Exercise 4.4

Applications of Differential Equation.

Q#1 Thomas Malthus in 1798 proved that increase in population of a country or a city at a certain time is proportional to the total population — — — — — — — — — — —

What would be the population of that city after 12 years?

Increase in population

Total population

Sol Given $\frac{dP}{dt} \propto P$

$$\frac{dP}{dt} = KP$$

Seperating variables

$$\frac{dP}{P} = K\,dt$$

Integrating b.sides

$$\int \frac{1}{P}\,dP = K\int 1\,dt$$

$$\ln P = Kt + C \quad \text{—(i)}$$

- At present time, initially $t=0$, $P=20$

$$\ln 20 = k\cdot 0 + c$$

$$c = \ln 20 \quad \text{Put in (1)}$$

$$\boxed{\ln P = kt + \ln 20} \text{ — (2)}$$

- After 4 years $t=4$, $P=25$ Put in (2)

$$\ln 25 = k(4) + \ln 20$$

$$\ln 25 - \ln 20 = 4k$$

$$\ln \frac{25}{20} = 4k$$

$$\ln \frac{5}{4} = 4k \Rightarrow k = \frac{1}{4}\ln\left(\frac{5}{4}\right)$$

Put in (2)

$$\ln P = \frac{1}{4}\ln\left(\frac{5}{4}\right)t + \ln 20 \text{ — (3)}$$

$P = ?$ $t = 12$ years.

$$\ln P = \frac{1}{4}\ln\left(\frac{5}{4}\right) 12 + \ln 20$$

$$\ln P = \frac{1}{4}\cdot 12 \ln\left(\frac{5}{4}\right) + \ln 20$$

$$\ln P = 3\ln\left(\frac{5}{4}\right) + \ln 20$$

$$= \ln\left(\frac{5}{4}\right)^3 + \ln 20$$

$$\ln P = \ln\left(\frac{5}{4}\right)^3 \cdot 20$$

Taking Antilog

$$e^{\ln P} = e^{\ln (5/4)^3 \cdot 20}$$

$$P = \left(\frac{5}{4}\right)^3 \times 20$$

$$= \frac{125}{64} \times 20$$

$$P \approx 39 \text{ million}$$

Q#2

Ayesha was preparing a pizza in a baking oven. She observed that temperature of the cooked pizza was 150°C. Four minutes after removing from the oven, the temperature — — — — — — — — to a temperature of 40°C if room temperature of 20°C.

Sol

- observed = initial = $t = 0$

  Temperature = 150°C
- $t = 4$, $T = 90°C$
- $t = ?$, $T = 40°C$

$T_r = 20$

Let Temperature $= T$
Room $"$ $= T_r$
Time $= t$

Given Model

$$\frac{d}{dt} T \propto (T - T_r)$$

$$\frac{dT}{dt} = k(T - T_r)$$

Seperating.

$$\frac{1}{T-T_r} dT = k\,dt$$

Integrating:

$$\int \frac{1}{T-T_r} dT = k\,dt$$

Room temperature
$T_r = 20$

$$\int \frac{1}{T-20} dT = \int k\,dt$$

$$\ln|T-20| = kt + c$$

Taking Antilog

$$e^{\ln|T-20|} = e^{kt+c}$$

$$T - 20 = e^{kt} \cdot e^{c}$$

$$T - 20 = e^{kt} \cdot C \qquad \because C = e^{c}$$

$$\boxed{T = 20 + e^{kt} \cdot c} \text{ --- (i)}$$

Initially $t=0$, $T=150°$

$$150 = 20 + e^{k(0)} c$$

$$150-20 = e^{0} \cdot c$$

$$c = 130 \quad \text{Put in (1)}$$

$$T = 20 + e^{kt} \cdot 130$$

$$T = 20 + 130 e^{kt} \quad \text{--- (2)}$$

$t=4$ $T = 90°$

$$90 = 20 + 130 e^{k \cdot 4}$$

$$90 - 20 = 130 e^{4k}$$

$$\frac{70}{130} = e^{4k}$$

$$e^{4k} = \frac{7}{13}$$

Apply Ln on b.sides

$$\ln e^{4k} = \ln(7/13)$$

$$4k \ln e = \ln(7/13)$$

$$4k = \ln(7/13)$$

$$K = \frac{1}{4} \ln\left(7/13\right)$$

Put in (2)

$$T = 20 + 130\, e^{t\left(\frac{1}{4} \ln(7/13)\right)} \quad — (3)$$

$t = ?$, $T = 40^\circ C$

$$40 = 20 + 130 e^{t\left(1/4 \ln 7/13\right)}$$

$$40 - 20 = 130 e^{t/4 \ln 7/13}$$

$$20 = 130\, e^{t/4 \ln 7/13}$$

$$\frac{2\cancel{0}}{13\cancel{0}} = \cancel{e}^{\cancel{\ln}\left(7/13\right)^{t/4}}$$

$$\frac{2}{13} = \left(\frac{7}{13}\right)^{t/4}$$

Taking Ln on b.sids

$$\ln\left(\frac{2}{13}\right) = \ln\left(\frac{7}{13}\right)^{t/4}$$

$$\ln\left(\frac{2}{13}\right) = \frac{t}{4} \ln\left(7/13\right)$$

$$\frac{4 \ln(2/13)}{\ln(7/13)} = t$$

$$t = 12.09 \text{ min}$$

Q#3 In a culture, the rate of growth of bacteria is proportional to the population present. If the population of bacteria becomes four times in two days, - - - -

— — — — — .

If the initial population was 20?

Sol Let Present Population $= P(t)$

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = KP$$

$$\frac{1}{P} dP = K\,dt$$

Apply integration

$$\int \frac{1}{P} dP = K \int 1\,dt$$

$$\ln P = Kt + C \quad \text{--- (i)}$$

initially $t = 0,\quad P = 20$

$$\ln 20 = K(0) + C$$

$$C = \ln 20$$

Put in (i)

$$\ln P = Kt + \ln 20 \quad \text{--- (ii)}$$

Given Population 4 times in 2 days

$P = 20 \times 4 = 80$ $\quad t = 2$

Put in (ii)

$\ln 80 = 2k + \ln 20$

$\ln 80 - \ln 20 = 2k$

$\ln \frac{80}{20} = 2k$

$\ln 4 = 2k$

$\frac{1}{2} \ln 2^2 = k$

$k = \ln (2^{\cancel{2}})^{1/\cancel{2}}$

$k = \ln 2$

Put in (ii)

$\ln P = t \ln 2 + \ln 20$

$P = ?$ $\quad t = 10$

$\ln P = 10 \ln 2 + \ln 20$

$\ln P = \ln 2^{10} + \ln 20$

$\ln P = \ln 2^{10} \times 20$

Apply Anti log

$e^{\ln P} = e^{\ln 2^{10} \cdot 20}$

$P = 2^{10} \cdot 20 = 20480$

(4) Most of the radioactive substances disintegrate at the rate proportional to the amount present. If the amount of a radioactive substances is 50 grams and its half Life is 1000 years, find the amount of substance present after the amount of 800 years.

Sol Let radioactive element Present Amount = A(t)

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = kA$$

$$\frac{1}{A} dA = k\,dt$$

Apply integral

$$\int \frac{1}{A} dA = k \int 1\,dt$$

$$\ln A = kt + C \quad \text{--- (1)}$$

initial : $t=0$ If the amount of radioactive substance is 50 gm

$$\ln 50 = k(0) + C$$

$C = \ln 50$ Put in (1)

$$\ln A = kt + \ln 50 \quad \text{--- (2)}$$

And its half Life 1000 years

$A = \frac{50}{2} = 25$, $t = 1000$ years.

$\ln 25 = K \cdot 1000 + \ln 50$

$\ln 25 - \ln 50 = 1000\,K$

$\ln \frac{25}{50} = 1000\,K$

$\ln \frac{1}{2} = 1000\,K$

$\frac{1}{1000} \cdot \ln \frac{1}{2} = K$ Put in (2)

$\ln A = t \cdot \frac{1}{1000} \ln \frac{1}{2} + \ln 50$

$A = ?$ $t = 800$

$\ln A = \frac{\cancel{800}^{4}}{\cancel{1000}_{5}} \ln \frac{1}{2} + \ln 50$

$\ln A = 4/5 \ln (1/2) + \ln 50$

$\ln A = \ln (1/2)^{4/5} + \ln 50$

$\ln A = \ln \left(\left(\frac{1}{2}\right)^{4/5} \times 50 \right)$

Apply Antilog

$$e^{\ln A} = e^{\ln (1/2)^{4/5} \times 50}$$

$$A = \left(\frac{1}{2}\right)^{4/5} \times 50$$

$$A = 28.71 \quad \text{Ans.}$$

(5) A thermometer showing room temperature of $80°F$ is placed on a block of ice with a temperature of $30°F$. After one min the temperature of thermometer is $40°F$. How Long will it — — — — — of $70°F$?

Sol Let Temperature $= T = 80°F$

Block of ice temperature $= T_i = 30F$

Time $= t$

Given Model:

$$\frac{dT}{dt} \propto T - T_i$$

$$\frac{dT}{dt} = k(T - T_i)$$

$$\frac{dT}{dt} = K(T - 30)$$

$$\frac{1}{T-30}\, dT = K\, dt$$

Apply integration

$$\int \frac{1}{T-30}\, dT = k \int 1\, dt$$

$$\ln|T-30| = k\,t + c \quad —(1)$$

- initially $\quad t = 0 \qquad T = 80\,F$

$$\ln|80-30| = k(0) + c$$

$$C = \ln 50 \qquad \text{Put in (1)}$$

$$\ln|T-30| = k\,t + \ln 50 \quad —(2)$$

- After 1 min the temp of thermometer is $40^\circ F$

$t = 1 \qquad T = 40$

$$\ln|40-30| = k(1) + \ln 50$$

$$\ln 10 - \ln 50 = k$$

$$\ln \frac{10}{50} = k$$

$$k = \ln \frac{1}{5} \qquad \text{Put in (2)}$$

$$\ln|T-30| = \ln \frac{1}{5} \cdot t + \ln 50 \quad —(3)$$

- $t = ? \qquad T = 70$

$$\ln|70-30| = \ln \frac{1}{5} \cdot t + \ln 50$$

$$\ln 40 - \ln 50 = t \ln \frac{1}{5}$$

$$\ln \frac{40}{50} = \ln \left(\tfrac{1}{5}\right)^t$$

$$\ln \frac{4}{5} = \ln \left(\tfrac{1}{5}\right)^t$$

Apply Antilog

$$e^{\ln 4/5} = e^{\ln (1/5)^t}$$

$$\frac{4}{5} = \left(\frac{1}{5}\right)^t$$

Taking Ln on b.sids

$$\ln \frac{4}{5} = \ln \left(\frac{1}{5}\right)^t$$

$$\ln \left(\frac{4}{5}\right) = t \ln \left(\frac{1}{5}\right)$$

$$\frac{\ln (4/5)}{\ln (1/5)} = t$$

$$t = 0.14 \text{ min}$$

or

$$t = 0.14 \times 60 \text{ sec}$$

$$t = 8.3 \text{ second.}$$

Q#6 A ball is thrown upward with a velocity of 40m/sec. Develop a differential equation representing the flow phenomenon and find the velocity — — — — — —. Neglect the air resistance.

Sol Since ball is thrown vertically upward

So $\boxed{a = -g}$

$$\frac{dv}{dt} = -g \qquad \because \text{acceleration } a = \frac{v}{t}$$

For upward ↑ $a = -g$

$$dv = -g\,dt$$

Apply integral

$$\int 1\,dv = -g\int 1.dt$$

$$v = -g.t + C \quad —(1)$$

- initial $V = 40\,m/sec$ $t = 0$

$$40 = -g(0) + C$$

$$C = 40 \quad \text{Put in (1)}$$

$$V = -gt + 40 \quad —(2)$$

$\boxed{g = 9.8\,m/sec^2}$

$$V = -9.8t + 40$$

- Velocity of ball after 1 second

$V = ?$ $t = 1$

$V = -9.8 \times 1 + 40$

$V = 30.2$ m/sec

- Maximum Height ?

$a = \frac{dv}{dt}$

$V = \frac{ds}{dt}$

$V = \frac{ds}{dt}$

$\frac{ds}{dt} = (-9.8t + 40)$ using (2)

$ds = (-9.8t + 40)\,dt$

Apply integral

$\int 1\,ds = \int (-9.8t + 40)\,dt$

$S = -9.8 \cdot \frac{t^2}{2} + 40t + C$

$S = -4.9t^2 + 40t + C$ ——(3)

- Initial Condition $t = 0$, $S = 0$

$0 = -4.9(0) + 40(0) + C$

$C = 0$ Put in (3)

$S = -4.9t^2 + 40t$ — (4)

To find S first we find t.

For maximum height velocity is Zero

$V = 0$

$-9.8t + 40 = 0$ $\quad V = -9.8t + 40$

$-9.8t = -40$

$t = \frac{40}{9.8}$

$t = 4$ Put in (4)

$S = -4.9(4)^2 + 40(4)$

$S = 81.6m$ Ans.

Complete