Exc 5.3

Vector Valued Function

A vector valued function is an function where the domain is the subset of real numbers and the range is a vector.
In three dimensions

$$r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

where x, y and z are the functions of 't' or may be constants.

Exp:

$$\vec{r}(t) = 3\hat{i} + 7\hat{j} + t\hat{k}$$

This is vector valued function or position vector in three dimensions.

Domain and Range of vector valued function.

Let $\vec{r}(t) = t\hat{i} + (t+1)\hat{j} + 5t\hat{k}$

Domain: Put $t = 1$ $\quad 1 \in R$

$r(1) = \hat{i} + 2\hat{j} + 5\hat{k}$

Here

$t = 1$ Domain

$r(t)$ or $r(1)$ is range.

Q #1 If $r(t)$ is the position of the Particle. Find its domain and the range at given point. Also Find its 1st and 2nd derivative.

i) $\vec{r}(t) = (t+1)\hat{i} + (t^2-1)\hat{j} + 5t\hat{k}$ — (i), $t=2$

- **Domain** = $\boxed{(-\infty, +\infty)}$

- **Range** = At $t=2$

  Put $t=2$ in (i)

  $$\boxed{r(2) = 3\hat{i} + 3\hat{j} + 10\hat{k}}$$

- **First derivative At $t=2$**

  $$r'(t) = 1\hat{i} + 2t\hat{j} + 5\hat{k}$$

  $$\boxed{r'(2) = \hat{i} + 4\hat{j} + 5\hat{k}}$$

- **Second derivative At $t=2$**

  $$r''(t) = 0\hat{i} + 2\hat{j} + 0\hat{k}$$

  $$\boxed{r''(2) = 0\hat{i} + 2\hat{j} + 0\hat{k}}$$

(ii) $\vec{r}(t) = \frac{t}{t+1}\hat{i} + \frac{1}{t}\hat{j} + t^3\hat{k}$ —(i) $\quad t = -\frac{1}{2}$

• **Domain** =

$$\text{Dom } r(t) = \boxed{R - \{-1, 0\}}$$

• **Range** = At $t = -\frac{1}{2}$

Put $t = -\frac{1}{2}$ in (i)

$$r\left(-\frac{1}{2}\right) = \frac{-1/2}{-\frac{1}{2}+1}\hat{i} + \frac{1}{-1/2}\hat{j} + \left(-\frac{1}{2}\right)^3\hat{k}$$

$$= \frac{-1}{\cancel{2}} \times \frac{\cancel{2}}{1}\hat{i} + \frac{1}{1} \times -\frac{2}{1}\hat{j} - \frac{1}{8}\hat{k}$$

$$\boxed{r\left(-\frac{1}{2}\right) = -\hat{i} - 2\hat{j} - \frac{1}{8}\hat{k}}$$

• **First derivative At $t = -\frac{1}{2}$**

$$r'(t) = \frac{(t+1)\frac{d}{dt}t - t\frac{d}{dt}(t+1)}{(t+1)^2}\hat{i} + \frac{d}{dt}t^{-1}\hat{j} + \frac{d}{dt}t^3\hat{k}$$

$$r'(t) = \frac{(t+1)\cdot 1 - t\cdot(1+0)}{(t+1)^2}\hat{i} + -1\,t^{-1-1}\hat{j} + 3t^{3-1}\hat{k}$$

$$r'(t) = \frac{\cancel{t}+1-\cancel{t}}{(t+1)^2}\hat{i} - \frac{1}{t^2}\hat{j} + 3t^2\hat{k}$$

$$r'\left(\frac{-1}{2}\right) = \frac{1}{\left(\frac{-1}{2}+1\right)^2}\hat{i} - \frac{1}{\left(\frac{-1}{2}\right)^2}\hat{j} + 3\left(\frac{-1}{2}\right)^2\hat{k}$$

$$r'\left(-\frac{1}{2}\right) = \frac{1}{(1/2)^2}\hat{i} - \frac{1}{(-1/2)^2}\hat{j} + 3\left(\frac{-1}{2}\right)^2\hat{k}$$

$$r'\left(-\frac{1}{2}\right) = \frac{1}{(1/4)}\hat{i} - \frac{1}{(1/4)}\hat{j} + \frac{3}{4}\hat{k}$$

$$\boxed{r'\left(\frac{-1}{2}\right) = 4\hat{i} - 4\hat{j} + \frac{3}{4}\hat{k}}$$

- **Second Derivative: At $t = \frac{-1}{2}$**

As

$$r'(t) = \frac{1}{(t+1)^2}\hat{i} - \frac{1}{t^2}\hat{j} + 3t^2\hat{k}$$

Again Diff.

$$r''(t) = \frac{d}{dt}(t+1)^{-2}\hat{i} - \frac{d}{dt}t^{-2}\hat{j} + 3\frac{d}{dt}t^2$$

$$= -2(t+1)^{-3}\hat{i} - (-2)t^{-3}\hat{j} + 3\cdot 2t$$

$$r''(t) = \frac{-2}{(t+1)^3}\hat{i} + \frac{2}{t^3}\hat{j} + 6t$$

$$r''\left(\frac{-1}{2}\right) = \frac{-2}{\left(\frac{-1}{2}+1\right)^3}\hat{i} + \frac{2}{(-1/2)^3} + 6\cdot\left(\frac{-1}{2}\right)$$

$$r''\left(\frac{-1}{2}\right) = \frac{-2}{+\frac{1}{8}}\hat{i} + \frac{2}{\frac{-1}{8}} - 3$$

$= \frac{-2}{1} \times \frac{+8}{1} \hat{i} + \frac{2}{1} \times \frac{8}{-1} \hat{j} - 3\hat{k}$

$(\frac{1}{1}) = \boxed{-16\hat{i} - 16\hat{j} - 3\hat{k}}$

(iii) $r(t) = e^{t}\hat{i} + \frac{2}{9} e^{2t}\hat{j} + 5e^{-t}\hat{k} \qquad t = \ln 2$

Sol

- Domain = $\boxed{(-\infty, +\infty)}$

- Range = At $t = \ln 2$

$r(\ln 2) = e^{\ln 2}\hat{i} + \frac{2}{9} e^{2\ln 2}\hat{j} + 5e^{-\ln 2}\hat{k}$

$= e^{\ln 2}\hat{i} + \frac{2}{9} e^{\ln 2^{2}}\hat{j} + 5e^{\ln 2^{-1}}\hat{k}$

$= 2\hat{i} + \frac{2}{9} \cdot 4\hat{j} + 5 \cdot 2^{-1}\hat{k}$

$\boxed{r(\ln 2) = 2\hat{i} + \frac{8}{9}\hat{j} + \frac{5}{2}\hat{k}}$

- First Derivative At $t = \ln 2$

$r(t) = e^{t}\hat{i} + \frac{2}{9} e^{2t}\hat{j} + 5e^{-t}\hat{k}$

$r'(t) = e^{t}\hat{i} + \frac{2}{9} e^{2t} \cdot 2\hat{j} + 5e^{-t} \cdot -1\hat{k}$

$$r'(t) = e^{t}\hat{i} + \frac{4}{9}e^{2t}\hat{j} - 5e^{-t}\hat{k}$$

$$r'(\ln 2) = e^{\ln 2}\hat{i} + \frac{4}{9}e^{2\ln 2}\hat{j} - 5e^{-\ln 2}\hat{k}$$

$$= \cancel{e}^{\cancel{\ln} 2}\hat{i} + \frac{4}{9}\cancel{e}^{\cancel{\ln} 2^{2}}\hat{j} - 5\cancel{e}^{\cancel{\ln} 2^{-1}}\hat{k}$$

$$= 2\hat{i} + \frac{4}{9}\cdot 4\hat{j} - 5\cdot 2^{-1}\hat{k}$$

$$\boxed{r'(\ln 2) = 2\hat{i} + \frac{16}{9}\hat{j} - \frac{5}{2}\hat{k}}$$

- **Second derivative At $t = \ln 2$**

As

$$r'(t) = e^{t}\hat{i} + \frac{4}{9}e^{2t}\hat{j} - 5e^{-t}\hat{k}$$

Again Diff

$$r''(t) = e^{t}\hat{i} + \frac{4}{9}e^{2t}\cdot 2\hat{j} - 5e^{-t}\cdot -1\hat{k}$$

$$r''(t) = e^{t}\hat{i} + \frac{8}{9}e^{2t}\hat{j} + 5e^{-t}\hat{k}$$

$$r''(\ln 2) = e^{\ln 2}\hat{i} + \frac{8}{9}e^{2\ln 2}\hat{j} + 5e^{-\ln 2}\hat{k}$$

$$r''(\ln 2) = \cancel{e}^{\cancel{\ln} 2}\hat{i} + \frac{8}{9}\cancel{e}^{\cancel{\ln} 2^{2}}\hat{j} + 5\cancel{e}^{\cancel{\ln} 2^{-1}}\hat{k}$$

$$r''(\ln 2) = 2\hat{i} + \frac{8}{9}\cdot 4\hat{j} + 5\cdot 2^{-1}\hat{k}$$

$$\boxed{r''(\ln 2) = 2\hat{i} + \frac{32}{9}\hat{j} + \frac{5}{2}\hat{k}}$$

(iv) $r(t) = (\cos 2t)\hat{i} + (3\sin 2t)\hat{j} + 5t\hat{k}$ $t = 0$

- **Domain =** $\boxed{(-\infty, +\infty)}$

- **Range = At $t = 0$**

$r(0) = (\cos 0)\hat{i} + (3\sin 0)\hat{j} + 5(0)\hat{k}$

$$\boxed{r(0) = \hat{i} + 0\hat{j} + 0\hat{k}}$$

- **First derivative at $t = 0$.**

$r'(t) = -\sin 2t \cdot 2\hat{i} + 3 \cdot \cos 2t \cdot 2\hat{j} + 5\hat{k}$

$r'(t) = -2\sin 2t\hat{i} + 6\cos 2t\hat{j} + 5\hat{k}$

$r'(0) = -2\sin 0\hat{i} + 6\cos 0\hat{j} + 5\hat{k}$

$$\boxed{r'(0) = 0\hat{i} + 6\hat{j} + 5\hat{k}}$$

- **Second derivative at $t = 0$**

As

$r'(t) = -2\sin 2t\,\hat{i} + 6\cos 2t\,\hat{j} + 5\hat{k}$

Again Diff

$r''(t) = -2\cos 2t \cdot 2\,\hat{i} - 6\sin 2t \cdot 2\,\hat{j} + 0\hat{k}$

$r''(t) = -4\cos 2t\,\hat{i} - 12\sin 2t\,\hat{j} + 0\,\hat{k}$

$r''(0) = -4\cos 0\,\hat{i} - 12\sin 0\,\hat{j} + 0\,\hat{k}$

$\boxed{r''(0) = -4\hat{i} - 0\hat{j} + 0\hat{k}}$

Q #2

Find the velocity and acceleration of the Function at $t=0$

i) $r(t) = (3t+1)\hat{i} + \sqrt{3}\,t\,\hat{j} + t^2\hat{k}$

Sol

To find the velocity and acceleration, we differentiate the given Function.

Velocity

- $V = \dfrac{d}{dt}\,r(t)$

$V = r'(t) = \dfrac{d}{dt}(3t+1)\hat{i} + \dfrac{d}{dt}\sqrt{3}\,t\,\hat{j} + \dfrac{d}{dt}t^2\hat{k}$

$V = r'(t) = 3\hat{i} + \sqrt{3}\,\hat{j} + 2t\,\hat{k}$

$V = r'(0) = 3\hat{i} + \sqrt{3}\hat{j} + 2(0)\hat{k}$

$\boxed{V = 3\hat{i} + \sqrt{3}\hat{j} + 0\hat{k}}$

Acceleration

- $a(t) = \frac{dv}{dt}$

As $V = 3\hat{i} + \sqrt{3}\hat{j} + 2t\hat{k}$

$r''(t) = a(t) = \frac{dv}{dt} = \frac{d}{dt} 3\hat{i} + \frac{d}{dt}\sqrt{3} j + 2\frac{d}{dt} t\hat{k}$

$a(t) = 0\hat{i} + 0\hat{j} + 2\hat{k}$

$\boxed{a(0) = 2\hat{k}}$

---

ii) $r(t) = \left(\frac{t}{\sqrt{2}}\right)\hat{i} + \left(\frac{t}{\sqrt{2}} - 16t^2\right)\hat{j} - 2t\hat{k}$

Velocity

- $V = \frac{d}{dt} r(t)$

$V = r'(t) = \frac{d}{dt}\left(\frac{t}{\sqrt{2}}\right)\hat{i} + \frac{d}{dt}\left(\frac{t}{\sqrt{2}} - 16t^2\right)\hat{j} - \frac{d}{dt} 2t\hat{k}$

$$V = \frac{1}{\sqrt{2}} \frac{d}{dt} t \,\hat{i} + \left[\frac{1}{\sqrt{2}} \frac{d}{dt} t - 16 \frac{d}{dt} t^2\right] \hat{j} - 2 \frac{d}{dt} t \,\hat{k}$$

$$V = \frac{1}{\sqrt{2}} \hat{i} + \left(\frac{1}{\sqrt{2}} - 32t\right) \hat{j} + 2 \hat{k}$$

$$V = r'(0) = \frac{1}{\sqrt{2}} \hat{i} + \left(\frac{1}{\sqrt{2}} - 32(0)\right) \hat{j} + 2 \hat{k}$$

$$\boxed{V = r'(0) = \frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} + 2 \hat{k}}$$

Acceleration

- $a(t) = \dfrac{dV}{dt}$

$$a(t) = r''(t) = \frac{d}{dt}\left(\frac{1}{\sqrt{2}} \hat{i} + \left[\frac{1}{\sqrt{2}} - 32t\right] \hat{j} + 2 \hat{k}\right)$$

$$a(t) = r''(t) = 0\hat{i} - 32\hat{j} + 0\hat{k}$$

$$a(0) = r''(0) = 0\hat{i} - 32\hat{j} + 0\hat{k}$$

$$\boxed{a(0) = -32\hat{j}}$$

(iii) $r(t) = \ln(t^2+1)\hat{i} + (\tan^{-1} t)\hat{j} + \sqrt{t^2+1}\,\hat{k}$

Sol: To find velocity, Differentiate $r(t)$ w.r.to t.

Velocity

- $V = \frac{d}{dt} r(t)$

$V = r'(t) = \frac{d}{dt} \ln(t^2+1)\hat{i} + \frac{d}{dt}(\tan^{-1} t)\hat{j} + \frac{d}{dt}\sqrt{t^2+1}\,\hat{k}$

$V = r'(t) = \frac{1}{t^2+1} \cdot (2t)\hat{i} + \frac{1}{1+t^2}\hat{j} + \frac{1}{\cancel{2}}(t^2+1)^{-1/2} \cdot (\cancel{2}t)\hat{k}$

$V = r'(t) = \frac{2t}{t^2+1}\hat{i} + \frac{1}{1+t^2}\hat{j} + \frac{t}{\sqrt{t^2+1}}\hat{k}$

$V = r'(0) = \frac{2(0)}{0+1}\hat{i} + \frac{1}{1+0}\hat{j} + \frac{0}{\sqrt{0+1}}\hat{k}$

$\boxed{V = r'(0) = 0\hat{i} + \hat{j} + 0\hat{k}}$

Acceleration

- $a(t) = \frac{dv}{dt}$

$$a(t) = \frac{d}{dt}\left(\frac{2t}{t^2+1}\hat{i} + \frac{1}{1+t^2}\hat{j} + \frac{t}{\sqrt{t^2+1}}\hat{k}\right)$$

$$a(t) = \left[\frac{(t^2+1)\frac{d}{dt}2t - 2t\frac{d}{dt}(t^2+1)}{(t^2+1)^2}\hat{i} + (-1)(1+t^2)^{-1-1}(2t)\hat{j} + \frac{\sqrt{t^2+1}\cdot 1 - t\cdot\frac{1}{2}(t^2+1)^{-1/2}\cdot 2t}{(\sqrt{t^2+1})^2}\right.$$

$$a(t) = \left[\frac{(t^2+1)\cdot 2 - 2t(2t)}{(t^2+1)^2}\hat{i} - \frac{2t}{(1+t^2)^2}\hat{j} + \frac{\sqrt{t^2+1} - t^2(t^2+1)^{-1/2}}{t^2+1}\right.$$

At $t = 0$

$$a(0) = \left[\frac{2-0}{1}\hat{i} - 0\hat{j} + \hat{k}\right]$$

$$\boxed{a(0) = 2\hat{i} - 0\hat{j} + \hat{k}}$$

(IV) $r(t) = \frac{4}{9}(t+1)^{3/2}\hat{i} + \frac{4}{9}(1-t)^{3/2}\hat{j} + \frac{1}{3}t\hat{k}$

Sol

Velocity

- $V = \frac{d}{dt} r(t)$

$$v(t) = r'(t) = \frac{d}{dt}\left[\frac{4}{9}(t+1)^{3/2}\hat{i} + \frac{4}{9}(1-t)^{3/2}\hat{j} + \frac{1}{3}t\hat{k}\right]$$

$$v(t) = r'(t) = \frac{\cancel{4}^{2}}{\cancel{9}_{3}} \cdot \frac{\cancel{3}^{1}}{\cancel{2}_{1}} (t+1)^{\frac{3}{2}-1}\hat{i} + \frac{4}{9} \cdot \frac{3}{2} (1-t)^{\frac{3}{2}-1} (-1)\hat{j} + \frac{1}{3}\hat{k}$$

$$v(t) = r'(t) = \frac{2}{3}(t+1)^{1/2}\hat{i} - \frac{2}{3}(1-t)^{1/2}\hat{j} + \frac{1}{3}\hat{k}$$

At $t = 0$

$$\boxed{v(0) = r'(0) = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}}$$

Acceleration:

- $a(t) = \frac{dv}{dt}$

$$a(t) = \frac{d}{dt}\left(\frac{2}{3}(t+1)^{1/2}\hat{i} - \frac{2}{3}(1-t)^{\frac{1}{2}}\hat{j} + \frac{1}{3}\hat{k}\right)$$

$$a(t) = \left(\frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}}(t+1)^{-1/2}\hat{i} - \frac{\cancel{2}}{3}\frac{1}{\cancel{2}}(1-t)^{-1/2}(-1)\hat{j} + 0\hat{k}\right)$$

$$a(t) = \left(\frac{1}{3}(t+1)^{-1/2}\hat{i} + \frac{1}{3}(1-t)^{-1/2}\hat{j} + 0\hat{k}\right)$$

At $t = 0$

$$\boxed{a(0) = \frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} + 0\hat{k}}$$

Q #3 Find the scalar valued function in terms of the magnitude

i) $r(t) = (3t-7)\hat{i} + t\hat{j} - t^2\hat{k}$

Sol To make the vector $r(t)$ a scalar valued function we take its magnitude

$$|r(t)| = |(3t-7)\hat{i} + t\hat{j} - t^2\hat{k}|$$

$$= \sqrt{(3t-7)^2 + (t)^2 + (-t^2)^2}$$

$$|r(t)| = \sqrt{9t^2 + 49 - 42t + t^2 + t^4}$$

$$|r(t)| = \sqrt{t^4 + 10t^2 - 42t + 49}$$

(ii) $r(t) = \left(\frac{t}{\sqrt{2}}\right)\hat{i} + \left(\frac{t}{\sqrt{2}} + 16t^2\right)\hat{j} - 2t\hat{k}$

$$|r(t)| = \left|\left(\frac{t}{\sqrt{2}}\right)\hat{i} + \left(\frac{t}{\sqrt{2}} + 16t^2\right)\hat{j} - 2t\hat{k}\right|$$

$$= \sqrt{\left(\frac{t}{\sqrt{2}}\right)^2 + \left(\frac{t}{\sqrt{2}} + 16t^2\right)^2 + (-2t)^2}$$

$$= \sqrt{\frac{t^2}{2} + \frac{t^2}{2} + (16t^2)^2 + 2 \cdot \frac{t}{\sqrt{2}} \cdot 16t^2 + 4t^2}$$

$$= \sqrt{\frac{t^2}{2} + \frac{t^2}{2} + 256t^4 + 16\sqrt{2}\,t^3 + 4t^2}$$

$$= \sqrt{256t^4 + 16\sqrt{2}\,t^3 + 5t^2}$$ Ans.

(iii) $r(t) = \ln(t^2+1)\hat{i} + (\tan t)\hat{j} + \sec t\,\hat{k}$

$|r(t)| = \left| \ln(t^2+1)\hat{i} + (\tan t)\hat{j} + (\sec t)\hat{k} \right|$

$$= \sqrt{[\ln(t^2+1)]^2 + [\tan t]^2 + [\sec t]^2}$$

$$= \sqrt{\tan^2 t + \sec^2 t + [\ln(t^2+1)]^2}$$ Ans.

(iv) $r(t) = \frac{4}{9}(t+1)^{3/2}\hat{i} + \frac{4}{9}(1-t)^{3/2} j + \frac{1}{3} t^{3/2} k$

Sol

$|r(t)| = \left| \frac{4}{9}(t+1)^{3/2}\hat{i} + \frac{4}{9}(1-t)^{3/2}\hat{j} + \frac{1}{3} t^{3/2} \right|$

$$= \sqrt{\left[\frac{4}{9}(t+1)^{3/2}\right]^2 + \left[\frac{4}{9}(1-t)^{3/2}\right]^2 + \left[\frac{1}{3} t^{3/2}\right]^2}$$

$$= \sqrt{\frac{16}{81}(t+1)^3 + \frac{16}{81}(1-t)^3 + \frac{1}{9} t^3}$$ Ans.

Complete