Exc 3.3

Some Useful Substitution

i) $\sqrt{a^2-x^2}$ Put $x = a\sin\theta$

2) $\sqrt{a^2+x^2}$ Put $x = a\tan\theta$

3) $\sqrt{x^2-a^2}$ Put $x = a\sec\theta$

Rule - I

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

Rule - II

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C$$

Use some useful substition to evaluate the integral:

(i) $\int \frac{1}{x^2+9}\,dx$

Sol

$$\int \frac{1}{x^2+3^2}\,dx \quad \text{—(i)}$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{1}{x^2+3^2}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$$

$\sqrt{a^2+x^2}$ Put $x=a\tan\theta$

Put $x = 3\tan\theta$	$x = 3\tan\theta$
$\frac{dx}{d\theta} = 3 \cdot \sec^2\theta.$	$\tan\theta = \frac{x}{3}$
$dx = 3\sec^2\theta \, d\theta$	$\theta = \tan^{-1}\left(\frac{x}{3}\right)$

So eq (i) becomes:

$$\int \frac{dx}{x^2+9} = \int \frac{3\sec^2\theta}{(3\tan\theta)^2+9} d\theta$$

$$= \int \frac{3\sec^2\theta}{9\tan^2\theta+9} d\theta$$

$$= \int \frac{3\sec^2\theta}{9(\tan^2\theta+1)} d\theta$$

$$= \int \frac{\cancel{\sec^2\theta}}{3\cancel{\sec^2\theta}} d\theta$$

$$= \frac{1}{3}\int 1 \cdot d\theta$$

$$= \frac{1}{3}\theta + C$$

$$= \frac{1}{3} \cdot \tan^{-1}\left(\frac{x}{3}\right) + C \quad \text{Ans.}$$

(2) $\int \frac{dx}{\sqrt{5-x^2}}$

Sol

$$\int \frac{dx}{\sqrt{(\sqrt{5})^2 - x^2}}$$

$$\boxed{\sqrt{a^2 - x^2} \qquad x = a\sin\theta}$$

Put $x = \sqrt{5}\,\sin\theta$

$\frac{dx}{d\theta} = \sqrt{5}\,\cos\theta$

$dx = \sqrt{5}\,\cos\theta\, d\theta$

$\frac{x}{\sqrt{5}} = \sin\theta$

$\theta = \sin^{-1}\left(\frac{x}{\sqrt{5}}\right)$

$$\int \frac{\sqrt{5}\cos\theta\, d\theta}{\sqrt{(\sqrt{5})^2 - (\sqrt{5}\sin\theta)^2}}$$

$$\int \frac{\sqrt{5}\cos\theta\, d\theta}{\sqrt{5 - 5\sin^2\theta}}$$

$$\int \frac{\cancel{\sqrt{5}}\cos\theta\, d\theta}{\cancel{\sqrt{5}}\sqrt{1 - \sin^2\theta}}$$

$$\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\, d\theta$$

$$\int \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}}\, d\theta = \int 1\, d\theta$$

$= \theta + C$

$= \sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$

3 $\int (2x+7)(x^2+7x+3)^{4/5}\,dx.$

Sol $\int (x^2+7x+3)^{4/5}(2x+7)\,dx$

Put $t = x^2+7x+3$

$\frac{dt}{dx} = \frac{d}{dx}(x^2+7x+3)$

$\frac{dt}{dx} = 2x+7$

$dt = (2x+7)\,dx$

$\int (t)^{4/5} \cdot dt$

$= \frac{t^{4/5+1}}{4/5+1} + C$

$= \frac{t^{9/5}}{9/5} + C \Rightarrow \frac{5}{9} t^{9/5} + C$

$= \frac{5}{9}(x^2+7x+3)^{9/5} + C$

(4) $\int \frac{x^2}{x^3+1}\, dx$

Sol

Put $x^3+1 = y$

$$\frac{dy}{dx} = 3x^2$$

$$dy = 3x^2\, dx$$

$$\frac{dy}{3} = x^2\, dx$$

$$= \int \frac{1}{x^3+1} \cdot x^2\, dx$$

$$= \int \frac{1}{y} \cdot \frac{dy}{3}$$

$$= \frac{1}{3} \int \frac{1}{y}\, dy$$

$$= \frac{1}{3} \cdot \ln|y| + C$$

$$= \frac{1}{3} \ln|x^3+1| + C$$ Ans.

(5) $\int \frac{dy}{y^2+8y+20}$

Sol

$$\int \frac{dy}{y^2+2(y)(4)+(4)^2-(4)^2+20}$$

$$\int \frac{dy}{(y+4)^2+4}$$

$$\int \frac{dy}{(y+4)^2+(2)^2} = \int \frac{dy}{(2)^2+(y+4)^2}$$

$$\boxed{\sqrt{a^2+x^2} \qquad \text{Put } x = a\tan\theta}$$

Put $y+4 = 2\tan\theta$

$$\boxed{\begin{aligned} \frac{y+4}{2} &= \tan\theta \\ \theta &= \tan^{-1}\left(\frac{y+4}{2}\right) \end{aligned}}$$

Diff w.r.t to θ

$$\frac{dy}{d\theta} = 2\sec^2\theta$$

$$dy = 2\sec^2\theta\, d\theta$$

$$\int \frac{2\sec^2\theta\, d\theta}{4+4\tan^2\theta}$$

$$\int \frac{2\sec^2\theta}{4(1+\tan^2\theta)}\, d\theta$$

$$\int \frac{2\cancel{\sec^2\theta}}{4\cancel{\sec^2\theta}}\, d\theta$$

$$= \frac{1}{2}\int 1\, d\theta$$

$$= \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{y+4}{2}\right) + C$$

Ans

6 $$\int \frac{dx}{\sqrt{20-x^2-4x}}$$

Sol

$$\int \frac{dx}{\sqrt{20-(x^2+4x)}}$$

$$\int \frac{dx}{\sqrt{20-[(x)^2+2(x)(2)+(2)^2-(2)^2]}}$$

$$\int \frac{dx}{\sqrt{20-(x+2)^2+4}}$$

$$\int \frac{dx}{\sqrt{24-(x+2)^2}}$$

$$\int \frac{dx}{\sqrt{(\sqrt{24})^2-(x+2)^2}}$$

$\sqrt{a^2-x^2}$ $\qquad x = a\sin\theta$

Put $x+2 = \sqrt{24}\,\sin\theta$ $\qquad \sin\theta = \frac{x+2}{\sqrt{24}}$

$\frac{dx}{d\theta} = \sqrt{24}\,\cos\theta$ $\qquad \theta = \sin^{-1}\left(\frac{x+2}{\sqrt{24}}\right)$

$dx = \sqrt{24}\,\cos\theta\, d\theta$

$$\int \frac{\sqrt{24}\cos\theta \, d\theta}{\sqrt{24 - (\sqrt{24}\sin\theta)^2}}$$

$$\int \frac{\cancel{\sqrt{24}}\cos\theta \, d\theta}{\cancel{\sqrt{24}}\left(\sqrt{1 - \sin^2\theta}\right)}$$

$$\int \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}} \, d\theta$$

$$\int 1 \, d\theta$$

$$\theta + C$$

$$\sin^{-1}\left(\frac{x+2}{\sqrt{24}}\right) + C \quad \text{Ans.}$$

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(7) $$\int \frac{x}{(4x^2+1)^3} \, dx$$

Sol

$$\int (4x^2+1)^{-3} \cdot x \, dx$$

Put $4x^2 + 1 = t$

$$8x = \frac{dt}{dx}$$

$$dt = 8x \, dx$$

$$\frac{dt}{8} = x \, dx$$

$$= \int t^{-3} \frac{dt}{8}$$

$$= \frac{1}{8} \int t^{-3} \, dt$$

$$= \frac{1}{8} \cdot \frac{t^{-3+1}}{-3+1} + C$$

$$= \frac{1}{8} \frac{t^{-2}}{-2} + C$$

$$= \frac{-1}{16} t^{-2} + C$$

$$= \frac{-1}{16(4x^2+1)^2} + C$$

8 $\int x^4 \sqrt{3x^5 - 5} \, dx$

Sol $\int (3x^5-5)^{1/2} \cdot x^4 \, dx$

Put $3x^5 - 5 = t$

Diff w.r. to x

$$15x^4 = \frac{dt}{dx}$$

$$dt = 15x^4 \, dx$$

$$\frac{dt}{15} = x^4 \, dx$$

$$= \int (t)^{1/2} \cdot \frac{dt}{15}$$

$$= \frac{1}{15}\int t^{1/2}\, dt$$

$$= \frac{1}{15} \cdot \frac{t^{1/2+1}}{1/2+1} + C$$

$$= \frac{1}{15} \frac{t^{3/2}}{3/2} + C$$

$$= \frac{1}{15} \cdot \frac{2}{3} \cdot (3x^5-5)^{3/2} + C$$

$$= \frac{2}{45} (3x^5-5)^{3/2} + C$$

Ans.

(9) $$\int \frac{2ax+b}{ax^2+bx+c}\, dx$$

Sol

Put $t = ax^2+bx+c$

$$\frac{dt}{dx} = 2ax+b$$

$$dt = (2ax+b)\, dx$$

$$\int \frac{dt}{t} = \ln|t| + C$$

$$= \ln|ax^2+bx+c| + C \quad \text{Ans}$$

(10) $$\int \frac{dx}{(1-3x)^2}$$

Sol $$\int (1-3x)^{-2}\, dx$$

Put $t = 1-3x$

$$\frac{dt}{dx} = 0-3$$

$$dt = -3\,dx$$

$$\frac{dt}{-3} = dx$$

$$dx = \frac{-1}{3}\,dt$$

$$= \int (t)^{-2} \cdot \frac{-1}{3}\,dt$$

$$= \frac{-1}{3}\int t^{-2}\,dt$$

$$= \frac{-1}{3}\,\frac{t^{-2+1}}{-2+1} + C$$

$$= \frac{\cancel{-1}}{3}\,\frac{t^{-1}}{\cancel{-1}} + C$$

$$= \frac{1}{3t} + C$$

$$= \frac{1}{3(1-3x)} + C$$

(ii) $$\int \frac{z^3}{1+z^4}\,dz$$

Sol: Put $1+z^4 = t$

$$\frac{dt}{dZ} = 0 + 4Z^3$$

$$\frac{dt}{4} = Z^3\,dZ$$

$$\int \frac{1}{1+Z^4} \cdot Z^3\,dZ$$

$$= \int \frac{1}{t} \cdot \frac{1}{4}\,dt$$

$$= \frac{1}{4}\int \frac{1}{t}\,dt$$

$$= \frac{1}{4}\,Ln|t| + c$$

$$= \frac{1}{4}\,Ln|1+Z^4| + C \quad \text{Ans.}$$

(12) $\int \frac{\cot^{-1}x}{1+x^2}\,dx$

Sol

Put $t = \cot^{-1}x$

$$\frac{dt}{dx} = \frac{-1}{1+x^2}$$

$$-dt = \frac{dx}{1+x^2}$$

$$\int \cot^{-1}x \cdot \frac{1}{1+x^2}\,dx$$

$$= \int t \cdot -dt$$

$$= -\int t\,dt$$

$$= -\frac{t^{1+1}}{1+1} + C$$

$$= -\frac{t^2}{2} + C$$

$$= -\frac{(\cot^{-1}x)^2}{2} + C$$