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Exc 3.4

INTEGRATION BY PARTS

$$\int U \cdot V \, dx = U \int V dx - \int \int V dx \frac{d}{dx} U$$

$$\int I \cdot II \, dx = I \int II \, dx - \int \int II \, dx \frac{d}{dx}(I).$$

I Inverse Function ($\sin^{-1} x$, $\cos^{-1} x$, etc)

L Logarithm Function ($\ln x$ etc)

A Algebric Function (x, x^2 etc)

T Trigonometric Function ($\sin x$, $\cos x$, etc)

E Exponential Function (e^x, e^{2x} etc)

(i) $\int \ln x \, dx$

Sol $\int \underset{I}{\ln x} \cdot \underset{II}{1} \, dx$

Using $\int U \cdot V dx = U \int V dx - \int \int V dx \frac{d}{dx} U$

$$= \ln x \int 1\,dx - \int \left[\int 1\,dx \cdot \frac{d}{dx} \ln x\right]$$

$$= \ln x \cdot x - \int x \cdot \frac{1}{x}\,dx$$

$$= x \ln x - \int 1\,dx$$

$$= x \ln x - x + c \qquad \text{Ans.}$$

② $\int (\ln x)^2\,dx$

Sol

$$\int \underset{\text{I}}{(\ln x)^2} \cdot \underset{\text{II}}{1}\,dx$$

$$= (\ln x)^2 \cdot \int 1\,dx - \int \left[\int 1\,dx \cdot \frac{d}{dx} (\ln x)^2\right] dx$$

$$= (\ln x)^2 \cdot x - \int x \cdot \left[2 (\ln x)^{2-1} \cdot \frac{1}{x}\right] dx$$

$$= x (\ln x)^2 - \int 2 \ln x\,dx$$

$$= x (\ln x)^2 - 2 \int \ln x\,dx$$

$$= x (\ln x)^2 - 2 \int \underset{\text{II}}{1} \cdot \underset{\text{I}}{\ln x}\,dx$$

$$= x (\ln x)^2 - 2 \left[\ln x \cdot \int 1\,dx - \int \int 1\,dx \; \frac{d}{dx} \ln x\,dx\right.$$

$$= x(\ln x)^2 - 2\left[\ln x \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}}\,dx\right]$$

$$= x(\ln x)^2 - 2\left[x\ln x - \int 1\,dx\right]$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C \quad \text{Ans.}$$

(3) $I = \int \sin(\ln x)\,dx$

Sol. $\int \underset{I}{\sin(\ln x)} \cdot \underset{II}{1}\,dx$

Integrating by Parts:

$$= \sin(\ln x)\int 1\,dx - \int \int 1\,dx \, \frac{d}{dx}\sin(\ln x)\,dx$$

$$= \sin(\ln x)\cdot x - \int \cancel{x} \cdot \cos\ln x \cdot \frac{1}{\cancel{x}}\,dx$$

$$= x\sin(\ln x) - \int \cos\ln x\,dx$$

$$= x\sin(\ln x) - \int \underset{II}{1} \cdot \underset{I}{\cos\ln x}\,dx$$

$$= x\sin(\ln x) - \left[\cos\ln x\int 1\,dx - \int\int 1\,dx \, \frac{d}{dx}\cos\ln x\,dx\right]$$

$$I = x \sin \ln x - x \cos \ln x + \int \cancel{x} \cdot -\sin \ln x \cdot \frac{1}{\cancel{x}} \, dx$$

$$I = x \sin \ln x - x \cos \ln x - \int \sin \ln x \, dx.$$

$$I = x \sin \ln x - x \cos \ln x - I + C$$

$$2I = x \sin \ln x - x \cos \ln x + C$$

$$I = \frac{x}{2} \sin \ln x - \frac{x}{2} \cos \ln x + C$$

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$$\int \underset{\text{II}}{x^3} \underset{\text{I}}{\ln x} \, dx$$

Sol Integrating by Parts.

$$= \ln x \int x^3 dx - \int \left[\int x^3 dx \cdot \frac{d}{dx} \ln x \right] dx$$

$$= \ln x \cdot \frac{x^4}{4} - \int \frac{x^{\cancel{4}3}}{4} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} \, dx$$

$$= \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4 \ln x}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + C$$

$$= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$$ Ans

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5. $\int \underset{I}{y} \cdot \underset{II}{\sin 2y}\, dy$

Sol

$= y \int \sin 2y \, dy - \int \left[\int \sin 2y \, dy \; \frac{d}{dy} y \right] dy$

$= y \cdot \frac{-\cos 2y}{2} - \int \frac{-\cos 2y}{2} \cdot 1 \, dy$

$= -\frac{y \cos 2y}{2} + \frac{1}{2} \int \cos 2y \, dy$

$= -\frac{y \cos 2y}{2} + \frac{1}{2} \frac{\sin 2y}{2} + C$

$= -\frac{y}{2} \cos 2y + \frac{1}{4} \sin 2y + C$

6. $I = \int \underset{II}{e^x} \underset{I}{\cos x} \, dx$

Sol

$= \cos x \int e^x dx - \int \left(\int e^x dx \; \frac{d}{dx} \cos x \right) dx$

$= \cos x \cdot e^x - \int e^x \cdot (-\sin x) \, dx$

$= e^x \cos x + \int \underset{II}{e^x} \underset{I}{\sin x} \, dx$

$= e^x \cos x + \left[\int e^x \sin x \, dx \right]$

$= e^x \cos x + \left[\sin x \int e^x dx - \int \left(\int e^x dx \cdot \frac{d}{dx} \sin x \right) dx \right]$

$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$

$I = e^x \cos x + e^x \sin x - I + C$

$2I = e^x \cos x + e^x \sin x + C$

$I = \frac{e^x}{2} (\cos x + \sin x) + C$ Ans.

(7) $\int \underset{II}{x} \cdot \underset{I}{\sec^{-1} x} \, dx$

Sol

$\sec^{-1} x \cdot \int x \, dx - \int \left(\int x \, dx \cdot \frac{d}{dx} \sec^{-1} x \right) dx$

$\sec^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x\sqrt{x^2-1}} dx$

$\frac{x^2}{2} \sec^{-1} x - \frac{1}{2} \int \frac{x^{\cancel{2}}}{\cancel{x} \sqrt{x^2-1}} dx$

$\frac{x^2}{2} \sec^{-1} x - \frac{1}{2} \int (x^2-1)^{-1/2} \cdot x \, dx$ Using Rule.

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$\frac{x^2}{2} \sec^{-1} x - \frac{1}{2} \cdot \frac{1}{2} \int (x^2-1)^{-1/2} \cdot 2x \, dx$

$\frac{x^2}{2} \sec^{-1} x - \frac{1}{4} \int (x^2-1)^{-1/2} \cdot 2x \, dx$

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$$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{4}\frac{(x^2-1)^{-1/2+1}}{-1/2+1} + C$$

$$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{4}\frac{(x^2-1)^{1/2}}{1/2} + C$$

$$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{4}\cdot\frac{2}{1}(x^2-1)^{1/2} + C$$

$$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\sqrt{x^2-1} + C$$ Ans.

8 $$\int \ln(2x+3)\,dx$$

Sol $$\int \underset{\mathrm{I}}{\ln(2x+3)}\cdot \underset{\mathrm{II}}{1}\,dx$$

$$= \ln(2x+3)\int 1\,dx - \int\left(\int 1\,dx\ \frac{d}{dx}\ln(2x+3)\right)dx$$

$$= \ln(2x+3)\cdot x - \int\left(x\cdot\frac{1}{(2x+3)}\cdot 2\right)dx$$

$$= x\ln(2x+3) - \int\frac{2x}{2x+3}\,dx$$

Add and subtract 3

$$= x\ln(2x+3) - \int\frac{2x+3-3}{2x+3}\,dx$$

$$= x\ln(2x+3) - \int\frac{2x+3}{2x+3}\,dx + \int\frac{3}{2x+3}\,dx$$

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$= x\ln(2x+3) - \int 1\,dx + 3\int \frac{1}{2x+3}\,dx$

$= x\ln(2x+3) - x + \frac{3}{2}\int \frac{2}{2x+3}\,dx$

$= x\ln(2x+3) - x + \frac{3}{2}\ln|2x+3| + C$

(9) $\int \underset{I}{x^2}\,\underset{II}{e^x}\,dx$

Sol

Integrating by Parts

$= x^2\int e^x dx - \int\left(\int e^x dx\,\frac{d}{dx}x^2\right)dx$

$= x^2 \cdot e^x - \int e^x \cdot 2x\,dx$

$= x^2 e^x - 2\int \underset{I}{x}\,\underset{II}{e^x}\,dx$

$= x^2 e^x - 2\left[x\int e^x dx - \int\left(\int e^x dx\,\frac{d}{dx}x\right)dx\right]$

$= x^2 e^x - 2\left[xe^x - \int e^x dx\right]$

$= x^2 e^x - 2xe^x + 2e^x + C$ Ans

(10) $\int \underset{I}{x}\,\underset{II}{\cos x}\,dx$

Integrating by Parts.

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$= x \int \cos x \, dx - \int \left(\int \cos x \, dx \cdot \frac{d}{dx} x \right) dx$

$= x \cdot \sin x - \int \sin x \, dx$

$= x \cdot \sin x - (-\cos x) + C$

$= x \sin x + \cos x + C$

(11) $\int \cos^{-1} x \, dx$

Sol: $\int \underset{\mathrm{I}}{\cos^{-1} x} \cdot \underset{\mathrm{II}}{1} \, dx$

$= \cos^{-1} x \cdot \int 1 \, dx - \int \left(\int 1 \, dx \cdot \frac{d}{dx} \cos^{-1} x \right) dx$

$= \cos^{-1} x \cdot x - \int x \cdot \frac{-1}{\sqrt{1-x^2}} \, dx$

$= x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} \, dx$

$= x \cos^{-1} x + \int (1-x^2)^{-1/2} \cdot x \, dx$

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$= x \cos^{-1} x + \frac{1}{-2} \int (1-x^2)^{-1/2} \cdot -2x \, dx$

$= x \cos^{-1} x - \frac{1}{2} \cdot \frac{(1-x^2)^{-1/2+1}}{\frac{-1}{2}+1} + C$

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$$= x\cos^{-1}x - \frac{1}{2}\cdot\frac{(1-x^2)^{1/2}}{\cancel{1/2}} + C$$

$$= x\cos^{-1}x - \sqrt{1-x^2} + C \quad \text{Ans.}$$

(12) $\int \tan^{-1}x \, dx$

Sol $\int \underset{\text{I}}{\tan^{-1}x} \cdot \underset{\text{II}}{1} \, dx$

$$= \tan^{-1}x \cdot \int 1\,dx - \int\left(\int 1\,dx \; \frac{d}{dx}\tan^{-1}x\right)dx$$

$$= \tan^{-1}x \cdot x - \int x \cdot \frac{1}{1+x^2}\,dx$$

$$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C$$

(13) $\int \underset{\text{I}}{x} \cdot \underset{\text{II}}{\sec^2 x} \, dx$

Sol Integrating by Parts

$$\int u\cdot v\,dx = u\int v\,dx - \int\left(\int v\,dx \; \frac{d}{dx}u\right)dx$$

$$= x\int \sec^2 x\,dx - \int\left(\int \sec^2 x\,dx\ \frac{d}{dx}x\right)dx$$

$$= x\tan x - \int \tan x\,dx$$

$$= x\tan x - \ln|\sec x| + c \quad \text{Ans.}$$

Q#14 v.imp

$$\int \underset{\text{II}}{x^2}\ \underset{\text{I}}{\sin^{-1}x}\,dx$$

Integrating by Parts

$$= \sin^{-1}x \cdot \int x^2 dx - \int\left(\int x^2 dx\ \frac{d}{dx}\sin^{-1}x\right)dx$$

$$= \sin^{-1}x \cdot \frac{x^{2+1}}{2+1} - \int\left(\frac{x^{2+1}}{2+1} \cdot \frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx \quad \text{---(i)}$$

Consider $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

Put $(1-x^2) = y$ | $1-x^2 = y$

Diff w.r.t to x | $1-y = x^2$

$$0 - 2x = \frac{dy}{dx}$$

$$-2x\,dx = dy$$

$$x\,dx = \frac{-1}{2}\,dy$$

$$= \int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{x^2 \cdot x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{(1-y)}{\sqrt{y}} \cdot \frac{-1}{2}\,dy$$

$$= \frac{-1}{2}\int \frac{1-y}{\sqrt{y}}\,dy$$

$$= \frac{-1}{2}\int \left(\frac{1}{\sqrt{y}} - \frac{y}{\sqrt{y}}\right) dy$$

$$= \frac{-1}{2}\int \left(y^{-1/2} - y^{1/2}\right) dy$$

$$= \frac{-1}{2}\left[\frac{y^{-1/2+1}}{-1/2+1} - \frac{y^{1/2+1}}{1/2+1}\right] + C$$

$$= \frac{-1}{2} \cdot \left[\frac{y^{1/2}}{1/2} - \frac{y^{3/2}}{3/2}\right] + C$$

$$= \frac{-1}{2}\left[2\,y^{1/2} - \frac{2}{3}\,y^{3/2}\right] + C$$

$$= -y^{1/2} + \frac{1}{3}\,y^{3/2} + C$$

$$= -\sqrt{1-x^2} + \frac{1}{3}\,(1-x^2)^{3/2} + C$$

Put in (i)

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left(-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2}\right) + C$$

$$= \frac{x^3}{3}\sin^{-1}x + \frac{\sqrt{1-x^2}}{3} - \frac{(1-x^2)^{3/2}}{9} + C$$

Ans.

(15) $$\int \ln\left[x+\sqrt{1+x^2}\right] dx$$

Sol

$$\int \underset{\text{I}}{\ln\left[x+\sqrt{1+x^2}\right]} \cdot \underset{\text{II}}{1}\, dx$$

Integrating by Parts.

$$\ln\left[x+\sqrt{1+x^2}\right]\cdot\int 1\,dx - \int\left(\int 1\,dx\ \frac{d}{dx}\ln\left[x+\sqrt{1+x^2}\right]\right)dx$$

$$\ln\left[x+\sqrt{1+x^2}\right]\cdot x - \int\left(x\cdot\frac{1}{x+\sqrt{1+x^2}}\ \frac{d}{dx}\left[x+\sqrt{1+x^2}\right]\right)dx$$

$$x\ln\left[x+\sqrt{1+x^2}\right] - \int\left(\frac{x}{x+\sqrt{1+x^2}}\cdot\left[1+\frac{1}{2}(1+x^2)^{-1/2}(2x)\right]\right)dx$$

$$x\ln\left[x+\sqrt{1+x^2}\right] - \int\frac{x}{x+\sqrt{1+x^2}}\cdot 1+\frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}x\ dx$$

$$x\ln\left[x+\sqrt{1+x^2}\right] - \int\frac{x}{\cancel{x+\sqrt{1+x^2}}}\cdot\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}\,dx$$

$$x\ln\left[x+\sqrt{1+x^2}\right] - \int\frac{x}{\sqrt{1+x^2}}\,dx$$

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$$x \ln\left[x+\sqrt{1+x^2}\right] - \frac{1}{2}\int (1+x^2)^{-1/2} \cdot 2x \, dx$$

$$x \ln\left[x+\sqrt{1+x^2}\right] - \frac{1}{2} \frac{(1+x^2)^{-1/2+1}}{-1/2+1} + C$$

$$x \ln\left[x+\sqrt{1+x^2}\right] - \frac{1}{2} \frac{(1+x^2)^{1/2}}{1/2} + C$$

$$x \ln\left[x+\sqrt{1+x^2}\right] - \sqrt{1+x^2} + C \quad \text{Ans.}$$

(16) $\int x^3 \cdot e^{x^2} dx$

Sol

$$\int x^2 \cdot x e^{x^2} dx$$

$$\int x^2 e^{x^2} \cdot x \, dx$$

Put $x^2 = y$

$$2x \, dx = dy$$

$$x \, dx = \frac{dy}{2}$$

$$= \int y e^y \frac{dy}{2}$$

$$= \frac{1}{2}\int \underset{I}{y} \underset{II}{e^y} dy$$

$$= \frac{1}{2}\left[y \int e^y dy - \int \left(\int e^y dy \frac{d}{dy} y\right) dy\right]$$

$$= \frac{1}{2}\left[y \cdot e^y - \int e^y dy\right]$$

$= \frac{1}{2} y e^{y} - \frac{1}{2} e^{y} + c$

$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C$ Ans.

(17) $\int \underset{I}{x^2} \underset{II}{\sin x}\, dx$

Sol: Integrating by Parts

$= x^2 \int \sin x\, dx - \int \left(\int \sin x\, dx \, \frac{d}{dx} x^2 \right) dx$

$= x^2 \cdot -\cos x - \int -\cos x \cdot 2x \; dx$

$= -x^2 \cos x + 2 \int \underset{I}{x} \underset{II}{\cos x}\, dx$

$= -x^2 \cos x + 2 \left[x \int \cos x\, dx - \int \left(\int \cos x\, dx \, \frac{d}{dx} x \right) dx \right]$

$= -x^2 \cos x + 2 \left[x \cdot \sin x - \int \sin x\, dx \right]$

$= -x^2 \cos x + 2x \sin x - 2(-\cos x) + C$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$ Ans.

(18) $\int \frac{\ln x}{\sqrt{x}}\, dx$

Sol: $\int \underset{I}{\ln x} \cdot \underset{II}{x^{-1/2}}\, dx$

Integrating by Parts

$$= \ln x \cdot \int x^{-1/2}\,dx - \int\left(\int x^{-1/2}\,dx \cdot \frac{d}{dx}\ln x\right)dx$$

$$= \ln x \cdot \frac{x^{-1/2+1}}{-1/2+1} - \int\left(\frac{x^{-1/2+1}}{-1/2+1} \cdot \frac{1}{x}\right)dx$$

$$= \ln x \cdot \frac{x^{1/2}}{1/2} - \int \frac{x^{1/2}}{1/2} \cdot \frac{1}{x}\,dx$$

$$= 2\ln x \cdot \sqrt{x} - 2\int x^{1/2} \cdot x^{-1}\,dx$$

$$= 2\ln x \cdot \sqrt{x} - 2\int x^{1/2-1}\,dx$$

$$= 2\ln x\sqrt{x} - 2\int x^{-1/2}\,dx$$

$$= 2\ln x\sqrt{x} - 2 \cdot \frac{x^{-1/2+1}}{-1/2+1} + C$$

$$= 2\ln x\sqrt{x} - 2 \cdot \frac{x^{1/2}}{1/2} + C$$

$$= 2\ln x\sqrt{x} - 4\sqrt{x} + C \quad \text{Ans.}$$

Complete