Algebraic Function:
A function $y = f(x)$ is algebraic if it is obtained from x using a finite number of algebraic operations such as addition, subtraction, multiplication, division, and nth roots.
e.g.
$f(x) = 2x^2 - 3x$, $g(x) = \frac{x^2+1}{x-3}$, $h(x) = (x+1)^{\frac{2}{3}}$, $k(x) = \sqrt{x+1}$

Transcendental Function:
A function is called a transcendental function if it is not algebraic. $y = e^x$, $y = \ln x$, $y = \sin x$, $y = sin^{-1}x$, $y = \sin x + \ln x$, $y = e^{x^2}$ are examples of transcendental functions.

Fundamental transcendental Function:
Fundamental transcendental functions are the commonly used standard transcendental functions like $y = a^x$, $y = \ln x$, $y = \sin x$, $y = sin^{-1}x$, which form the base for more complex transcendental functions.

Non-Fundamental transcendental Function:
Functions formed by combining transcendental functions with algebraic functions, or by applying transformations and composition, are non-fundamental transcendental functions.
$y = e^x$, $y = x\sin x$, $y = \ln 2x$, $y = e^{x^2}$ are examples of non-fundamental transcendental functions.

EXERCISE # 1.2

Q.1: Which of the following functions are algebraic?

Algebraic Functions are:

$g(x) = \frac{2x-3}{x+1}$, $h(x) = \sqrt{x^2+2x}$, $r(x) = (x-1)^3$

Non-Algebraic Functions:

$f(x) = \sec x$, $k(x) = \frac{e^x - e^{-x}}{2}$.

Q.2: Which of the following functions are transcendental?

Solution: Transcendental Functions are:

$g(x) = cot\ x$, $h(x) = e^{2x} + \sqrt{x^2+2x}$, $k(x) = \frac{e^x+e^{-x}}{2}$.

Q.3: Which of the following functions are fundamental functions?

Solution: Fundamental Transcendental Functions:

$g(x) = sin\ x$, $h(x) = e^{2x}$, $k(x) = \frac{e^x+e^{-x}}{2}$, $r(x) = 15^x$.

Q.4: Which of the following functions are non-fundamental functions?

Solution: Non-Fundamental Transcendental Functions are:
$f(x) = e^{2x}$, $k(x) = \cos x^2$, $r(x) = 2^{-x}$.

Q.5: Discuss domain, range, and graph of the function $f(x) = 3^x$.

Solution: Domain:
Since the given function is defined for all values of x therefore,
Domain of $f(x) = (-\infty, \infty)$.

Range:
Range of $f(x) = [0, +\infty)$

Graph:

x	-3	-2	-1	0	1	2	3
$f(x)$	0.0370	0.1111	0.3333	1	3	9	27

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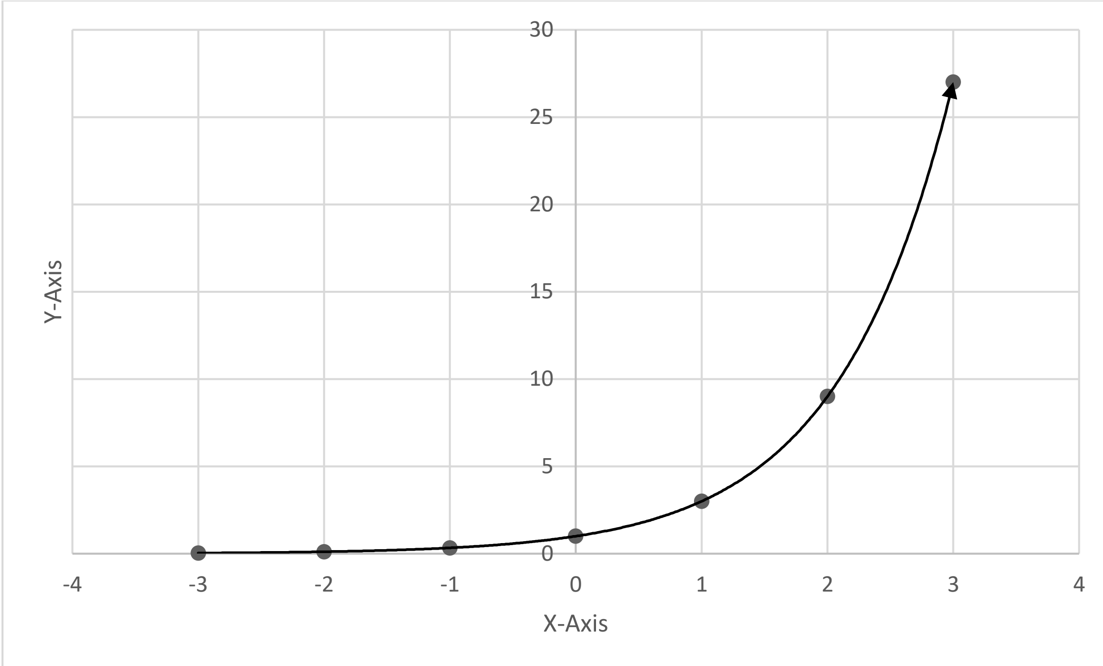


Q.6: Discuss domain, range, and graph of the function $f(x)=\left(\frac{1}{3}\right)^{x}$.

Solution: Domain:

Since the given function is defined for all values of x therefore,

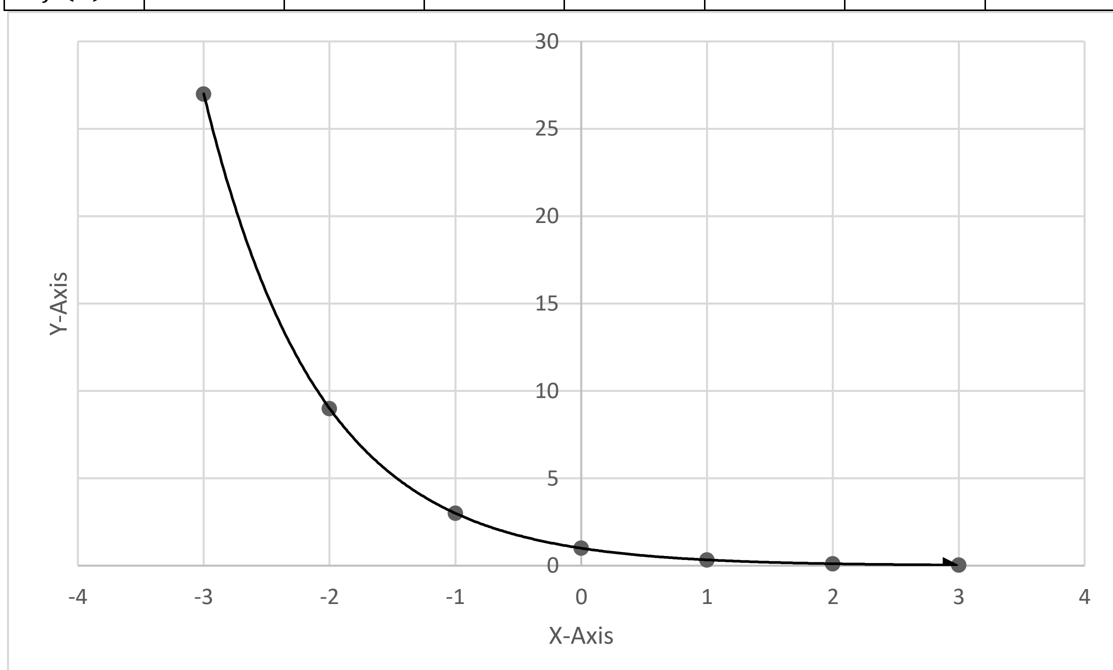
Domain of $f(x)=(-\infty,\infty)$.

Range:

Range of $f(x)=(0,+\infty)$

Graph:

x	-3	-2	-1	0	1	2	3
$f(x)$	27	9	3	1	0.3333	0.1111	0.0370



Q.7: Prove the following hyperbolic identities:

(i) $1 - tanh^2x = sech^2x$

$$L.H.S. = 1 - \left(\frac{e^x - e^{-x}}{e^x - e^{-x}}\right)^2$$
$$L.H.S. = 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$
$$L.H.S. = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$
$$L.H.S. = \frac{e^{2x} + e^{-2x} + 2e^{2x}e^{-2x} - (e^{2x} + e^{-2x} - 2e^{2x}e^{-2x})}{(e^x + e^{-x})^2}$$
$$L.H.S. = \frac{e^{2x} + e^{-2x} + 2e^{2x}e^{-2x} - e^{2x} - e^{-2x} + 2e^{2x}e^{-2x}}{(e^x + e^{-x})^2}$$
$$L.H.S. = \frac{4e^0}{(e^x + e^{-x})^2}$$
$$L.H.S. = \frac{4(1)}{(e^x + e^{-x})^2}$$
$$L.H.S. = \frac{4}{(e^x + e^{-x})^2}$$
$$L.H.S. = \left(\frac{2}{e^x + e^{-x}}\right)^2$$
$$L.H.S. = sech^2x = R.H.S.$$

(ii) $csch^2x = coth^2x - 1$

$$R.H.S. = coth^2x - 1$$
$$R.H.S. = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - 1$$
$$R.H.S. = \frac{(e^x + e^{-x})^2}{(e^x - e^{-x})^2} - 1$$
$$R.H.S. = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x - e^{-x})^2}$$
$$R.H.S. = \frac{e^{2x} + e^{-2x} + 2e^{2x}e^{-2x} - (e^{2x} + e^{-2x} - 2e^{2x}e^{-2x})}{(e^x - e^{-x})^2}$$
$$R.H.S. = \frac{e^{2x} + e^{-2x} + 2e^{2x}e^{-2x} - e^{2x} - e^{-2x} + 2e^{2x}e^{-2x}}{(e^x - e^{-x})^2}$$
$$R.H.S. = \frac{4e^0}{(e^x - e^{-x})^2}$$
$$R.H.S. = \frac{4(1)}{(e^x - e^{-x})^2}$$
$$R.H.S. = \frac{4}{(e^x - e^{-x})^2}$$
$$R.H.S. = \left(\frac{2}{e^x - e^{-x}}\right)^2$$
$$R.H.S. = csch^2x = L.H.S.$$

(iii) $\sinh 2x = 2\sinh x \cosh x$

Solution:

$$R.H.S. = 2\sinh x \cosh x$$

$$R.H.S. = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$
$$R.H.S. = \frac{(e^x - e^{-x})(e^x + e^{-x})}{2}$$
$$R.H.S. = \frac{(e^x)^2 - (e^{-x})^2}{2}$$
$$R.H.S. = \frac{e^{2x} - e^{-2x}}{2}$$
$$R.H.S. = \sinh 2x$$
$$R.H.S. = L.H.S.$$

(iv) $\cosh 2x = cosh^2 x + sinh^2 x$

Solution:

$$R.H.S. = cosh^2 x + sinh^2 x$$
$$R.H.S. = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$R.H.S. = \frac{(e^x + e^{-x})^2}{4} + \frac{(e^x - e^{-x})^2}{4}$$
$$R.H.S. = \frac{e^{2x} + e^{-2x} + 2e^x e^{-x}}{4} + \frac{e^{2x} + e^{-2x} - 2e^x e^{-x}}{4}$$
$$R.H.S. = \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} + e^{2x} + e^{-2x} - 2e^x e^{-x}}{4}$$
$$R.H.S. = \frac{2e^{2x} + 2e^{-2x}}{4}$$
$$R.H.S. = \frac{2(e^{2x} + e^{-2x})}{4}$$
$$R.H.S. = \frac{e^{2x} + e^{-2x}}{2}$$
$$R.H.S. = \cosh 2x = L.H.S.$$

Q.8: Prove the following inverse hyperbolic formulae.

(i) $\boldsymbol{tanh^{-1}x = \frac{1}{2} ln\left(\frac{1+x}{1-x}\right)}$

Solution: Let,

$$y = tanh^{-1}x$$
$$\tanh y = x$$
$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x$$
$$e^y - e^{-y} = x(e^y + e^{-y})$$
$$e^y - e^{-y} = xe^y + xe^{-y}$$
$$e^y - \frac{1}{e^y} = xe^y + \frac{x}{e^y}$$
$$\frac{e^{2y} - 1}{e^y} = \frac{xe^{2y} + x}{e^y}$$
$$e^{2y} - 1 = xe^{2y} + x$$
$$e^{2y} - 1 - xe^{2y} - x = 0$$
$$e^{2y} - xe^{2y} - 1 - x = 0$$
$$(1-x)e^{2y} - 1 - x = 0$$
$$(1-x)e^{2y} = 1 + x$$
$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \ln\left(\frac{1+x}{1-x}\right)$$
$$2y = \ln\left(\frac{1+x}{1-x}\right)$$
$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
$$tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

(ii) $\boldsymbol{coth^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$

Solution: Let,

$$y = coth^{-1}x$$
$$\coth y = x$$
$$\frac{e^y + e^{-y}}{e^y - e^{-y}} = x$$
$$e^y + e^{-y} = x(e^y - e^{-y})$$
$$e^y + e^{-y} = xe^y - xe^{-y}$$
$$e^y + \frac{1}{e^y} = xe^y - \frac{x}{e^y}$$
$$\frac{e^{2y}+1}{e^y} = \frac{xe^{2y}-x}{e^y}$$
$$e^{2y} + 1 = xe^{2y} - x$$
$$e^{2y} + 1 - xe^{2y} + x = 0$$
$$e^{2y} - xe^{2y} + 1 + x = 0$$
$$(1-x)e^{2y} = -1 - x$$
$$e^{2y} = -\frac{(1+x)}{(1-x)}$$
$$e^{2y} = \frac{x+1}{x-1}$$
$$2y = \ln\left(\frac{x+1}{x-1}\right)$$
$$y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$
$$coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

(iii) $\boldsymbol{sech^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}$

Solution: Let,

$$y = sech^{-1}x$$
$$\text{sech}\, y = x$$
$$\frac{2}{e^y + e^{-y}} = x$$
$$2 = x(e^y + e^{-y})$$
$$2 = xe^y + xe^{-y}$$
$$2 = xe^y + \frac{x}{e^y}$$
$$2 = \frac{xe^{2y} + x}{e^y}$$
$$2e^y = xe^{2y} + x$$
$$0 = xe^{2y} + x - 2e^y$$

$$xe^{2y} - 2e^y + x = 0$$
$$e^y = \frac{-(-2) \pm \sqrt{(2)^2 - 4(x)(x)}}{2(x)}$$
$$e^y = \frac{2 \pm \sqrt{4 - 4x^2}}{2x}$$
$$e^y = \frac{2 \pm \sqrt{4(1 - x^2)}}{2x}$$
$$e^y = \frac{2 \pm 2\sqrt{1 - x^2}}{2x}$$
$$e^y = \frac{2\left(1 \pm \sqrt{1 - x^2}\right)}{2x}$$
$$e^y = \frac{1 \pm \sqrt{1 - x^2}}{x}$$
$$y = \ln\left[\frac{1 \pm \sqrt{1 - x^2}}{x}\right]$$
$$sech^{-1}x = \ln\left[\frac{1 \pm \sqrt{1 - x^2}}{x}\right]$$

(iv) $\boldsymbol{csch^{-1}x = \ln\left[\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right]}$

Solution: Let,

$$y = csch^{-1}x$$
$$\operatorname{csch} y = x$$
$$\frac{2}{e^y - e^{-y}} = x$$
$$2 = x(e^y - e^{-y})$$
$$2 = xe^y - xe^{-y}$$
$$2 = xe^y - \frac{x}{e^y}$$
$$2 = \frac{xe^{2y} - x}{e^y}$$
$$2e^y = xe^{2y} - x$$
$$0 = xe^{2y} - x - 2e^y$$
$$xe^{2y} - 2e^y - x = 0$$
$$e^y = \frac{-(-2) \pm \sqrt{(2)^2 - 4(x)(-x)}}{2(x)}$$
$$e^y = \frac{2 + \sqrt{4 + 4x^2}}{2x}$$
$$e^y = \frac{2 + \sqrt{4(1 + x^2)}}{2x}$$
$$e^y = \frac{2 + 2\sqrt{1 + x^2}}{2x}$$
$$e^y = \frac{2\left(1 + \sqrt{1 + x^2}\right)}{2x}$$
$$e^y = \frac{1 + \sqrt{1 + x^2}}{x}$$

$$y = \ln\left[\frac{1}{x} + \frac{\sqrt{1+x^2}}{x}\right]$$

$$y = \ln\left[\frac{1}{x} + \sqrt{\frac{1+x^2}{x^2}}\right]$$

$$y = \ln\left[\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right]$$

$$csch^{-1}x = \ln\left[\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right]$$