Exc 2.9

Critical value:- A critical value of a Function 'F' is a number in 'C' in its domain For which $F'(c)=0$ or $F'(c)$ does not exist

Q#1 Find the critical values of the Function

i) $F(x) = 2x^2 - 6x + 8$

Sol

Diff w.r.to x.

$$\frac{d}{dx}F(x) = \frac{d}{dx}(2x^2 - 6x + 8)$$

$$F'(x) = \frac{d}{dx}2x^2 - \frac{d}{dx}6x + \frac{d}{dx}8$$

$$= 2\frac{d}{dx}x^2 - 6\frac{d}{dx}x + \frac{d}{dx}8$$

$$= 2 \cdot 2x - 6 \cdot 1 + 0$$

$$F'(x) = 4x - 6$$

For critical values Put $F'(x) = 0$

$$4x - 6 = 0$$

$$4x = 6$$

$x = \frac{6}{4}$ $\quad \boxed{x = \frac{3}{2}}$

$x = \frac{3}{2}$ is the critical value where $f'(x) = 0$

2. $f(x) = x^3 + x - 2$

Sol Diff w.r.to x.

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x^3 + x - 2)$$

$$= \frac{d}{dx}x^3 + \frac{d}{dx}x - \frac{d}{dx}2$$

$$f'(x) = 3x^2 + 1$$

For critical values put $f'(x) = 0$.

$$3x^2 + 1 = 0$$

$$3x^2 = -1$$

$$x^2 = \frac{-1}{3}$$

$$\sqrt{x^2} = \sqrt{\frac{-1}{3}} = x = \frac{i}{\sqrt{3}}$$

There is no critical value

3 $F(x) = \dfrac{x}{x^2+2}$

Sol Diff w.r.to x:

$$\frac{d}{dx}F(x) = \frac{d}{dx}\frac{x}{x^2+2}$$

$$F'(x) = \frac{(x^2+2)\frac{d}{dx}x - x\frac{d}{dx}(x^2+2)}{(x^2+2)^2}$$

$$F'(x) = \frac{(x^2+2)\cdot 1 - x\,(2x+0)}{(x^2+2)^2}$$

$$= \frac{x^2+2-2x^2}{(x^2+2)^2}$$

$$F'(x) = \frac{2-x^2}{(x^2+2)^2}$$

Put Critical values Put $F'(x) = 0$

$$\frac{2-x^2}{(x^2+2)^2} = 0.$$

$$2-x^2 = 0$$

$$\cancel{-}x^2 = \cancel{-}2$$

$$\sqrt{x^2} = \sqrt{2}$$

$$\boxed{x = \pm\sqrt{2}}$$

(4)

$$F(x) = \cos 4x$$

Sol

Diff w.r.t. to x.

$$\frac{d}{dx} F(x) = \frac{d}{dx} \cos 4x$$

$$F'(x) = -\sin 4x \cdot 4$$

$$F'(x) = -4 \sin 4x$$

For critical values put $F'(x) = 0$

$$-4 \sin 4x = 0$$

$$\sin 4x = 0$$

$$4x = \sin^{-1}(0)$$

$$4x = n\pi \quad ; n \in Z$$

$$\boxed{x = \frac{n\pi}{4}}$$

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(5)

$$F(x) = (4x-3)^{1/3}$$

Sol

Diff w.r.t. to x.

$$\frac{d}{dx} F(x) = \frac{d}{dx} (4x-3)^{1/3}$$

$$F'(x) = \frac{1}{3} (4x-3)^{\frac{1}{3}-1} \frac{d}{dx} (4x-3)$$

$$F'(x) = \frac{1}{3}(4x-3)^{-2/3}.(4)$$

$$F'(x) = \frac{4}{3}(4x-3)^{-2/3}$$

For critical values put $F'(x) = 0$

$$\frac{4}{3}(4x-3)^{-2/3} = 0$$

$$4(4x-3)^{-2/3} = 0$$

$$(4x-3)^{-2/3} = 0$$

$$\frac{1}{(4x-3)^{+2/3}} = 0$$

Put $(4x-3)^{2/3} = 0$

$$\left[(4x-3)^{\cancel{2}/\cancel{3}}\right]^{\cancel{3}/\cancel{2}} = (0)^{3/2}$$

$$4x - 3 = 0$$

$$4x = 3$$

$$x = 3/4$$

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⑥

$$F(x) = x^2 (x+1)^3$$

Sol

Diff w.r.t. x

$$\frac{d}{dx} f(x) = \frac{d}{dx} x^2 (x+1)^3$$

$$= x^2 \frac{d}{dx} (x+1)^3 + (x+1)^3 \frac{d}{dx} x^2$$

$$= x^2 \left[3(x+1)^{3-1} \cdot (1+0) \right] + (x+1)^3 \cdot 2x$$

$$f'(x) = 3x^2 (x+1)^2 + 2x (x+1)^3$$

$$f'(x) = x(x+1)^2 \left[3x + 2(x+1) \right]$$

$$= x(x+1)^2 \left[3x + 2x + 2 \right]$$

$$f'(x) = x(x+1)^2 (5x+2)$$

For critical values put $f'(x) = 0$

$$x(x+1)^2(5x+2) = 0$$

$\boxed{x = 0}$; $\sqrt{(x+1)^2} = \sqrt{0}$ $\quad$ $5x+2=0$

$x+1 = 0$ $\quad$ $5x = -2$

$\boxed{x = -1}$ $\quad$ $\boxed{x = \frac{-2}{5}}$

Ans.

Absolute Maximum
- Function ka Highest point
- Maximum value

Absolute Minimum
- Function ka Lowest Point
- Minimum value.

ABSOLUTE EXTREMA

OR

GLOBAL EXTREMA

i) A number $F(c)$ is an absolute maximum of a function 'F' if $F(x) \leq F(c)$ for every 'x' in the domain of F.

ii) A number $F(c)$ is absolute minima of a function if $F(x) \geq F(c)$ for every 'x' in the domain of F.

Absolute Extrema = Maximum value and Minimum value

Q#2 Find absolute extrema of the function on the indicated interval.

i) $F(x) = -x^2 + 6x \qquad [1, 4]$

(سب سے پہلے critical value معلوم کرنی ہے۔)

Sol

$$F(x) = -x^2 + 6x \quad \text{---(i)}$$

Diff w.r.to x.

$$F'(x) = \frac{d}{dx} -x^2 + \frac{d}{dx} 6x$$

$$F'(x) = -2x + 6$$

For critical value put $F'(x) = 0$

$$-2x + 6 = 0$$

$$-2x = -6$$

$$x = 3 \in [1, 4]$$

اگر critical value دیے گئے interval سے باہر ہے تو اس کو نہیں لینا.

$x = 1$ $\qquad$ $x = 3$ $\qquad$ $x = 4$

Put these values in given Function $F(x)$

$F(1) = -(1)^2 + 6(1)$	$F(3) = -(3)^2 + 6(3)$	$F(4) = -(4)^2 + 6(4)$
$= -1 + 6$	$= -9 + 18$	$= -16 + 24$
$= 5$	$= 9$	$= 8$

Maximum value $= 9$ at $x = 3$

Minimum value $= 5$ at $x = 1$

(ii) $F(x) = (x-1)^2 \quad -(i) \quad ; \quad [2, 5]$

Sol

Diff w.r.to x

$$F'(x) = 2(x-1)\frac{d}{dx}(x-1)$$

$$F'(x) = 2x - 2$$

For critical value Put $F'(x) = 0$

$$2x - 2 = 0$$

$$2x = 2$$

$$x = 1 \notin [2, 5]$$

$x = 2$

$F(2) = (2-1)^2$

$F(2) = 1$

$x = 5$

$F(5) = (5-1)^2$

$F(5) = 16$

Maximum Value = 16 at $x = 5$

Minimum Value = 1 at $x = 2$

(viii) $f(x) = x^{2/3}$; $[-1, 8]$

Sol

Diff w.r.to x:

$$\frac{d}{dx} f(x) = \frac{d}{dx} x^{2/3}$$

$$f'(x) = \frac{2}{3} x^{2/3 - 1}$$

$$f'(x) = \frac{2}{3} x^{-1/3}$$

$$f'(x) = \frac{2}{3x^{1/3}}$$

For critical value put $f'(x) = 0$.

$$\frac{2}{3x^{1/3}} = 0$$

$$\frac{2}{x^{-1/3}} = 0$$

For $x = 0$ Function is undefined

$x = 0$ is the critical value

$$x = 0 \in [-1, 8]$$

$x=0$	$x=-1$	$x=8$
	$F(x)= x^{2/3}$	
$F(0)= 0^{2/3}$	$F(-1)= [(-1)^2]^{1/3}$	$F(8)= [8]^{2/3}$
$F(0)=0$	$F(-1)= [1]^{1/3}$	$F(8)= (2^3)^{2/3}$
	$F(-1)= 1$	$F(8) = 4$

Maximum value $= 4$ at $x=8$
Minimum value $= 0$ at $x=0$.

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iv) $F(x) = x^3 - 6x^2 + 2$ $[-3, 2]$

Diff w.r.t on x

$F'(x) = 3x^2 - 12x$

For critical value put $F'(x)=0$

$3x^2 - 12x = 0$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x=0 \in [-3,2]$ $x=4 \notin [-3,2]$ (Not Consider)

$x=0$	$x=-3$	$x=2$
$F(0)= 0-0+2$	$F(-3)= (-3)^3 - 6(-3)^2 + 2$	$F(2)= 2^3 - 6 \cdot 2^2 + 2$
$F(0)= 2$	$= -27 - 54 + 2$	$= 8 - 24 + 2$
	$F(-3) = 81 - 79$	$F(2) = -14$

Maximum value = 2 at $x = 0$
Minimum value = -79 at $x = -3$

(v) $F(x) = 1 + 5\sin 3x \quad ; \quad \left[0, \frac{\pi}{2}\right]$

Sol: Diff w.r.t x.

$$F'(x) = 0 + 5\cos 3x \cdot 3$$

$$F'(x) = 15\cos 3x$$

For critical value put $F'(x) = 0$

$$15\cos 3x = 0$$
$$\cos 3x = 0$$
$$3x = \cos^{-1}(0)$$
$$3x = \frac{\pi}{2}$$
$$x = \frac{\pi}{6} \in \left[0, \frac{\pi}{2}\right]$$

$x = 0$	$x = \pi/6$	$x = \pi/2$
$F(0) = 1 + 5\sin 3(0)$	$F(\pi/6) = 1 + 5\sin 3\cdot\left(\frac{\pi}{6}\right)$	$F(\pi/2) = 1 + 5\sin\frac{3\pi}{2}$
$F(0) = 1$	$= 1 + 5\sin\frac{\pi}{6}$	$= 1 + 5(-1)$
	$= 1 + 5$	$= 1 - 5$
	$= 6$	$= -4$

Maximum value $= 6$ at $x = \frac{\pi}{6}$

Minimum value $= -4$ at $x = \pi/2$

(vi) $F(x) = 2\cos 2x - 4\cos x$; $[0, 2\pi]$.

Diff w.r.t on

$F'(x) = 2 \cdot -\sin 2x \cdot 2 - 4 \cdot -\sin x$.

$= -4\sin 2x + 4\sin x$

For Critical value put $f'(x) = 0$.

$-4\sin 2x + 4\sin x = 0$.

$4(-2\sin x\cos x + \sin x) = 0$

$-2\sin x\cos x + \sin x = 0$

$\sin x(-2\cos x + 1) = 0$

$\sin x = 0$

$x = \sin^{-1}(0)$

$x = \boxed{0}, \boxed{\pi}, \boxed{2\pi} \in [0, 2\pi]$

$-2\cos x + 1 = 0$

$\cos x = \frac{+1}{\cancel{-}2}$

$\cos x = \frac{1}{2}$

$x = \cos^{-1}\left(\frac{1}{2}\right)$

$x = \boxed{\frac{\pi}{3}}, \boxed{\frac{5\pi}{3}} \in [0, 2\pi]$

$x = 0, \pi, 2\pi, \pi/3, 5\pi/3$

$x = 0$

$= 2\cos^2(0) - 4\cos(0)$

$= -2$

$x = \pi$

$= 2\cos 2\pi - 4\cos\pi$

$= 2 + 4$

$= 6$

$x = 2\pi$

$= 2\cos 2(2\pi) - 4\cos(2\pi)$

$= 2\cos 4\pi - 4\cos 2\pi$

$= 2 - 4$

$= -2$

$x = \pi/3$

$= 2\cos\frac{2\pi}{3} - 4\cos\frac{\pi}{3}$

$= 2 \cdot \frac{-1}{2} - 4\,\frac{1}{2}$

$= -1 - 2$

$= -3$

$x = 5\pi/3$

$= 2\cos\frac{10\pi}{3} - 4\cos\frac{5\pi}{3}$

$= 2\cdot\left(-\frac{1}{2}\right) - 4\left(\frac{1}{2}\right) = -3$

Maximum value $= 6$ at $x = \pi$

Minimum value $= -3$ at $x = \pi/3$ and $5\pi/3$

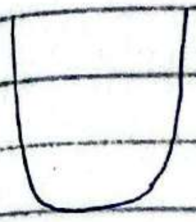
Second Derivative Test for Relative Extrema

OR

Concavity

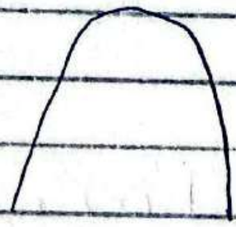
a) Concave up

Graph upwards



b) Concave down

Graph downward



Definition test for Concavity.

Let F be a function for which $F''(x)$ exist on (a,b) and

i) If $F''(x) > 0$ for all $x \in (a,b)$, then the graph is concave up on (a,b)

ii) If $F''(x) < 0$ for all $x \in (a,b)$, then the graph is concave downward on (a,b)

Q#3 Use the second derivative to determine the intervals on which function is concave upward and concave downward.

i) $F(x) = -x^2 + 7x$

Sol

Diff w.r.t x.

$$\frac{d}{dx} F(x) = \frac{d}{dx}(-x^2 + 7x)$$

$$= -\frac{d}{dx} x^2 + 7 \frac{d}{dx} x$$

$$F'(x) = -2x + 7$$

Again Diff w.r.t x.

$$F''(x) = -2 \frac{d}{dx} x + \frac{d}{dx} 7$$

$$= -2 \cdot 1 + 0$$

$$F''(x) = -2 < 0$$

So As $F''(x) < 0$

So Function is concave downward.

ii) $F(x) = -x^3 + 6x^2 + x - 1$

Sol

Diff w.r.t x

$$F'(x) = \frac{d}{dx}(-x^3 + 6x^2 + x - 1)$$

$$= \frac{d}{dx}(-x^3) + \frac{d}{dx} 6x^2 + \frac{d}{dx} x - \frac{d}{dx} 1$$

$$= -3x^2 + 12x + 1 - 0$$

$$f'(x) = -3x^2 + 12x + 1$$

Again Diff w.r.to x.

$$F''(x) = -3\frac{d}{dx}x^2 + 12\frac{d}{dx}x + \frac{d}{dx}1$$

$$= -3 \cdot 2x + 12 + 0$$

$$F''(x) = -6x + 12 \quad \text{--- (i)}$$

Put $F''(x) = 0$ (For Concave upward and downward)

$$-6x = -12$$

$$\boxed{x = 2}$$

So intervals are

$(-\infty, 2)$, $(2, \infty)$

interval $(-\infty, 2)$	Interval $(2, \infty)$
Put $x = 0$ in (i)	Put $x = 3$ in (i)
$F''(x) = -6x + 12$	$F''(x) = -6x + 12$
$F''(0) = 12$	$F''(3) = -18 + 12$
$F''(x) > 0$	$= -6 < 0$
So Function is Concave upward at $(-\infty, 2)$	$F''(x) < 0$
	So function is Concave downward at $(2, \infty)$

(iii) $F(x) = (x+5)^5$

Sol Diff w.r.to x

$$\frac{d}{dx} F(x) = \frac{d}{dx} (x+5)^5$$

$$F'(x) = 5(x+5)^{5-1} \cdot \frac{d}{dx}(x+5)$$

$$= 5(x+5)^4 (1+0)$$

$$F'(x) = 5(x+5)^4$$

Again Diff w.r.to x.

$$\frac{d}{dx} F'(x) = 5 \frac{d}{dx} (x+5)^4$$

$$F''(x) = 5 \cdot 4(x+5)^{4-1} \frac{d}{dx}(x+5)$$

$$F''(x) = 20(x+5)^3 \quad \text{—(i)}$$

Put $F''(x) = 0$ For Concave upward and downward

$$20(x+5)^3 = 0$$

$$(x+5)^3 = 0$$

$$x+5 = 0$$

$$x = -5$$

So intervals are $(-\infty, -5)$ $(-5, +\infty)$

Intervals

$(-\infty, -5)$	$(-5, +\infty)$
$x = -6$	$x = 0$
Put in (i)	Put in (i)
$f''(x) = 20(x+5)^3$	$F''(x) = 20(x+5)^3$
$f''(-6) = 20(-6+5)^3$	$f''(0) = 20(5)^3$
$= 20(-1)^3$	$F''(0) = 2500 > 0$
$= -20 < 0$	So Function is Concave upward at $(-5, \infty)$
So Function is Concave downward at $(-\infty, -5)$	

125
$\times 20$
000
$250\times$

(iv) $F(x) = x(x-4)^3$

Sol

Diff w.r.to x

$$\frac{d}{dx} F(x) = \frac{d}{dx} x(x-4)^3$$

$$F'(x) = x \frac{d}{dx}(x-4)^3 + (x-4)^3 \frac{d}{dx} x$$

$$= x \cdot 3(x-4)^2 + (x-4)^3$$

$$F'(x) = 3x(x-4)^2 + (x-4)^3$$

Again Diff w.r.to x

$$F''(x) = 3\frac{d}{dx}x(x-4)^2 + \frac{d}{dx}(x-4)^3$$

$$F''(x) = 3\left[x\frac{d}{dx}(x-4)^2 + (x-4)^2\frac{d}{dx}x\right] + \left[3(x-4)^2\right]$$

$$F''(x) = 3\left[x \cdot 2(x-4) + (x-4)^2\right] + 3(x-4)^2$$

$$= 6x(x-4) + 3(x-4)^2 + 3(x-4)^2$$

$$F''(x) = 6x(x-4) + 6(x-4)^2$$

$$F''(x) = 6(x-4)[x + x - 4]$$

$$F''(x) = 6(x-4)(2x-4)$$

$$F''(x) = 12(x-4)(x-2) \quad \text{— (i)}$$

Put $F''(x) = 0$ (For concave up or downward)

$$12(x-4)(x-2) = 0$$

$x - 4 = 0$ $\quad$ $x - 2 = 0$

$x = 4$ $\quad$ $x = 2$

So intervals are $(-\infty, 2), (2, 4), (4, \infty)$

Intervals

$(-\infty, 2)$	$(2, 4)$	$(4, \infty)$
$x = 0$	$x = 3$	$x = 6$
Put in (i)	Put in (i)	Put in (i)
$F''(0) = 12(-4)(-2)$	$F''(3) = 12(-1)(1)$	$F''(6) = 12(6-4)(6-2)$
$F''(0) = 96 > 0$	$= -12 < 0$	$= 12(2)(4)$
So Function is concave upwards in $(-\infty, 2)$	So Function is concave downward in $(2, 4)$	$= 96 > 0$
		So Function is concave upwards at $(4, \infty)$.

(v) $F(x) = x^{-1/2} + 2x$

Sol

Diff w.r.to x

$$F'(x) = \frac{1}{2} x^{1/2 - 1} + 2$$

$$F'(x) = \frac{1}{2} x^{-1/2}$$

Again Diff w.r.to x

$$F''(x) = \frac{1}{2} \cdot \frac{-1}{2} x^{-\frac{1}{2} - 1}$$

$$F''(x) = \frac{-1}{4} x^{-3/2} \quad \text{—(i)}$$

Put $F''(x) = 0$ $\qquad \frac{-1}{4} x^{-3/2} = 0$

$$x^{-3/2} = 0$$

$$x = 0$$

So intervals are $(-\infty, 0)$ $(0, \infty)$

Intervals

$(-\infty, 0)$	$(0, \infty)$
$x = -1$	$x = 1$
Put in (i)	Put in (i)
Not possible Complex answer	$F''(x) = \frac{-1}{4} x^{-3/2}$
	$F''(1) = \frac{-1}{4} < 0$
	Concave downward at $(0, \infty)$.

vi) $F(x) = x + \frac{9}{x}$

Sol Diff w.r to x.

$$F'(x) = \frac{d}{dx} x + 9 \frac{d}{dx} x^{-1}$$

$$= 1 + 9 \cdot -1 \, x^{-2}$$

$$= 1 - 9x^{-2}$$

Again Diff

$$F''(x) = \frac{d}{dx}(1) - 9 \frac{d}{dx} x^{-2}$$

$$= 0 - 9 \cdot -2 \, x^{-3}$$

$$F''(x) = 18x^{-3} = \frac{18}{x^3}$$

Put $F''(x) = 0$

$$18x^{-3} = 0$$

$$x = 0$$

So Intervals are

$(-\infty, 0)$ $(0, \infty)$

$(-\infty, 0)$	$(0, \infty)$
Put $x = -1$	Put $x = 1$
$F''(x) = \frac{18}{x^3}$	$F''(x) = \frac{18}{x^3}$
$= \frac{18}{(-1)^3}$	$= \frac{18}{(1)^3}$
$= -18 < 0$	$= 18 > 0$
So Function is Concave downward	So Function is Concave upwards.

Point of Inflection

A Point on the graph of a Function where the Concavity changes from upward or downward or reverse is called point of inflection.

Note: To Find Point of Inflection

i) First we find 2nd derivative $F''(x)$.
ii) Put $F''(x) = 0$ and Find the value of x is called point of inflection.

Q #4 Use the Second derivative to Locate all points of InFlection:

i) $F(x) = x^4 - x^3 + 2x^2 + x - 1$

Sol.

Diff w.r.to x

$$\frac{d}{dx} F(x) = \frac{d}{dx}(x^4 - x^3 + 2x^2 + x - 1)$$

$$F'(x) = 4x^3 - 3x^2 + 4x + 1$$

Again Diff w.r.to x:

$$\frac{d}{dx} F'(x) = \frac{d}{dx}(4x^3 - 3x^2 + 4x + 1)$$

$$F''(x) = 12x^2 - 6x + 4$$

Put $F''(x) = 0$

$$12x^2 - 6x + 4 = 0$$

$$2(6x^2 - 3x + 2) = 0$$

$$6x^2 - 3x + 2 = 0$$

$a = 6 \quad b = -3 \quad c = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(6)(2)}}{2(6)}$$

$$x = \frac{3 \pm \sqrt{9 - 48}}{12}$$

$$x = \frac{3 \pm \sqrt{-39}}{12}$$

$$x = \frac{3 \pm \sqrt{39}\, i}{12}$$

Hence no real point of inflection.

(iii) $F(x) = x^{5/3} + 4x$.

Sol

Diff w.r. to x.

$$\frac{d}{dx} F(x) = \frac{d}{dx} x^{5/3} + \frac{d}{dx} 4x$$

$$F'(x) = \frac{5}{3} x^{2/3} + 4$$

Again Diff w.r.t. x

$$\frac{d}{dx} F'(x) = \frac{5}{3} \frac{d}{dx} x^{2/3} + 0$$

$$F''(x) = \frac{5}{3}\left(\frac{2}{3}x^{2/3-1}\right)$$

$$F''(x) = \frac{10}{9}x^{-1/3}$$

$$F''(x) = \frac{10}{9x^{1/3}}$$

Put $F''(x) = 0$

$$\frac{10}{9x^{1/3}} = 0$$

Hence no point ~~[illegible]~~ of Inflection.

(iii) $F(x) = \sin x$

Sol

Diff w.r to x

$$\frac{d}{dx}F(x) = \frac{d}{dx}\sin x$$

$$F'(x) = \cos x$$

$$F''(x) = -\sin x$$

Put $F''(x) = 0$

$$-\sin x = 0$$

$$\sin x = 0$$

$$x = \sin^{-1}(0)$$

$$x = n\pi \qquad n \in Z$$

Hence $x = n\pi$ is the required point of inflection.

(iv) $F(x) = \cos x$

Sol

Diff w.r.to x.

$$F'(x) = -\sin x.$$

$$F''(x) = -\cos x.$$

Put $F''(x) = 0$

$$-\cos x = 0$$

$$\cos x = 0$$

$$x = \cos^{-1}(0)$$

$$x = (2n+1)\frac{\pi}{2} \quad ; \quad n \in Z$$

Hence $x = (2n+1)\frac{\pi}{2}$ is the required point of inflection.

(v) $F(x) = x - \sin x$

Sol

$$F'(x) = 1 - \cos x$$

$$F''(x) = 0 - (-\sin x)$$

$$F''(x) = \sin x$$

Put $F''(x) = 0$

$$\sin x = 0$$

$$x = \sin^{-1}(0)$$

$$x = n\pi \quad n \in Z$$

$x = n\pi$ be the required point of inflection.

vi) $F(x) = \tan x$

$\tan x$ is undefined
$x = (2n+1)\frac{\pi}{2} \quad n \in Z$

Sol:

$F'(x) = \sec^2 x$

$F''(x) = 2 \sec x \cdot \sec x \tan x$

$F''(x) = 2 \sec^2 x \tan x$

$F''(x) = 2 \frac{\tan x}{\cos^2 x}$

Put $F''(x) = 0$

$\cos^2 x \neq 0$

$\therefore \quad 2 \tan x = 0$

$\tan x = 0$

$\frac{\sin x}{\cos x} = 0$

$\cos x \neq 0$

$x = n\pi \qquad n \in Z$

$\sin x = 0$

$x = \sin^{-1}(0)$

$x = n\pi$ is the required point of inflection.

Second Derivative test For Relative Extrema

Let F be Function for which F'' exists on interval (a,b) that Contains critical number C.

i) IF $F''(c) > 0$, then $F(c)$ is a relative minima.

ii) IF $F''(c) < 0$ then $F(c)$ is a relative maxima.

Main Step:-

i) Find $\frac{dy}{dx}$ or $F'(x)$

ii) Put $F'(x) = 0$ and Find critical values.

iii) Find 2nd derivative $\frac{d^2y}{dx^2}$ or $F''(x)$.

iv) Put the value of x (critical value) in 2nd derivative and check its sign

IF sigin is +ve then relative minimum. IF sign is -ve then relative maximum.

Q#5 Use 2nd derivative test to Find the relative extrema of the Function.

i) $F(x) = -(-2x-5)^2$

Sol

Diff w.r.to x

$$\frac{d}{dx}F(x) = \frac{d}{dx} - (-2x-5)^2$$

$$F'(x) = -\left[2(-2x-5)^{2-1}\frac{d}{dx}(-2x-5)\right]$$

$$= -2(-2x-5)(-2(1)-0)$$

$$= -2(-2x-5)(-2)$$

$$F'(x) = 4(-2x-5) \quad \text{---(i)}$$

Put $F'(x) = 0$ For Critical value

$$4(-2x-5) = 0$$

$$-2x = 5$$

$$\boxed{x = \frac{-5}{2}}$$

Again Diff (i) w.r.to x

$$\frac{d}{dx}F'(x) = 4\frac{d}{dx}(-2x-5)$$

$$F''(x) = 4(-2)$$

$$F''(x) = -8$$

$F''(x) < 0$

So $F(x)$ is relative maximum at $x = -\frac{5}{2}$

(ii) $F(x) = x^3 + 3x^2 + 3x + 1$

Sol

Diff w.r.t x.

$$\frac{d}{dx} F(x) = \frac{d}{dx} x^3 + 3x^2 + 3x + 1$$

$$F'(x) = 3x^2 + 6x + 3 \quad \text{--- (1)}$$

For critical values put $F'(x) = 0$

$$3x^2 + 6x + 3 = 0$$

$$x^2 + 2x + 1 = 0$$

$$(x+1)^2 = 0$$

$$x + 1 = 0$$

$$\boxed{x = -1}$$

Again Diff (1) w.r.t x.

$$F''(x) = 6x + 6$$

at $x = -1$

$$F''(-1) = 6(-1) + 6$$

$$= -6 + 6$$

$$= 0$$

Not Maximum and not minimum.

(iii) $f(x) = 6x^5 - 10x^2$

Sol

Diff w.r.to x.

$$\frac{d}{dx} f(x) = \frac{d}{dx}(6x^5 - 10x^2)$$

$$f'(x) = 30x^4 - 20x \quad \text{—(i)}$$

For critical values put $f'(x) = 0$

$$30x^4 - 20x = 0$$

$$10x(3x^3 - 2) = 0$$

$10x = 0$ $\qquad$ $3x^3 - 2 = 0$

$\boxed{x = 0}$ $\qquad$ $3x^3 = 2$

$$x^3 = \frac{2}{3}$$

$$x = \left(\frac{2}{3}\right)^{1/3}$$

Again Diff w.r.to x.

$$\frac{d}{dx} f'(x) = \frac{d}{dx} 30x^4 - 20x.$$

$$f''(x) = 120x^3 - 20$$

At $x = 0$

$$f''(x) = 0 - 20 = -20 < 0$$

So $F(x)$ is relative maximum at $x=0$

At $x = \left(\frac{2}{3}\right)^{1/3}$

$$F''(x) = 120\left[\left(\frac{2}{3}\right)^{1/\cancel{3}}\right]^{\cancel{3}} - 20$$

$$= \overset{40}{\cancel{120}} \cdot \frac{2}{\cancel{3}_1} - 20$$

$$F''(x) = 60 > 0$$

So $F(x)$ is relative minimum at $x = \left(\frac{2}{3}\right)^{1/3}$

(iv) $F(x) = x^2 + \frac{1}{x^2}$

Sol Diff w.r.to x

$$\frac{d}{dx}F(x) = \frac{d}{dx}x^2 + \frac{d}{dx}x^{-2}$$

$$F'(x) = 2x - 2x^{-3} \quad \text{—(i)}$$

For critical values put $F'(x) = 0$

$$2x - 2x^{-3} = 0$$

$$2x - \frac{2}{x^3} = 0$$

$$\frac{2x^4 - 2}{x^3} = 0$$

$$2x^4 - 2 = 0$$

$$2x^4 = 2$$

$$x^4 = 1$$

$$\boxed{x = 1}$$

Again Diff w.r to x.

$$\frac{d}{dx} F'(x) = \frac{d}{dx}(2x) - \frac{d}{dx} 2x^{-3}$$

$$= 2 + 6x^{-4}$$

$$F''(x) = 2 + \frac{6}{x^4}$$

at $x = 1$

$$F''(x) = 2 + \frac{6}{1} = 2 + 6 = 8 > 0$$

So $f(x)$ is relative minimum at $x = 1$.

(v) $F(x) = \cos 3x$ $[0, 2\pi]$

Diff w.r to x.

$$\frac{d}{dx} f(x) = \frac{d}{dx} \cos 3x$$

$$F'(x) = -\sin 3x \cdot 3$$

$$F'(x) = -3 \sin 3x \quad \text{--- (1)}$$

For critical values Put $F'(x) = 0$

$$-3\sin 3x = 0$$

$$\sin 3x = 0$$

$$3x = \sin^{-1}(0)$$

$$3x = 0, \pi, 2\pi \ldots$$

$$3x = n\pi$$

$$x = \frac{n\pi}{3} \quad n \in \mathbb{Z}$$

$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$$

Again Diff w.r.t o x.

$$\frac{d}{dx} F'(x) = \frac{d}{dx}(-3\sin 3x)$$

$$F''(x) = -3\cos 3x \cdot 3$$

$$F''(x) = -9\cos 3x$$

$x = 0$

$F''(0) = -9 < 0$

relative Maximum at $x = 0$

$x = \pi/3$

$F''(\pi/3) = 9 > 0$

relative minimum at $x = \pi/3$

$x = 2\pi/3$

$F''(2\pi/3) = -9 < 0$

relative maximum at $x = 2\pi/3$

$x = \pi$

$F''(\pi) = 9 > 0$

relative minimum at $x = \pi$

$x = 4\pi/3$

$F''(4\pi/3) = -9 < 0$

relative maximum at $x = 4\pi/3$

$x = 5\pi/3$

$F''(5\pi/3) = 9 > 0$

relative minimum at $x = 5\pi/3$

$x = 2\pi$

$F''(2\pi) = -9 < 0$

relative maxima at $x = 2\pi$

vi) $F(x) = \cos x + \sin x \qquad [0, 2\pi]$

Sol

Diff w.r.to x:

$$\frac{d}{dx} f(x) = \frac{d}{dx} \cos x + \sin x.$$

$$f'(x) = -\sin x + \cos x.$$

For critical values put $f'(x) = 0$.

$$-\sin x + \cos x = 0$$

$$\cancel{-}\sin x = \cancel{-}\cos x$$

$$\sin x = \cos x$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$x = \tan^{-1}(1)$$

$$x = \pi/4, \ \frac{5\pi}{4}$$

Again Diff (i) w.r.to x.

$$\frac{d}{dx} f'(x) = \frac{d}{dx}(-\sin x + \cos x)$$

$$f''(x) = -\cos x - \sin x.$$

At $x = \pi/4$

$$f''(\pi/4) = -\sqrt{2} < 0$$

relative maximum at $x = \pi/4$

At $x = 5\pi/4$

$$f''(5\pi/4) = \sqrt{2} > 0$$

relative minimum at $x = 5\pi/4$.

Q#6 Determine whether the given function has a relative extremum at the indicated points.

1) $F(x) = \cos x \sin x$ $\quad x = \frac{\pi}{4}$ (critical value)

Sol Diff w.r.to x:

$$\frac{d}{dx} F(x) = \frac{d}{dx} \cos x \sin x$$

$$F'(x) = \cos x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} \cos x$$

$$F'(x) = \cos x \cdot \cos x + \sin x \cdot -\sin x$$

$$F'(x) = \cos^2 x - \sin^2 x$$

$$F'(x) = \cos 2x \quad \text{--- (i)}$$

Check : Put $x = \pi/4$

$$F'(\pi/4) = \cos 2 \cdot \pi/4$$

$$= \cos \frac{\pi}{2} = 0 \quad \text{verified.}$$

Again Diff w.r.to n

$$F''(x) = -2\sin 2x$$

At $x = \frac{\pi}{4}$

$$F''(\pi/4) = -2 \cdot \sin 2 \cdot \frac{\pi}{4}$$

$$= -2\sin\frac{\pi}{2}$$

$$= -2 < 0$$

So $F(x)$ is relative maximum at $x = \pi/4$

Critical value.

(iii) $F(x) = x\sin x$ $\quad x = 0$

Sol Diff w.r. to n.

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x\sin x)$$

$$F'(x) = x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x$$

$$= x \cdot \cos x + \sin x \cdot 1$$

$$f'(x) = x\cos x + \sin x$$

Check put $x = 0$

$$F'(0) = 0 \cdot \cos(0) + \sin(0)$$

$$= 0 + 0$$

$$= 0$$

Verified

Again Diff w.r.to x

$$F''(x) = \frac{d}{dx} x \cos x + \frac{d}{dx} \sin x$$

$$F''(x) = x \frac{d}{dx} \cos x + \cos x \frac{d}{dx} x + \frac{d}{dx} \sin x$$

$$= x \cdot -\sin x + \cos x \cdot 1 + \cos x$$

$$= -x \sin x + \cos x + \cos x$$

$$f''(x) = -x \sin x + 2 \cos x$$

At $x = 0$

$$F''(0) = -0 \sin(0) + 2 \cos(0)$$

$$= 0 + 2(1)$$

$$F''(0) = 2 > 0$$

So $F(x)$ is relative minimum at $x = 0$.

(iii) $F(x) = \tan^2 x$ $\quad x = \pi$ (critical value)

Sol Diff w.r.to x

$$\frac{d}{dx} F(x) = \frac{d}{dx} \tan^2 x$$

$$F'(x) = 2 \tan x \cdot \frac{d}{dx} \tan x$$

$$F'(x) = 2 \tan x \cdot \sec^2 x$$

Check $x = \pi$

$F'(x) = 2\tan\pi\,(\sec\pi)^2$

$F'(\pi) = 0$ verified.

Again Diff w.r.to x.

$$\frac{d}{dx}F'(x) = 2\frac{d}{dx}\tan x\sec^2 x.$$

$$F''(x) = 2\cdot\left[\tan x\frac{d}{dx}\sec^2 x + \sec^2 x\frac{d}{dx}\tan x\right]$$

$$F''(x) = 2\left[\tan x\cdot 2\sec x\cdot\sec x\tan x + \sec^2 x\cdot\sec^2 x\right]$$

$$F''(x) = 4\sec^2 x\tan^2 x + 2\sec^4 x$$

At $x = \pi$

$$F''(\pi) = 4(\sec\pi)^2(\tan\pi)^2 + 2(\sec\pi)^4$$

$$= 0 + \frac{2}{(-1)^4} = \frac{2}{1} = 2 > 0$$

So $F(x)$ has relative minimum at $x = \pi$.

(v) $F(x) = (1+\sin x)^3$ $\quad x = \frac{\pi}{8}$ (critical value)

Diff w.r.to x.

$$\frac{d}{dx}F(x) = \frac{d}{dx}(1+\sin x)^3$$

$$F'(x) = 3(1+\sin x)^{3-1}\frac{d}{dx}(1+\sin x)$$

$$= 3(1+\sin x)^2 \cdot (0+\cos x)$$

$$F'(x) = 3\cos x\,(1+\sin x)^2$$

At $x = \pi/8$

$$F'(x) = 3\cos\frac{\pi}{8}\left(1+\sin\frac{\pi}{8}\right)^2$$

$F'(x) \neq 0$ at $x = \pi/8$

Hence $x = \pi/8$ is not a critical value. So we cannot find local or relative extrema at $x = \pi/8$.

Complete