) if and only if statements

(c) Statement is proven by a theorem (d) based on general principles

(ii) Which of the following sentences describe deductive reasoning?

(a) general conclusions from a limited number of observations

(b) based on repeated experiments

(c) based on units of information that are accurate

✓(d) draw conclusion from well-known facts

(iii) Which one of the following statements is true?

(a) The set of integers is finite

(b) The sum of the interior angles of any quadrilateral is always 180°

✓(c) $\frac{22}{7} \notin Q'$

(d) All isosceles triangles are equilateral triangles

(iv) Which of the following statements is the best to represent the negation of the statement "The stove is burning"?

✓(a) the stove is not burning.

(b) the stove is dim

(c) the stove is turned to low heat

(d) it is both burning and not burning.

(v) The conjunction of two statements p and q is true when:

(a) both p and q are false. ✓(b) both p and q are true.

(c) only q is true. (d) only p is true

(vi) A conditional is regarded as false only when:

✓(a) antecedent is true and consequent is false.
(b) consequent is true and antecedent is false.
(c) antecedent is true only.
(d) consequent is false only.

(vii) Contrapositive of $q \to p$ is

(a) $q \to \sim p$ (b) $\sim q \to p$ ✓(c) $\sim p \to \sim q$ (d) $\sim q \to \sim p$

(viii) The statement "Every integer greater than 2 is a sum of two prime numbers" is:

(a) theorem ✓(b) conjecture (c) axiom (d) postulates

(ix) The statement "A straight line can be drawn between any two points" is :

(a) theorem (b) conjecture ✓(c) axiom (d) logic

(x) The statement "The sum of the interior angle of a triangle is 180°" is:

(a) converse ✓(b) theorem (c) axiom (d) conditional

2. Write the converse, inverse and contrapositive of the following conditionals:

(i) $\sim p \to q$ (ii) $q \to p$ (iii) $\sim p \to \sim q$ (iv) $\sim q \to \sim p$

Solution

Conditional	**Converse**	**Inverse**	**Contra Positive**
$\sim p \to q$	$q \to \sim p$	$p \to \sim q$	$\sim q \to p$
$q \to p$	$p \to q$	$\sim q \to \sim p$	$\sim p \to \sim q$
$\sim p \to \sim q$	$\sim q \to \sim p$	$p \to q$	$q \to p$
$\sim q \to \sim p$	$\sim p \to \sim q$	$q \to p$	$p \to q$

3. Write the truth table of the following

(i) $\sim(p \vee q) \vee (\sim q)$ (ii) $\sim(\sim q \vee \sim p)$ (iii) $(p \vee q) \leftrightarrow (p \wedge q)$

Solution

3(i) $\sim(\mathbf{p} \vee \mathbf{q}) \vee (\sim\mathbf{q})$

p	q	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim(p \vee q) \vee (\sim q)$
T	T	F	T	F	F
T	F	T	T	F	T
F	T	F	T	F	F
F	F	T	F	T	T

3(ii) $\sim(\sim q \vee \sim p)$

p	q	~p	~q	(~q ∨ ~p)	~(~q ∨ ~p)
T	T	F	F	F	T
T	F	*F*	T	T	*F*
F	T	*T*	F	T	F
F	F	T	T	T	F

3(iii) $(p \vee q) \leftrightarrow (p \wedge q)$

p	q	(p ∨ q)	(p ∧ q)	(p ∨ q) ↔ (p ∧ q)
T	T	T	T	T
T	F	T	F	*F*
F	T	T	F	F
F	F	F	F	T

4. Differentiate between a mathematical statement and its proof. Given two examples.

Solution

A **"mathematical statement"** is simply a declarative sentence that may be either true or false but not both, while a **"mathematical proof"** is a logical argument that demonstrates the truth of a mathematical statement using established axioms and theorems, effectively showing why a statement is true; essentially, a statement is the claim itself, and a proof is the process of verifying that claim is true.

Examples of Mathematical Statement:

$\frac{22}{7} \notin Q'$; $Q \subseteq R$ these are true statements

$3 + 4 = 8$; $Z \subseteq W$ these are false statements

Examples of Mathematical Proof:

If x is an odd integer, then x^2 is also an odd integer

The sum of two odd numbers is an even number

5. What is the difference between an axiom and a theorem? Give examples of each.

Solution: Theorem:

A **theorem** is a mathematical statement that has been proved true based on previously known facts.

Example of Theorems:

Theorem: The sum of the interior angles of a quadrilateral is 360 degrees.

The Fundamental Theorem of Arithmetic: Every integer greater than 1 can be uniquely expressed as a product of prime numbers up to the order of the factors.

Fermat's Last Theorem: There are no three positive integers a, b, c, which satisfy the equation $a^n + b^n = c^n$, where $n \in N$ and $n > 2$

Axiom:

An **axiom** is a mathematical statement that we believe to be true without any evidence or requiring any proof.

Example Axioms:

Axiom: Through a given point, infinitely many lines can pass.

Euclid Axioms: A straight line can be drawn between any two points.

Peano Axioms: Every natural number has a successor, which is also a natural number.

Axiom of Extensionality: Two sets are equal if they have the same elements.

Axiom of Power Set: Any set has a set of all its subsets.

6. What is the importance of logical reasoning in mathematical proofs? Give an example to illustrate your point.

Solution

Logic is a systematic method of reasoning that enable you to interpret the meaning of statement, examine the truth of statements, and deduce new information from existing facts. Logic play a key role in problem solving and decision making. We generally use logic in our daily life and certainly while engaging in mathematics. For example, we often draw general conclusions from a limited number of observations or experiences. A person gets penicillin injection once or twice and experiences reaction soon afterwards. He generalizes that he is allergic to penicillin.

7. Indicate whether it is an axiom, conjecture or theorem and explain your reasoning.
   (i) There is exactly one straight line through any two points.
   (ii) Every even number greater than 2 can be written as the sum of two prime numbers."
   (iii) The sum of the angles in a triangle is 180°.

Solution

7(i) There is exactly one straight line through any two points.

This statement is a **Euclidean Axiom**. And it is believe to be true without any evidence or requiring any proof.

7(ii) Every even number greater than 2 can be written as the sum of two prime numbers.

This statement is a **Conjecture**, specifically known as **Goldbach's Conjecture**. And it has not been formally proven or disproven.

7(iii) The sum of angles in a triangle is 180°.

This statement is a **Theorem**. And it has been formally proven using established axioms and definitions of geometry.

8. Formulate simple deductive proofs for each of the following algebraic expressions, prove that the L.H.S is equal to the R.H.S:
   (i) prove that $(x-4)^2+9=x^2-8x+25$
   (ii) prove that $(x+1)^2-(x-1)^2=4x$
   (iii) prove that $(x+5)^2-(x-5)^2=20x$

Solution

Deductive Proof: L.H.S. = ……… (solve)

Conclusion: the L.H.S. is exactly same as the R.H.S.

8(i) $L.H.S.=(x-4)^2+9=x^2-8x+16+9=x^2-8x+25=R.H.S.$

8(ii) $L.H.S.=(x+1)^2-(x-1)^2=(x^2+2x+1)-(x^2-2x+1)$

$$=x^2+2x+1-x^2+2x-1=4x=R.H.S.$$

8(iii) $L.H.S.=(x+5)^2-(x-5)^2=(x^2+10x+25)-(x^2-10x+25)$

$$=x^2+10x+25-x^2+10x-25=20x=R.H.S.$$

9. Prove the following by justifying each step:

(i) $\frac{4+16x}{4}=1+4x$ (ii) $\frac{6x^2+18x}{3x^2-27}=\frac{2x}{x-3}$

(iii) $\frac{x^2+7x+10}{x^2-3x-10}=\frac{x+5}{x-5}$

Solution

(i) $\boldsymbol{\frac{4+16x}{4}=1+4x}$

$\frac{4+16x}{4}=\frac{1}{4}\times(4+16x)$ $\because \frac{a}{b}=\frac{1}{b}\times a$

$=\frac{1}{4}\times(4\times 1+4\times 4x)$ ∵ multiplicative identity

$=\frac{1}{4}\times 4\times(1+4x)$ ∵ distributive Law

$=\left(\frac{1}{4}\times 4\right)\times(1+4x)$ ∵ associative Law

$=1\times(1+4x)$ ∵ multiplicative inverse

$=1+4x$ ∵ multiplicative identity

(ii) $\boldsymbol{\frac{6x^2+18x}{3x^2-27}=\frac{2x}{x-3}}$

$\frac{6x^2+18x}{3x^2-27}=\frac{6x(x+3)}{3(x^2-9)}$ ∵ left distributive property

$=\frac{2x(x+3)}{(x-3)(x+3)}$ ∵ Factorization

$=\frac{2x}{x-3}$ ∵ cancellation property

(iii) $\boldsymbol{\frac{x^2+7x+10}{x^2-3x-10}=\frac{x+5}{x-5}}$

$\frac{x^2+7x+10}{x^2-3x-10}=\frac{(x+2)(x+5)}{(x+2)(x-5)}$ ∵ Factorization

$=\frac{x+5}{x-5}$ ∵ cancellation property

10. Suppose x is an integer. Then x is odd if and only if $9x + 4$ is odd.

Solution

If x is odd, then 9x + 4 is odd

Let x be an odd integer. Then, x = 2k + 1 for some integer k.

9x + 4 = 9(2k + 1) + 4= 18k + 9 + 4= 18k + 13= 2(9k + 6) + 1

Since 9k + 6 is an integer, 2(9k + 6) + 1 is odd. Therefore, 9x + 4 is odd.

If 9x + 4 is odd, then x is odd

Let 9x + 4 be odd. Then, 9x + 4 = 2m + 1 for some integer m.

9x = 2m - 3

x = (2m - 3)/9

Since 2m - 3 is odd, (2m - 3)/9 is either an integer or a non-integer.

If x is a non-integer, then 9x + 4 is not an integer, which contradicts the assumption. Therefore, x must be an integer.

x = (2m - 3)/9 = 2((m - 1)/9) - 1/3 (since m is odd, m - 1 is even)

Since (m - 1)/9 is an integer, x is odd.

Therefore, if x is an integer, then x is odd if and only if 9x + 4 is odd.

11. Suppose x is an integer. If x is odd, then $7x + 5$ is even.

Solution

Let x be an odd integer. Then, x = 2k + 1 for some integer k.

7x + 5 = 7(2k + 1) + 5 = 14k + 7 + 5 = 14k + 12 = 2(7k + 6)

Since 7k + 6 is an integer, 2(7k + 6) is even.

Therefore, 7x + 5 is even.

12. Prove the following statements

(a) If x is an odd integer, then show that it $x^2 - 4x + 6$ is odd.

(b) If x is an even integer then show that $x^2 + 2x + 4$ is even.

Solution

12(a) Let x be an odd integer. Then, $x = 2k + 1$ for some integer k.

$x^2 - 4x + 6 = (2k + 1)^2 - 4(2k + 1) + 6$

$= 4k^2 + 4k + 1 - 8k - 4 + 6 = 4k^2 - 4k + 3 = 4k(k - 1) + 3$ **odd integer**

Therefore, $x^2 - 4x + 6$ is odd.

12(b) Let x be an even integer. Then, $x = 2k$ for some integer k.

$x^2 + 2x + 4 = (2k)^2 + 2(2k) + 4 = 4k^2 + 4k + 4 = 4(k^2 + k + 1)$

Since $k^2 + k + 1$ is an integer, $4(k^2 + k + 1)$ is even. Therefore, $x^2 + 2x + 4$ is even.

13. Prove that for any two non-empty sets A and B, $(A \cap B)' = A' \cup B'$.

Solution

Let $x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B) \Rightarrow x \notin A$ or $x \notin B \Rightarrow x \in A'$ or $x \in B' \Rightarrow x \in A' \cup B'$

$\Rightarrow (A \cap B)' \subseteq A' \cup B'$…………….. (i)

Again, let $y \in A' \cup B'$

$\Rightarrow y \in A'$ or $y \in B' \Rightarrow y \notin A$ or $y \notin B \Rightarrow y \notin (A \cap B) \Rightarrow y \in (A \cap B)'$

$\Rightarrow A' \cup B' \subseteq (A \cap B)'$…………….. (ii)

From (i) and (ii) we have

$(A \cap B)' = A' \cup B'$

14. If x and y are positive real numbers and $x^2 < y^2$ then $x < y$.

Solution

Given: $x^2 < y^2$

$\sqrt{x^2} < \sqrt{y^2}$

$x < y$

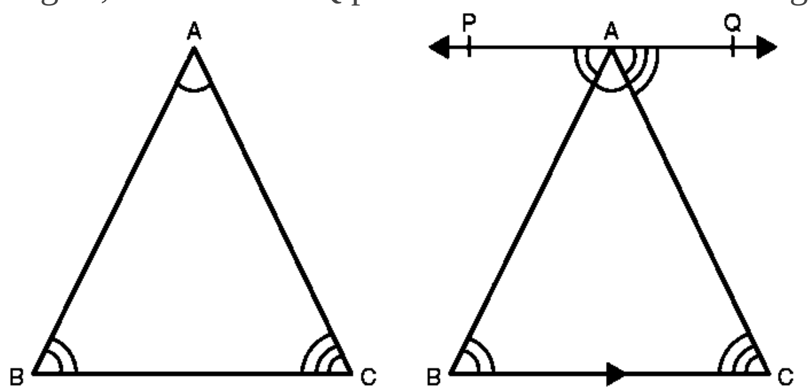
Therefore, if $x^2 < y^2$, then $x < y$.

Note: This proof relies on the fact that the square root function is monotonically increasing for positive real numbers.

15. The sum of the interior angles of a triangle is 180°

Solution

Consider a ΔABC, as shown in the figure below. To prove the above property of triangles, draw a line PQ parallel to the side BC of the given triangle.



Since PQ is a straight line, it can be concluded that:

∠PAB + ∠BAC + ∠QAC = 180° ………(1)

Since PQ||BC and AB, AC are transversals,

Therefore, ∠QAC = ∠ACB (a pair of alternate angle)

Also, ∠PAB = ∠CBA (a pair of alternate angle)

Substituting the value of ∠QAC and ∠PAB in equation (1),

∠ACB + ∠BAC + ∠CBA= 180°

Thus, the sum of the interior angles of a triangle is 180°.

16. If a, b and c are non-zero real numbers, prove that:

(a) $\frac{a}{b}=\frac{c}{d}\Leftrightarrow ad=bc$ (b) $\frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}$ (c) $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$

Solution

(a) $\boldsymbol{\frac{a}{b}=\frac{c}{d}\Leftrightarrow ad=bc}$

$\frac{a}{b}=\frac{c}{d}$

$\frac{a}{b}\times bd=\frac{c}{d}\times bd$ ∵ left multiplication

$\frac{b}{b}\times ad=\frac{d}{d}\times bc$ ∵ associative property

$1\times ad=1\times bc$ ∵ identity

$ad=bc$

(b) $\boldsymbol{\frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}}$

$\frac{a}{b}.\frac{c}{d}$

$=a\times\frac{1}{b}.c\times\frac{1}{d}$

$=a\times\left(\frac{1}{b}.c\right)\times\frac{1}{d}$ ∵ associative property

$=a\times\left(c.\frac{1}{b}\right)\times\frac{1}{d}$ ∵ commutative property

$=ac\times\frac{1}{bd}$ ∵ associative property

$=\frac{ac}{bd}$ ∵ multiplication property

(c) $\boldsymbol{\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}}$

$\frac{a}{b}+\frac{c}{b}$

$=a\times\frac{1}{b}+c\times\frac{1}{b}$ ∵ $\frac{a}{b}=a\times\frac{1}{b}$

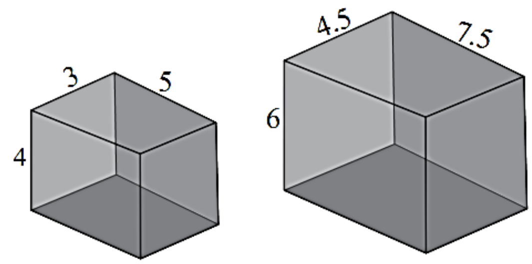
$=(a+c)\times\frac{1}{b}$ ∵ distributive property

$=\frac{a+c}{b}$ ∵ multiplication property

Unit 9 Similar Figures

EXERCISE 9.1

1. Find whether the solids are similar. All lengths are in cm.



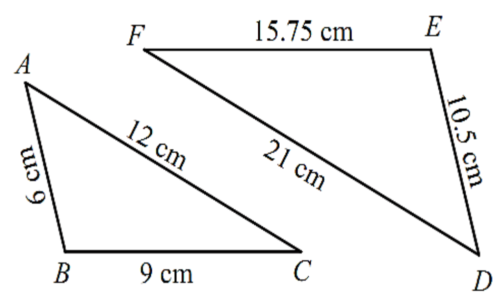
Solution

Ratio of corresponding sides are given as follows;

$\frac{4.5}{3} = \frac{7.5}{5} = \frac{6}{4} = \mathbf{1.5}$

Hence given Figures are similar.

2. In triangle ABC, the sides are given as $m\overline{AB} = 6$ cm, $m\overline{BC} = 9$ cm and $m\overline{CA} = 12$ cm. In triangle DEF, the sides are given as $m\overline{DE} = 10.5$ cm, $m\overline{EF} = 15.75$ cm, and $m\overline{FD} = 21$ cm. Prove that the triangles are similar.



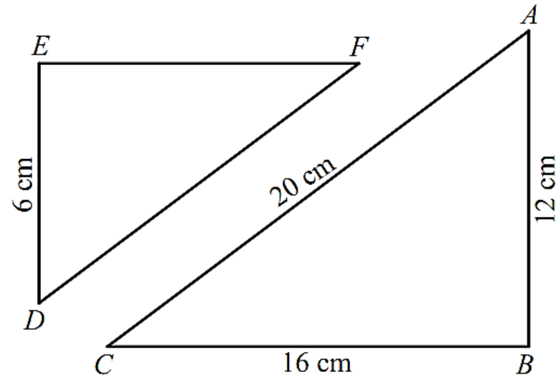
Solution

Ratio of corresponding sides are given as follows;

$\frac{21}{12} = \frac{10.5}{6} = \frac{15.75}{9} = \mathbf{1.75}$

Hence given triangles are similar.

3. In the given figure, $\Delta ABC \sim \Delta DEF$, $m\overline{AB} = 12$ cm, $m\overline{AC} = 20$ cm and $m\overline{BC} = 16$ cm. In ΔDEF, $m\overline{DE} = 6$ cm. Find $m\overline{DF}$ and $m\overline{EF}$.



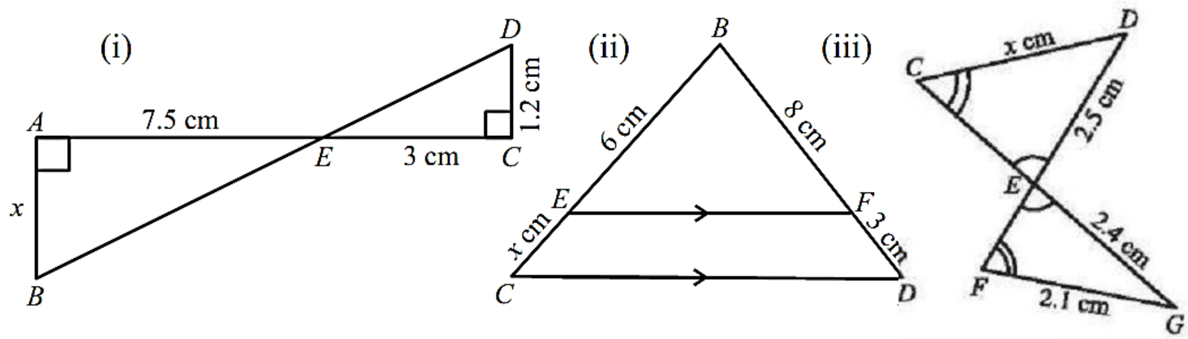
Solution

Since $\Delta ABC \sim \Delta DEF$

Therefore corresponding sides have same ratio.

Using $\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{BC}}{m\overline{EF}}$ $\frac{12}{6} = \frac{16}{m\overline{EF}}$ $m\overline{EF} = 8cm$	Similarly $\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{AC}}{m\overline{DF}}$ $\frac{12}{6} = \frac{20}{m\overline{DF}}$ $m\overline{DF} = 10cm$

4. Find the value of x in each of the following:



Solution

4(i) since figures are similar, therefore

$\frac{7.5}{3} = \frac{x}{1.2} \Rightarrow x = \frac{7.5}{3} \times 1.2 \Rightarrow x = 3cm$

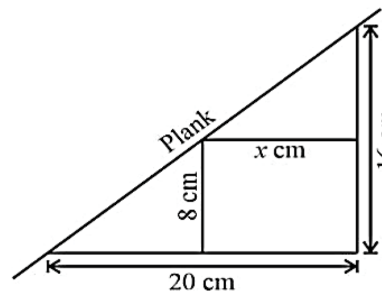
4(ii) since figures are similar, therefore

$\frac{8+3}{8} = \frac{6+x}{6} \Rightarrow 6 + x = \frac{11}{8} \times 6 \Rightarrow x = 2.25cm$

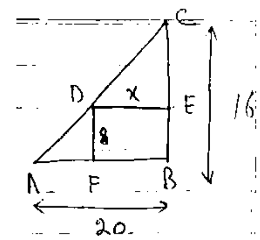
4(iii) since figures are similar, therefore

$\frac{x}{2.1} = \frac{2.5}{2.4} \Rightarrow x = \frac{2.5}{2.4} \times 2.1 \Rightarrow x = 2.19cm$

5. A plank is placed straight upstairs that 20 cm wide and 16 cm deep. A rectangular box of height 8 cm and width x cm is placed on a stair under the plank. Find the value of x.



Solution



In $\Delta ABC \sim \Delta AFD$

$\frac{BC}{DF} = \frac{AB}{AF} \Rightarrow \frac{16}{8} = \frac{20}{AF} \Rightarrow AF = 10\ cm$

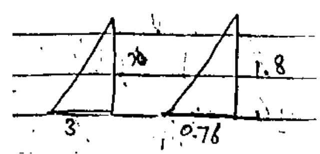
Since $AB = AF + FB$

$\Rightarrow 20 = 10 + FB \Rightarrow FB = 10cm$

$\Rightarrow x = 10cm \qquad \because DE = FB$

6. A man who is 1.8 m tall casts a shadow of a 0.76 m in length. If at the same time a telephone pole costs a 3 m shadow, find the height of the pole.

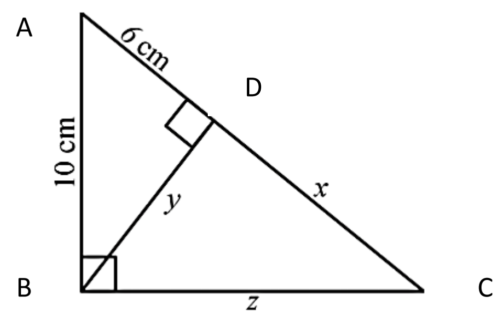
Solution



Since figures are similar, therefore

$\frac{3}{0.76} = \frac{x}{1.8} \Rightarrow x = \frac{3}{0.76} \times 1.8 \Rightarrow x = 7.11m$

7. Find the values of x, y and z in the given figure.



Solution

In $\Delta ABC \sim \Delta ABD$

$\frac{AB}{AD} = \frac{AC}{AB} \Rightarrow \frac{10}{6} = \frac{x+6}{10} \Rightarrow 6x + 36 = 100 \Rightarrow x = 10\frac{2}{3}cm = 10.667cm$

In ΔABD using Pythagoras Theorem

$H^2 = P^2 + B^2 \Rightarrow (10)^2 = (6)^2 + y^2 \Rightarrow 100 = 36 + y^2 \Rightarrow y^2 = 64 \Rightarrow y = 8cm$

In ΔABC we have $Hyp = AC = x + 6 \Rightarrow AC = 10.667 + 6 \Rightarrow AC = 16.667cm$

In ΔABC using Pythagoras Theorem

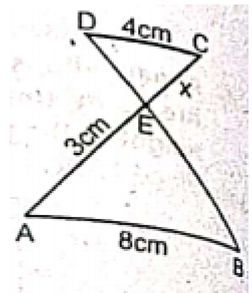
$H^2 = P^2 + B^2 \Rightarrow (16.667)^2 = (10)^2 + z^2 \Rightarrow 277.778 = 100 + z^2$

$\Rightarrow z^2 = 177.778 \Rightarrow z = 13\frac{1}{3}cm = 13.334cm$

8. Draw an isosceles trapezoid $ABCD$ where $\overline{AB} \parallel \overline{CD}$ and $m\overline{AB} > m\overline{CD}$. Draw diagonals $\overline{AC}$ and $\overline{BD}$, intersecting at E. Prove that $\triangle ABE$ is similar to $\triangle CDE$. If $m\overline{AB}$ = 8 cm, $m\overline{CD}$ = 4 cm, and $m\overline{AE}$ = 3 cm, find the length of $\overline{CE}$.

Solution

Given that $\Delta ABE \sim \Delta CDE$



$\frac{AB}{CD} = \frac{AE}{CE}$

$\Rightarrow \frac{8}{4} = \frac{3}{x}$

$\Rightarrow x = 1.5\ cm$

9. A regular dodecagon has its side lengths decreased by a factor of $\frac{1}{\sqrt{2}}$. If the perimeter of the original dodecagon is 72 cm. What is the side length of scaled dodecagon?

Solution

Dodecagon is 12 sided figure.

Perimeter = 72cm

Since perimeter is sum of all sides, so let x is length of sides, then

x + x + x + … 12 times = 72

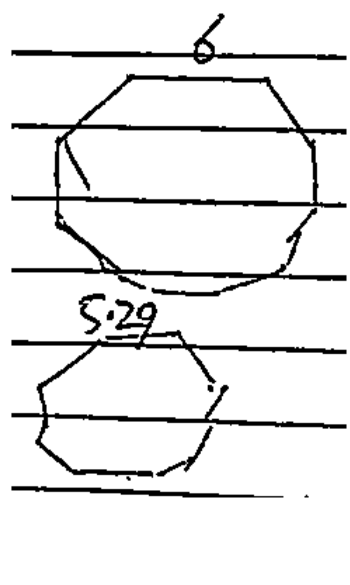
12x = 72

x = 6cm

after $\frac{1}{\sqrt{2}}$ decrease it becomes

Scaled side length $= S' = x \times \frac{1}{\sqrt{2}}$

$S' = 6 \times \frac{1}{\sqrt{2}} = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$ cm



EXERCISE 9.2

1. Find the ratio of the areas of similar figures if the ratio of their corresponding lengths are: (i) 1:3 (ii) 3:4 (iii) 2:7 (iv) 8:9 (v) 6:5

Solution

1(i): $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9} \Rightarrow \boldsymbol{A_1 : A_2 = 1 : 9}$

1(ii): $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16} \Rightarrow \boldsymbol{A_1 : A_2 = 9 : 16}$

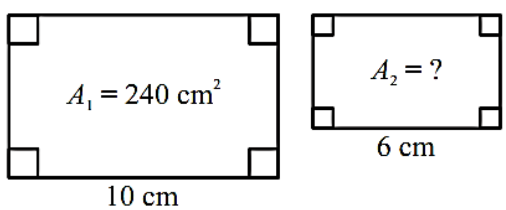
1(iii): $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 = \left(\frac{2}{7}\right)^2 = \frac{4}{49} \Rightarrow \boldsymbol{A_1 : A_2 = 4 : 49}$

1(iv): $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 = \left(\frac{8}{9}\right)^2 = \frac{64}{81} \Rightarrow \boldsymbol{A_1 : A_2 = 64 : 81}$

1(v): $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 = \left(\frac{6}{5}\right)^2 = \frac{36}{25} \Rightarrow \boldsymbol{A_1 : A_2 = 36 : 25}$

2. Find the unknowns in the following figures:

(i)

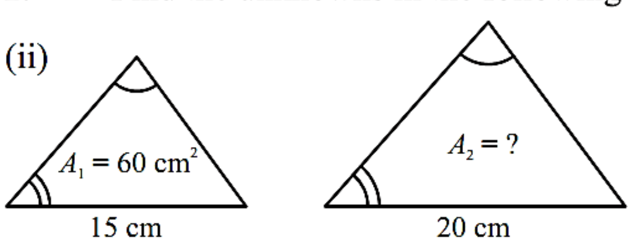


Solution

$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{240}{A_2} = \left(\frac{10}{6}\right)^2 \Rightarrow \frac{240}{A_2} = \frac{100}{36} \Rightarrow A_2 = \frac{240\times 36}{100} \Rightarrow \boldsymbol{A_2 = 86.4\text{cm}^2}$

2. Find the unknowns in the following figures:

(ii)

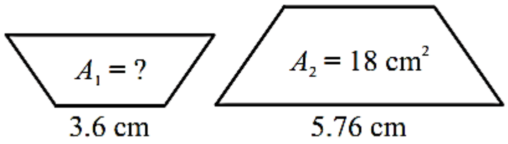


Solution

$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{60}{A_2} = \left(\frac{15}{20}\right)^2 \Rightarrow \frac{60}{A_2} = \frac{225}{400} \Rightarrow A_2 = \frac{60\times 400}{225} \Rightarrow \boldsymbol{A_2 = 106.67\text{cm}^2}$

2. Find the unknowns in the following figures:

(iii)

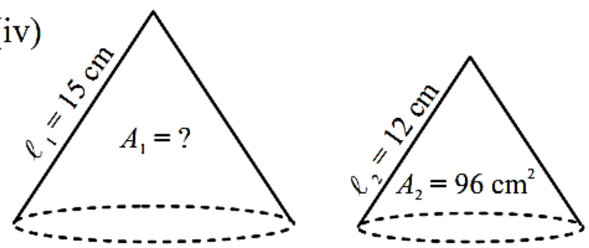


Solution

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{A_1}{18} = \left(\frac{3.6}{5.76}\right)^2 \Rightarrow \frac{A_1}{18} = \frac{12.96}{33.1776} \Rightarrow A_1 = \frac{12.96\times 18}{33.1776} \Rightarrow \boldsymbol{A_1 = 7.03125\text{cm}^2}$$

2. Find the unknowns in the following figures:

(iv)

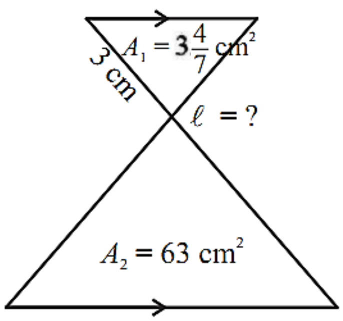


Solution

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{A_1}{96} = \left(\frac{15}{12}\right)^2 \Rightarrow \frac{A_1}{96} = \frac{225}{144} \Rightarrow A_1 = \frac{225\times 96}{144} \Rightarrow \boldsymbol{A_1 = 150\text{cm}^2}$$

2. Find the unknowns in the following figures:

(v)



Solution

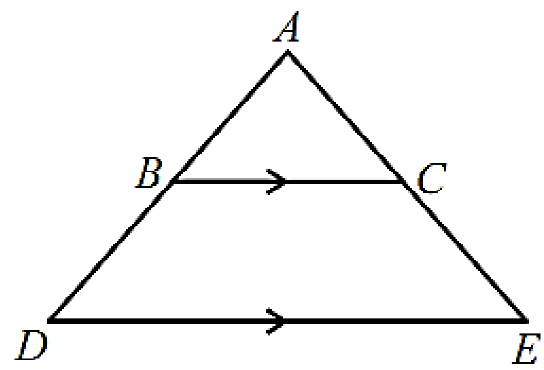
$$A_1 = 3\tfrac{4}{7} = 3.57, A_2 = 63, l_1 = 3, l_2 = ?$$

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{3.57}{63} = \left(\frac{3}{l_2}\right)^2 \Rightarrow 0.0587 = \left(\frac{3}{l_2}\right)^2 \Rightarrow 0.239 = \frac{3}{l_2} \Rightarrow l_2 = \frac{3}{0.239} \Rightarrow \boldsymbol{l_2 = 12.55\text{cm}^2}$$

3. Given that area of $\Delta ABC = 36\ \text{cm}^2$ and $m\overline{AB} = 6\ \text{cm}$, $m\overline{BD} = 4\ \text{cm}$. Find

(a) the area of ΔADE

(b) the area of trapezium $BCED$



Solution

(a) the area of ΔADE

$l_1 = m\overline{AB} = 6\text{cm}, l_2 = m\overline{AD} = m\overline{AB} + m\overline{BD} = 6\text{cm} + 4\text{cm} = 10\text{cm}$

$A_1 = 36\text{cm}^2\ , A_2 = ?$

$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{36}{A_2} = \left(\frac{6}{10}\right)^2 \Rightarrow \frac{36}{A_2} = \frac{36}{100} \Rightarrow A_2 = \frac{36\times100}{36} \Rightarrow \boldsymbol{A_2 = 100\text{cm}^2}$

(b) the area of trapezium $BCED$

Area of trapezium BCED = Area of ΔADE − Area of ΔABC

Area of trapezium BCED = $100 - 36$

Area of trapezium BCED = 64cm^2

4. Given that ΔABC and ΔDEF are similar, with a scale factor of $k = 3$. If the area of ΔABC is $50\ \text{cm}^2$, find the area of triangle ΔDEF?

Solution

Scale Factor = k = 3 ; Area of ΔABC = 50cm^2

Using formula: $\frac{\text{Area of }\Delta\text{DEF}}{\text{Area of }\Delta\text{ABC}} = \text{k}^2$

Area of ΔDEF = $(\text{Area of }\Delta\text{ABC})(\text{k}^2) = (50)(3^2) = (50)(9)$

Area of ΔDEF = 450cm^2

5. Quadrilaterals $ABCD$ and $EFGH$ are similar, with a scale factor of $k = \frac{1}{4}$. If the area of quadrilateral $ABCD$ is 64 cm^2, find the area of quadrilateral $EFGH$.

Solution

Scale Factor = k = $\frac{1}{4}$; Area of Quadrilateral ABCD = $64cm^2$

Using formula: $\frac{\text{Area of Quadrilateral EFGH}}{\text{Area of Quadrilateral ABCD}} = k^2$

$$\text{Area of Quadrilateral EFGH} = (\text{Area of Quadrilateral ABCD})(k^2)$$

$$\text{Area of Quadrilateral EFGH} = (64)\left(\frac{1}{4}\right)^2 = (64)\left(\frac{1}{16}\right) = 4cm^2$$

6. The areas of two similar triangles are 16 cm^2 and 25 cm^2. What is the ratio of a pair of corresponding sides?

Solution

$$\left(\frac{l_1}{l_2}\right)^2 = \frac{A_1}{A_2} \Rightarrow \left(\frac{l_1}{l_2}\right)^2 = \frac{16}{25} \Rightarrow \frac{l_1}{l_2} = \sqrt{\frac{16}{25}} \Rightarrow \frac{l_1}{l_2} = \frac{4}{5} \Rightarrow \boldsymbol{l_1 : l_2 = 4 : 5}$$

7. The areas of two similar triangles are 144 cm^2 and 81 cm^2. If the base of the large triangle is 30 cm, find the corresponding base of the smaller triangle.

Solution

We have to find here l_2.

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{144}{81} = \left(\frac{30}{l_2}\right)^2 \Rightarrow \frac{12}{9} = \frac{30}{l_2} \Rightarrow l_2 = \frac{9\times30}{12} \Rightarrow \boldsymbol{l_2 = 22.5}$$

8. A regular heptagon is inscribed in a larger regular heptagon and each side of the larger heptagon is 1.7 times the side of the smaller heptagon. If the area of the smaller heptagon is known to be 100 cm², find the area of the larger heptagon.

Solution

Scale Factor = k = 1.7 ; Area smaller heptagon = $100cm^2$

Using formula: $\frac{\text{Area larger heptagon}}{\text{Area smaller heptagon}} = k^2$

$$\text{Area larger heptagon} = (\text{Area smaller heptagon})(k^2)$$

$$\text{Area larger heptagon} = (100)(1.7)^2 = (100)(2.89) = 289cm^2$$

EXERCISE 9.3

1. The radii of two spheres are in the ratio 3 : 4. What is the ratio of their volumes?

Solution

$\frac{V_1}{V_2} = \left(\frac{r_1}{r_2}\right)^3 \Rightarrow \frac{V_1}{V_2} = \left(\frac{3}{4}\right)^3 \Rightarrow \frac{V_1}{V_2} = \frac{27}{64} \Rightarrow \boldsymbol{V_1 : V_2 = 27 : 64}$

2. Two regular tetrahedrons have volumes in the ratio 8 : 27. What is the ratio of their sides?

Solution

$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3 \Rightarrow \frac{8}{27} = \left(\frac{l_1}{l_2}\right)^3 \Rightarrow \sqrt[3]{\frac{8}{27}} = \sqrt[3]{\left(\frac{l_1}{l_2}\right)^3} \Rightarrow \frac{2}{3} = \frac{l_1}{l_2} \Rightarrow \boldsymbol{l_1 : l_2 = 2 : 3}$

3. Two right cones have volumes in the ratio 64 : 125. What is the ratio of:

(a) their heights (b) their base areas?

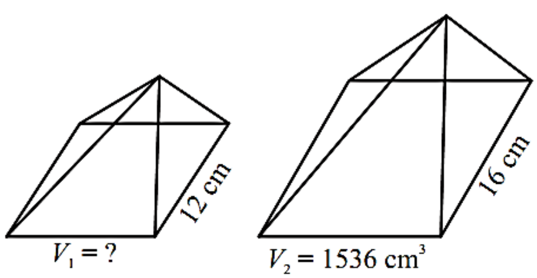
Solution

(a) $\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3 \Rightarrow \frac{64}{125} = \left(\frac{h_1}{h_2}\right)^3 \Rightarrow \sqrt[3]{\frac{64}{125}} = \sqrt[3]{\left(\frac{h_1}{h_2}\right)^3} \Rightarrow \frac{4}{5} = \frac{h_1}{h_2} \Rightarrow \boldsymbol{h_1 : h_2 = 4 : 5}$

(b) $\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2 \Rightarrow \frac{A_1}{A_2} = \left(\frac{4}{5}\right)^2 \Rightarrow \frac{A_1}{A_2} = \frac{16}{25} \Rightarrow \boldsymbol{A_1 : A_2 = 16 : 25}$

4. Find the missing value in the following similar solids.

(i)



Solution

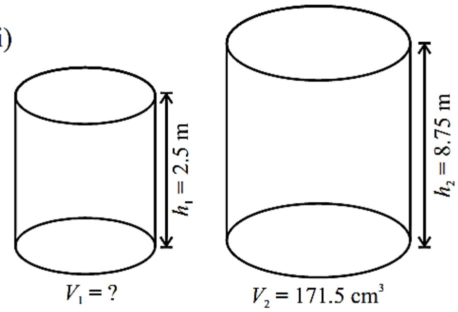
$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3 \Rightarrow \frac{V_1}{1536} = \left(\frac{12}{16}\right)^3 \Rightarrow \frac{V_1}{1536} = \left(\frac{3}{4}\right)^3 \Rightarrow \frac{V_1}{1536} = \frac{27}{64}$

$\Rightarrow V_1 = \frac{1536 \times 27}{64}$

$\Rightarrow \boldsymbol{V_1 = 648\text{cm}^3}$

4. Find the missing value in the following similar solids.

(ii)



Solution

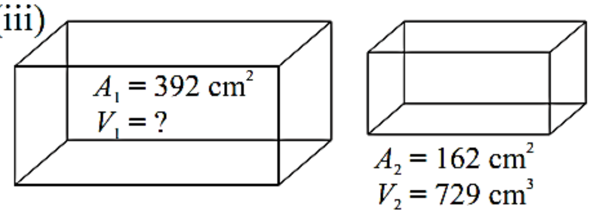
$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3 \Rightarrow \frac{V_1}{171.5} = \left(\frac{2.5}{8.75}\right)^3 \Rightarrow \frac{V_1}{171.5} = \frac{15.625}{669.921}$

$\Rightarrow V_1 = \frac{15.625 \times 171.5}{669.921}$

$\Rightarrow \boldsymbol{V_1 = 4\text{cm}^3}$

4. Find the missing value in the following similar solids.

(iii)



Solution

Using formula: $\frac{A_1}{A_2} = \text{k}^2$

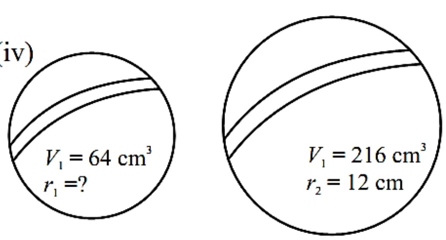
$\Rightarrow \frac{392}{162} = \text{k}^2 \Rightarrow \text{k}^2 = 2.420 \Rightarrow \text{k} = 1.56$

Again using formula: $\frac{V_1}{V_2} = \text{k}^3$

$\Rightarrow \frac{V_1}{729} = (1.56)^3 \Rightarrow V_1 = (729)(3.80) \Rightarrow \boldsymbol{V_1 \approx 2744\text{cm}^3}$

4. Find the missing value in the following similar solids.

(iv)



Solution

$\frac{V_1}{V_2} = \left(\frac{r_1}{r_2}\right)^3 \Rightarrow \frac{64}{216} = \left(\frac{r_1}{12}\right)^3 \Rightarrow \sqrt[3]{\frac{64}{216}} = \sqrt[3]{\left(\frac{r_1}{12}\right)^3} \Rightarrow \frac{4}{6} = \frac{r_1}{12} \Rightarrow r_1 = \frac{4 \times 12}{6}$

$\Rightarrow \boldsymbol{r_1 = 8\text{ cm}}$

5. The ratio of the corresponding lengths of two similar canonical cans is 3 : 2.
   (i) The larger canonical can have surface area of 96 m^2. Find the surface area of the smaller canonical can.
   (ii) The smaller canonical can have a volume of 240 m^3. Find the volume of larger canonical can.

Solution

(a) $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{96}{A_2} = \left(\frac{3}{2}\right)^2 \Rightarrow \frac{96}{A_2} = \frac{9}{4} \Rightarrow A_2 = \frac{96\times4}{9} \Rightarrow \boldsymbol{A_2 = 42.67\mathrm{m}^2}$

(b) $\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3 \Rightarrow \frac{V_1}{240} = \left(\frac{3}{2}\right)^3 \Rightarrow \frac{V_1}{240} = \frac{27}{8} \Rightarrow V_1 = \frac{240\times27}{8} \Rightarrow \boldsymbol{V_1 = 8107\mathrm{m}^3}$

6. The ratio of the heights of two similar cylindrical water tanks is 5 : 3.
   (i) If the surface area of the larger tank is 250 square metres, find the surface area of the smaller tank.
   (ii) If the volume of the smaller tank is 270 cubic metres, find the volume of the larger tank.

Solution

(a) $\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2 \Rightarrow \frac{250}{A_2} = \left(\frac{5}{3}\right)^2 \Rightarrow \frac{250}{A_2} = \frac{25}{9} \Rightarrow A_2 = \frac{250\times9}{25} \Rightarrow \boldsymbol{A_2 = 90\mathrm{m}^2}$

(b) $\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3 \Rightarrow \frac{V_1}{270} = \left(\frac{5}{3}\right)^3 \Rightarrow \frac{V_1}{270} = \frac{125}{27} \Rightarrow V_1 = \frac{270\times125}{27} \Rightarrow \boldsymbol{V_1 = 1250\mathrm{m}^3}$

EXERCISE 9.4

1. (i) What is the sum of the interior angles of a decagon (10-sided polygon)?
   (ii) Calculate the measure of each interior angle of a regular hexagon.
   (iii) What is each exterior angle of a regular pentagon?
   (iv) If the sum of the interior angles of a polygon is 1260°, how many sides does the polygon have?

Solution

(i) Sum of the Interior Angle $= (n-2)\times 180° = (10-2)\times 180° = 8\times 180° = 1440°$

(ii) Measure of the Interior Angle $= \frac{\text{Sum of the Interior Angle}}{\text{n}} = \frac{(\text{n}-2)\times 180°}{\text{n}} = \frac{(6-2)\times 180°}{6} = 120°$

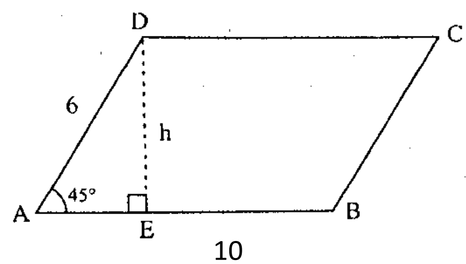
(iii) Measure of each Exerior Angle $= \frac{360°}{\text{n}} = \frac{360°}{5} = 72°$

(iv) Sum of the Interior Angle of Polygon $= (n-2)\times 180°$

$\Rightarrow 1260° = (n-2)\times 180° \Rightarrow \frac{1260°}{180°} = \text{n}-2 \Rightarrow 7 = \text{n}-2 \Rightarrow \text{n} = 9$

2. In a parallelogram $ABCD$, $m\overline{AB} = 10$ cm, $m\overline{AD} = 6$ cm and $m\angle BAD = 45°$. Calculate the area of $ABCD$.

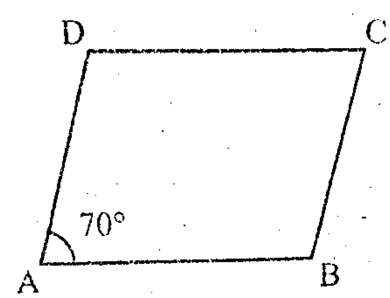
Solution



Given ABCD is a parallelogram.
Also $m\overline{AB} = 10\text{cm}, m\overline{AD} = 6\text{cm}, m\angle BAD = 45°$
Area of parallelogram ABCD= Base × Height
Area of parallelogram ABCD= $\overline{AB} \times \overline{ED}$
Area of parallelogram ABCD= $\overline{AB} \times \overline{AD} sin\theta$
Area of parallelogram ABCD= $10 \times 6 sin45°$
Area of parallelogram ABCD= 42.43cm^2

3. In a parallelogram $ABCD$ if $m\angle DAB = 70°$, find the measures of all other angles in the parallelogram.

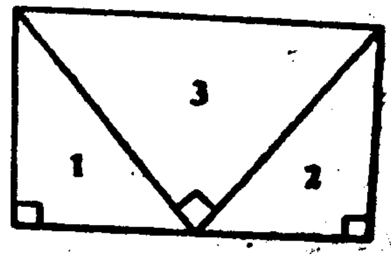
Solution



$m\angle DAB = 70°$
$m\angle DAB = m\angle BCD = 70°$ ∵opposite angles of ‖gram
$m\angle DAB + m\angle ABC = 180°$ $\because AD \parallel BC$
$70° + m\angle ABC = 180°$
$m\angle ABC = 110°$
also $m\angle CDA = 110°$ $\because A m\angle ABC = m\angle CDA$

4. A shape is created by cutting a square in half diagonally and then attaching a right-angled triangle to the hypotenuse of each half. Explain why this shape can tessellate and calculate the interior angle of the new shape.

Solution



The new shape can tessellate because it is composed of triangles which are inherently able to tessellate when arranged approximately that they fit together perfectly without gaps.
Also, the sum of interior angles is 360°.

Explanation

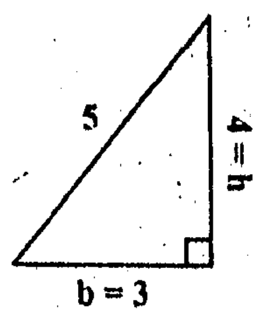
The new shape can tessellate because it is composed of triangles which are inherently able to tessellate when arranged approximately.
The original square when certain half diagonally, forms two congruent right angled triangles.
Attaching right angled triangles to the hypotenuse of these two halves does not change their overall symmetry. The new shape remains geometrically compatible with tessellation because the triangles angles ensure that they fit together perfectly without gaps.

Interior angles of the new shape

The sum of interior angles of a hexagon is (6 - 2) × 180° = 720°.
Since the shape is composed of two identical parts, the sum of interior angles of one part is 720° ÷ 2 = 360°.
Since the shape has 6 sides, the interior angle of the new shape is 360° ÷ 6 = 60° (for the equilateral triangle) and 120° (for the isosceles triangle).

5. A tessellation is created by repeatedly reflecting a basic shape. The basic shape is a right-angled triangle with sides of length 3, 4, and 5 units. Find: The minimum number of reflections needed to create a tessellation that covers a square with an area of 3600 square units.

Solution



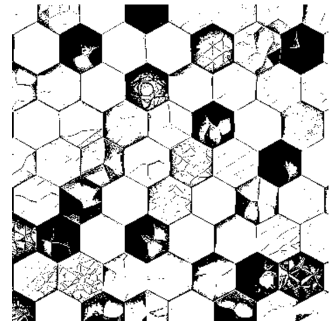
Area of triangle $= \frac{1}{2} \times \text{Base} \times \text{Height}$

Area of triangle $= \frac{1}{2} \times 3 \times 4 = 6$ units

Number of triangles $= \frac{\text{area of square}}{\text{area of triangles}} = \frac{3600}{6} = 600$

So 600 reflections needed to cover the square.

6. A tessellation is created using regular hexagons. Each hexagon has a side length of 5 cm. Find the total area of the tessellation if it consists of 25 hexagons and total perimeter of the outer edge of the tessellation, assuming it's a perfect hexagon.



Solution

Area of hexagon $= \frac{3\sqrt{3}}{2}a^2 = \frac{3\sqrt{3}}{2} \times 5^2 = \frac{3\sqrt{3}}{2} \times 25 = 64.95\text{cm}^2$

Area of 25 hexagon $= 64.95 \times 25 = 1623.75\text{cm}^2$

Perimeter $= 6 \times (5 \times 5) = 6 \times 25 = 150\text{cm}$

Note:

length of 1 side is 5×5 since each side consists of 5 hexagons

7. A rectangular floor is 12 m by 15 m. How many square tiles, each 1 m by 1 m, are needed to cover the floor?

Solution

Number of tiles $= \frac{\text{area of floor}}{\text{area of one tile}} = \frac{12\times 15}{1\times 1} = \frac{180\text{m}^2}{1\text{m}^2} = 180$ tiles

8. A rectangular wall is 10 m tall and 12 m wide. How many gallons of paint are needed to cover the wall, if one gallon covers 35 m^2?

Solution

Number of gallons $= \frac{\text{area of rectangular wall}}{\text{coverage per gallon}} = \frac{10\times 12}{350} = \frac{120\text{m}^2}{350\text{m}^2} = 0.343$ gallons

9. A rectangular wall has a length of 10 m and a width of 4 meters. If 1 litre of paint covers 7 m^2, how many liters of paint are needed to cover the wall?

Solution

Paint needed $= \frac{\text{area of the wall}}{\text{paint per liter}} = \frac{10\times 4}{7} = \frac{40\text{m}^2}{7\text{m}^2} = 5.71 \approx 6$ liters

10. A window has a trapezoidal shape with parallel sides of 3 m and 1.5 m and a height of 2 m. Find the area of the window.

Solution

Area of trapezoidal $= \frac{\text{sum of parallel sides}}{2} \times \text{height} = \frac{3+1.5}{2} \times 2 = 4.5\text{m}^2$

REVIEW EXERCISE 9

1. Four options are given against each statement. Encircle the correct one.

(i) If two polygons are similar, then:
- (a) ✓ their corresponding angles are equal.
- (b) their areas are equal.
- (c) their volumes are equal.
- (d) their corresponding sides are equal.

(ii) The ratio of the areas of two similar polygons is:
- (a) equal to the ratio of their perimeters.
- (b) ✓ equal to the square of the ratio of their corresponding sides.
- (c) equal to the cube of the ratio of their corresponding sides.
- (d) equal to the sum of their corresponding sides.

(iii) If the volume of two similar solids is 125 cm^3 and 27 cm^3, the ratio of their corresponding heights is -----------.

(a) 3:5 (b) ✓ 5:3 (c) 25:9 (d) 9:25

(iv) The exterior angle of regular pentagon is:

(a) 40° (b) 45° (c) 60° (d) ✓ 72°

(v) A parallelogram has an area of 64 cm^2 and a similar parallelogram has an area of 144 cm^2. If a side of the smaller parallelogram is 8 cm, the corresponding side of the larger parallelogram is:

(a) 10 cm (b) 12 cm (c) ✓ 18 cm (d) 16 cm

(vi) The total number of diagonals in a polygon with 9 sides is:

(a) 18 (b) 21 (c) 25 (d) ✓ 27

(vii) Two spheres are similar, and their radii are in the ratio 4:5. If the surface area of the larger sphere is 500π cm^2, what is the surface area of the smaller sphere?

(a) 256π cm^2 (b) 320π cm^2 (c) ✓ 400π cm^2 (d) 405π cm^2

(viii) A regular polygon has an exterior angle of 30°. How many diagonals does the Polygon have?

(a) ✓ 54 (b) 90 (c) 72 (d) 108

(ix) In a regular hexagon, the ratio of the length of a diagonal to the side length is:

(a) $\sqrt{3}:1$ (b)✓ $2:1$ (c) $3:2$ (d) $2:3$

(x) A regular polygon has an interior angle of 165°. How many sides does it have?

(a) 15 (b) 16 (c) 20 (d)✓ 24

2. If the sum of the interior angles of a polygon is 1080°, how many sides does the polygon has?

Solution

Sum of the Interior Angle of Polygon $= (n-2) \times 180°$

$\Rightarrow 1080° = (n-2) \times 180° \Rightarrow \frac{1080°}{180°} = \mathrm{n} - 2 \Rightarrow 6 = \mathrm{n} - 2 \Rightarrow \mathrm{n} = 8$

3. Two similar bottles are such that one is twice as high as the other. What is the ratio of their surface areas and their capacities?

Solution

(a) $\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2 \Rightarrow \frac{A_1}{A_2} = \left(\frac{2x}{1x}\right)^2 = \left(\frac{2}{1}\right)^2 = \frac{4}{1} \Rightarrow \boldsymbol{A_1 : A_2 = 4 : 1}$

(b) $\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3 \Rightarrow \frac{V_1}{V_2} = \left(\frac{2}{1}\right)^3 = \frac{8}{1} \Rightarrow \boldsymbol{V_1 : V_2 = 8 : 1}$

4. Each dimension of a model car is $\frac{1}{10}$ of the corresponding car dimension. Find the ratio of:

(a) the areas of their windscreens (b) the capacities of their boots

(c) the widths of the cars (d) the number of wheels they have.

Solution

(a) $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{A_1}{A_2} = \left(\frac{1}{10}\right)^2 = \frac{1}{100} \Rightarrow \boldsymbol{A_1 : A_2 = 1 : 100}$

(b) $\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3 \Rightarrow \frac{V_1}{V_2} = \left(\frac{1}{10}\right)^3 = \frac{1}{1000} \Rightarrow \boldsymbol{V_1 : V_2 = 1 : 1000}$

(c) $\frac{l_1}{l_2} = \frac{1}{10} \Rightarrow \boldsymbol{l_1 : l_2 = 1 : 10}$

(d) ratio of wheels of car $= \frac{4}{4} = \frac{1}{1}$

$\Rightarrow$ **ratio of number of wheels of car $= 1 : 1$**

5. Three similar jugs have heights 8 cm, 12 cm and 16 cm. If the smallest jug holds $\frac{1}{2}$ litre, find the capacities of the other two.

Solution

(a) $\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3 \Rightarrow \frac{\frac{1}{2}}{V_2} = \left(\frac{8}{12}\right)^3 = \left(\frac{2}{3}\right)^3 \Rightarrow \frac{\frac{1}{2}}{V_2} = \frac{8}{27} \Rightarrow V_2 = \frac{1\times 27}{2\times 8} \Rightarrow \boldsymbol{V_2 = 1.69}$ **liter**

(b) $\frac{V_1}{V_3} = \left(\frac{h_1}{h_3}\right)^3 \Rightarrow \frac{\frac{1}{2}}{V_3} = \left(\frac{8}{16}\right)^3 = \left(\frac{1}{2}\right)^3 \Rightarrow \frac{\frac{1}{2}}{V_3} = \frac{1}{8} \Rightarrow V_3 = \frac{1\times 8}{2\times 1} \Rightarrow \boldsymbol{V_3 = 4}$ **liter**

6. Three similar drinking glasses have heights 7.5 cm, 9 cm and 10.5 cm. If the tallest glass holds 343 millilitres, find the capacities of the other two.

Solution

(a) $\frac{V_1}{V_3} = \left(\frac{h_1}{h_3}\right)^3 \Rightarrow \frac{V_1}{0.343} = \left(\frac{7.5}{10.5}\right)^3 = \left(\frac{1.5}{2.1}\right)^3 \Rightarrow \frac{V_1}{0.343} = \frac{3.375}{9.261} \Rightarrow V_1 = \frac{3.375\times 0.343}{9.261}$

$\Rightarrow \boldsymbol{V_1 = 0.125}$ **liter** $= \boldsymbol{125}$ **mL**

(b) $\frac{V_2}{V_3} = \left(\frac{h_2}{h_3}\right)^3 \Rightarrow \frac{V_2}{0.343} = \left(\frac{9}{10.5}\right)^3 \Rightarrow \frac{V_2}{0.343} = \frac{729}{1157.625} \Rightarrow V_2 = \frac{729\times 0.343}{1157.625}$

$\Rightarrow \boldsymbol{V_2 = 0.216}$ **liter** $= \boldsymbol{216}$ **mL**

7. A toy manufacturer produces model cars which are similar in every way to the actual cars. If the ratio of the door area of the model to the door area of the car is 1 cm to 2500 cm, find:
   (a) the ratio of their lengths
   (b) the ratio of the capacities of their petrol tanks
   (c) the width of the model, if the actual car is 150 cm wide
   (d) the area of the rear window of the actual car if the area of the rear window of the model is 3 cm^2.

Solution

(a) $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{1}{2500} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \sqrt{\frac{1}{2500}} = \sqrt{\left(\frac{l_1}{l_2}\right)^2} \Rightarrow \frac{1}{50} = \frac{l_1}{l_2} \Rightarrow \boldsymbol{l_1 : l_2 = 1 : 50}$

(b) $\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3 \Rightarrow \frac{V_1}{V_2} = \left(\frac{1}{50}\right)^3 = \frac{1}{125000} \Rightarrow \boldsymbol{V_1 : V_2 = 1 : 125000}$

(c) $\frac{w_1}{w_2} = \frac{l_1}{l_2} \Rightarrow \frac{w_1}{150} = \frac{1}{50} \Rightarrow w_1 = \frac{150}{50} \Rightarrow \boldsymbol{w_1 = 3\text{cm}}$

(d) $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \Rightarrow \frac{3}{A_2} = \left(\frac{1}{50}\right)^2 \Rightarrow \frac{3}{A_2} = \frac{1}{2500} \Rightarrow A_2 = 3 \times 2500 \Rightarrow \boldsymbol{A_2 = 7500\text{cm}^2}$

8. The ratio of the areas of two similar labels on two similar jars of coffee is 144 : 169. Find the ratio of

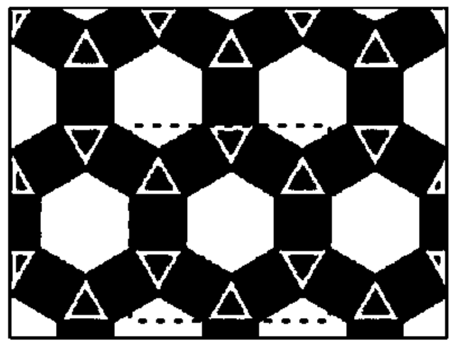
(a) the heights of the two jars (b) their capacities.

Solution

(a) $\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2 \Rightarrow \frac{144}{169} = \left(\frac{h_1}{h_2}\right)^2 \Rightarrow \sqrt{\frac{144}{169}} = \sqrt{\left(\frac{h_1}{h_2}\right)^2} \Rightarrow \frac{12}{13} = \frac{h_1}{h_2} \Rightarrow \boldsymbol{h_1 : h_2 = 12 : 13}$

(b) $\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3 \Rightarrow \frac{V_1}{V_2} = \left(\frac{12}{13}\right)^3 = \frac{1728}{2197} \Rightarrow \boldsymbol{V_1 : V_2 = 1728 : 2197}$

9. A tessellation of tiles on a floor has been made using a repeating pattern of a regular hexagon, six squares and six equilateral triangles. Find the total area of a single pattern with side length $\frac{1}{2}$ metre of each polygon.



Solution

The pattern will be 12 sided regular polygon (Dodecagon)

Area of Dodecagon $= 3(2+\sqrt{3})a^2$

Area of Dodecagon $= 3(2+\sqrt{3}) \times \left(\frac{1}{2}\right)^2$

Area of Dodecagon $= 3(2+\sqrt{3}) \times \frac{1}{4}$

Area of Dodecagon $= 2.8\text{m}^2$

Unit 10 Graphs of Functions

EXERCISE 10.1

1. Sketch the graph of the following linear functions:

(i) $y = 3x - 5$ (ii) $y = -2x + 8$

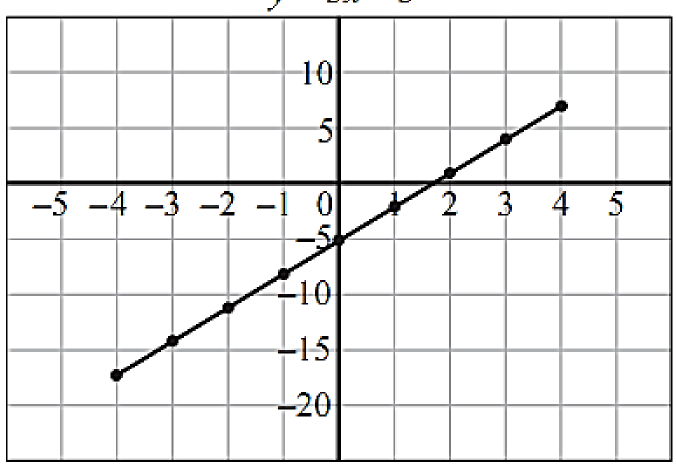
(iii) $y = 0.5x - 1$

Solution

(i)

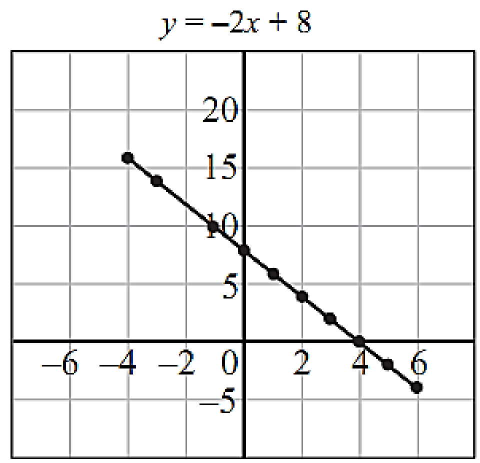
x	-2	-1	0	1	2	3
y = 3x-5	-11	-8	-5	-2	1	4

$y = 3x - 5$



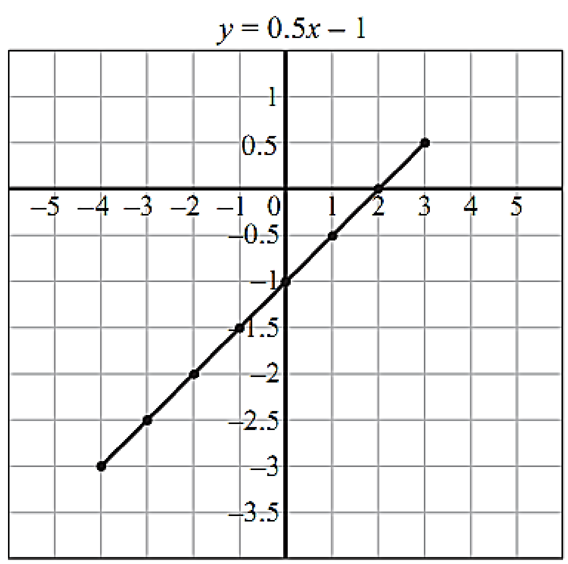
(ii)

x	-2	-1	0	1	2	3
y = -2x+8	12	10	8	6	4	2



(iii)

x	-2	-1	0	1	2	3
y = 0.5x-1	-2	-1.5	-1	-0.5	0	0.5



2. Plot the graph of the following quadratic and cubic functions:

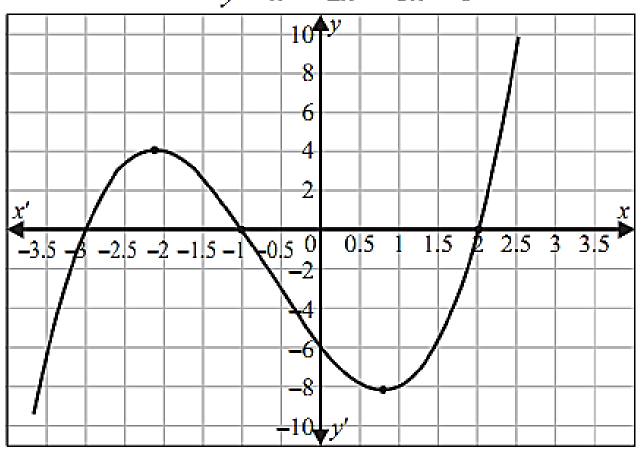
(i) $y = x^3 + 2x^2 - 5x - 6; -3.5 \le x \le 2.5$ (ii) $y = x^2 + x - 2$

(iii) $y = x^3 + 3x^2 + 2x; -2.5 \le x \le 0.5$ (iv) $y = 5x^2 - 2x - 3$

Solution

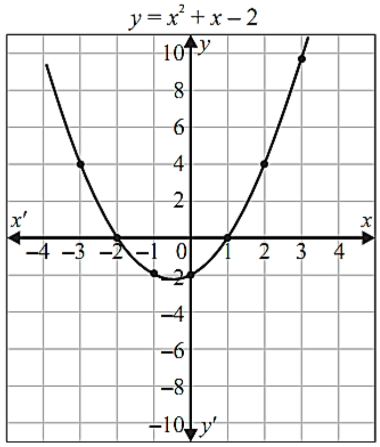
(i)

x	-3.5	-3	-2	-1	0	1	2	2.5
y	-6.88	0	4	0	-6	-8	0	9.63



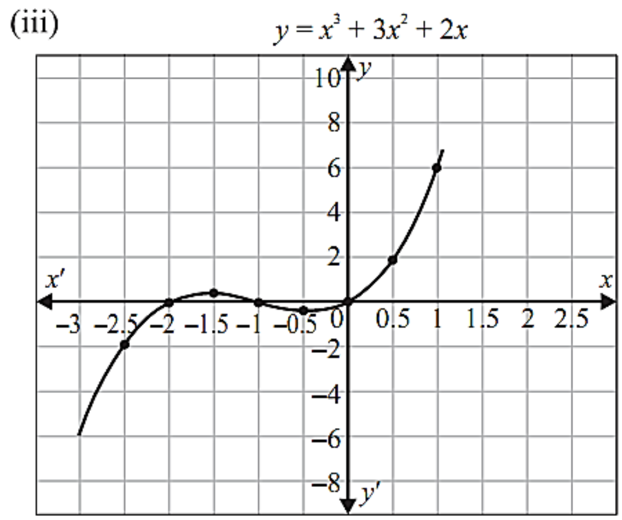
(ii)

x	-4	-3	-2	-1	0	1	2	3	4
y	10	4	0	-2	-2	0	4	10	18



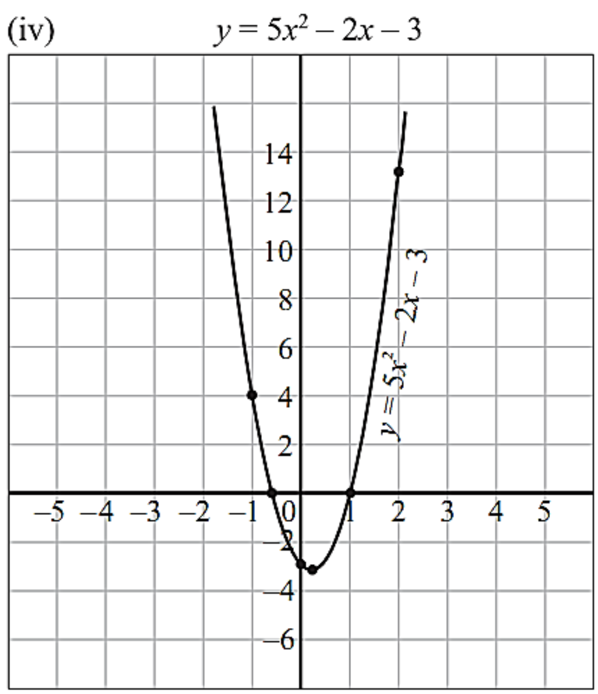
(iii)

x	-2.5	-2	-1.5	-1	-0.5	0	0.5	1
y	-1.875	0	0.375	0	-0.375	0	1.875	6



(iv)

x	-2	-1	0	1	2
y	21	4	-3	0	13



3. Plot the graph of the following functions:

(i) $y = 4^x$ (ii) $y = 5^{-x}$ (iii) $y = \frac{1}{x-3}\ x \neq 3$

(iv) $y = \frac{2}{x} + 3, x \neq 0$ (v) $y = x^{\frac{1}{2}}$ (vi) $y = 3x^{\frac{1}{3}}$

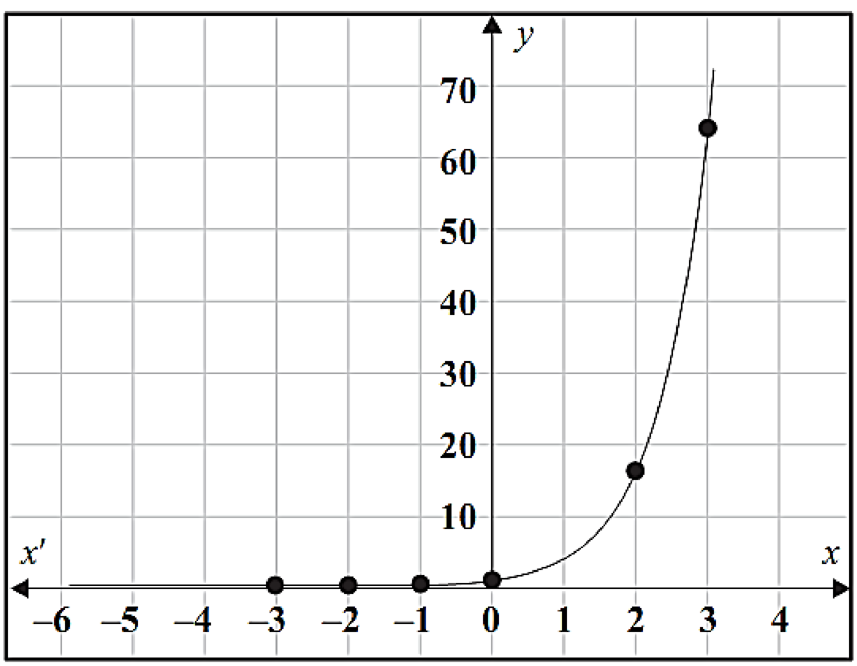
(vii) $y = 2x^{-2}$

Solution

(i)

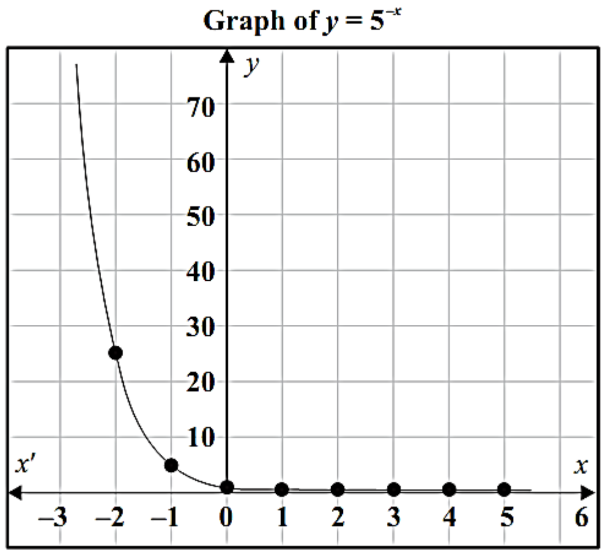
x	-3	-2	-1	0	1	2	3
y	0.02	0.06	0.25	1	4	16	64

Graph of $y = 4^x$



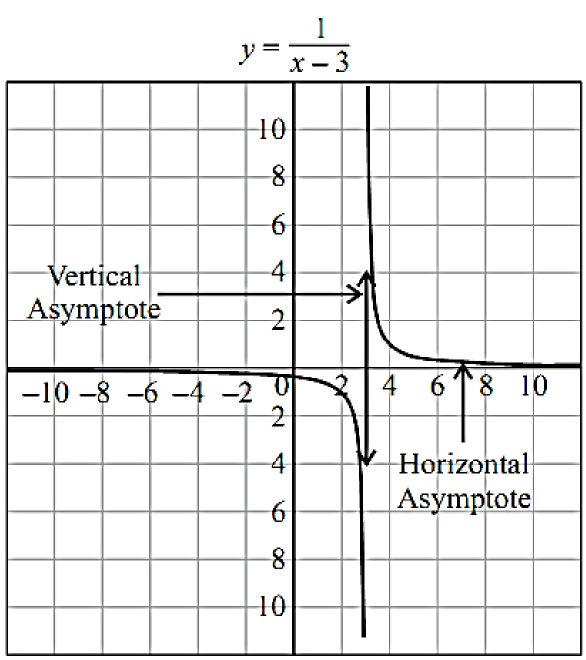
(ii)

x	-3	-2	-1	0	1	2	3
y	125	25	5	1	0.2	0.04	0.008



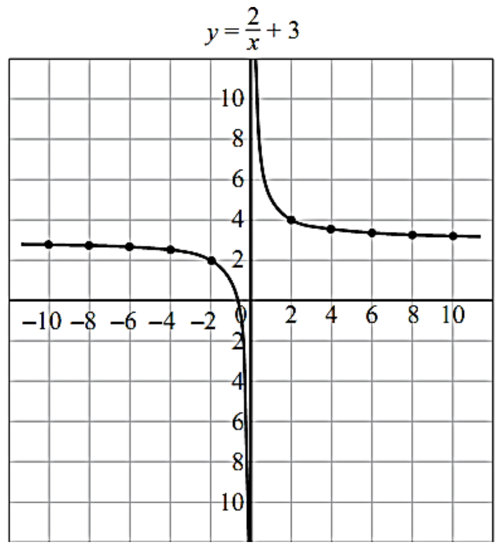
(iii)

x	-2	-1	0	1	2	4	5
y	-0.2	-0.25	-0.3	-0.5	-1	1	0.5



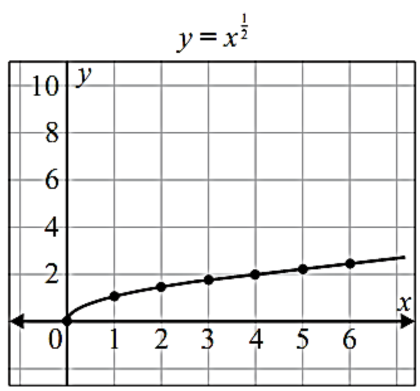
(iv)

x	-3	-2	-1	1	2	3
y	2.3	2	1	5	4	3.7



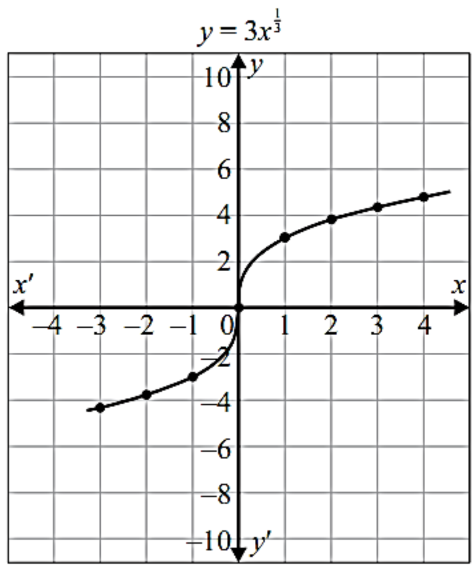
(v)

x	0	1	2	3	4	5	6
y	0	1	1.4	1.7	2	2.2	2.4



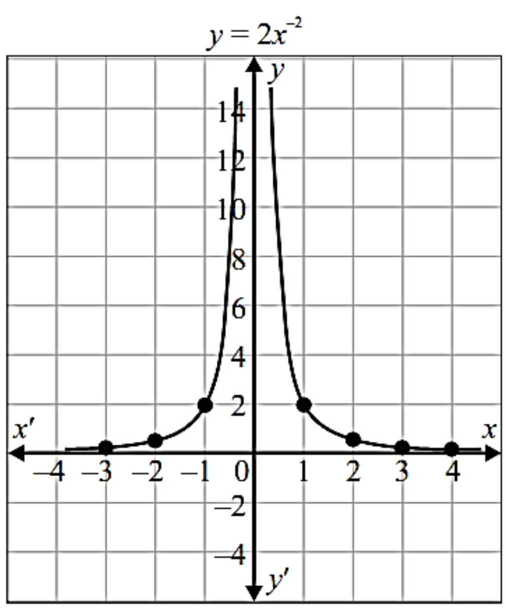
(vi)

x	-1	0	1	2	3	4
y	-3	0	3	3.6	4.2	4.5



(vii)

x	-2	-1.5	-1	-0.5	0.5	1	1.5	2
y	0.5	0.9	2	8	8	2	0.9	0.5



EXERCISE 10.2

1. Plot the graph of $y = 2x^2 - 4x + 3$ for x from -1 to 3. Draw tangent at (2, 3) and find the gradient.

Solution

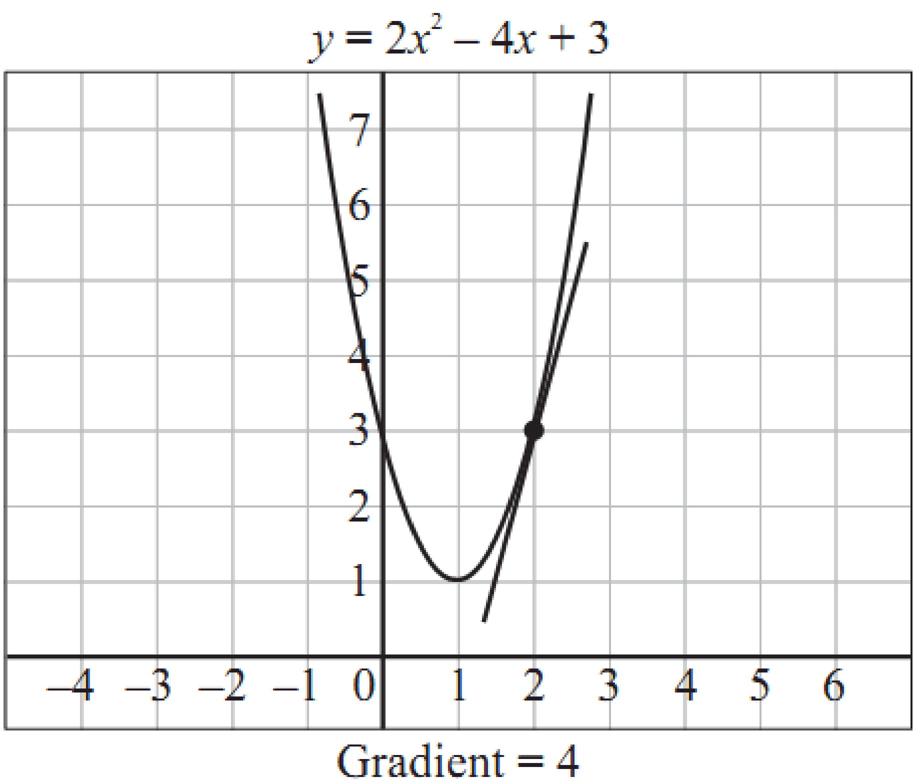
x	-1	-0.5	0	0.5	1	1.5	2	2.5	3
y	9	5.5	3	1.5	1	1.5	3	5.5	9

Consider (2.5,5.5)&(1.5,1.5)

$\text{Gradient} = \frac{1.5-5.5}{1.5-2.5} = \frac{-4}{-1}$

$\text{Gradient} = 4$

Graph



2. Plot the graph of $y = 3x^2 + x + 1$ and draw tangent at (1, 5). Also find gradient of the tangent line at this point.

Solution

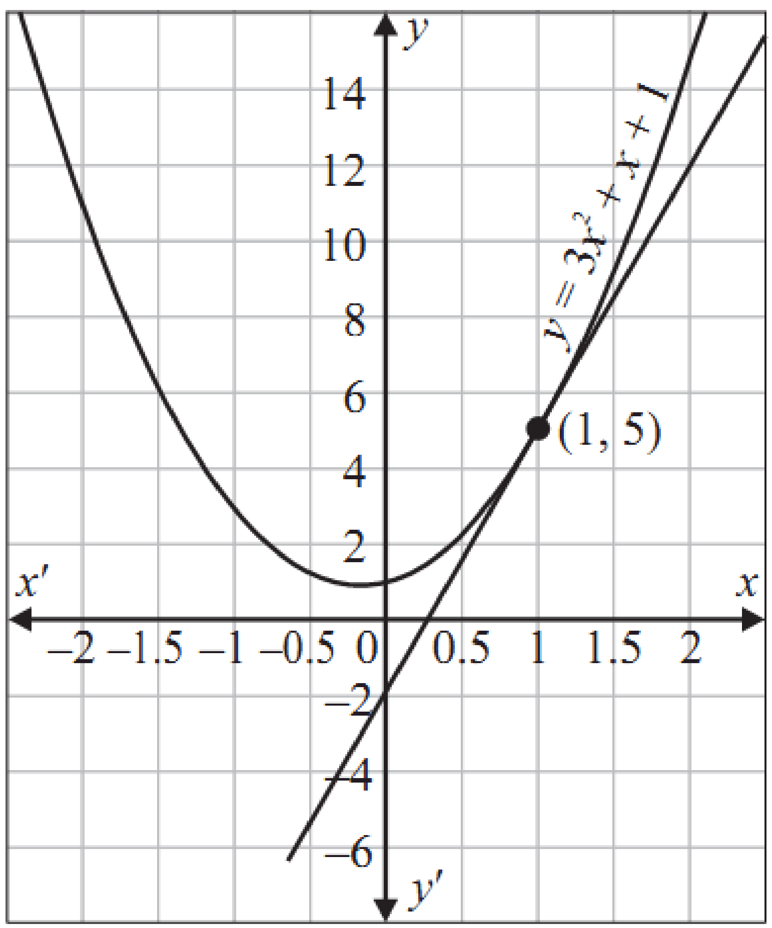
x	-1	-0.5	0	0.5	1	1.5
y	3	1.25	1	2.25	5	9.25

Consider $(1.5, 8.5) \& (0.5, 1.5)$

$$\text{Gradient} = \frac{1.5-8.5}{0.5-1.5} = \frac{-7}{-1}$$

$$\text{Gradient} = 7$$

Graph



Gradient = 7

3. The strength of students in a school was 1000 in 2016. If the strength decay according to the equation $S = 1000\ e^{-t}$, where S is the number of students at time t.

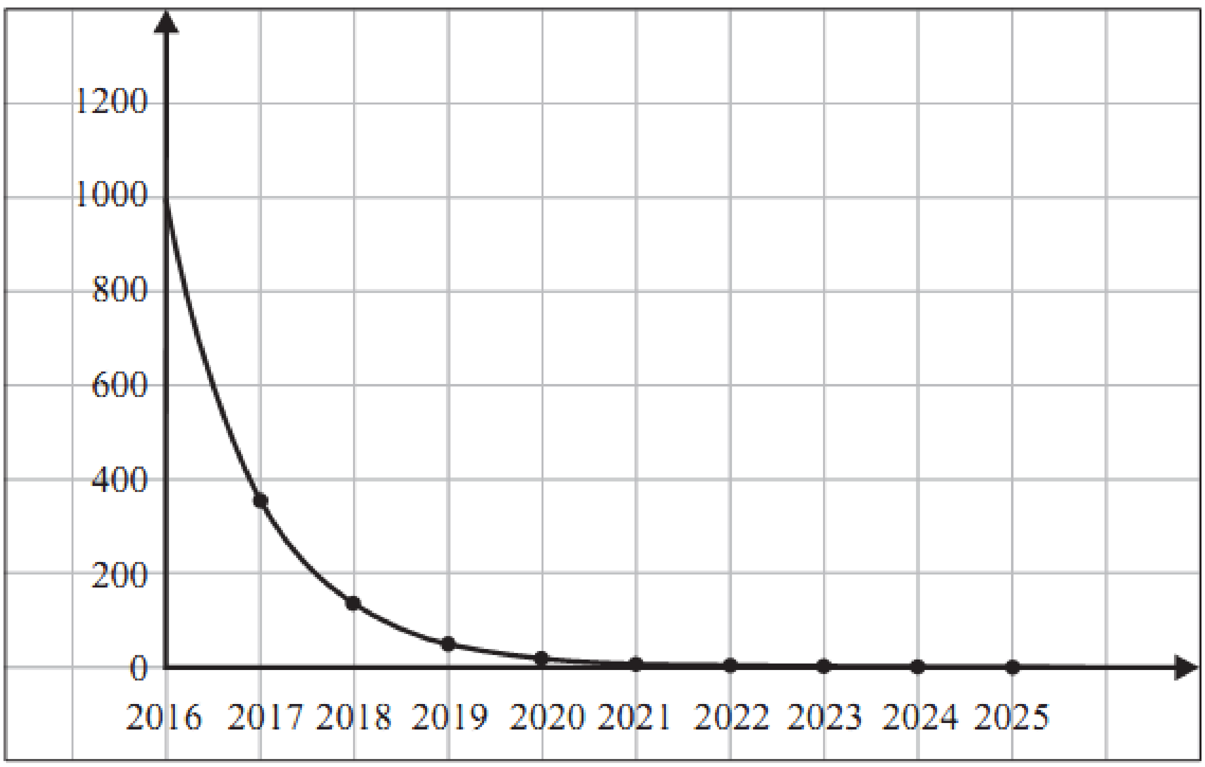
(a) Graph the given equation for $t = 0$ (in 2016) to $t = 9$ (*in* 2025).

(b) From the graph, estimate the student's strength in 2019 and in 2023.

Solution

x	0	1	2	3	4	5	6	7	8	9
y	1000	368	135	50	18	7	2	0.9	0.3	0.1

$y = 100e^{-t}$



(b) From the graph, students' strength in 2019 is approximately 50, and in 2023 approximately 1.

4. The demand and supply functions for a product are given by the equations $P_d = 400 - 5Q$, $P_s = 3Q + 24$:

Plot the graph of each function over the interval $Q = 0$ to $Q = 300$.

Solution

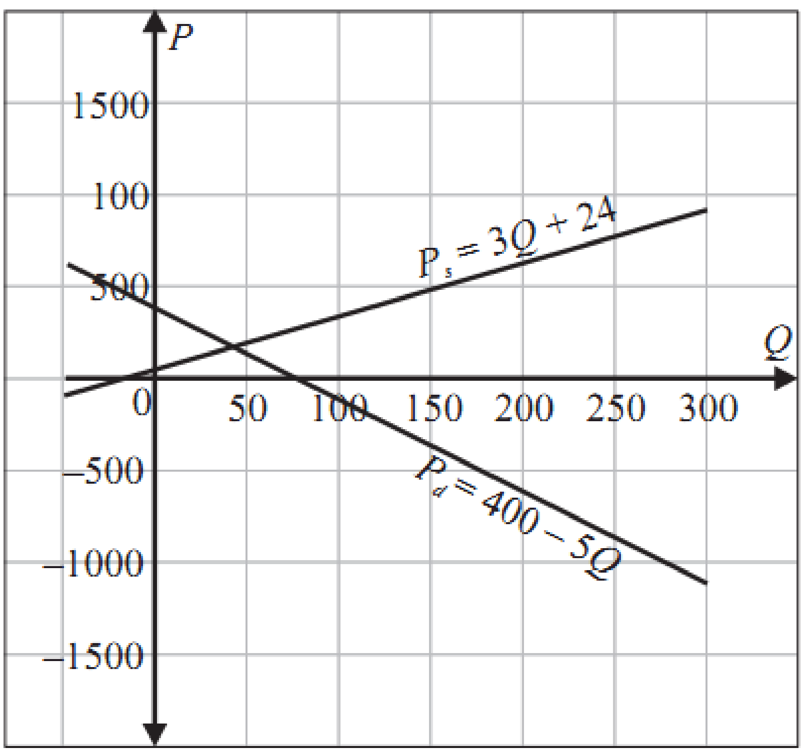
$P_d = 400 - 5Q$

Q	0	50	100	150	200	250	300
P_d	400	150	-100	-350	-600	-850	-1100

$P_s = 3Q + 24$

Q	0	50	100	150	200	250	300
P_s	24	174	324	474	624	774	924

Graph



5. Shahid's salary S(x) in rupees is based on the following formula:

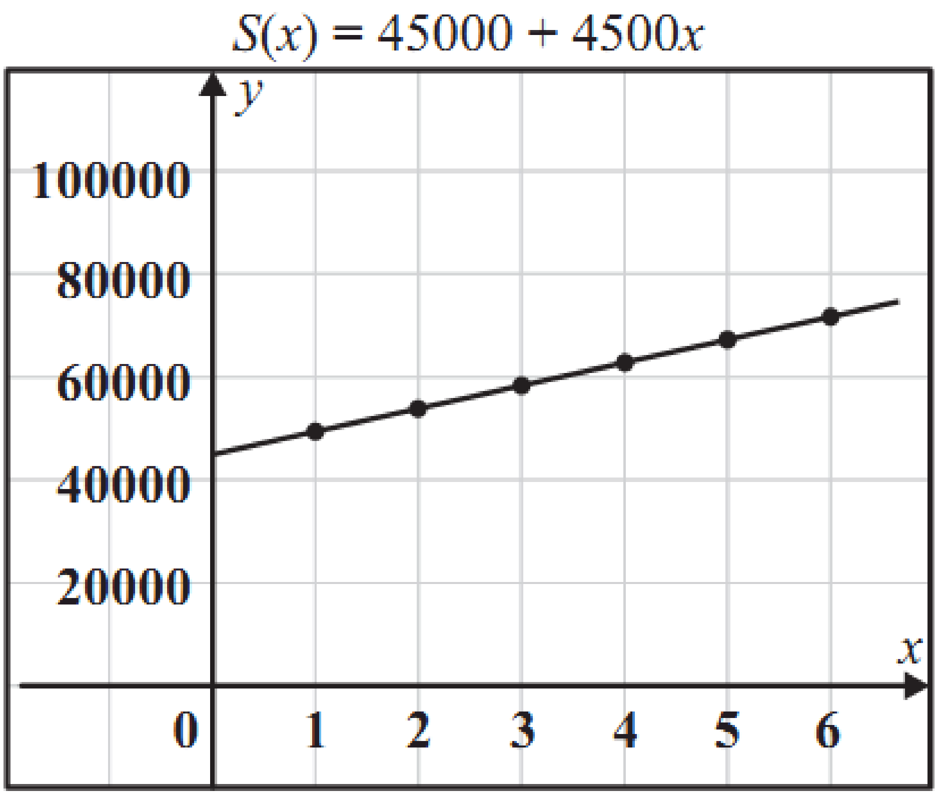
$$S(x) = 45000 + 4500x,$$

where x is the number of years he has been with the company. Sketch and interpret the graph of salary function for $0 \le x \le 5$.

Solution

S	0	1	2	3	4	5
S(x)	45000	49500	54000	58500	63000	67500

Graph



Shahid's salary increases linearly with years of service and rises by Rs. 4500 for every year.

6. A company manufactures school bags. The cost function of producing x bags is $C(x) = 1200 + 20x$ and the revenue from selling x bags is $R(x) = 50x$.

(a) Find the break-even point.

(b) Determine the profit or loss when 250 bags are sold.

(c) Plot the graphs of both the functions and identify the break-even point.

Solution

(a) The break – even point

The break – even point occur when R(x) = C(x)

$50x = 1200 + 20x$

$50x - 20x = 1200$

$30x = 1200$

$x = 40$ bags

(b) Profit or Loss after Sale

$P(x) = R(x) - C(x)$

$P(x) = 50x - 1200 - 20x$

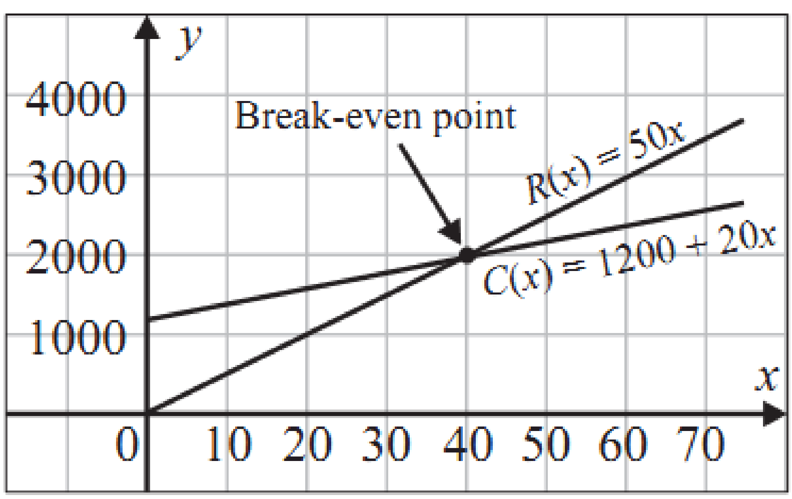
$P(x) = 30x - 1200$

$P(250) = 30(250) - 1200$

Profit = Rs. 6300

(c) Graph

x	0	30	60	90	120	150	180	210
C(x)	1200	1800	2400	3000	3600	4200	4800	5400
R(x)	0	1500	3000	4500	6000	7500	9000	10500



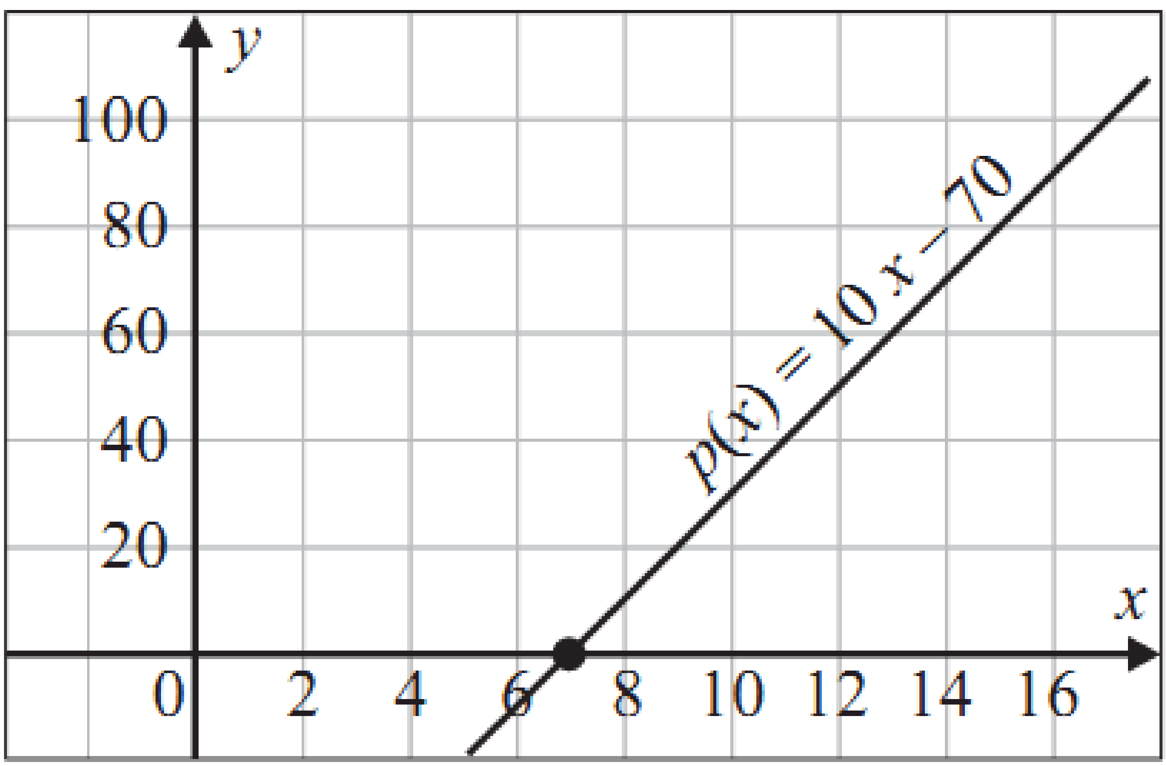
7. A newspaper agency fixed cost of Rs. 70 per edition and marginal printing and distribution costs of Rs. 40 per copy. Profit function is $p(x) = \mathbf{10\,x - 70}$, where x is the number of newspapers. Plot the graph and find profit for 500 newspapers.

Solution

x	6	7	8	10
P(x)	-10	0	10	30

$P(500) = 10(500) - 70 = 5000 - 70 = 4930$

Graph



Profit for 500 newspapers = Rs. 4930

8. Ali manufactures expensive shirts for sale to a school. Its cost (in rupees) for x shirts is $C(x) = 1500 + 10x + 0.2x^2$, $0 \le x \le 150$. Plot the graph and find the cost of 200 shirts.

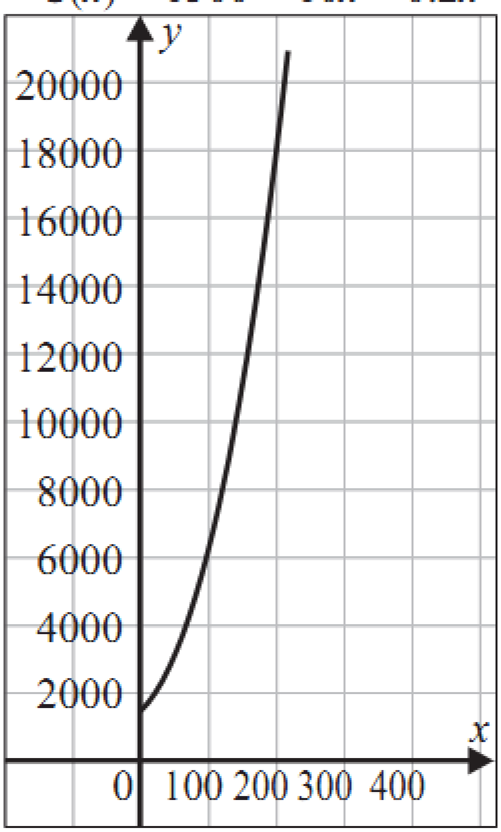
Solution

x	0	50	100	150	200
P(x)	1500	2500	4500	7500	11500

$$C(x) = 1500 + 10x + 0.2x^2$$
$$C(200) = 1500 + 10(200) + 0.2(200)^2$$
$$C(200) = 1500 + 2000 + 8000$$
$$C(200) = 11500$$

Graph

$C(x) = 1500 + 10x + 0.2x^2$



Cost of 200 shirts = Rs. 11500

REVIEW EXERCISE 10

1. Four options are given against each statement. Encircle the correct option.

(i) $x = 5$ represents:

(a) x-axis (b) y-axis
(c) line || to x-axis (d) line || to y-axis

(ii) Slope of the line $y = 5x + 3$ is:

(a) 3 (b) –3 (c) 5 (d) –5

(iii) The y- intercepts of $y = -2x - 1$ is:

(a) –2 (b) 2
(c) –1 (d) 1

(iv) The graph of $y = x^3$, cuts the x-axis at:

(a) $x = 0$ (b) $x = 1$ (c) $x = -1$ (d) $x = 2$

(v) The graph of 3^x represents:

(a) growth (b) decay (c) both(a)and(b) (d) a line

(vi) The graph of $y = -x^2 + 5$ opens:

(a) upward (b) downward (c) left side (d) right side

(vii) The graph of $y = x^2 - 9$ opens:

(a) upward (b) downward (c) left side (d) right side

(viii) $y = 5^x$ is ___________function.

(a) linear (b) quadratic (c) cubic (d) exponential

(ix) Reciprocal function is:

(a) $y = 7^x$ (b) $y = \frac{2}{x}$ (c) $y = 2x^2$ (d) $y = 5x^3$

(x) $y = -3x^3 + 7$ is______________function.

(a) exponential (b) cubic (c) linear (d) reciprocal

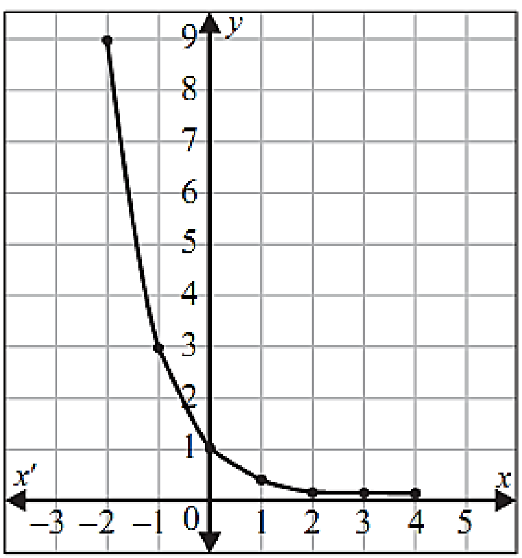
2. Plot the graph of the following functions:

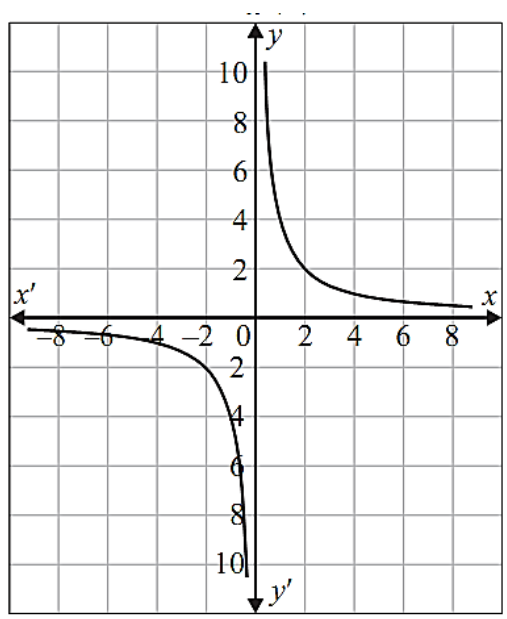
(i) $y = 3^{-x}$ for x from -2 to 4 (ii) $y = \frac{2}{x}, x \neq 0$

Solution

x	y = 3^{-x}
-2	9
-1	3
0	1
1	0.33
2	0.11
3	0.04
4	0.01

x	y = 2/x
-3	-2/3
-2	-1
-1	-2
-0.5	-4
-0.2	-10
0.2	10
0.5	4
1	2
2	1
3	2/3





3. Sales for a new magazine are expected to grow according to the equation: $S = 200000\,(1 - e^{-0.05t})$, where t is given in weeks.

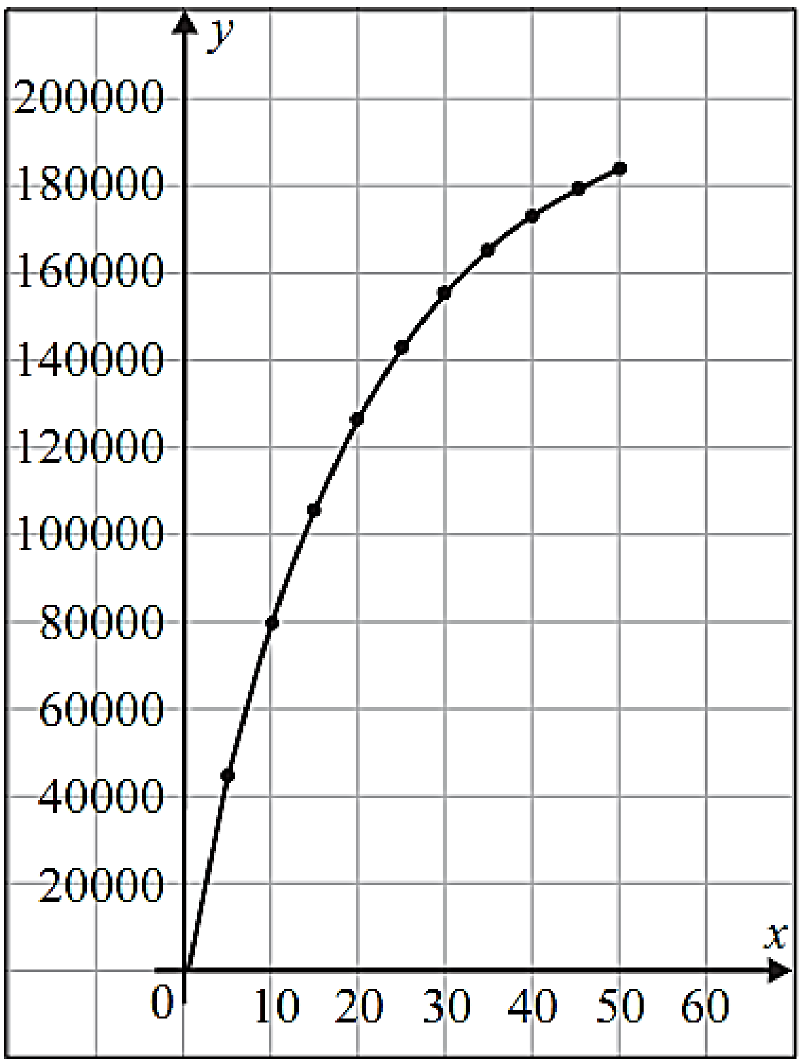
(a) Plot graph of sales for the first 50 weeks.

(b) Calculate the number of magazines sold, when $t = 5$ and $t = 35$.

Solution

t	0	10	20	30	40	50
S(t)	0	78694	126424	155374	172933	183583

(a)



(b) For t = 5, S = 44239.84 and for t = 35, S = 165245.2

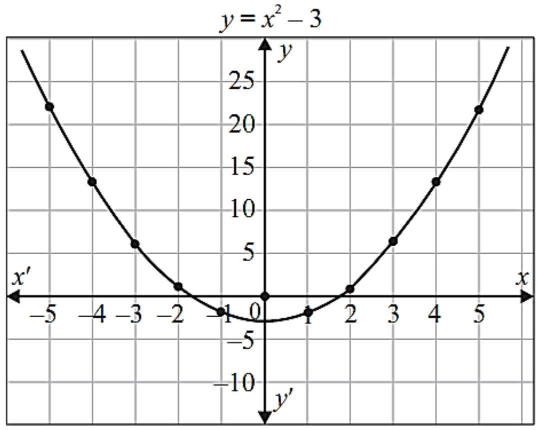
4. Plot the graph of following for x from -5 to 5:

(i) $y = x^2 - 3$ (ii) $y = 15 - x^2$

Solution

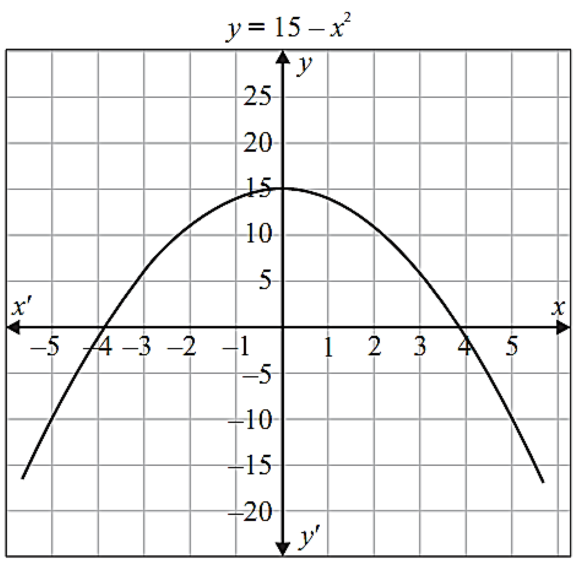
4(i)

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	22	13	6	1	-2	-3	-2	1	6	13	22



4(ii)

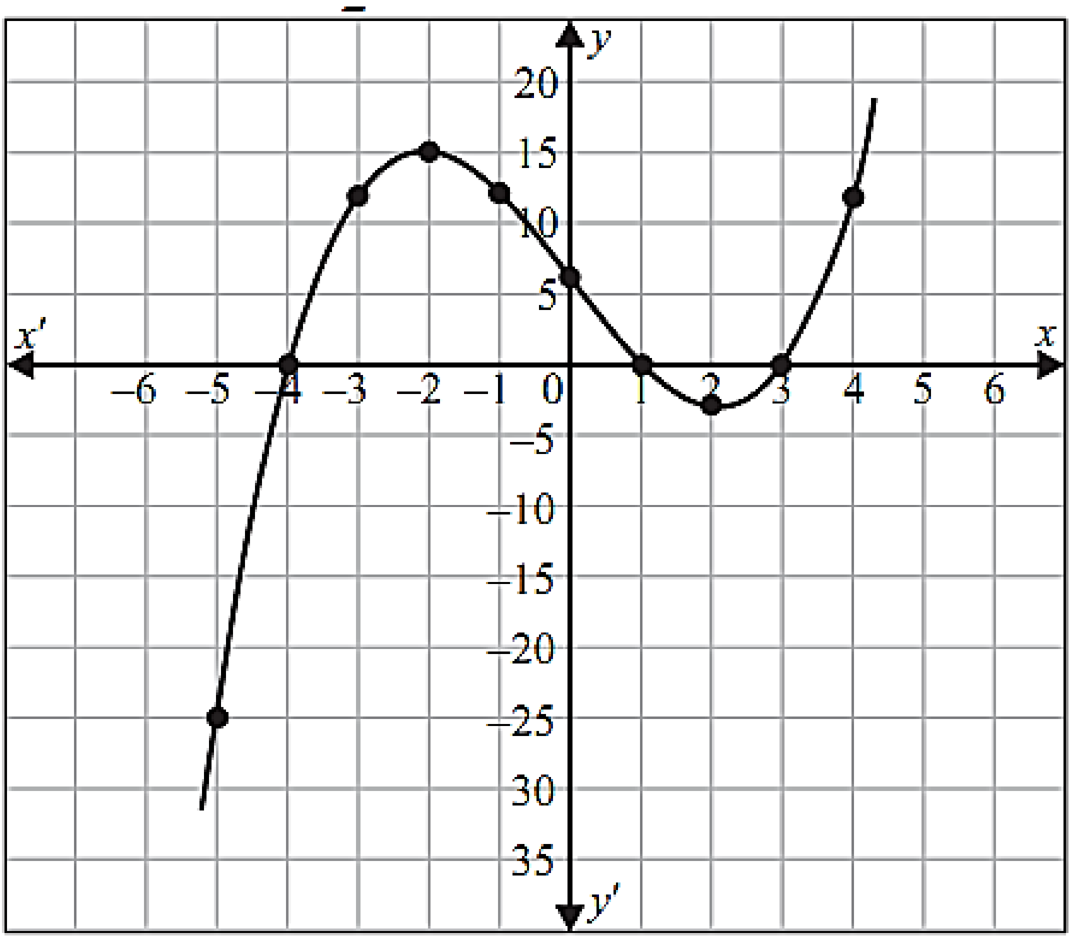
x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	-10	-1	6	11	14	15	14	11	6	-1	-10



5. Plot the graph of $y = \frac{1}{2}(x+4)(x-1)(x-3)$ for x from –5 to 4

Solution

x	-5	-4	-3	-2	-1	0	1	2	3	4
y	-24	0	12	15	12	6	0	-3	0	12



6. The supply and demand functions for a particular market are given by the equations:

$P_s = Q^2 + 5$ and $P_d = Q^2 - 10Q$, where P represents price and Q represents quantity,

Sketch the graph of each function over the interval $Q = -20$ to $Q = 20$.

Solution

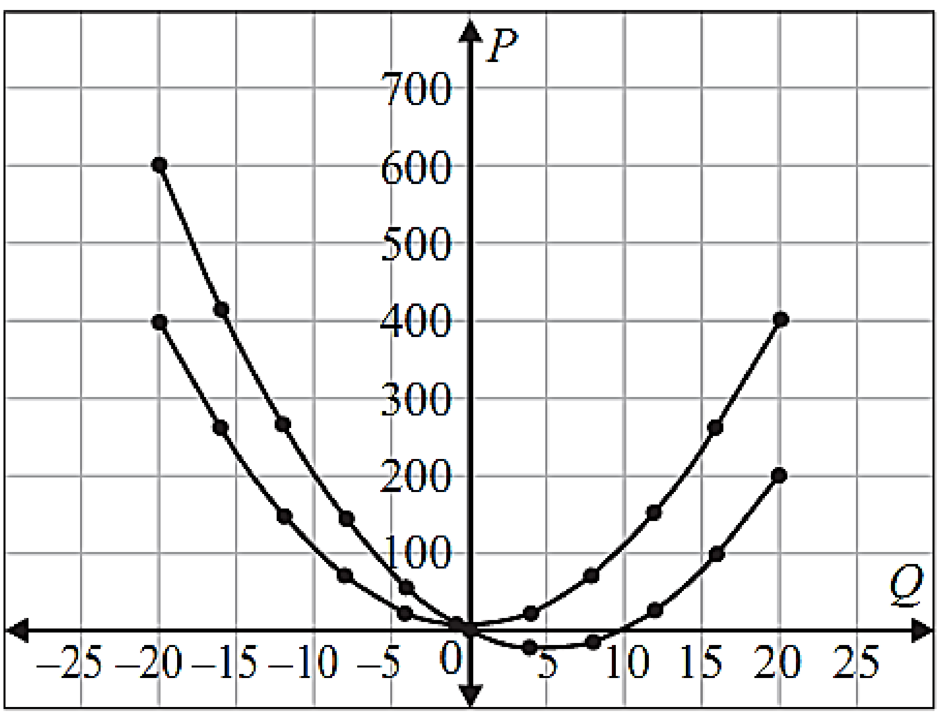
$P_s = Q^2 + 5$

Q	-20	-15	-10	-5	0	5	10	15	20
P_s	405	230	105	30	5	30	105	230	405

$P_d = Q^2 - 10Q$

Q	-20	-15	-10	-5	0	5	10	15	20
P_d	600	375	200	75	0	-25	0	75	200

Graph



7. A television manufacturer company make 40 inches LEDs. The cost of manufacturing x LEDs is $C(x) = 60,000 + 250x$ and the revenue from selling x LEDs is $R(x) = 1200x$. Find the break-even point and find the profit or loss when 100 LEDs are sold. Identify the break-even point graphically.

Solution

(a) The break – even point (no profit or loss)

The break – even point occur when R(x) = C(x)

$1200x = 60000 + 250x$

$x = 63.16$ LED′s

(b) Profit or Loss after Sale

$P(x) = R(x) - C(x)$

$P(x) = 1200x - 60000 - 250x$

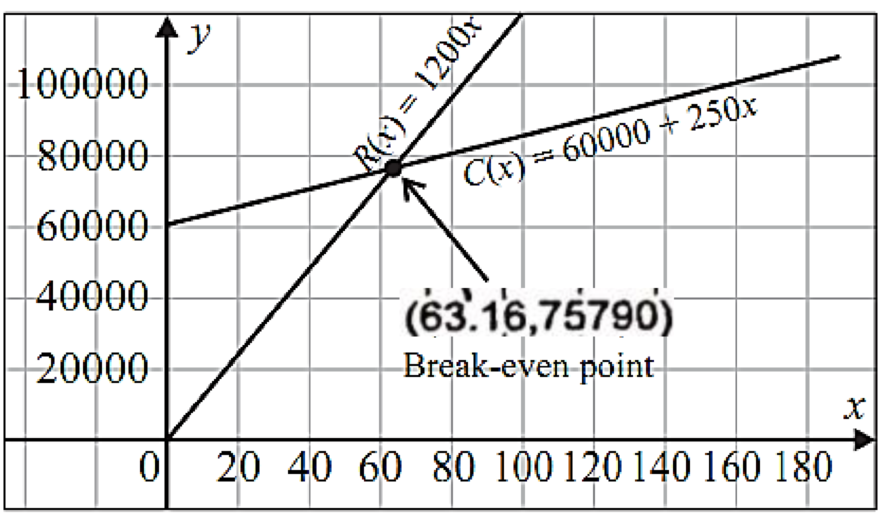
$P(x) = 950x - 60000$

$P(100) = 950(100) - 60000$

Profit = Rs. 35000

(c) Graph

x	20	40	60	80	100	120	140
C(x)	65000	70000	75000	80000	85000	90000	95000
R(x)	24000	48000	72000	96000	120000	144000	168000



Profit = Rs. 35000

Unit 11 Loci and Construction

EXERCISE 11.1

1. Construct ΔABC with the given measurements and verify that the perpendicular bisectors of the triangle are concurrent.

(i) $m\overline{AB} = 5$ cm, $m\overline{BC} = 6$ cm and $m\overline{AC} = 7$ cm

Solution

(i) Draw a line segment AB of length 5cm long.

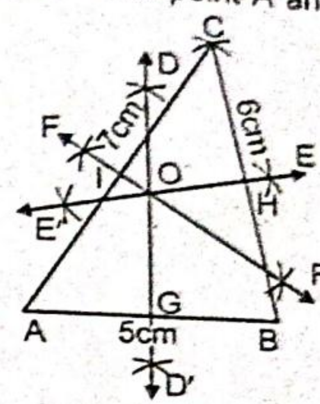
(ii) Using a pair of compasses, draw two arcs with centres at point A and B of radii 7cm and 6cm respectively.

(iii) These two arcs intersect each other at point C.

(iv) Join C with A and B. Hence, ΔABC is the required triangle.

(v) Draw two arcs above and below AB with more Than half radius of mAB with centre at A.

(vi) Draw two arcs above and below AB with radius more than half of mAB with centre at B.



(vii) Draw a line through the points of intersection of the arcs in step (v) and (vi), we get the perpendicular bisector DD′ of the side AB at G.

(viii) Draw two more perpendicular bisectors EE′ and FF′ of the sides BC and AC at H and I respectively.

Hence, we see that the perpendicular bisectors $\overline{DD'}$, $\overline{EE'}$ and $\overline{FF'}$ are concurrent at point O.

1. Construct ΔABC with the given measurements and verify that the perpendicular bisectors of the triangle are concurrent.

(ii) $m\overline{AB} = 7.1$ cm, $m\angle B = 135°$ and $m\overline{BC} = 6.5$ cm

Solution

(i) Draw a line segment AB of length 7.1cm.

(ii) Draw an angle 135° at point B using a pair of compasses and draw a ray $\overrightarrow{BD}$ through this angle.

(iii) Draw an arc of radius 6.5cm with centre at point B intersecting $\overline{BD}$ at point C.

(iv) Join C and A.

(v) Hence, ABC is the required triangle.

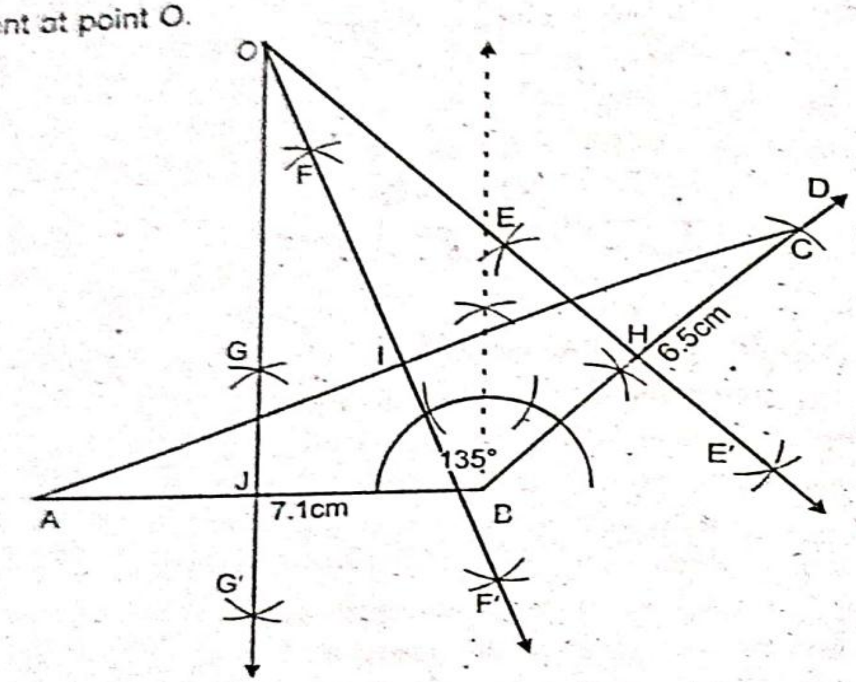
(vi) Draw two arcs above and below $\overline{AB}$ with more than half of $\overline{AB}$ with centre at A.

(vii) Draw two arcs above and below $\overline{AB}$ with radius more than half of $m\overline{AB}$ with centre at B.

(viii) Draw a line through the points of intersection of the arcs in step (vi) and (vii), we get the perpendicular bisector EGG' of the side $\overline{AB}$ at J.

(ix) Draw two more perpendicular bisectors EE' and FF' of the sides $\overline{BC}$ and $\overline{AC}$ at H and I respectively.

Hence, we see that the perpendicular bisector EE', FF' and GG' are concurrent at point O.



2. Construct ΔLMN of the following measurements and verify that the medians of the triangle are concurrent.

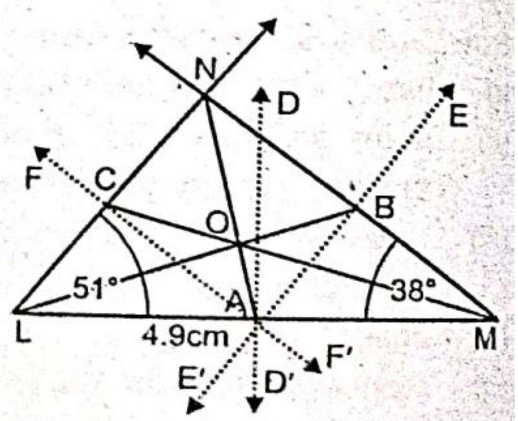
(i) $m\overline{LM} = 4.9$ cm, $m\angle L = 51°$ and $m\angle M = 38°$

Solution

(i) Construct ΔLMN using the given measurements.

(ii) Draw two arcs above and below $\overline{LM}$ with more than half of $\overline{LM}$ with centre at L.

(iii) Draw two arcs above and below $\overline{LM}$ with radius more than half of $m\overline{LM}$ with centre at M.



(iv) Draw a line through the points of intersection of the arcs in step (ii) and (iii), we get the perpendicular bisector DD' of the side $\overline{LM}$ at A.

(v) Draw two more perpendicular bisectors EE' and FF' of the sides $\overline{MN}$ and $\overline{LN}$ at B and C respectively.

(vi) Join point L with opposite midpoint B, so LB is a median.

(vii) Join the point M with opposite midpoint C, we get the median MC and join N with opposite mid point A, we get median $\overline{NA}$.

Hence, we see that the medians $\overline{LB}$, $\overline{MC}$ and $\overline{NA}$ are concurrent at O.

2. Construct ΔLMN of the following measurements and verify that the medians of the triangle are concurrent.

(ii) $m\overline{MN} = 4.8$ cm, $m\angle N = 30°$ and $m\overline{LM} = 8.1$ cm

Solution

(i) Construct ΔLMN using the given measurements.

(ii) Draw two arcs above and below $\overline{MN}$ with more than half of $\overline{MN}$ with centre at M.

(iii) Draw two arcs above and below $\overline{MN}$ with radius more than half of $m\overline{MN}$ with centre at N.

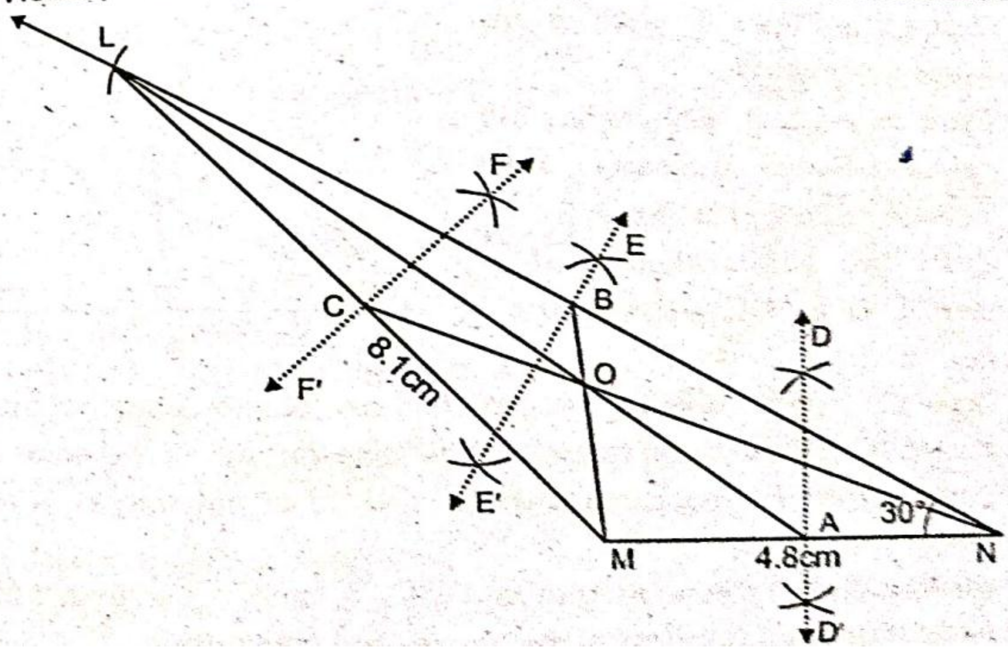
(iv) Draw a line through the points of intersection of the arcs in step (ii) and (iii), we get the perpendicular bisector DD′ of the side $\overline{MN}$ at A.

(v) Draw two more perpendicular bisectors EE′ and FF′ of the sides $\overline{LN}$ and $\overline{LM}$ at B and C respectively.

(vi) Join point N with opposite midpoint C, so $\overline{NC}$ is a median.

(vii) Join the point M with opposite midpoint B, we get the median $\overline{MB}$ and join L with opposite mid point A, we get median $\overline{LA}$.

Hence, we see that the medians $\overline{NC}$, $\overline{MB}$ and $\overline{LA}$ are concurrent at O.



3. Verify that the angle bisectors of ΔABC are concurrent with the following measurement:

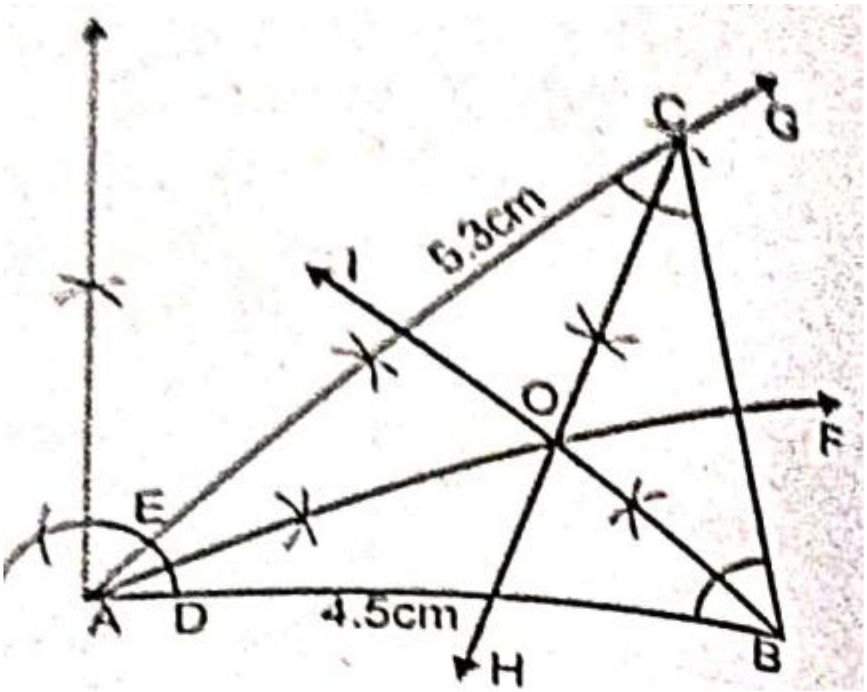
(i) $m\overline{AB} = 4.5$ cm, $m\angle A = 45°$ and $m\overline{AC} = 5.3$ cm

Solution

(i) Construct ΔABC with given lengths and angle.

(ii) Draw an arc of suitable radius with centre at point A intersecting sides $\overline{AB}$ and $\overline{AC}$ at points D and E.

(iii) Draw two arcs with centres at points D and E with suitable radius.



(iv) Draw a ray from A passing through the point of intersection of the arcs in step (iii). Which is the required angle bisector $\overrightarrow{AF}$ of the angle A.

(v) Draw two more angle bisectors BI and CH of the angles B and C respectively.

We see that all the angle bisectors $\overline{AF}$, $\overrightarrow{BI}$ and $\overline{CH}$ intersect at one point O. i.e, the angle bisectors of the triangle are concurrent.

3. Verify that the angle bisectors of ΔABC are concurrent with the following measurement:

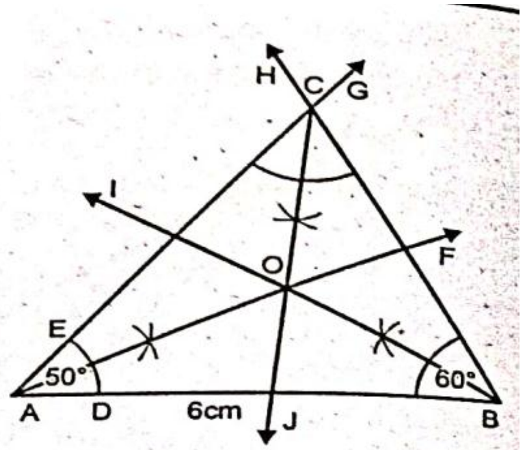
(ii) $m\overline{AB} = 6$ cm, $m\angle A = 150^\circ$ and $m\angle B = 60^\circ$

Solution

(i) Construct ΔABC with given lengths and angle.

(ii) Draw an arc of suitable radius with centre at point A intersecting sides $\overline{AB}$ and $\overline{AC}$ at points D and E.

(iii) Draw two arcs with centres at points D and E with suitable radius.



(iv) Draw a ray from A passing through the point of intersection of the arcs in step (iii). Which is the required angle bisector $\overrightarrow{AF}$ of the angle A.

(v) Draw two more angle bisectors BI and CJ of the angles B and C respectively.

We see that all the angle bisectors $\overrightarrow{AF}$, $\overrightarrow{BI}$ and $\overrightarrow{CJ}$ intersect at one point O. i.e, the angle bisectors of the triangle are concurrent.

4. Given the measurements of ΔDEF : $m\overline{DE} = 4.8$ cm, $m\overline{EF} = 4$ cm and $m\angle E = 45°$, draw altitudes of ΔDEF and find orthocentre.

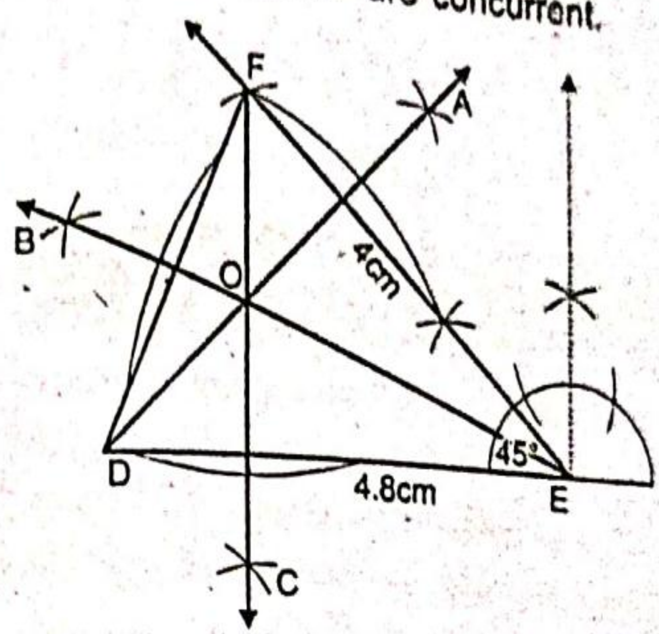
Solution

Construct ΔDEF using the given measurements.

(i) Draw perpendicular $\overrightarrow{DA}$ from D to the opposite side $\overline{EF}$.

(ii) Draw two more perpendiculars $\overrightarrow{EB}$ and $\overrightarrow{FC}$. The first is from point E to the opposite side $\overline{FD}$ and the other is from point F to the opposite side $\overline{DE}$.

(iii) So, $\overrightarrow{DA}$, $\overrightarrow{EB}$ and $\overrightarrow{FC}$ are the altitudes of ΔDEF and they intersect at one point O. i.e., the altitudes of ΔDEF are concurrent.



5. Construct the following triangles and find whether there exists any ambiguous case.

(i) ΔBCD ; $m\overline{BC} = 5$ cm, $m\angle B = 62°$ and $m\overline{CD} = 4.7$ cm

Solution

(i) Draw $m\overline{BC}$=5cm.

(ii) Construct an angle 62° at point B using a protractor and draw $\overrightarrow{BA}$ through this angle.

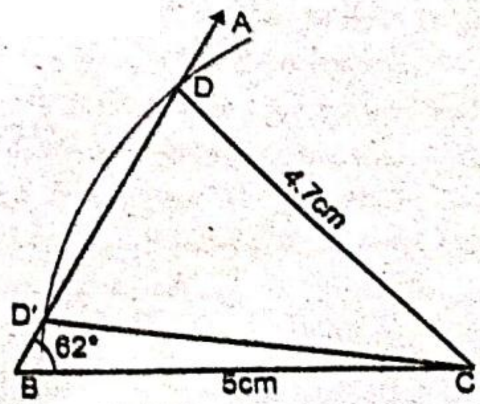
(iii) Draw an arc of radius 4.7 cm with centre at point C.

(iv) This arc intersects $\overrightarrow{BA}$ at two points D and D′.

(v) Join D and D′ with C.

We get two triangles BCD and DCD′.

This is known as ambiguous case.



5. Construct the following triangles and find whether there exists any ambiguous case.

(ii) ΔKLM ; $m\overline{LM} = 6$ cm, $m\angle M = 42°$ and $m\overline{LN} = 5$ cm

Solution

(i) Draw $m\overline{LM}$=6cm.

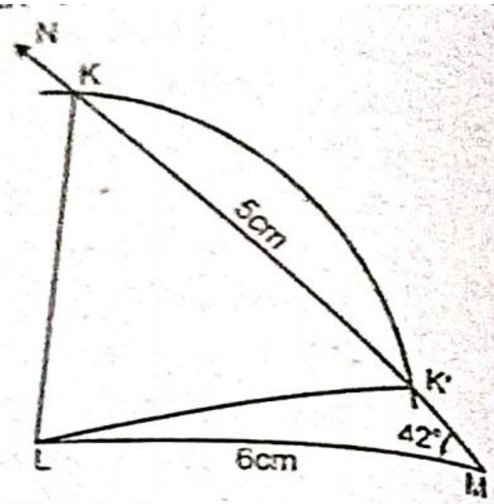
(ii) Construct an angle 42° at point M using a protractor and draw $\overrightarrow{MN}$ through this angle.

(iii) Draw an arc of radius 5cm with centre at point L.

(iv) This arc intersects $\overrightarrow{MN}$ at two points K and K'.

(v) Join K and K' with N.

We get two triangles KLM and K'LM. This is known as ambiguous case.



EXERCISE 11.2

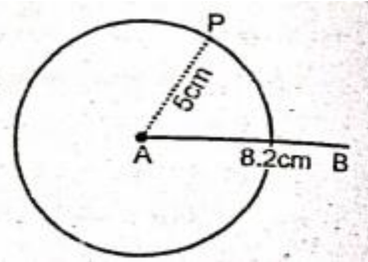
1. Two points *A* and *B* are 8.2 cm apart. Construct the locus of points 5 cm from point *A*.

Solution

(i) Draw a line segment $m\overline{AB}=8.2cm$.

(ii) Draw a circle of radius 5cm with centre at point A.

The locus is 5cm from point A. Any point on this circle is exactly 5cm away from point A.



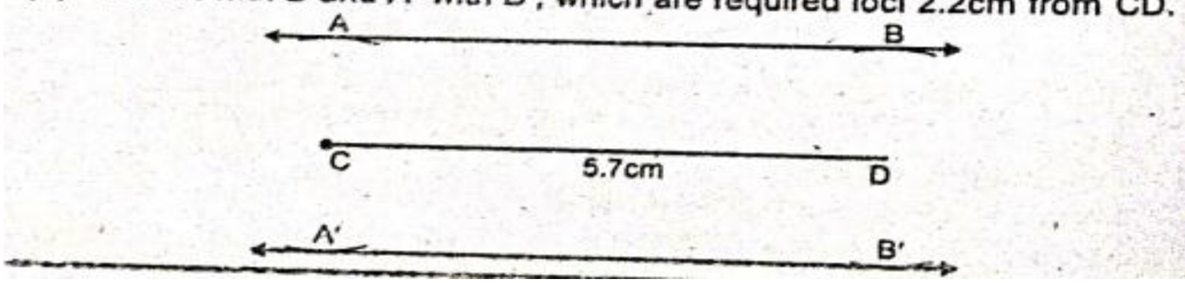
2. Construct a locus of point 2.2 cm from line segment CD of measure 5.7cm.

Solution

(i) Draw a line segment $m\overline{CD} = 5.7cm$.

(ii) Draw two arcs A,A′ and B,B′ above and below $\overline{CD}$ of radius 2.2cm.

(iii) Join A with B and A′ with B′, which are required loci 2.2cm from $\overline{CD}$.



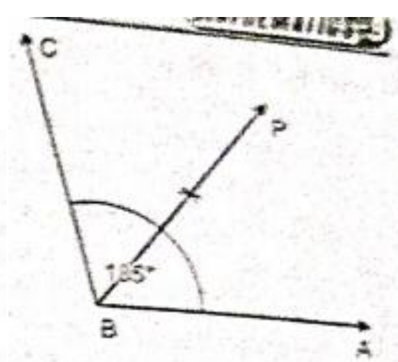
3. Construct an angle $ABC = 105°$. Construct a locus of a point P which moves such that it is equidistant from $\overline{BA}$ and $\overline{BC}$.

Solution

(i) Draw $m\angle\overline{AB}=105°$.

(ii) Draw an angle bisector $\overline{BP}$ of $m\angle ABC$.

All the points on $\overline{BP}$ are equidistant from $\overline{BA}$ and $\overline{BC}$.



4. Two points E and F are 5.4 cm apart. Construct a locus of a point P which moves such that it is equidistant from E and F.

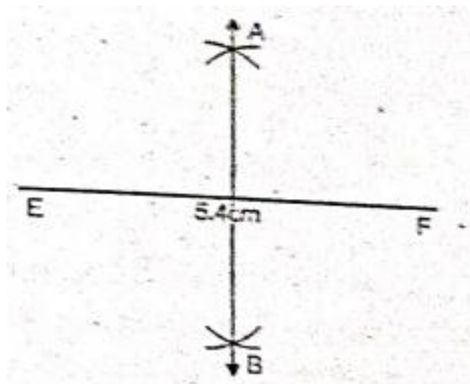
Solution

it is ...

(i) Draw $m\overline{EF} = 5.4cm$.

(ii) Draw a right bisector $\overline{AB}$ of line segment $\overline{EF}$.

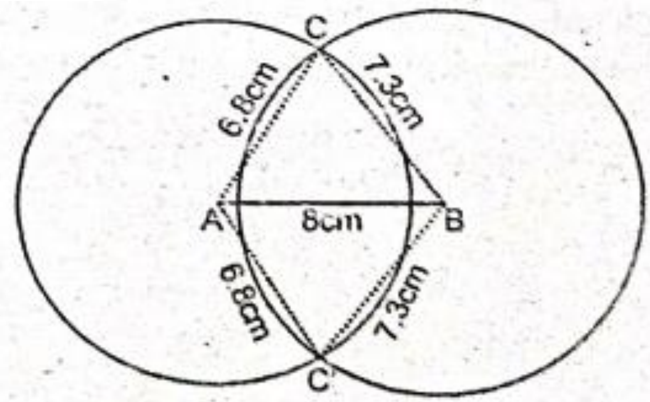
(iii) All the points on $\overline{AB}$ are equidistant from E and F.



5. The island has two main cities A and B 8 km apart. Kashif lives on the island exactly 6.8 km from city A and exactly 7.3 km from city B. Mark with a cross the points on the island where Kashif could live.

Solution

tion: (Here 1km=1cm)



(i) Draw mAB=8cm

(ii) With centre at point A, draw a circle of radius 6.8cm.

(iii) With centre at point B, draw a circle of radius 7.3cm.

(iv) Both the circles intersect each other at point C and C'.

So, C and C are the points on the island where Kashif could live.

6. Construct a triangle CDE with $m\overline{CD} = 7.6$ cm, $m\angle D = 45°$ and $mDE = 5.9$ cm. Draw the locus of all points which are:

(a) equidistant from C and D (b) equidistant from $\overline{CD}$ and $\overline{CE}$

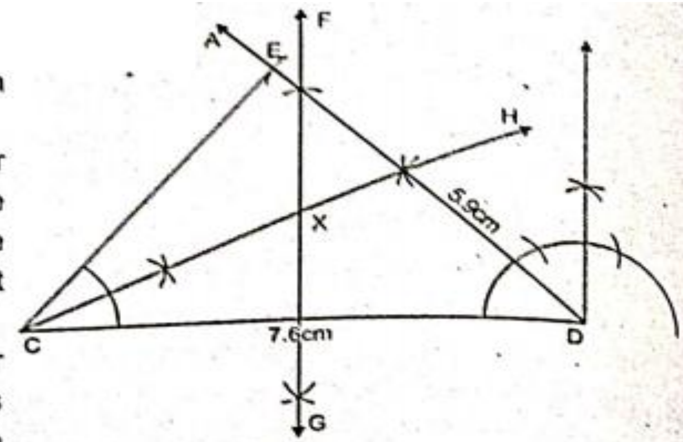
Mark the point X where the two loci intersect.

Solution

(i) Construct a $\triangle CDE$ with the given measurements.

(ii) Draw perpendicular bisector of $\overline{CD}$. All the points located on the line FG are equidistant from C and D.

Draw the angle bisector of $\angle DCE$. All the points located on $\overline{CH}$ are equidistant from $\overline{CD}$ and $\overline{CE}$. Both loci intersect each other at point X.



7. Construct a triangle LMN with $m\overline{LM} = 7$ cm, $m\angle L = 70°$ and $m\angle M = 45°$. Find a point within the triangle LMN which is equidistant from L and M and 3 cm from L.

Solution

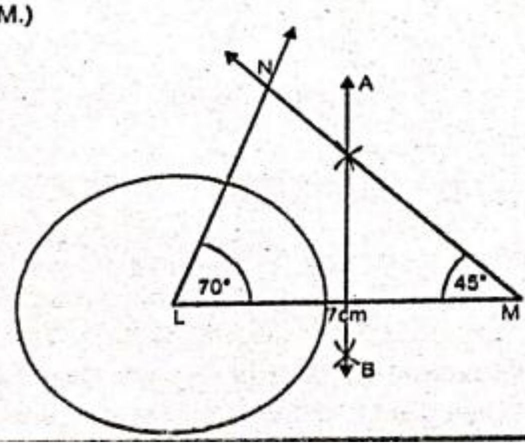
(i) Draw a triangle LMN from the given measurements.

(ii) Draw a right bisector $\overline{AB}$ of line segment LM.

(iii) All the points on $\overline{AB}$ are equidistant from L and M.

(iv) Draw a circle of radius 3cm with centre at point L which is 3cm from L.

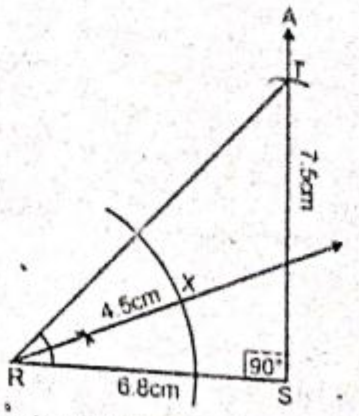
(Note: From the given measurements, it is not possible to draw a point 3cm from L and equistant from L and M.)



8. Construct a right angled triangle RST with $m\overline{RS} = 6.8$ cm, $m\angle S = 90°$ and $m\overline{ST} = 7.5$ cm. Find a point within the triangle RST which is equidistant from $\overline{RS}$ and $\overline{RT}$ and 4.5 cm from R.

Solution

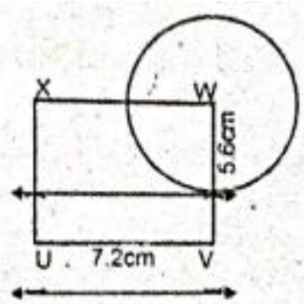
(i) Draw $m\overline{RS} = 6.8$cm.
(ii) Construct $m\angle S = 90°$ and draw $\overline{SA}$.
(iii) Draw an arc of radius 7.5cm with centre at point S intersecting $\overline{SA}$ at point T.
(iv) Joint T and R. Hence ΔRST is the triangle.
(v) Draw the angle bisector of $\angle SRT$.
(vi) Draw an arc of radius 4.5cm with centre at R inside the triangle.
(vii) The point of intersection X of angle bisector and the arc inside the circle is the required point which is equidistant from $\overline{RS}$ and $\overline{RT}$ and 4.5cm from R.



9. Construct a rectangle $UVWX$ with $m\overline{UV} = 7.2$ cm and $m\overline{VW} = 5.6$ cm. Draw the locus of points at a distance of 2 cm from $\overline{UV}$ and 3.5 cm from W.

Solution

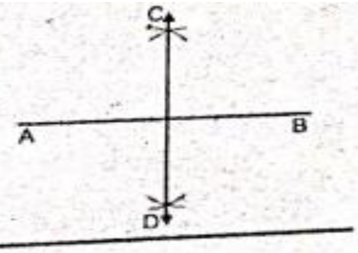
(i) Construct a rectangle UVWX with given lengths.
(ii) Draw a line parallel to $\overline{UV}$ at 2cm. It is a required locus at a distance of 2cm from $\overline{UV}$.
(iii) With centre at W draw a circle of radius 3.5cm. It is required locus which is at the distance of 3.5cm from W.



10. Imagine two cell towers located at points A and B on a coordinate plane. The GPS-enabled device, positioned somewhere on the plane, receives signals from both towers. To ensure accurate navigation, the device is placed equidistant from both towers to estimate its position. Draw this locus of navigation.

Solution

The locus of points equidistant from two points A and B is the perpendicular bisector of the line segment joining A and B. This perpendicular bisector will extend infinitely in both directions and serve as the locus of all points equidistant from two towers.

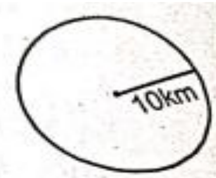


11. Epidemiologists use loci to determine infection zones, especially for contagious diseases, to predict the spread and take containment measures. In the case of a disease outbreak, authorities might determine a quarantine zone within 10 km of the infection source. Draw the locus of all points 10 km from the source defining the quarantine area to monitor and control the disease's spread.

Solution

The locus of all points 10km from the infection source would be a circle with a radius of 10km, centered at the infection source. This circle would define the quarantine zone.

In the below figure, O is the infection source and any point within the circle will be quarantine zone.



12. There is a treasure buried somewhere on the island. The treasure is 24 kilometres from A and equidistant from B and C. Using a scale of 1cm to represent 10 km, find where the treasure could be buried.

Solution

could be buried.

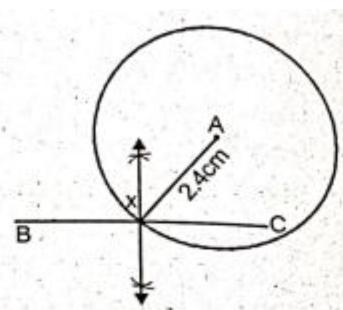
(i) Plot points A,B and C, using appropriate scale (i.e., 1cm=10km).

(ii) With centre at point A, draw a circle of radius 2.4cm.

(iii) Connect points B and C with a straight line.

(iv) Draw perpendicular bisector of $\overline{BC}$.

(v) The treasure lies at point X where the circle centred at A intersects the perpendicular bisector of BC.



13. There is an apple tree at a distance of 90 metres from banana tree in the garden of Sara's house. Sara wants to plant a mango tree M which is 64 metres from apple tree and between 54 and 82 metres from the banana tree. Using a scale of 1cm to represent 10m, Find the points where the mango tree should be planted.

Solution

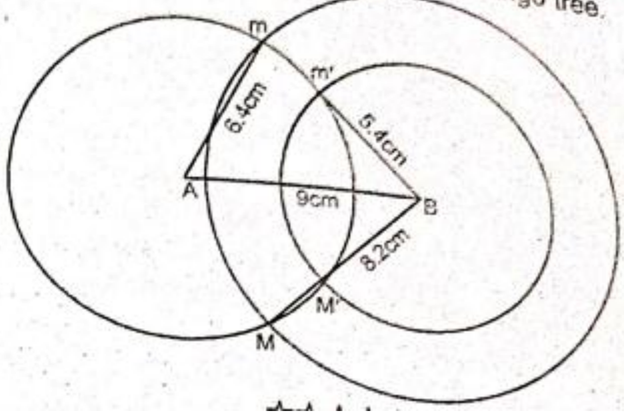
(i) Draw a line segment AB=9cm which is a distance between apple and banana tree.

(ii) With centre at A draw a circle of radius 6.4cm. This circle represents all points that are 64m away from the apple tree.

(iii) With centre at B draw a circle of radius 5.4cm.

(iv) With centre at B draw another arc of radius 8.2cm. These circles represent the range of distances (between 54m and 82m) from the banana tree.

(v) The points where the circle around A intersects with the two circles around B are the possible locations for the mango tree.



REVIEW EXERCISE 11

1. Four options are given against each statement. Encircle the correct option.

   (i) A triangle can be constructed if the sum of the measure of any two sides is ____________ the measure of the third side.

   (a) less than　(b) ✓ greater than
   (c) equal to　(d) greater than and equal to

   (ii) An equilateral triangle ____________.

   (a) ✓ can be isosceles　(b) can be right angled
   (c) can be obtuse angled　(d) has each angle equal to 50^0.

   (iii) If the sum of the measures of two angles is less than 90^0, then the triangle is ____________.

   (a) equilateral　(b) acute angled
   (c) ✓ obtuse angled　(d) right angled

   (iv) The line segment joining the midpoint of a side to its opposite vertex in a triangle is called ____________.

   (a) ✓ median　(b) perpendicular bisector
   (c) angle bisector　(d) circle

   (v) The angle bisectors of a triangle intersect at ____________.

   (a) ✓ one point　(b) two points
   (c) three points　(d) four points

   (vi) Locus of all points equidistant from a fixed point is ____________.

   (a) ✓ circle　(b) perpendicular bisector
   (c) angle bisector　(d) parallel lines

   (vii) Locus of points equidistant from two fixed points is -----------------

   (a) circle　(b) ✓ perpendicular bisector
   (c) angle bisector　(d) parallel lines

   (viii) Locus of points equidistant from a fixed line is/are -------------------

   (a) circle　(b) perpendicular bisector
   (c) angle bisector　(d) ✓ parallel lines

(ix) Locus of points equidistant from two intersecting lines is ____________.

(a) circle (b) perpendicular bisector
(c) ✓ angle bisector (d) parallel lines

(x) The set of all points which is farther than 2 km from a fixed point B is a region outside a circle of radius ____________ and centre at B.

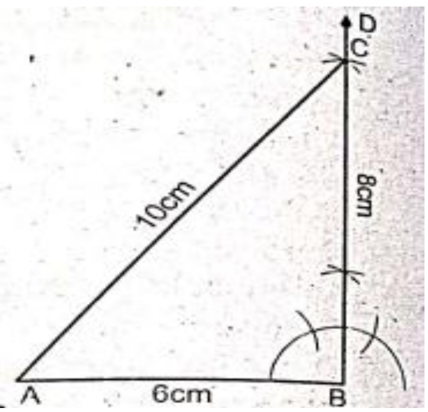
(a) 1 km (b) 1.9 km
(c) ✓ 2 km (d) 2.1 km

2. Construct a right angled triangle with measures of sides 6 cm, 8 cm and 10 cm.

Solution

8cm and 10cm.

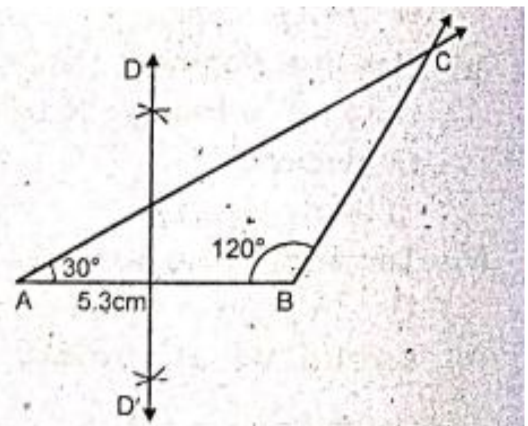
(i) Draw $m\overline{AB}$=6cm.
(ii) Construct mB=90° using compass and draw $\overline{BD}$ through this angle.
(iii) Draw an arc of radius 8cm with centre at point B intersecting $\overline{BD}$ at point C.
(iv) With centre at A, draw an arc of radius 10cm intersecting $\overline{BD}$ at point C.
(v) join C with A.
Hence ΔABC is the required right angled triangle.



3. Construct a triangle ABC with $m\overline{AB}$ = 5.3 cm, $m\angle A$ = 30° and $m\angle B$ = 120°. Draw the locus of all points which are equidistant from A and B.

Solution

(i) Draw $m\overline{AB}$=5.3 cm.
(ii) Draw angles 30° and 120° at points A and B respectively and draw two rays through these angles from A and B.
(iii) These two rays intersect each other at point C.
Hence, ΔABC is the required triangle.



(iv) Draw a perpendicular bisector $\overline{DD'}$ of $\overline{AB}$. The locus of all the points on $\overline{DD'}$ are equidistant from A and B.

4. Construct a triangle with $m\overline{DE} = 7.3$ cm, $m\angle D = 42°$ and $m\overline{EF} = 5.4$ cm.

Solution

Steps of Construction:

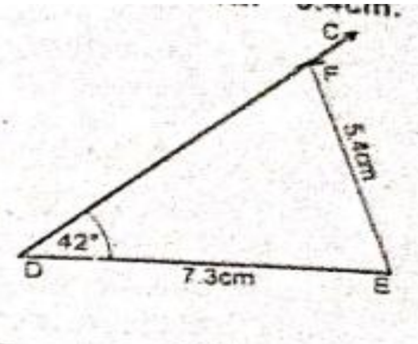
(i) Draw a line segment $\overline{DE}$ of length 7.3cm.

(ii) Draw an angle 42° at point D using a [illegible] and draw a ray $\overline{DC}$ through this angle.

(iii) Draw an arc of radius 5.4cm with centre at point E intersecting $\overline{DC}$ at point F.

(iv) Join E and F.

Hence, ΔDEF is the required triangle.



5. Construct a triangle XYZ with $m\overline{YX} = 8$ cm, $m\overline{YZ} = 7$ cm and $m\overline{XZ} = 6.5$ cm. Draw the locus of all points which are equidistant from $\overline{XY}$ and $\overline{XZ}$.

Solution

Steps [illegible]

(i) Draw a line segment YX of length 8cm long.

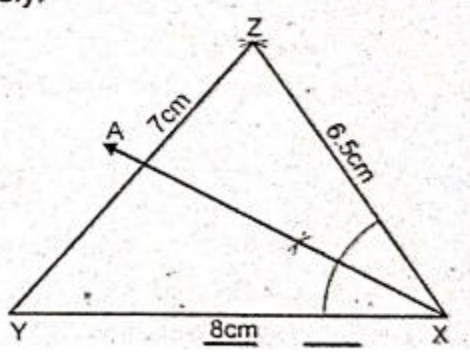
(ii) Using a pair of compasses, draw two arcs with centres at point Y and X of radii 7cm and 6.5cm respectively.

(iii) These two arcs intersect each other at point Z.

(iv) Join Z with X and Y.

Hence, ΔXYZ is the required triangle.

(v) Draw angle bisector of angle XYZ. All the points on $\overrightarrow{XA}$ are equidistant from $\overline{XY}$ and $\overline{XZ}$.



6. Construct a triangle FGH such that $m\overline{FG} = m\overline{GH} = 6.4$ cm, $m\angle G = 122°$. Draw the locus of all points which are:

(a) equidistant from F and G,

(b) equidistant from $\overline{FG}$ and $\overline{GH}$.

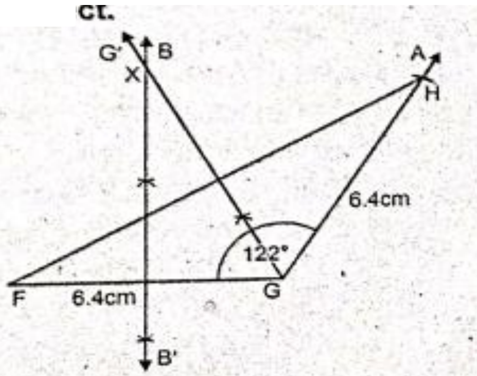
(c) Mark the point where the two loci intersect.

Solution

(i) Draw a line segment FG of length 6.4cm.

(ii) Draw an angle 122° at point G using a protractor and draw a ray $\overrightarrow{GA}$ through this angle.

(iii) Draw an arc of radius 6.4cm with centre at point G intersecting $\overrightarrow{GA}$ at point H.



(iv) Join H and F.

Hence, FGH is the required triangle.

(v) Draw the bisector of $\overline{GF}$. All the points on $\overline{AB}$ are equidistant from F and G.

(vi) Draw angle bisector of m∠FGH. All the points of $\overline{GG'}$ are equidstant from $\overline{FG}$ and $\overline{GH}$.

(vii) Two loci intersect each other at point X.

7. Two houses Q and R are 73 metres apart. Using a scale of 1 cm to represent 10 m, construct the locus of a point P which moves such that it is:

(i) at a distance of 32 metres from Q

(ii) at a distance of 48 metres from the line joining Q and R.

Solution

To construct the locus of a point P from Q, using a scale of 1cm to represent 10m.

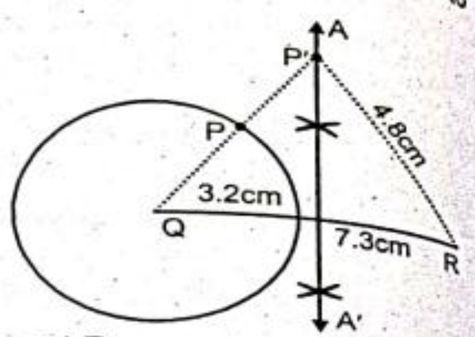
(i) Draw QR=7.3cm.

(ii) The locus of P is a circle centered at Q with a radius of 3.2cm.

(iii) All the points on circle centered at Q are equidistant from Q.

(iv) Draw the perpendicular bisector of $\overline{QR}$.

(v) Mark a point P' on $\overline{AA'}$ such that point P' is 4.8cm from the line joining Q and R.



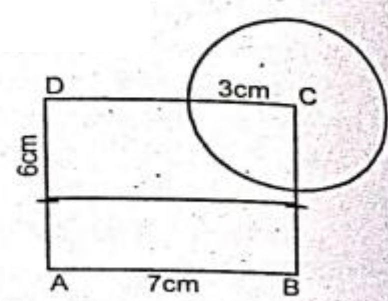
8. The field is in the form of a rectangle $ABCD$ with $m\overline{AB} = 70$m and $m\overline{BC} = 60$ m. Construct the rectangle $ABCD$ using a scale of 1cm to represent 10 m. Show the region inside the field which is less than 30 m from C and farther than 25 m from $\overline{AB}$.

Solution

To construct a locus use the scale of 1cm equals to 10m.

(i) Construct a rectangle ABCD from the given measurements.

(ii) With centre at point C draw a circle of radius 3cm. The region inside the circle represents the point less than 3cm from C.



(iii) Draw a line parallel to AB which is 2.5cm from $\overline{AB}$. The region above this line represents the points farther than 2.5cm from $\overline{AB}$.

Unit 12 Information Handling

EXERCISE 12.1

1. The following distribution represents the scores achieved by a group of chemistry students in the chemistry laboratory.

Scores	24 – 28	29 – 33	34 – 38	39 – 43	44 – 48	49 – 53	Total
No. of students	3	6	12	23	15	6	65

Answer the following questions.

(i) What is the upper limit of the last class?

(ii) What is the lower limit of the class 39 – 43?

(iii) What is the midpoint of the class (34 – 38)?

(iv) What are the class frequencies of the classes 29 – 33 and 44 – 48?

(v) What is the size of the class limits in the above frequency distribution?

(vi) In which class or group does minimum number of students fall?

(vii) What is the lower limit of the class having 15 as its class frequency?

(viii) What is the number of students having scores between 24 and 43?

Solution

(i) 53 (ii) 39 (iii)

$$\text{Midpoint} = \frac{\text{Lower class limit} + \text{Upper class limit}}{2}$$

$$= \frac{34 + 38}{2} = \frac{72}{2}$$

$$= 36$$

(iv) 6 and 15 (v) 5 (vi) (24 – 28) (vii) 44 (viii) $3 + 6 + 12 + 23 = 44$

2. For a school staff, the following expenditures (rupees in hundred) are required for the repair of chairs.

145,	152,	153,	156,	158,	160,	146,	152,	155,	159,
161,	163,	165,	147,	148,	151,	154,	156,	158,	160,
144,	167,	151,	150,	152,	149,	145,	153,	152,	155

Prepare a frequency distribution by tally bar method using 3 as the size of class limits and also write down what are the frequencies of the last three classes?

Solution: Smallest value = 144, Largest value = 167

Class limits	Tally marks	f
144 – 146	||||	4
147 – 149	|||	3
150 – 152	卌 ||	7
153 – 155	卌	5
156 – 158	||||	4
159 – 161	||||	4
162 – 164	|	1
165 – 167	||	2
Total		$\Sigma f = 30$

Frequencies of last three classes are

4,1,2

3. Given below are the weights in kg of 30 students of a high school.

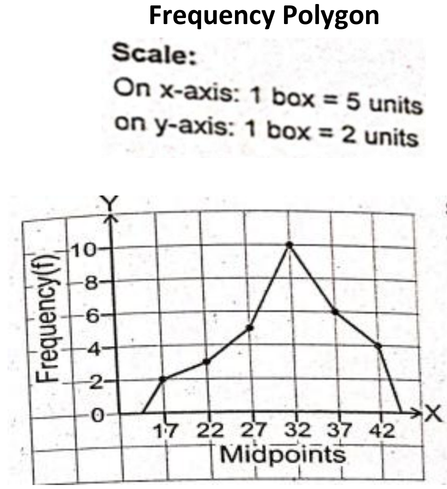
30,	33,	24,	21,	15,	39,	37,	44,	42,	33,
33,	28,	29,	32,	31,	28,	26,	32,	34,	35,
38,	36,	41,	30,	35,	41,	23,	26,	18,	34

Taking 5 as the size of the class limit, prepare a frequency table and construct a frequency polygon.

Solution: Smallest value = 15, Largest value =44

Class limits	Tally marks	f
15 – 19	||	2
20 – 24	|||	3
25 – 29	卌	5
30 – 34	卌 卌	10
35 – 39	卌 |	6
40 – 44	||||	4
Total		$\Sigma f = 30$

Frequency Polygon



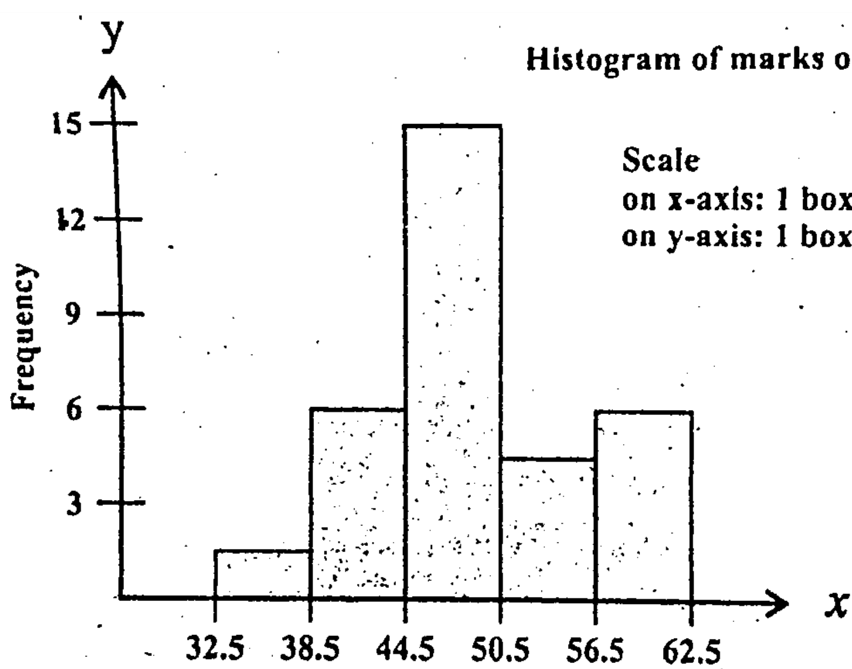
4. A group of Grade - 10 students obtained the following marks out of 100 marks in English test.

58,	59,	58,	33,	40,	58,	45,	46,	43,	45,	45,
50,	52,	49,	50,	57,	52,	55,	49,	50,	62,	49,
48,	44,	42,	47,	46,	47,	46,	53,	40,	44	

Classify the data into a frequency distribution by (direct method) taking 6 as the size of class limit. Also find the class limit with least class frequency and construct histogram for the data.

Solution

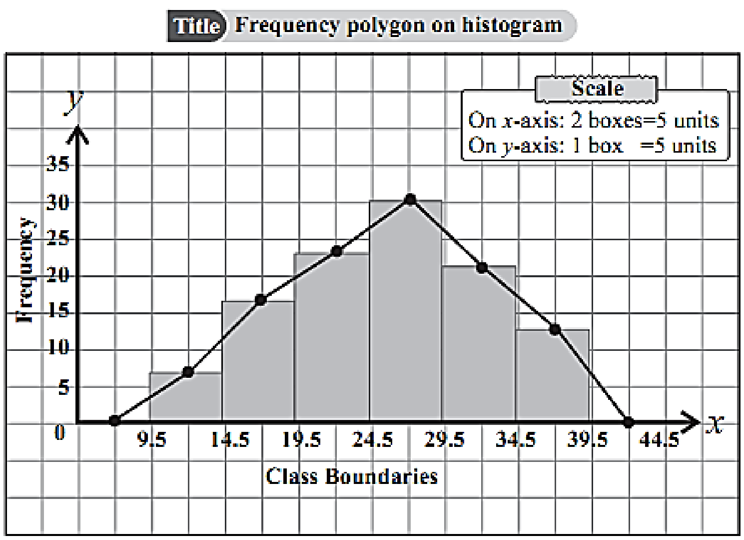
Class limits	Tally marks	f
33 – 38	|	1
39 – 44	𝍸 |	6
45 – 50	𝍸 𝍸 𝍸	15
51 – 56	||||	4
57 – 62	𝍸 |	6
Total		$\Sigma f =$ 32



5. From the table given below. Draw a frequency polygon on histogram for the given frequency distribution.

Weight (kg)	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34	35 – 39
Frequency (*f*)	06	17	23	30	22	13

Solution



6. The following data shows the number of heads in an experiment of 50 sets of tossing a coin 5 times. Make a discrete frequency distribution from the information.

3, 3, 4, 0, 5, 4, 3, 3, 1, 2, 4, 5, 0, 3, 2, 4, 4, 0, 0, 0, 5, 5, 3, 2, 1
4, 3, 2, 5, 3, 2, 1, 3, 5, 4, 3, 2, 1, 3, 2, 1, 3, 1, 3, 1, 4, 3, 2, 2, 4

Solution

No. of heads	Tally marks	*F*
0	𝍸	5
1	𝍸 ||	7
2	𝍸 ||||	9
3	𝍸 𝍸 ||||	14
4	𝍸 ||||	9
5	𝍸 |	6
Total		$\sum f = 50$

7. The marks obtained by the students of Grade - 10 in mathematics test were grouped into the following frequency distribution.

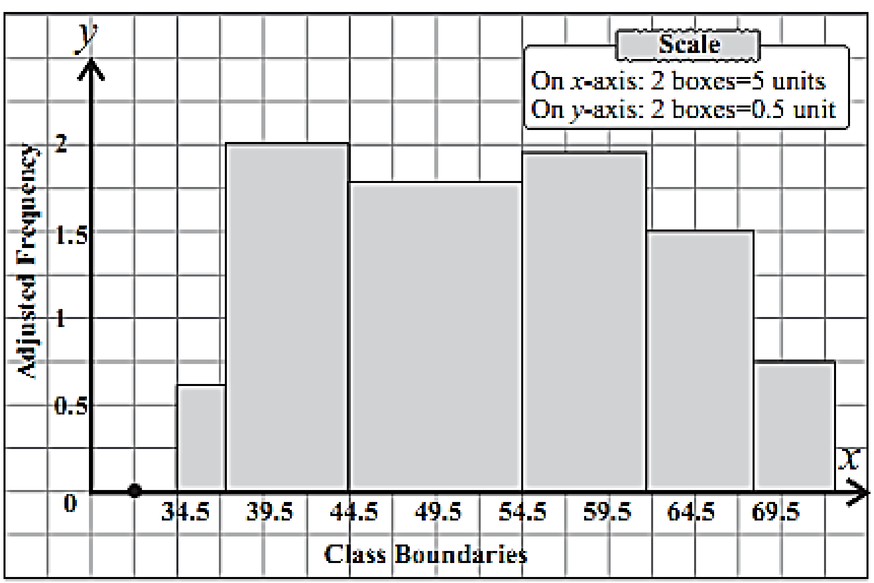
Marks	35 – 37	38 – 44	45 – 54	55 – 61	62 – 67	68 – 72
Frequency	2	12	16	13	9	3

Draw a histogram for the above distribution.

Solution

Marks	Class boundaries	Frequency (f)	Width of Class	Height of rectangles
35 – 37	34.5 – 37.5	2	37.5 – 34.5 = 3	$\frac{2}{3} = 0.67$
38 – 44	37.5 – 44.5	12	44.5 – 37.5 = 7	$\frac{12}{7} = 1.71$
45 – 54	44.5 – 54.5	16	54.5 – 44.5 = 10	$\frac{16}{10} = 1.6$
55 – 61	54.5 – 61.5	13	61.5 – 54.5 = 7	$\frac{13}{7} = 1.86$
62 – 67	61.5 – 67.5	9	67.5 – 61.5 = 6	$\frac{9}{6} = 1.5$
68 – 72	67.5 – 72.5	3	72.5 – 67.5 = 5	$\frac{3}{5} = 0.6$

Title Histogram of marks obtained by students



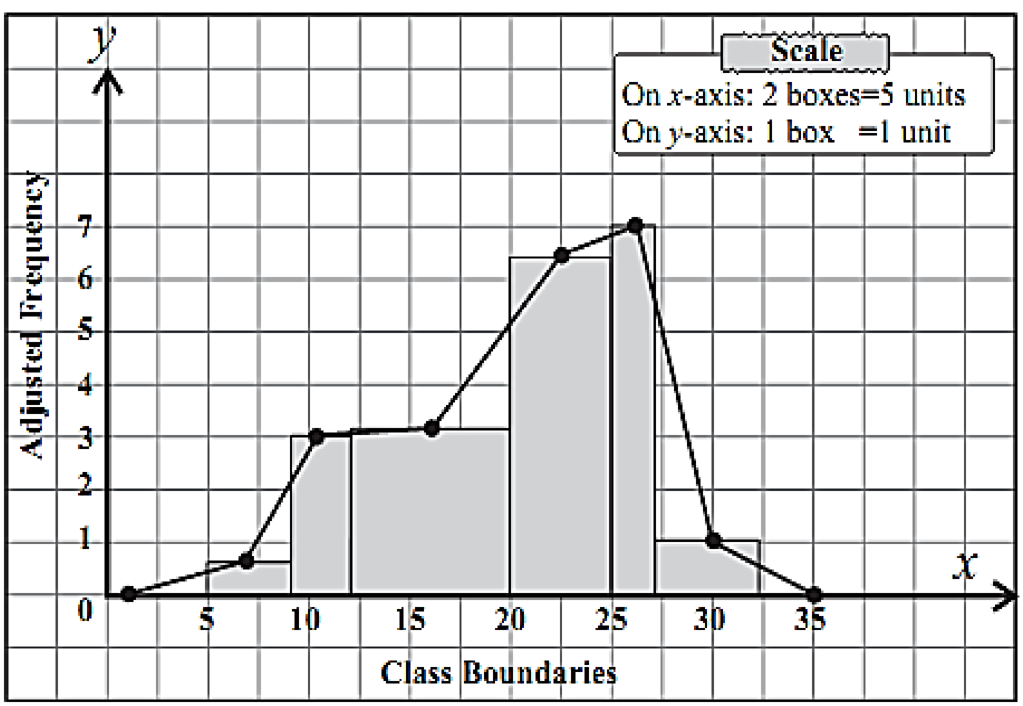
8. Make a frequency polygon on histogram for the following grouped data:

Class limits	5 – 8	8 – 12	12 – 20	20 – 25	25 – 27	27 – 32
Frequency (f)	2	12	25	32	14	5

Solution

Marks	Frequency(f)	Width of class	Height of rectangle
5 – 8	2	8 – 5 = 3	$\frac{2}{3} = 0.67$
8 – 12	12	12 – 8 = 4	$\frac{12}{4} = 3$
12 – 20	25	20 – 12 = 8	$\frac{25}{8} = 3.125$
20 – 25	32	25 – 20 = 5	$\frac{32}{5} = 6.4$
25 – 27	14	27 – 25 = 2	$\frac{14}{2} = 7$
27 – 32	5	32 – 27 = 5	$\frac{5}{5} = 1$

- **Title** Frequency polygon on histogram



EXERCISE 12.2

1. Find the arithmetic mean in each of the following:

(i) 4, 6, 10, 12, 15, 20, 25, 28, 30.

(ii) 12, 18 19, 0, –19, –18, –12

(iii) 6.5, 11, 12.3, 9, 8.1, 16, 18, 20.5, 25

(iv) 8, 10, 12, 14, 16, 20, 22

Solution

(i) $\overline{X} = \frac{\sum X}{n} = \frac{4+6+10+12+15+20+25+28+30}{9} = \frac{150}{9} = 16.67$

(ii) $\overline{X} = \frac{\sum X}{n} = \frac{12+18+19+0-19-18-12}{7} = \frac{0}{7} = 0$

(iii) $\overline{X} = \frac{\sum X}{n} = \frac{6.5+11+12.3+9+8.1+16+18+20.5+25}{9} = \frac{126.4}{9} = 14.04$

(iv) $\overline{X} = \frac{\sum X}{n} = \frac{8+10+12+14+16+20}{7} = \frac{102}{7} = 14.57$

2. Following are the heights in (inches) of 12 students. Find the median height.

55, 53, 54, 58, 60, 61, 62, 56, 57, 52, 51, 63.

Solution

51,52,53,54,55,**56,57**,58,60,61,62,63

Median height $= \frac{1}{2}\left(6^{\text{th}} \text{ term} + 7^{\text{th}} \text{ term}\right) = \frac{1}{2}(56+57) = \frac{113}{2} = 56.5$

3. Following are the earnings (in Rs.) of ten workers:

88, 70, 72, 125, 115, 95, 81, 90, 95, 90. Calculate

(i) Arithmetic Mean (ii) Median (iii) Mode

Solution

70,72,81,88,90,90,95,95,115,125

(i) Arithmetic Mean $= \overline{X} = \frac{\sum X}{n} = \frac{70+72+81+88+90+90+95+95115+125}{10} = \frac{921}{10} = 92.1$

(ii) Median $= \frac{1}{2}\left(5^{\text{th}} \text{ term} + 6^{\text{th}} \text{ term}\right) = \frac{1}{2}(90+90) = \frac{180}{2} = 90$

(iii) Modes $= 90,95$ **most repeated values in data**

4. The Marks obtained by the students in the subject of English are given below.

Marks obtained	15 – 19	20 – 24	25 – 29	30 – 34	35 – 39
Frequency	9	18	35	17	5

Find: (i) Arithmetic mean of their marks by direct and short formula.

(ii) Median of their marks.

Solution

(i) Arithmetic Mean by direct method:

Class limits	Frequency f	mid-point x	fx
15 – 19	9	17	153
20 – 24	18	22	396
25 – 29	35	27	945
30 – 34	17	32	544
35 – 39	5	37	185
Total	**84**		**2223**

$\overline{X} = \frac{\sum fX}{\sum f} = \frac{2223}{84} = 24.46$

Arithmetic Mean by short formula:

Take A = 27

Class Limits	frequency f	mid-points x	$y = x - A$	fy
15 – 19	9	17	– 10	– 90
20 – 24	18	22	– 5	– 90
25 – 29	35	27	0	0
30 – 34	17	32	5	85
35 – 39	5	37	10	50
Total	84			– 45

$\overline{Y} = \frac{\sum fY}{\sum f} = \frac{-45}{84} = -0.54$

$\overline{X} = \overline{Y} + A - 0.54 + 27$

$\overline{X} = 24.46$

(ii) Median of the Marks:

... are explained after the table.

Class Boundaries	frequency f	cumulative frequency
14.5 – 19.5	9	9
19.5 – 24.5	18	27 → C
24.5 – 29.5	35	62 → median class
29.5 – 34.5	17	79
34.5 – 39.5	5	84

For median class: $\frac{n}{2} = \frac{\sum f}{2} = \frac{84}{2} = 42$

class containing 42 is median class. i.e. 24.5 – 29.5

Median $= l + \frac{h}{f}\left(\frac{n}{2} - c\right) = 24.5 + \frac{5}{35}\left(\frac{84}{2} - 27\right) = 26.64$

5. Given below is a frequency distribution.

Class Interval	5 – 9	10 – 14	15 – 19	20 – 24	25 – 29
Frequency	1	8	18	11	2

Find the mode of the frequency distribution.

Solution

Class Limits	frequency
5 – 9	1
10 – 14	8 → f_1
15 – 19	18 → modal class
20 – 24	11 → f_2
25 – 29	2

Mode $= l + \frac{f_m - f_1}{(f_m - f_1) + (f_m - f_2)} \times h = 14.5 + \frac{18-8}{(18-8)+(18-11)} \times 5$

Mode $= 17.44$

6. Ten boys work on a petrol pump station. They get weekly wages as follows:
Wages (in Rs.) 4250, 4350, 4400, 4250, 4350, 4410,4500, 4300, 4500, 4390.
Find the arithmetic mean by short formula, median and mode of their wages.

Solution

4250,4250,4300,4350,4350,4390,4400,4410,4500,4500

Arithmetic Mean by Short Formula

Let A = 4350

$\mathrm{Y} = \mathrm{X} - \mathrm{A}$ is $-100, -100, -50, 0, 0, 40, 50, 60, 150, 150$

$\overline{\mathrm{Y}} = \frac{\sum f\mathrm{Y}}{\sum f} = \frac{-100-100-50+40+50+60+150+150}{10} = \frac{200}{10} = 20$

$\overline{\mathrm{X}} = \overline{\mathrm{Y}} + \mathrm{A} = 20 + 4350 = 4370$

Median

Median $= \frac{1}{2}\left(5^{\text{th}} \text{ term} + 6^{\text{th}} \text{ term}\right) = \frac{1}{2}(4350 + 4390) = \frac{8740}{2} = 4370$

Mode

Modes $= 4250, 4350, 4500$ **most repeated values in data**

7. The arithmetic mean of 45 numbers is 80. Find their sum.

Solution

$\overline{\mathrm{X}} = \frac{\sum \mathrm{X}}{\mathrm{n}} \Rightarrow 80 = \frac{\sum \mathrm{X}}{45} \Rightarrow \sum \mathrm{X} = 80 \times 45 \Rightarrow \text{Sum} = 3600$

8. Five numbers are 1, 4, 0, 7, 9. Find their mean, median and mode.

Solution

0,1,**4**,7,9

Mean $= \overline{\mathrm{X}} = \frac{\sum \mathrm{X}}{\mathrm{n}} = \frac{0+1+4+7+9}{5} = \frac{21}{5} = 4.2$

Median $= 4$ (middle term out of 5)

Mode = no mode (no entry is repeated)

9. A set of data contains the values as 148, 145, 160. 157, 156, 160.

Show that Mode > Median > Mean.

Solution

145,148,156,157,160,160

$$\text{Mean} = \overline{X} = \frac{\sum X}{n} = \frac{145+148+156+157+160+160}{6} = \frac{926}{6} = 154.33$$

$$\text{Median} = \frac{1}{2}\left(3^{rd}\text{ term} + 4^{th}\text{ term}\right) = \frac{1}{2}(156+157) = \frac{313}{2} = 156.5$$

Mode = 160

As, $160 > 156.5 > 154.33$

So, $\text{Mode} > \text{Median} > \text{Mean}$

10. The monthly attendance of 10 students for their lunch in the hostel is recorded as: 21, 15, 16, 18, 14, 17, 15, 12, 13, 11.

Find the median and mode of the attendance. Also find the mean if $D = A - 20$.

Solution

11,12,13,14,**15,15,**16,17,18,21

$$\text{Median} = \frac{1}{2}\left(5^{th}\text{ term} + 6^{th}\text{ term}\right) = \frac{1}{2}(15+15) = \frac{30}{2} = 15$$

Mode = 15

Let A = 20

$Y = X - 20$ is $-9, -8, -7, -6, -5, -5, -4, -3, -2, 1$

$$\overline{Y} = \frac{\sum fY}{\sum f} = \frac{-1-9-8-7-6-5-5-4-3-2+1}{10} = -\frac{48}{10} = -4.8$$

$$\overline{X} = \overline{Y} + A = -4.8 + 20 = 20.2$$

11. On a prize distribution day, 50 students brought pocket money as under:

Rupees	5 – 10	10 – 15	15 – 20	20 – 25	25 – 30
Frequency (*f*)	12	9	18	7	4

(i) Find the median and mode of the above data.

(ii) Find the arithmetic mean of the data given above using coding method.

Solution

(i) Median and Mode of the data

Classes	*f*	C.*f*.
5 – 10	12	12
10 – 15	9	21
15 – 20	18	39
20 – 25	7	46
25 – 30	4	50
Total	**50**	

$\text{Median} = l + \frac{h}{f}\left(\frac{n}{2} - c\right) = 15 + \frac{5}{18}(25 - 21) = 16.11$

$\text{Mode} = l + \frac{f_m - f_1}{(f_m - f_1) + (f_m - f_2)} \times h = 15 + \frac{9}{9+11} \times 5 = 17.25$

(ii) Arithmetic Mean using Coding method

Classes	*f*	mid point *x*	$y = x - 17.5$	*fy*
5 – 10	12	7.5	– 10	– 120
10 – 15	9	12.5	– 5	– 45
15 – 20	18	17.5	0	0
20 – 25	7	22.5	5	35
25 – 30	4	27.5	10	40
Total	**50**			– 90

Let A = 17.5

$\overline{Y} = \frac{\sum fY}{\sum f} = -\frac{90}{50} = -1.8$

$\overline{X} = \overline{Y} + A = -1.8 + 17.5 = 15.70$

12. The arithmetic mean of the ages of 20 boys is 13 years, 4 months and 5 days. Find the sum of their ages. If one of the boys is of age exactly 15 years. What is the average age of the remaining boys?

Solution

Sum of all ages = 20(13 years 4 months and 5 days)

Sum of all ages = 20×13 years $+ 20 \times 4$ months $+ 20 \times 5$ days

Sum of all ages = 260 years + 6 years 8 months + 3months 10 days

Sum of all ages = 266 years 11 months and 10 days

Total age of remaining students excluding age of 15 years old one

= 251 years 11 months and 10 days

Total age of remaining 19 students = $251 \times 360 + 11 + 30 + 10 = 90700$ days

Average age of remaining 19 students = $\frac{\text{total age}}{\text{no. of students}} = \frac{90700}{19} \approx 4774$ days

Average age of remaining 19 students = 13 years 3 months and 4 days

13. Calculate the arithmetic mean from the following information:

(i) If $D = X - 140$, $\Sigma D = 500$ and $n = 10$

(ii) If $U = \frac{x-130}{6}$, $\Sigma U = -150$ and $n = 15$

(iii) If $D = x - 25$, $\Sigma fD = 300$ and $\Sigma f = 20$

(vi) If $U = \frac{x-120}{5}$, $\Sigma fU = 60$ and $\Sigma f = 100$

Solution

(i) $\overline{D} = \frac{\sum D}{n} = \frac{500}{10} = 50$ then we have $\overline{X} = \overline{D} + 140 = 50 + 140 = 190$

(ii) $\overline{U} = \frac{\sum U}{n} = -\frac{150}{50} = -10$ then we have $\overline{X} = 6\overline{U} + 30 = -60 + 130 = 70$

(iii) $\overline{D} = \frac{\sum fD}{\sum f} = \frac{300}{20} = 15$ then we have $\overline{X} = \overline{D} + 25 = 15 + 25 = 40$

(iv) $\overline{U} = \frac{\sum fU}{\sum f} = \frac{60}{100} = 0.6$ then we have $\overline{X} = 5\overline{U} + 120 = 3 + 120 = 123$

14. The three children Haris, Maham and Minal made the following scores in a game conducted by a group of teachers in the school.

Haris scores	50	55	70	85	90
Maham scores	75	60	60	45	53
Minal scores	80	77	66	42	48

It is decided that the candidate who gets the highest average score will be awarded rupees 1000. Who will get the awarded amount?

Solution

Average of Haris $= \frac{50+55+70+85+90}{5} = \frac{350}{5} = 70$ **winner with highest average**

Average of Maham $= \frac{75+60+60+45+53}{5} = \frac{293}{5} = 58.6$

Average of Minal $= \frac{80+77+66+42+48}{5} = \frac{313}{5} = 62.6$

15. Given below is a frequency distribution derived by making a substitution as $D = X - 20$. Calculate the arithmetic mean.

D	–6	–4	–2	0	2	4	6
f	1	3	6	20	26	12	2

Solution

D	*f*	*f*D
– 6	1	– 6
– 4	3	– 12
– 2	6	– 12
0	20	0
2	26	52
4	12	48
6	2	12
Total	**70**	**82**

$\overline{\text{D}} = \frac{\sum f\text{D}}{\sum f} = \frac{82}{70} = 1.17$

$\overline{\text{X}} = \overline{\text{D}} + 20 = 1.17 + 20 = 21.17$

16. Being partners Hafsa and Fatima took part in a quiz programme. They made the following number of points 45, 51, 58, 61, 74. 48, 46 and 50. Compute the average number of points using deviation $D = x - 58$.

Solution

$D = X - 58$ is $-13, -7, 0, 3, 16, -10, -12, -8$

$\overline{D} = \frac{-13-7+0+3+16-10-12-8}{8} = \frac{-31}{8} = -3.87$

$\overline{X} = \overline{D} + 58 = -3.87 + 58 = 54.13$

17. A person purchased the following food items:

Food item	Quantity (in Kg)	Cost per Kg (in Rs.)
Rice	10	96
Flour	12	48
Ghee	4	190
Sugar	3	49
Mutton	2	650

What is the weighted mean of cost of food items per kg?

Solution

Food Items	Quantity in kg W	Price x	Wx
Rice	10	96	960
Flour	12	48	576
Ghee	4	190	760
Sugar	3	49	147
Mutton	2	650	1300
Total	**31**		**3743**

$\overline{X} = \frac{\sum wX}{\sum w} = \frac{3743}{31} = 120.74$

18. For the following data, find the weighted mean.

Item	Quantity	Cost of item (in thousands)
Washing Machine	5	35
Heater	3	5
Stove	2	13
Dispenser	6	18

Solution

Item	Quantity w	Cost (in thousands) x	wx
Working Machine	5	35	175
Heater	3	5	15
Stove	2	13	26
Dispenser	6	18	108
Total	**16**		**324**

$\overline{X} = \frac{\sum wX}{\sum w} = \frac{324}{16}$

$\overline{X} = 20.25$ thousands

19. A company is planning its next year marketing budget across five years: yearly budgets (in million) are: 5, 7, 8, 6, 7. Find the average budget for the next year.

Solution

Average Budget $= \frac{5+6+6+7+8}{5} = \frac{33}{5} = 6.6$ millions

20. Ahmad obtained the following marks in a certain examination. Find the weighted mean if weights 5, 4, 2, 3, 2, 4 respectively are allotted to the subjects.

Urdu	English	Science	Math	Islamiyat	Computer
78	65	80	90	85	72

Solution

Subject	Marks x	Weights w	wx
Urdu	78	5	390
English	65	4	260
Science	80	2	160
Math	90	3	270
Islamiat	85	2	170
Computer	72	4	288
Total		$\Sigma w = 20$	$\Sigma wx = 1538$

$\overline{X} = \frac{\sum wX}{\sum w} = \frac{1538}{20}$

$\overline{X} = 76.9$

REVIEW EXERCISE 12

1. Four options are given against each statement. Encircle the correct option.

(i) Which data takes only some specific values?

(a) continuous data (b)✓ discrete data
(c) grouped data (d) ungrouped data

(ii) The number of times a value occurs in a data is called:

(a) ✓ frequency (b) relative frequency
(b) class limit (d) class boundaries.

(iii) Midpoint is also known as:

(a) mean (b) median
(c) class limit (d)✓ class mark

(iv) Frequency polygon is also drawn /constructed by using:

(a)✓ histogram (b) bar graph
(c) class boundaries (d) class limit

(v) The difference between the greatest value and the smallest value is called:

(a) class limits (b) midpoint
(c) relative frequency (d)✓ range

(vi) Measure of central tendency is used to find out the_______ of a data set.

(a) class boundaries (b) cumulative frequency
(c) ✓ middle or centre value (d) frequency

(vii) If the mean of 5, 7, 8, 9 and x is 7.5, what will be the value of x?

(a) 10 (b) 8 (c)✓ 8.5 (d) 5.8

(viii) Find the mode of the given data: 2, 5, 8, 9, 0, 1, 3, 7 and 10

(a) 5 (b) 7 (c) 0 (d)✓ no mode

(ix) In a data the values (observations) which appears or occurs most often is called:

(a) mean (b) ✓ mode
(c) median (d) weighted mean

(x) Find the median of the given data: 110, 125, 122, 130, 124, 127 and 120

(a)✓ 124 (b) 120 (c) 125 (d) 127

2. Define the following:

(i) frequency distribution (ii) histogram (unequal class limits)

(iii) mean (iv) median

Solution

Frequency Distribution

A distribution of table that represents classes or groups along with their respective class frequencies is called frequency distribution.

Histogram (with unequal class limits)

This is a graph of adjacent rectangles constructed on xy – plane.
In this type class intervals have varying width, and the area of each bar represents the frequency density, calculated by dividing the frequency by the class width.

Mean

It is defined as a value of variables which is obtained by dividing the sum of all the values by their numbers. i.e. $\overline{X} = \frac{\sum x}{n}$

Median

Median is the middle most value in an arranged data. i.e.

$$\tilde{X} = \begin{cases} \left(\frac{n+1}{2}\right)^{th} \text{ term} & ;\textbf{if n is odd} \\ \frac{1}{2}\left(\left(\frac{n}{2}\right)^{th} \text{ term} + \left(\frac{n+2}{2}\right)^{th} \text{ term}\right) & ;\textbf{if n is even} \end{cases}$$

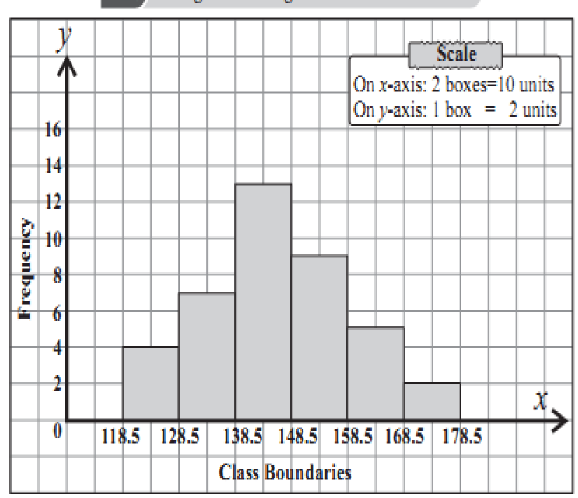
3. Following are the weights of 40 students recorded to the nearest (lbs).
138, 164, 150, 132, 144, 125, 149, 157, 146, 158, 140, 147, 136, 148, 152, 144, 168, 126, 138, 176, 163, 119, 154, 165, 146, 173, 142, 147, 135, 153, 140, 135 161, 145, 135, 142, 150, 156, 145, 128, make a frequency table taking size of class limits as 10. Also draw histogram and frequency polygon of the given data.

Solution

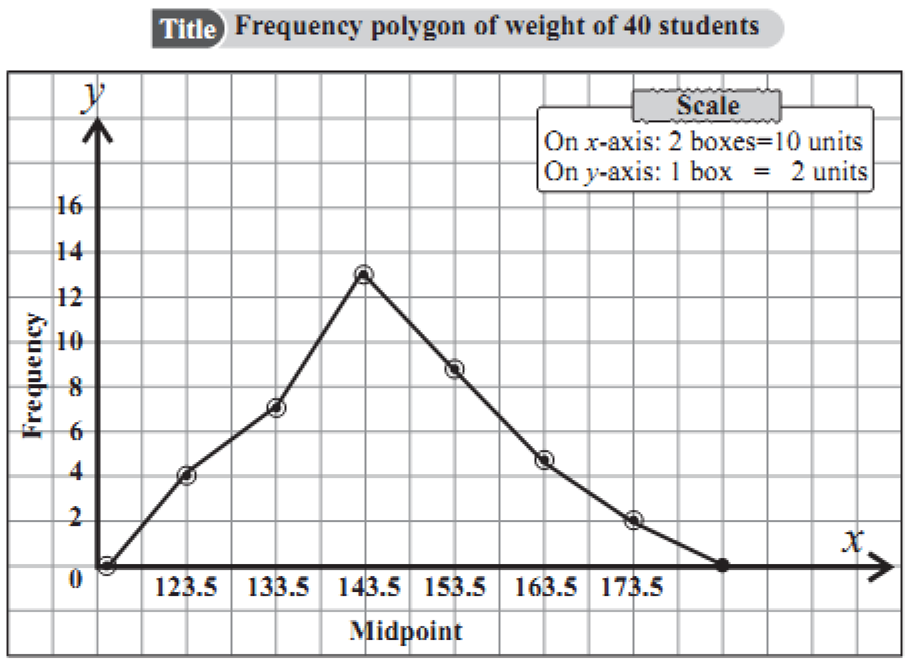
(a) Title: Frequency table taking size of class limits as 10

Class limits	Tally marks	f
119 – 128	IIII	4
129 – 138	卌 II	7
139 – 148	卌 卌 III	13
149 – 158	卌 IIII	9
159 – 168	卌	5
169 – 178	II	2
Total		$\sum f = 40$

(b) Title: Histogram of weight of 40 students



(c)

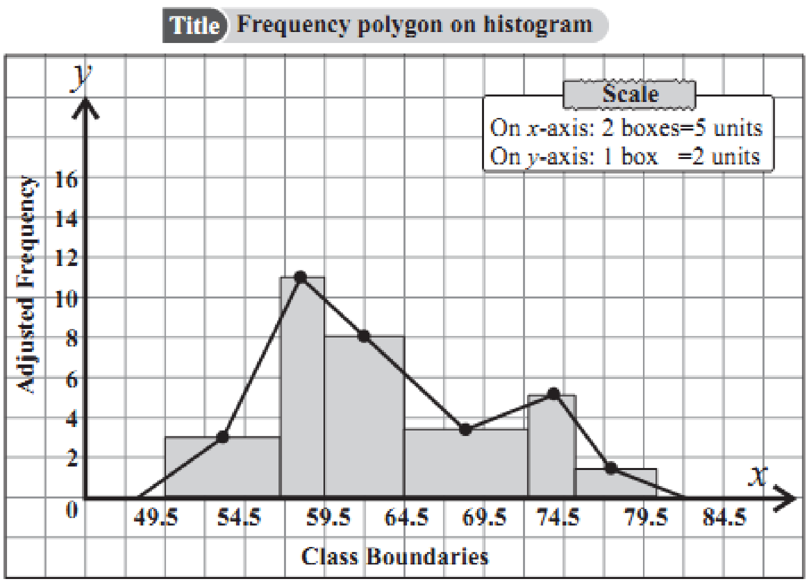


4. From the table given below. Draw a frequency polygon on histogram for the given frequency distribution.

Weight (kg)	50 – 56	57 – 59	60 – 64	65 – 72	73 – 75	76 – 80
Frequency (f)	25	32	40	30	15	8

Solution

Class Boundaries	frequency f	Class size h	Adjusted frequency $= \frac{f}{h}$
49.5 – 56.5	25	7	$\frac{25}{7} = 3.86$
56.5 – 59.5	32	3	$\frac{32}{3} = 10.67$
59.5 – 64.5	40	5	$\frac{40}{5} = 8$
64.5 – 72.5	30	8	$\frac{30}{8} = 3.75$
72.5 – 75.5	15	3	$\frac{15}{3} = 5$
75.5 – 80.5	8	5	$\frac{8}{5} = 1.6$



5. Given below are marks obtained by 45 students in the monthly test of Biology:

Marks	20 – 24	25 – 29	30 – 34	35 –39	40 –44	45 – 49
No. of students	05	08	12	15	03	02

With reference to the above table find the following:

(i) upper class boundary of the 5th class.

(ii) lower class boundaries of all the classes.

(iii) midpoint of all the classes.

(iv) the class interval with the least frequency.

Solution

Marks	Class boundaries	Frequency	Mid-Points
20 – 24	19.5 – 24.5	5	$\frac{20+24}{2} = 22$
25 – 29	24.5 – 29.5	8	$\frac{25+29}{2} = 27$
30 – 34	29.5 – 34.5	12	$\frac{30+34}{2} = 32$
35 – 39	34.5 – 39.5	15	$\frac{35+39}{2} = 37$
40 – 44	39.5 – 44.5	3	$\frac{40+44}{2} = 42$
45 – 49	44.5 – 49.5	2	$\frac{45+49}{2} = 47$

(i) 44 (ii) 19.5, 24.5, 29.5, 34.5, 39.5, 44.5 (iii) 22, 27, 32, 37, 42, 47 (iv) 5

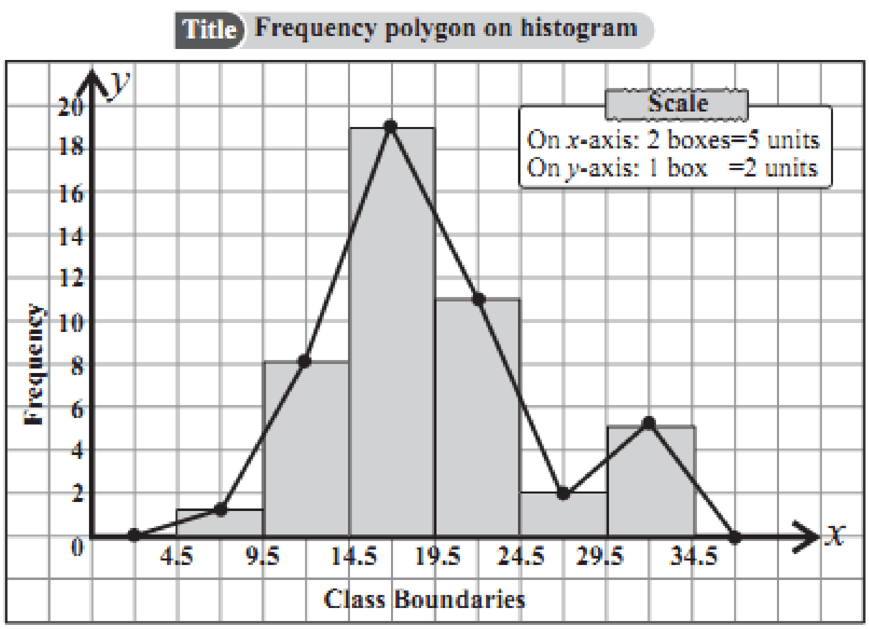
6. Given below is frequency distribution.

Draw frequency polygon and histogram for the distribution.

Class limits	5 – 9	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34
Frequency	1	8	18	11	2	5

Solution

Class Boundaries	Midpoints	Frequency(f)
4.5 – 9.5	7	1
9.5 – 14.5	12	8
14.5 – 19.5	17	18
19.5 – 24.5	22	11
24.5 – 29.5	27	2
29.5 – 34.5	32	5



7. For the following data, find the weighted mean.

Item	Quantity	Cost of item (Rs.)
Chair	20	500
Table	20	400
Black board	10	750
Tube light	25	230
Cupboard	09	950

Solution

Item	Quantity w	Cost of Item x	wx
Chair	20	500	10000
Table	20	400	8000
Black Board	10	750	7500
Tube Light	25	230	5750
Cupboard	9	950	8550
Total	**84**		**39800**

$\overline{X} = \frac{\sum wX}{\sum w} = \frac{39800}{84} = 473.81$ rupees

8. A principal of a school allocates funds of Rs.50, 000 to five different sectors:

(i) chairs: Rs. 15000 (ii) tables: Rs. 12,000

(iii) black boards: Rs.6,000 (iv) room renovation: Rs. 10,000

(v) gardening: Rs. 7,000

Find the average of funds allocation in each sector of the school.

Solution

$\overline{X} = \frac{\sum X}{n} = \frac{15000+12000+6000+10000+7000}{5}$

$\overline{X} = \frac{50000}{5} = \text{Rs.}\,10000$

9. The marks of a student Saad in six tests were 84, 91, 72, 68, 87, 78. Find the arithmetic mean of his marks.

Solution

$\overline{X} = \frac{\sum X}{n} = \frac{84+91+72+68+87+78}{6} = \frac{480}{6} = 80$ marks

10. Adjoining distribution showed maximum load (in kg) supported by certain ropes. Find the mean load using short method.

Max-Load kg	93 – 97	98 – 102	103 – 107	108 – 112	113 – 117	118 – 122
No. of ropes	2	5	8	12	6	2

Solution

Let D = 110

Max Load classes	No. of Ropes f	Midpoint x	$y = x - 110$	fy
93 – 97	2	95	– 15	– 30
98 – 102	5	100	– 10	– 50
103 – 107	8	105	– 5	– 40
108 – 112	12	110	0	0
113 – 117	6	115	5	30
118 – 122	2	120	10	20
Total	**35**			**– 70**

$\overline{Y} = \frac{\sum fY}{\sum f} = \frac{-70}{35} = -2$

$\overline{X} = \overline{Y} + 110 = -2 + 110 = 108$ kg

11. Usman rolled a fair dice eight times. Each time their sum was recorded as 8, 5, 6, 6, 9, 4, 3, 11. Find the median and mode of the sum.

Solution

3,4,5,6,6,8,9,11

Median $= \frac{1}{2}\left(4^{\text{th}} \text{ term} + 5^{\text{th}} \text{ term}\right) = \frac{1}{2}(6+6) = \frac{12}{2} = 6$

Mode = 6 **most repeated term**

12. Two partners Mr. Aslam and Mrs. Kalsoom run a company. In the following data the weekly wages (in Rs.) of employees who work in the company are given:

Wages (Rs.)	600 – 700	700 – 800	800 – 900	900 – 1000	1000 – 1100
Employees	3	5	7	21	11

Find mean, median and mode.

Solution
Mean

Wages Classes	No. of Employees f	Mid point x	fx
600 – 700	3	650	1950
700 – 800	5	750	3750
800 – 900	7	850	5950
900 – 1000	21	950	19950
1000 – 1100	11	1050	11550
Total	**47**		**43150**

$\overline{X} = \frac{\sum fX}{\sum f} = \frac{43150}{47} = 918.09$

Median

Wages (Rs.)	Frequency (f)	C. f
600 – 700	3	3
700 – 800	5	5 + 3 = 8
800 – 900	7	7 + 8 = 15 → c
900 – 1000	21	21 + 15 = 36 Median class
1000 – 1100	11	11 + 36 = 47 → n
Total	$\sum f = 47$	

$\text{Median} = l + \frac{h}{f}\left(\frac{n}{2} - c\right) = 900 + \frac{100}{21}(23.5 - 15) = 940.48$

Mode

Wages (Rs.)	Frequency
600 – 700	3
700 – 800	5
800 – 900	7 → f_1
900 – 1000	21 → f_m, here h = 100
1000 – 1100	11 → f_2
Total	$\sum f = 47$

$\text{Mode} = l + \frac{f_m - f_1}{(f_m - f_1) + (f_m - f_2)} \times h = 900 + \frac{14}{14+10} \times 100 = 958.33$

Unit 13 Probability

EXERCISE 13.1

1. Arshad rolls a dice, with sides labelled L, M, N, O, P, U. What is the probability that the dice lands on consonant?

Solution

$S = \{L, M, N, O, P, U\}$; $n(S) = 6$

$A = \{L, M, N, P\}$; $n(A) = 4$

$P(A) = \frac{n(A)}{n(S)} = \frac{4}{6} = \frac{2}{3}$

2. Shazia throws a pair of fair dice. What will be the probability of getting:

(i) sum of dots is at least 4.

(ii) product of both dots is between 5 to 10.

(iii) the difference between both the dots is equal to 4.

(iv) number at least 5 on the first dice and the number at least 4 on the second dice.

Solution

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$n(S) = 36$

(i) Sum of dots is at least 4
When a pair of fair dice is rolled, the sample space is as follows;
Let A be the even, when Sum of dots is at least 4

A = {(1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3),
(2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3),
(3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3),
(4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3),
(5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3),
(6, 4), (6, 5), (6, 6)}.

$n(A) = 33$
$\text{P(A)} = \frac{n(A)}{n(S)} = \frac{33}{36} = \frac{11}{12}$

(ii) Product of both dots is between 5 to 10
$B = \{(1,5), (1,6), (2,3), (2,4), (2,5), (3,2), (3,3), (4,2), (5,1), (5,2), (6,1)\}$
$n(B) = 11$
$\text{P(B)} = \frac{n(B)}{n(S)} = \frac{11}{36}$

(iii) The difference between both the dots is equal to 4
$C = \{(1,5), (2,6), (5,1), (6,2)\}$
$n(C) = 4$
$\text{P(C)} = \frac{n(C)}{n(S)} = \frac{4}{36} = \frac{1}{9}$

(iv) Number at least 5 on the first dice and the number at least 4 on the second dice
$D = \{(5,4), (5,5), (5,6), (6,4), (6,5), (6,6)\}$
$n(D) = 4$
$\text{P(D)} = \frac{n(D)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

3. One alphabet is selected at random from the word "MATHEMATICS". Find the probability of getting:

(i) vowel	(ii) consonant	(iii) an E
(iv) an A	(v) not M	(vi) not T

Solution
Given word **MATHEMATICS**
Total words are 11
$n(S) = 11$

(i) Vowel	(ii) Consonant
$A = \{A, E, A, I\}$; $n(A) = 4$ $\text{P(A)} = \frac{n(A)}{n(S)} = \frac{4}{11}$	$B = \{M, T, H, M, T, C, S\}$; $n(B) = 7$ $\text{P(B)} = \frac{n(B)}{n(S)} = \frac{7}{11}$
(iii) an E $C = \{E\}$; $n(C) = 1$ $\text{P(C)} = \frac{n(C)}{n(S)} = \frac{1}{11}$	**(iv) an A** $D = \{A, A\}$; $n(D) = 2$ $\text{P(D)} = \frac{n(D)}{n(S)} = \frac{2}{11}$
(v) not M $E = \{A, T, H, E, T, I, C, S\}$; $n(E) = 9$ $\text{P(E)} = \frac{n(E)}{n(S)} = \frac{9}{11}$	**(vi) not T** $F = \{M, A, H, E, M, A, I, C, S\}$; $n(F) = 9$ $\text{P(F)} = \frac{n(F)}{n(S)} = \frac{9}{11}$

4 Aslam rolled a dice. What is the probability of getting the numbers 3 or 4? Also find the probability of not getting the numbers 3 or 4.

Solution

$S = \{1,2,3,4,5,6\}$ $; n(S) = 6$

$A = \{3,4\}$ $; n(A) = 2$

$\text{P(A)} = \frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$

Probability of not gettig 3 or 4 $= \text{P(A}') = 1 - \text{P(A)} = 1 - \frac{1}{3} = \frac{2}{3}$

5 Abdul Hadi labelled cards from 1 to 30 and put them in a box. He selects a card at random. What is the probability that selected card containing:

(i) the number 25
(ii) number between 17 to 22
(iii) number at least 20
(iv) number not 27 and 29
(v) number not between 12 – 15

Solution

Since the cards are labeled from 1 to 30, therefore $n(S) = 30$

(i) $A = \{25\}$ $; n(A) = 1 \Rightarrow \text{P(A)} = \frac{n(A)}{n(S)} = \frac{1}{30}$

(ii) $A = \{17,18,19,20,21,22\}$ $; n(A) = 6 \Rightarrow \text{P(A)} = \frac{n(A)}{n(S)} = \frac{6}{30} = \frac{1}{5}$

(iii) $A = \{20,21,22,23,24,25,26,27,28,29,30\}$ $; n(A) = 11 \Rightarrow \text{P(A)} = \frac{n(A)}{n(S)} = \frac{11}{30}$

(iv) $A = \{27,29\}$ $; n(A) = 2 \Rightarrow \text{P(A)} = \frac{n(A)}{n(S)} = \frac{2}{30} = \frac{1}{15}$

Probability of not gettig 27 and 29 $= \text{P(A}') = 1 - \text{P(A)} = 1 - \frac{1}{15} = \frac{14}{15}$

(v) $A = \{12,13,14,15\}$ $; n(A) = 4 \Rightarrow \text{P(A)} = \frac{n(A)}{n(S)} = \frac{4}{30} = \frac{2}{15}$

Probability of not gettig 12 to 15 $= \text{P(A}') = 1 - \text{P(A)} = 1 - \frac{2}{15} = \frac{13}{15}$

6. The probability that Ayesha will pass the examination is 0.85. What will be the probability that Ayesha will not pass the examination?

Solution

Ayesha will pass the examinition $= \mathrm{P(A)} = 0.85$

Ayesha will not pass the examinition $= \mathrm{P(A')} = 1 - \mathrm{P(A)} = 1 - 0.85 = 0.15$

7 Taabish tossed a fair coin and rolled a fair dice once. Find the probability of the following events:

(i) tail on coin and at least 4 on dice.

(ii) head on coin and the number 2,3 on dice.

(iii) head and tail on coin and the number 6 on dice.

(iv) not tail on coin and the number 5 on dice.

(v) not head on coin and the number 5 and 2 on dice.

Solution

When a fair coin is tossed and fair dice is rolled, the sample space is as follows;

Die / Coin	1	2	3	4	5	6
H	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
T	T, 1	T, 2	T, 3	T, 4	T, 5	T, 6

Sample space = {(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)
(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)}

Therefore $n(S) = 12$

(i) $A = \{(T,4),(T,5),(T,6)\}$ $;n(A) = 3 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{3}{12} = \frac{1}{4}$

(ii) $A = \{(H,2),(H,3)\}$ $;n(A) = 2 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{2}{12} = \frac{1}{6}$

(iii) $A = \{(H,5),(T,5)\}$ $;n(A) = 2 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{2}{12} = \frac{1}{6}$

(iv) $A = \{(T,5)\}$ $;n(A) = 1 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{1}{12}$

$\mathrm{P(A')} = 1 - \mathrm{P(A)} = 1 - \frac{1}{12} = \frac{11}{12}$

(v) $A = \{1(H,5),(H,2)\}$ $;n(A) = 2 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{2}{12} = \frac{1}{6}$

$\mathrm{P(A')} = 1 - \mathrm{P(A)} = 1 - \frac{1}{6} = \frac{5}{6}$

8. A card is selected at random from a well shuffled pack of 52 plying cards. What will be the probability of selecting:

(i) a queen (ii) neither a queen nor a jack

Solution

Since there are 52 playing cards, so $n(S) = 52$

(i) $A = 4$ queen $;n(A) = 4 \Rightarrow \text{P(A)} = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$

(ii) $A = 4$ queen and 4 Jack $;n(A) = 8 \Rightarrow \text{P(A)} = \frac{n(A)}{n(S)} = \frac{8}{52} = \frac{2}{13}$

Probability of selecting neither a queen nor a jack $= \text{P(A}') = 1 - \text{P(A)}$

Probability of selecting neither a queen nor a jack $= \text{P(A}') = 1 - \frac{2}{13} = \frac{11}{13}$

Probability of selecting neither a queen nor a jack $= \text{P(A}') = \frac{11}{13}$

9. A card is chosen at random from a pack of 52 playing cards. Find the probability of getting:

(i) a jack (ii) no diamond

Solution

Since there are 52 playing cards, so $n(S) = 52$

(i) $A = 4$ jack $;n(A) = 4 \Rightarrow \text{P(A)} = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$

(ii) $A = 13$ diamond $;n(A) = 13 \Rightarrow \text{P(A)} = \frac{n(A)}{n(S)} = \frac{13}{52} = \frac{1}{4}$

Probability of selecting no diamond $= \text{P(A}') = 1 - \text{P(A)}$

Probability of selecting no diamond $= \text{P(A}') = 1 - \frac{1}{4}$

Probability of selecting no diamond $= \text{P(A}') = \frac{3}{4}$

EXERCISE 13.2

1. A researcher collected data on number of deaths from Horse-Ricks in Russian Army crops over to years. The table is as follows:

No. of death	0	1	2	3	4	5	6
Frequency	60	50	87	40	32	15	10

Find the relative frequency of the given data.

Solution: using the formula $r.f = \frac{f}{\Sigma f}$

No. of death	f	$r.f.$
0	60	$\frac{60}{294} = \frac{10}{49}$
1	50	$\frac{25}{147}$
2	87	$\frac{29}{98}$
3	40	$\frac{20}{147}$
4	32	$\frac{16}{147}$
5	15	$\frac{5}{98}$
6	10	$\frac{5}{147}$
Total	$\Sigma f = 294$	

2. The frequency of defective products in 750 samples are shown in the following table. Find the relative frequency for the given table.

No. of defectives per sample	0	1	2	3	4	5	6	7	8
No. of sample	120	140	94	85	105	50	40	66	50

Solution: using the formula $\boldsymbol{r.f} = \frac{f}{\sum f}$

No. of defective per sample	f	$r.f.$
0	120	$\frac{4}{25}$
1	140	$\frac{14}{75}$
2	94	$\frac{47}{375}$
3	85	$\frac{17}{150}$
4	105	$\frac{21}{150}$
5	50	$\frac{1}{15}$
6	40	$\frac{4}{75}$
7	66	$\frac{66}{750} = \frac{33}{375}$
8	50	$\frac{1}{15}$
Total	$\sum f = 750$	

3. A quiz competition on general knowledge is conducted. The number of corrected answers out of 5 questions for 100 sets of questions is given below.

X	0	1	2	3	4	5
f	10	23	15	25	18	9

Find the relative frequencies for the given data.

Solution: using the formula $\boldsymbol{r.f} = \frac{f}{\Sigma f}$

X	f	$r.f.$
0	10	$\frac{1}{10}$
1	23	$\frac{23}{100}$
2	15	$\frac{3}{20}$
3	25	$\frac{1}{4}$
4	18	$\frac{9}{50}$
5	09	$\frac{9}{100}$
Total	$\Sigma f = 100$	

4. A survey was conducted from the 50 students of a class and asked about their favourite food. The responses are as under:

Name of food item	Biryani	Fresh Juice	Chicken	Bar. B.Q	Sweets
No. of students	40	07	21	15	25

(i) how many percentages of students like biryani?

(ii) how many percentages of students like chicken?

(iii) which food is the least like by the students?

(iv) which food is the most prefer by the students?

Solution

Total number of students = 50, , so $n(S) = 50$

(i) Relative frequency of students who like biryani $= \frac{40}{50} = 0.8 = 80\%$

(ii) Relative frequency of students who like chicken $= \frac{21}{50} = 0.42 = 42\%$

(iii) Fresh Juice is the least like by the students. i.e. 7 students out of 50.

(iv) Biryani is the most prefer by the students. i.e. 40 students out of 50.

5. In 500 trials of a thrown of two dice, what is expected frequency that the sum will be greater than 8?

Solution

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$$
$$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$$
$$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),$$
$$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$n(S) = 36$

Let A be the event that sum will be greater than 8;

$A = \{(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\}$

$n(A) = 10$

$\mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{10}{36} = \frac{5}{18}$

$\text{Expected Frequency} = \mathrm{E(A)} = \mathrm{N} \times \mathrm{P(A)} = 500 \times \frac{5}{18} = 138.89 \approx 139$

6. What is the expectation of a person who is to get Rs. 120 if he obtains at least 2 heads in single toss of three coins?

Solution

$\mathrm{S} = \{\mathrm{HHH, HTH, HHT, THH, TTH, THT, HTT, TTT}\}$; $n(S) = 8$

$\mathrm{A} = \{\mathrm{HHH, HTH, HHT, THH}\}$; $n(S) = 4$

$\mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

$\text{Expected Frequency} = \mathrm{E(A)} = \mathrm{N} \times \mathrm{P(A)} = 120 \times \frac{1}{2} = 60$

7. Find the expected frequencies of the given data if the experiment is repeated 200 times.

x	0	1	2	3	4	5	6
$P(x)$	0.11	0.21	0.17	0.18	0.09	0.17	0.07

Solution: using the formula $\mathbf{EF} = \mathbf{P(X)} \times \mathbf{200}$

X	0	1	2	3	4	5	6
$P(X)$	0.11	0.21	0.17	0.18	0.09	0.17	0.07
Expected Frequency	22	42	34	36	18	34	14

8. The probability of getting 5 sixes while tossing six dice is $\frac{2}{5}$. How many times would you expect it to show 5 sixes?

Solution

$\mathrm{N} = 200$; $\mathrm{P(A)} = \frac{2}{5}$

$\text{Expected Frequency} = \mathrm{E(A)} = \mathrm{N} \times \mathrm{P(A)} = 200 \times \frac{2}{5} = 80 \text{ times}$

REVIEW EXERCISE 13

1. Four options are given against each statement. Encircle the correct option.

(i) Each element of the sample space is called:

(a) event (b) experiment
(c) ✓ sample point (d) outcomes

(ii) An outcome which represents how many times we expect the things to be happened is called:

(a) outcomes (b) ✓ favourable outcome
(c) sample space (d) sample point

(iii) Which one tells us how often a specific event occurs relative to the total number of frequency event or trials?

(a) expected frequency (b) sum of relative frequency
(c) ✓ relative frequency (d) frequency

(iv) Estimated probability of an event occurring is also known as:

(a) ✓ relative frequency (b) expected frequency
(c) class boundaries (d) sum of expected frequency

(v) The sum of all expected frequencies is equal to the fixed number of:

(a) ✓ trials (b) relative frequencies
(c) outcomes (d) events

(vi) The chance of occurrence of a particular event is called:

(a) sample space (b) estimated probability
(c) ✓ probability (d) expected frequency

(vii) An event which will probably occur. It has greater chance to occur is called:

(a) equally likely event (b) ✓ likely event
(c) unlikely event (d) certain event

(viii) Find out the total number of possible sample space when 4 dice are rolled.:

(a) 6^2 (b) 6^3 (c) ✓ 6^4 (d) 6^6

(ix) While rolling a pair of dice, what will be the probability of double 2?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{5}{6}$ (d)✓ $\frac{1}{36}$

(x) A card is chosen from a pack of 52 playing cards, find the probability of getting no jack and king:

(a) $\frac{2}{13}$ (b)✓ $\frac{11}{13}$ (c) $\frac{2}{52}$ (d) $\frac{11}{52}$

2. Define the following:

(i) relative frequency (ii) expected frequency

Solution

Relative Frequency: Relative Frequency is an estimated probability of an event occurring when an experiment is repeated a fixed number of times.

Expected Frequency: Expected Frequency is a measure that estimates how often an event should be occur depended on probability. Expected frequency is found by using the following method;

Expected Frequency = Total number of trials × Probability of an event

Expected Frequency = E(A) = N × P(A)

3. An urn contains 10 red balls, 5 green balls and 8 blue balls. Find the probability of selecting at random.

(i) a green ball (ii) a red ball (iii) a blue ball

(iv) not a red ball (v) not a green ball

Solution

Since total balls are 23, therefore $n(S) = 10 + 5 + 8 = 23$

(i) $A = \text{Green balls} \quad ; n(A) = 5 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{5}{23}$

(ii) $A = \text{Red balls} \quad ; n(A) = 10 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{10}{23}$

(iii) $A = \text{Blue balls} \quad ; n(A) = 8 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{8}{23}$

(iv) $A = \text{Red balls} \quad ; n(A) = 10 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{10}{23}$

Probability of not gettig a red ball $= \mathrm{P(A')} = 1 - \mathrm{P(A)} = 1 - \frac{10}{23} = \frac{13}{23}$

(v) $A = \text{Green balls} \quad ; n(A) = 5 \Rightarrow \mathrm{P(A)} = \frac{n(A)}{n(S)} = \frac{5}{23}$

Probability of not gettig a green ball $= \mathrm{P(A')} = 1 - \mathrm{P(A)} = 1 - \frac{5}{23} = \frac{18}{23}$

4. Three coins are tossed together. what is the probability of getting:

(i) exactly three heads

(ii) at least two tails

(iii) not at least two heads

(iv) not exactly two heads

Solution

$S = \{HHH, HTH, HHT, THH, TTH, THT, HTT, TTT\}$; $n(S) = 8$

(i) $A = \{HHH\}$; $n(A) = 1 \Rightarrow$ $P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$

(ii) $A = \{TTH, THT, HTT, TTT\}$; $n(A) = 4 \Rightarrow$ $P(A) = \frac{n(A)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

(iii) $A = \{HHH, HTH, HHT, THH\}$; $n(A) = 4 \Rightarrow$ $P(A) = \frac{n(A)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

Probability of not gettig at least two Heads $= P(A') = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$

(iv) $A = \{HTH, HHT, THH\}$; $n(A) = 3 \Rightarrow$ $P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$

Probability of not gettig exactly two Heads $= P(A') = 1 - P(A) = 1 - \frac{3}{8} = \frac{5}{8}$

5. A card is drawn from a well shuffled pack of 52 playing cards. What will be the probability of getting:

(i) king or jack of red colour

(ii) not "2" of club and spade

Solution

Since there are 52 playing cards, so $n(S) = 52$

(i) $A =$ king or jack of red colour ; $n(A) = 2 + 2 = 4 \Rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$

(ii) $A = 2$ of club and spade ; $n(A) = 2 \Rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{2}{52} = \frac{1}{26}$

Probability of selecting not 2 of club and spade $= P(A') = 1 - P(A)$

Probability of selecting not 2 of club and spade $= P(A') = 1 - \frac{1}{26} = \frac{25}{26}$

Probability of selecting not 2 of club and spade $= P(A') = \frac{25}{26}$

6. Six coins are tossed 600 times. The number of occurrence of tails are recorded and shown in the table given below:

No. of tails	0	1	2	3	4	5	6
Frequency	110	90	105	80	76	123	16

Find the relative frequency of given table.

Solution: using the formula $\boldsymbol{r.f} = \frac{\boldsymbol{f}}{\sum \boldsymbol{f}}$

No. of tails	0	1	2	3	4	5	6	Total
f	110	90	105	80	76	123	16	$\Sigma f = 600$
Relative Frequency	$\frac{11}{60}$	$\frac{3}{20}$	$\frac{7}{40}$	$\frac{2}{15}$	$\frac{19}{150}$	$\frac{41}{200}$	$\frac{2}{75}$	

7. From a lot containing 25 items, 8 items are defective. Find the relative frequency of non-defective items, also find the expected frequency of non-defective items.

Solution

Total Items = 25

Defective Items = 8

Non – Defective Items = 25 – 8 = 17

So expected frequency of Non – Defective Items is 17

$$\text{Relative Frequency} = \frac{\text{frequency of Non – Defective Items}}{\text{Total Items}}$$

$$\text{Relative Frequency} = \frac{17}{25} = 0.68$$

حرفِ آخر (17-05-2025)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالٰی آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (اٰمین)

محمد عثمان حامد

چک نمبر 105 شمالی (گودھے والا) سرگودھا

PUNJAB, PAKISTAN