Chap # 4

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Differential Equation

Exe 4.1

Derivative :- The rate of change of Function is called derivative.

Differential Equation:- An equation containing the derivative is called differential equation.

Ordinary Differential Equation:- ODE
If in a differential equation, only one independent variable 'x' is involved, the equation is called an ordinary differential equation.

dependent

$$x\frac{dy}{dx} + y = x$$

Independent

Order of Differential equation:- The order of the differential equation is the order of highest derivative present in the equation.

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x \quad (\text{order is } 2)$$

Degree of Differential equation:- The Power of highest derivative is called degree of differential equation.

Examples:

i) $\frac{dy}{dx} = a$

order = 1 degree = 1

ii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 1$

order = 2 degree = 1

(iii) $\left(\frac{d^3y}{dx^3}\right)^2 + x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2x = 5$

order = 3 degree = 2

1. Find the order and degree of each of differential equation.

(i) $(1-x)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 5y = \cos x$

order = 2 degree = 1

(ii) $y\frac{dy}{dx} + 2y = 1 + x^2$

order = 1 degree = 1

(iii) $\left(\frac{d^2y}{dx^2}\right)^3 - 3\frac{dy}{dx} + 2y = x$

order = 2 degree = 3

(iv) $\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2} + 2 = 0$

order = 2 degree = 1

(v) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

order = 2 degree = 1

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vi) $\frac{dy}{dx} = \sqrt{1+\left(\frac{d^2y}{dx^2}\right)^2}$

order = 2 degree = 2

vii) $x^3 \frac{d^4y}{dx^4} - x^2 \frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^5 + 4xy - 3y = 0$

order = 4 degree 1.

② Eliminate the arbitrary constants from the equations:

i) $y = ae^x + be^{-x} + c$

Sol Diff w.r.to x

$$\frac{dy}{dx} = \frac{d}{dx} ae^x + \frac{d}{dx} be^{-x} + \frac{d}{dx} c$$

$$\frac{dy}{dx} = a\frac{d}{dx} e^x + b\frac{d}{dx} e^{-x} + 0$$

$$\frac{dy}{dx} = a \cdot e^x + b e^{-x} \cdot -1$$

$$\frac{dy}{dx} = ae^x - be^{-x} \quad \text{—(i)}$$

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Again Diff w.r.t. to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx} ae^{x} - \frac{d}{dx} be^{-x}$$

$$\frac{d^2y}{dx^2} = a\frac{d}{dx} e^{x} - b\frac{d}{dx} e^{-x}$$

$$\frac{d^2y}{dx^2} = a.e^{x} - b\,e^{-x}.-1$$

$$\frac{d^2y}{dx^2} = ae^{x} + be^{-x}$$

Again Diff w.r.t. to x

$$\frac{d^3y}{dx^3} = \frac{d}{dx} ae^{x} + \frac{d}{dx} be^{-x}$$

$$\frac{d^3y}{dx^3} = a\frac{d}{dx} e^{x} + b\frac{d}{dx} e^{-x}$$

$$\frac{d^3y}{dx^3} = a\,e^{x} + b\,e^{-x}.-1$$

$$\frac{d^3y}{dx^3} = a\,e^{x} - b\,e^{-x}$$

From (i)

$$\frac{d^3y}{dx^3} = \frac{dy}{dx}$$

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$$ is the required differential Eq.

(ii) $y = \cos(x+b)$

Sol Diff w.r.to x

$$\frac{dy}{dx} = \frac{d}{dx}\cos(x+b)$$

$$\frac{dy}{dx} = -\sin(x+b).$$

Again Diff w.r.to x

$$\frac{d^2y}{dx^2} = -\cos(x+b)$$

$$\frac{d^2y}{dx^2} = -y$$

$\frac{d^2y}{dx^2} + y = 0$ is the req differential Eq.

(iii) $y = mx + c$

Sol Diff w.r.to x

$$\frac{dy}{dx} = \frac{d}{dx}mx + \frac{d}{dx}c$$

$$\frac{dy}{dx} = m + 0$$

$$\frac{dy}{dx} = m$$

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Again Diff w.r.to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx} m$$

$$\frac{d^2y}{dx^2} = 0$$

(iv) $y = bx^2 + 2ax$

Sol Diff w.r.to x

$$\frac{dy}{dx} = \frac{d}{dx} bx^2 + \frac{d}{dx} 2ax$$

$$\frac{dy}{dx} = b \frac{d}{dx} x^2 + 2a \frac{d}{dx} x$$

$$\frac{dy}{dx} = b \cdot 2x + 2a$$

$$\frac{dy}{dx} = 2bx + 2a$$

Again Diff w.r.to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx} 2bx + \frac{d}{dx} 2a$$

$$\frac{d^2y}{dx^2} = 2b \frac{d}{dx} x + 0$$

$$\frac{d^2y}{dx^2} = 2b$$

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Again Diff w.r.t on x

$$\frac{d^3y}{dx^3} = \frac{d}{dx} 2b$$

$$\frac{d^3y}{dx^3} = 0$$

Verify that the indicated Function is a solution of the given differential Eq

i) $2y' + y = 0$; $y = e^{-x/2}$

Sol

$$y = e^{-x/2}$$

$$y' = e^{-x/2} \cdot \frac{-1}{2}$$

$$y' = y \cdot \frac{-1}{2}$$

$$2y' = -y$$

$$2y' + y = 0 \quad \text{Ans.}$$

(ii) $\frac{dy}{dx} - 2y = e^{3x}$; $y = e^{3x} + 10e^{2x}$

Sol

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$$y = e^{3x} + 10e^{2x}$$

$$\frac{dy}{dx} = e^{3x} \cdot 3 + 10e^{2x} \cdot 2$$

$$\frac{dy}{dx} = 3e^{3x} + 2 \cdot 10e^{2x}$$

As $y = e^{3x} + 10e^{2x}$

$$y - e^{3x} = 10e^{2x}$$

$$\frac{dy}{dx} = 3e^{3x} + 2\left(y - e^{3x}\right)$$

$$\frac{dy}{dx} = 3e^{3x} + 2y - 2e^{3x}$$

$$\frac{dy}{dx} - 2y = e^{3x} \quad \text{Ans.}$$

(iii) $y' = 25 + y^2 \quad ; \quad y = 5\tan 5x$

Sol:
$$y = 5\tan 5x$$

$$y' = 5 \cdot \sec^2 5x \cdot 5$$

$$y' = 25\sec^2 5x$$

$$y' = 25\left[1 + \tan^2 5x\right]$$

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$$y' = 25\left[1 + \frac{y^2}{25}\right]$$

$$y' = \frac{\cancel{25}\left[25 + y^2\right]}{\cancel{25}}$$

$$y' = 25 + y^2$$ Proved.

(IV) $y' + y = \sin x$

Sol $y = \frac{1}{2}\sin x - \frac{1}{2}\cos x + 10e^{-x}$

Diff w.r.t. to x

$$y' = \frac{1}{2}\cos x - \frac{1}{2}(-\sin x) + 10e^{-x}\cdot -1$$

$$y' = \frac{1}{2}\cos x + \frac{1}{2}\sin x - 10e^{-x}$$

$$y' = \frac{1}{2}\cos x + \frac{1}{2}\sin x - \left(y - \frac{1}{2}\sin x + \frac{1}{2}\cos x\right)$$

$$y' = \frac{1}{2}\cancel{\cos x} + \frac{1}{2}\sin x - y + \frac{1}{2}\sin x - \cancel{\frac{1}{2}\cos x}$$

$$y' + y = \frac{\cancel{2}\sin x}{\cancel{2}}$$

$$y' + y = \sin x$$ Proved

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(v) $x^3 dy - 2 dx = 0$

$$y = -\frac{1}{x^2} + 6$$

Sol

$$y = -\frac{1}{x^2} + 6$$

$$y = -x^{-2} + 6$$

Diff w.r.to x

$$\frac{dy}{dx} = -.-2x^{-2-1} + 0$$

$$\frac{dy}{dx} = 2x^{-3}$$

$$dy = 2x^{-3} dx$$

$$\frac{dy}{x^{-3}} = 2 dx$$

$x^{+3} dy - 2dx = 0$ Proved.

(vi) $y' - \frac{1}{x} y = 1$

Sol

$y = x \ln x$

Diff w.r.to x:

$y' = x \frac{d}{dx} \ln x + \ln x \cdot \frac{d}{dx} x$

$y' = \cancel{x} \cdot \frac{1}{\cancel{x}} + \ln x \cdot 1$

$y' = 1 + \ln x$

$\ln x = \frac{y}{x}$

$y' = 1 + \frac{y}{x}$

$y' - \frac{1}{x} y = 1$ Proved.

Q#4 Find the order and degree, If defined For the differential equation $dy - \sin x\, dx = 0$

Sol

$dy - \sin x\, dx = 0$

$dy = \sin x\, dx$

$\frac{dy}{dx} = \sin x$

$\frac{dy}{dx} - \sin x = 0$

order = 1 degree = 1

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5) Verify that the function $y = a\cos x + b\sin x$ where $a, b \in R$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

Sol

$$y = a\cos x + b\sin x.$$

Diff w.r.t. x.

$$\frac{dy}{dx} = a\frac{d}{dx}\cos x + b\frac{d}{dx}\sin x.$$

$$\frac{dy}{dx} = a(-\sin x) + b(\cos x)$$

$$\frac{dy}{dx} = -a\sin x + b\cos x.$$

Diff w.r.t. x.

$$\frac{d^2y}{dx^2} = -a\cos x + b \cdot -\sin x.$$

$$\frac{d^2y}{dx^2} = -(a\cos x + b\sin x)$$

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{Proved.}$$

Q#6 Show that $y_1 = x^2$ and $y_2 = x^3$ are both Solutions of $x^2y'' - 4xy' + 6y = 0$

Sol

$y_1 = x^2$ $\qquad\qquad$ $y_2 = x^3$

$y' = 2x$ $\qquad\qquad$ $y' = 3x^2$

$y'' = 2$ $\qquad\qquad$ $y'' = 6x$

Put in given differential equation

$x^2y'' - 4xy' + 6y = 0$ $\qquad\qquad$ $x^2y'' - 4xy' + 6y = 0$

$x^2 \cdot 2 - 4x(2x) + 6x^2 = 0$ $\qquad\qquad$ $x^2 \cdot 6x - 4x \cdot 3x^2 + 6x^3 = 0$

$2x^2 - 8x^2 + 6x^2 = 0$ $\qquad\qquad$ $6x^3 - 12x^3 + 6x^3 = 0$

$8x^2 - 8x^2 = 0$ $\qquad\qquad$ $12x^3 - 12x^3 = 0$

$0 = 0$ $\qquad\qquad$ $0 = 0$

So y_1 and y_2 are the Solution:

- Are c_1y_1 and c_2y_2 also a Solution:

The differential equation is Linear and Homogenous So any Constant multiple of a Solution is also a Solution.
So c_1y_1 and c_2y_2 are also the Solution

- Check if $y_1 + y_2$ is also a Solution.

Let $y = y_1 + y_2$

$y = x^2 + x^3$

Diff w.r.t to x.

$y' = 2x + 3x^2$

Again Diff w.r.t to x.

$y'' = 2 + 6x$

Put in

$$x^2 y'' - 4x y' + 6y = 0$$

$$x^2(2+6x) - 4x(2x+3x^2) + 6(x^2+x^3) = 0$$

$$\cancel{2x^2} + \cancel{6x^3} - \cancel{8x^2} - \cancel{12x^3} + \cancel{6x^2} + \cancel{6x^3} = 0$$

$$0 = 0$$

So $y_1 + y_2$ is also a Solution.

Complete

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