Chap# 03

INTEGRATION

Reverse Process of Derivative is called integration.

$$\int f(x)\,dx$$

- $\int \longrightarrow$ Integral
- $f(x) \longrightarrow$ Function / Integrand
- $dx \longrightarrow$ Integrator

Derivative	Anti-Derivative
$\frac{d}{dx} x^n = n x^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$
$\frac{d}{dx} e^{ax} = e^{ax} \cdot a$	$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$
$\frac{d}{dx} e^{x} = e^{x}$	$\int e^{x}\,dx = e^{x} + C$ $\boxed{\int e^{x} \cdot 1\,dx = e^{x} + C}$ $\int e^{f(x)} \cdot f'(x)\,dx = e^{f(x)} + c$
$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
$\frac{d}{dx} x = 1$	$\int 1\,dx = x + C$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln x + c$

$\frac{d}{dx} a^x = a^x . \ln a$ | $\int a^x dx = \frac{a^x}{\ln a} + C$

$\int a^{bx} dx$

$= \frac{a^{bx}}{\ln a \cdot \frac{d}{dx} bx}$

$= \frac{a^{bx}}{\ln a . b}$

Exe 3.1

Evaluate the Following Integrals.

① $\int (x^2 - 3x + 9)\, dx$

Sol

$= \int x^2 dx - 3\int x\, dx + \int 9\, dx$

$= \int x^2 dx - 3\int x\, dx + 9\int 1\, dx$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$= \frac{x^{2+1}}{2+1} - 3\frac{x^{1+1}}{1+1} + 9x + C$

$= \frac{x^3}{3} - 3\frac{x^2}{2} + 9x + C$ Ans.

② $\int (y^2 + 8y + \sqrt{2})\, dy$

Sol

$= \int y^2 dy + \int 8y\, dy + \int \sqrt{2}\, dy$

$= \int y^2 dy + 8\int y\, dy + \sqrt{2}\int 1\, dy$

$$= \frac{y^{2+1}}{2+1} + 8\frac{y^{1+1}}{1+1} + \sqrt{2}\,y + C$$

$$= \frac{y^3}{3} + \overset{4}{\cancel{8}}\frac{y^2}{\cancel{2}_1} + \sqrt{2}\,y + C$$

$$= \frac{y^3}{3} + 4y^2 + \sqrt{2}\,y + C \quad \text{Ans.}$$

3

$$\int\left(\sqrt{y} + \frac{1}{y^2}\right) dy$$

Sol

$$= \int\left(y^{1/2} + y^{-2}\right) dy$$

$$= \int y^{1/2}\, dy + \int y^{-2}\, dy$$

$$= \frac{y^{1/2+1}}{1/2+1} + \frac{y^{-2+1}}{-2+1} + C$$

$$= \frac{y^{3/2}}{3/2} + \frac{y^{-1}}{-1} + C$$

$$= \frac{2}{3}y^{3/2} - \frac{1}{y} + C \quad \text{Ans.}$$

(4) $\int (4+x^2)^2 \, dx$

Sol. Using $(a+b)^2 = a^2 + b^2 + 2ab$

$= \int [(4)^2 + (x^2)^2 + 2(4)(x^2)] \, dx$

$= \int (16 + x^4 + 8x^2) \, dx$

$= \int 16 \, dx + \int x^4 \, dx + \int 8x^2 \, dx$

$= 16 \int 1 \, dx + \int x^4 \, dx + 8 \int x^2 \, dx$

$= 16 \cdot x + \frac{x^{4+1}}{4+1} + 8 \frac{x^{2+1}}{2+1} + C$

$= 16x + \frac{x^5}{5} + 8 \frac{x^3}{3} + C$ Ans.

(5) $\int (1+x)(1-x^2) \, dx$

Sol. $= \int (1 - x^2 + x - x^3) \, dx$

$= \int 1 \, dx - \int x^2 \, dx + \int x \, dx - \int x^3 \, dx$

$$= x - \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} - \frac{x^{3+1}}{3+1} + C$$

$$= x - \frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{4} + C \quad \text{Ans:}$$

6

$$\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}} \right) dx$$

Sol

$$= \int \left(x^{1/2} + \frac{1}{2} x^{-1/2} \right) dx$$

$$= \int x^{1/2}\, dx + \int \frac{1}{2} x^{-1/2}\, dx$$

$$= \frac{x^{1/2+1}}{1/2+1} + \frac{1}{2} \int x^{-1/2}\, dx + C$$

$$= \frac{x^{3/2}}{3/2} + \frac{1}{2} \cdot \frac{x^{-1/2+1}}{-1/2+1} + C$$

$$= \frac{2}{3} x^{3/2} + \frac{1}{2} \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3} x^{3/2} + \frac{1}{\cancel{2}} \cdot \cancel{2} x^{1/2} + C$$

$$= \frac{2}{3} x^{3/2} + x^{1/2} + C$$

(7) $\int (e^{4x} - e^{-1} + 1)\, dx$

Sol/ $= \int e^{4x} dx - \int e^{-1} dx + \int 1\, dx$

$= \int e^{4x} dx - e^{-1} \int 1\, dx + \int 1\, dx$

$\because \int e^{ax} dx = \frac{e^{ax}}{a} + c$

$= \frac{e^{4x}}{4} - e^{-1} \cdot x + x + C$

$= \frac{e^{4x}}{4} - \frac{x}{e} + x + C$ Ans.

(8) $\int \left(e^{\frac{9}{2}x} + \frac{1}{x}\right) dx$

Sol/ $\int e^{\frac{9}{2}x} dx + \int \frac{1}{x} dx$

$$\boxed{\int \frac{1}{x} dx = \ln x + C}$$

$$= \frac{e^{\frac{9}{2}x}}{\frac{9}{2}} + \ln x + c$$

$$= \frac{2}{9} e^{\frac{9}{2}x} + \ln x + c \quad \text{Ans.}$$

(9) $\int x e^{x^2} dx$

Sol

Put $x^2 = t$

$$2x\,dx = dt$$

$$x\,dx = \frac{dt}{2}$$

$$x^2 = t$$
$$2x = \frac{dt}{dx}$$
$$2x\,dx = dt$$

$$= \int e^t \cdot \frac{dt}{2}$$

$$= \frac{1}{2}\int e^t\,dt$$

$$= \frac{1}{2} e^t + c$$

$$= \frac{1}{2} e^{x^2} + c \quad \text{Ans.}$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$
$$\int 5^x dx = \frac{5^x}{\ln 5} + c$$
$$\int a^{2x} dx = \frac{a^{2x}}{2 \cdot \ln a}$$
$$\int 3^{5x} dx = \frac{3^{5x}}{5 \ln 3}$$

M.C.Q

(10) $\int 5^x dx$

Sol

$$\frac{5^x}{\ln 5 \cdot 1} + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

[illegible] derivative of Power [illegible]

M.C.Q

(11) $\int 7^{7y}\, dy$ ے d derivative K Power $(7y)$ نیچے لکھنا ہے۔

Sol $= \dfrac{7^{7y}}{\ln 7 \cdot 7} + C$

(12) $\int \left(x^3 + \dfrac{1}{2x} - \dfrac{1}{x^3}\right) dx$

Sol $= \int x^3\, dx + \int \dfrac{1}{2x}\, dx - \int \dfrac{1}{x^3}\, dx$

$= \int x^3\, dx + \dfrac{1}{2}\int \dfrac{1}{x}\, dx - \int x^{-3}\, dx$

$= \dfrac{x^{3+1}}{3+1} + \dfrac{1}{2} \cdot \ln x - \dfrac{x^{-3+1}}{-3+1} + C$

$= \dfrac{x^4}{4} + \dfrac{\ln x}{2} - \dfrac{x^{-2}}{-2} + C$

$= \dfrac{x^4}{4} + \dfrac{\ln x}{2} + \dfrac{1}{2x^2} + C$

(13) $\int \left(\dfrac{2x+1}{x^2+3}\right) dx$ V imp

Sol $\int \dfrac{f'(x)}{f(x)}\, dx = \ln f(x) + C$ (I)

$$\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C}$$

(2)

$$= \int \frac{2x}{x^2+3}\,dx + \int \frac{1}{x^2+3}\,dx$$

$$= \ln(x^2+3) + \int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= \ln(x^2+3) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) =$$

$$= \frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= \frac{1}{1+\frac{x^2}{a^2}}\cdot\frac{1}{a}$$

$$\frac{1}{\frac{a^2+x^2}{a^2}}\cdot\frac{1}{a}$$

$$\frac{a^{\cancel{2}}}{a^2+x^2}\cdot\frac{1}{\cancel{a}}$$

$$a\cdot\frac{1}{x^2+a^2}$$

$$\int d\left(\tan^{-1}\frac{x}{a}\right) = \int a\cdot\frac{1}{x^2+a^2}\,dx$$

$$\tan^{-1}\frac{x}{a} = a\int\frac{1}{x^2+a^2}\,dx$$

$$\frac{1}{a}\tan^{-1}\frac{x}{a} = \int\frac{1}{x^2+a^2}\,dx$$

Q#9 2nd Method

$$= \frac{1}{2}\int 2x e^{x^2}\,dx$$

$$= \frac{1}{2}\int e^{x^2}(2x)\,dx \qquad \boxed{\int e^{f(x)}\cdot f'(x)\,dx = e^{f(x)}+C}$$

$$= \frac{1}{2}e^{x^2}+C$$

(14)

$$\int \frac{e^{\tan^{-1}z}}{1+z^2}\,dz$$

Sol/

$$\int e^{\tan^{-1} z} \cdot \frac{1}{1+z^2}\, dz$$

Put $\tan^{-1} z = y$

$$\frac{d}{dz} \tan^{-1} z = \frac{dy}{dz}$$

$$\frac{1}{1+z^2}\, dz = dy$$

$$= \int e^{y} \cdot dy$$

$$= \frac{e^{y}}{1} = e^{y} + C$$

$$= e^{\tan^{-1} z} + C$$

(15) $\int (x^{3/2} + e^{3x} + x^{0})\, dx$

Sol $= \int x^{3/2}\, dx + \int e^{3x}\, dx + \int x^{0}\, dx$

$$= \frac{x^{3/2+1}}{\frac{3}{2}+1} + \frac{e^{3x}}{3} + \frac{x^{0+1}}{0+1} + C$$

$$= \frac{x^{5/2}}{5/2} + \frac{e^{3x}}{3} + \frac{x^{1}}{1} + C$$

$$= \frac{2}{5} x^{5/2} + \frac{e^{3x}}{3} + x + C \quad \text{Ans.}$$

16. $\int (3x^2 + 2x)(x^3 + x^2 + 9)^5 \, dx$

Sol.

$$\int (x^3 + x^2 + 9)^5 (3x^2 + 2x) \, dx$$

Put $x^3 + x^2 + 9 = y$

Diff w.r.t. to x

$$\frac{d}{dx}(x^3 + x^2 + 9) = \frac{dy}{dx}$$

$$(3x^2 + 2x) \, dx = dy$$

$$\int y^5 \cdot dy$$

$$= \frac{y^{5+1}}{5+1} + C = \frac{y^6}{6} + C$$

$$= \frac{(x^3 + x^2 + 9)^6}{6} + C \quad \text{Ans.}$$

CamScanner

17 $\int (5e^{5x} - x^{-3} + 3^{2x})\,dx$

Sol $= \int 5e^{5x}\,dx - \int x^{-3}\,dx + \int 3^{2x}\,dx$

$= 5\int e^{5x}\,dx - \int x^{-3}\,dx + \int 3^{2x}\,dx$

$= \cancel{5} \cdot \dfrac{e^{5x}}{\cancel{5}} - \dfrac{x^{-3+1}}{-3+1} + \dfrac{3^{2x}}{2 \cdot \ln 3} + C$

$= e^{5x} + \dfrac{x^{-2}}{2} + \dfrac{3^{2x}}{2 \cdot \ln 3} + C$

$= e^{5x} + \dfrac{1}{2x^2} + \dfrac{3^{2x}}{2 \cdot \ln 3} + C$ Ans.

18 $\int \left(z^{-1/4} + \sqrt{3z} + \dfrac{4}{z} - \dfrac{1}{e^{z}}\right) dz$

Sol $= \int z^{-1/4}\,dz + \int \sqrt{3z}\,dz + \int \dfrac{4}{z}\,dz - \int \dfrac{1}{e^{z}}\,dz$

$= \int z^{-1/4}\,dz + \int 3^{1/2} z^{1/2}\,dz + 4\int \dfrac{1}{z}\,dz - \int e^{-z}\,dz$

$= \int z^{-1/4}\,dz + 3^{1/2}\int z^{1/2}\,dz + 4\int \dfrac{1}{z} - \int e^{-z}\,dz$

$$= \frac{z^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + 3^{1/2} \cdot \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 4 \cdot \ln z - \frac{e^{-z}}{-1} + C$$

$$= \frac{z^{3/4}}{3/4} + \sqrt{3} \cdot \frac{z^{3/2}}{3/2} + 4 \ln z + e^{-z} + C$$

$$= \frac{4}{3} z^{3/4} + \frac{\sqrt{3} \cdot 2}{3} \cdot z^{3/2} + 4 \ln z + \frac{1}{e^{z}} + C$$

Complete.