Exc 2.7

Find the Derivative of Functions.

i) $y = \left(x - \frac{1}{x^2}\right)^5$

Sol: Diff w.r.t. x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x - \frac{1}{x^2}\right)^5$$

$$= 5\left(x - \frac{1}{x^2}\right)^{5-1} \cdot \frac{d}{dx}\left(x - \frac{1}{x^2}\right)$$

$$= 5\left(x - \frac{1}{x^2}\right)^4 \cdot \left[\frac{d}{dx}x - \frac{d}{dx}x^{-2}\right]$$

$$= 5\left(x - \frac{1}{x^2}\right)^4 \left(\left[1\right] - \left[-2x^{-2-1}\right]\right)$$

$$= 5\left(x - \frac{1}{x^2}\right)^4 \left(1 + 2x^{-3}\right)$$

$$= 5\left(x - \frac{1}{x^2}\right)^4 \left(1 + \frac{2}{x^3}\right) \quad \text{Ans.}$$

Q#2 $F(x) = \left(\frac{x^2-1}{-x^2+1}\right)^2$

Sol: Diff w.r.t. x

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)^2$$

$$F'(x) = 2\left(\frac{x^2-1}{x^2+1}\right)^{2-1} \cdot \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$= 2\left(\frac{x^2-1}{x^2+1}\right) \cdot \frac{(x^2+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= 2\left(\frac{x^2-1}{x^2+1}\right) \cdot \frac{(x^2+1)\cdot(2x-0) - (x^2-1)(2x+0)}{(x^2+1)^2}$$

$$= 2\left(\frac{x^2-1}{x^2+1}\right) \cdot \frac{\cancel{2x^3} + 2x - \cancel{2x^3} + 2x}{(x^2+1)^2}$$

$$= 2\left(\frac{x^2-1}{x^2+1}\right) \cdot \frac{4x}{(x^2+1)^2}$$

$$= \frac{8x(x^2-1)}{(x^2+1)^3} \quad \text{Ans}$$

(3) $y = (3x-1)^4(-2x+9)^5$

Sol

Diff w.r.t x.

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{(3x-1)^4}_{U}\underbrace{(-2x+9)^5}_{V}$$

$$\frac{dy}{dx} = (3x-1)^4 \frac{d}{dx}(-2x+9)^5 + (-2x+9)^5 \frac{d}{dx}(3x-1)^4$$

$$= (3x-1)^4 \left[5(-2x+9)^{5-1} \frac{d}{dx}(-2x+9)\right] + (-2x+9)^5 \left[4(3x-1)^{4-1} \frac{d}{dx}(3x-1)\right]$$

$$= (3x-1)^4 \left[5(-2x+9)^4(-2+0)\right] + (-2x+9)^5 \left[4(3x-1)^3 \cdot (3-0)\right]$$

$$= (3x-1)^4 \left[-10(-2x+9)^4\right] + (-2x+9)^5 \left[12(3x-1)^3\right]$$

$$= -10(3x-1)^4(-2x+9)^4 + 12(-2x+9)^5(3x-1)^3$$

$$= 2(3x-1)^3(-2x+9)^4\left[-5(3x-1) + 6(-2x+9)\right]$$

$$= 2(3x-1)^3(-2x+9)^4\left[-15x+5-12x+54\right]$$

$$= 2(3x-1)^3(-2x+9)^4\left[-27x+59\right] \quad \text{Ans.}$$

④ $F(\theta) = (2\theta+1)^3 \tan^2\theta.$

Sol

Diff w.r.to θ.

$$\frac{d}{d\theta}F(\theta) = \frac{d}{d\theta}(2\theta+1)^3 \tan^2\theta.$$

$$F'(\theta) = (2\theta+1)^3 \frac{d}{d\theta}\tan^2\theta + \tan^2\theta \frac{d}{d\theta}(2\theta+1)^3$$

$$F'(\theta) = (2\theta+1)^3\left[2\tan\theta \frac{d}{d\theta}\tan\theta\right] + \tan^2\theta\left[3(2\theta+1)^2 \frac{d}{d\theta}(2\theta+1)\right]$$

$$= (2\theta+1)^3\left[2\tan\theta \cdot \sec^2\theta\right] + \tan^2\theta\left[3(2\theta+1)^2 \cdot 2\right]$$

$= 2(2\theta+1)^3 \tan\theta \sec^2\theta + 6\tan^2\theta\,(2\theta+1)^2$

Ans.

(5) $y = \sin 2x \cos 3x$

Sol

Diff w.r to x.

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 2x \cos 3x)$$

$$= \sin 2x \frac{d}{dx}\cos 3x + \cos 3x \frac{d}{dx}\sin 2x$$

$$= \sin 2x\,[-\sin 3x \cdot 3] + \cos 3x\,[\cos 2x \cdot 2]$$

$$= -3\sin 2x \sin 3x + 2\cos 2x \cos 3x$$ Ans.

(6) $F(x) = (\sec 4x + \tan 2x)^5$

Sol

Diff w.r to x

$$\frac{d}{dx}F(x) = \frac{d}{dx}(\sec 4x + \tan 2x)^5$$

$$= 5(\sec 4x + \tan 2x)^{5-1}\frac{d}{dx}(\sec 4x + \tan 2x)$$

$$= 5(\sec 4x + \tan 2x)^4\left[\sec 4x \tan 4x \cdot \frac{d}{dx}4x + \sec^2 2x \frac{d}{dx}2x\right]$$

$$= 5(\sec 4x + \tan 2x)^4\left[4\sec 4x \tan 4x + 2\sec^2 2x\right]$$

(7) $h(t) = \dfrac{t + \sin^2 4t}{10 + \cos 3t}$

Sol Diff w.r.to t.

$$\frac{d}{dt} h(t) = \frac{d}{dt}\left(\frac{t + \sin^2 4t}{10 + \cos 3t}\right)$$

$$= \frac{(10 + \cos 3t)\frac{d}{dt}(t + \sin^2 4t) - (t + \sin^2 4t)\frac{d}{dt}(10 + \cos 3t)}{(10 + \cos 3t)^2}$$

$$= \frac{(10 + \cos 3t)\left[\frac{d}{dt}t + \frac{d}{dt}(\sin 4t)^2\right] - (t + \sin^2 4t)\left[0 + \frac{d}{dt}\cos 3t\right]}{(10 + \cos 3t)^2}$$

$$= \frac{(10 + \cos 3t)\left[1 + 2(\sin 4t)^{2-1}\frac{d}{dt}\sin 4t\right] - (t + \sin^2 4t)\left[-\sin 3t\frac{d}{dt}3t\right]}{(10 + \cos 3t)^2}$$

$$= \frac{(10 + \cos 3t)\left[1 + 2(\sin 4t).\cos 4t \cdot 4\right] - (t + \sin^2 4t)\left[-3\sin 3t\right]}{(10 + \cos 3t)^2}$$

$$= \frac{(10 + \cos 3t)\left[1 + 8\sin 4t\cos 4t\right] - (-3\sin 3t)(t + \sin^2 4t)}{(10 + \cos 3t)^2}$$

(8) $f(x) = \tan\left(\cos\frac{x}{2}\right)$

Sol Diff w.r.to x

$$\frac{d}{dx} F(x) = \frac{d}{dx} \tan\left(\cos\frac{x}{2}\right)$$

$$F'(x) = \sec^2\left(\cos\frac{x}{2}\right) \frac{d}{dx} \cos\frac{x}{2}$$

$$= \sec^2\left(\cos\frac{x}{2}\right) \cdot -\sin\frac{x}{2} \frac{d}{dx} \frac{x}{2}$$

$$= \sec^2\left(\cos\frac{x}{2}\right) \cdot -\sin\frac{x}{2} \cdot \frac{1}{2} \frac{d}{dx} x$$

$$= -\frac{1}{2} \sec^2\left(\cos\frac{x}{2}\right) \cdot \sin\frac{x}{2} \cdot 1$$

$$= \frac{-1}{2} \sin\frac{x}{2} \sec^2\left(\cos\frac{x}{2}\right) \quad \text{Ans.}$$

Implicit Differentiation

When the relationship between x and y is given equation but y is not isolated

e.g. $x + y^2 = 4$

Implicit

$$xy = \sin xy + y$$

Diff w.r.t x

$$\frac{d}{dx}(xy) = \frac{d}{dx} \sin xy + \frac{dy}{dx}$$

اس میں direct derivative اپلائی کرنا ہے
y الگ نہیں کرنا۔

Explicit

$$xy = x^2 + y$$

$$xy - y = x^2$$

$$y = \frac{x^2}{x-1}$$

اس میں y الگ ہوگا۔
یہاں پہ پہلے y کو الگ کرنا ہے۔ پھر derivative لینا ہے۔

Q#9 Use Implicit differentiation to find $\frac{dy}{dx}$.

$$4x^2 + y^2 = 8$$

Diff w.r.to x

$$\frac{d}{dx}4x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}8$$

$$4\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$4\left[2x^{2-1}\right] + \left[2y^{2-1}\frac{dy}{dx}\right] = 0$$

$$8x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -8x$$

$$\frac{dy}{dx} = \frac{-8x}{2y} = \boxed{\frac{-4x}{y} = \frac{dy}{dx}}$$

(10)

$$x + xy - y^2 - 20 = 0$$

Diff w.r.to x

$$\frac{d}{dx}x + \frac{d}{dx}xy - \frac{d}{dx}y^2 - \frac{d}{dx}20 = 0$$

$$\Rightarrow 1 + \left[x\frac{dy}{dx} + y\frac{d}{dx}x\right] - \left[2y^{2-1}\frac{dy}{dx}\right] - 0 = 0$$

$$\Rightarrow 1 + x\frac{dy}{dx} + y \cdot 1 - 2y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = -y - 1$$

$$\frac{dy}{dx}(x - 2y) = -y - 1$$

$$\frac{dy}{dx} = \frac{-y-1}{x-2y} \quad \text{Ans.}$$

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$$y^4 - y^2 = 10x - 3$$

Diff w.r.to x.

$$\frac{d}{dx}y^4 - \frac{d}{dx}y^2 = \frac{d}{dx}10x - \frac{d}{dx}3$$

$$\left[4y^{4-1}\frac{dy}{dx}\right] - \left[2y^{2-1}\frac{dy}{dx}\right] = 10\frac{d}{dx}x - 0$$

$$4y^3\frac{dy}{dx} - 2y\frac{dy}{dx} = 10 \cdot 1$$

$$\frac{dy}{dx}(4y^3 - 2y) = 10$$

$$\frac{dy}{dx} = \frac{10}{4y^3 - 2y}$$

$$\frac{dy}{dx} = \frac{5}{2y^3 - y} \quad \text{Ans.}$$

(12) $x^3y^2 = 2x^2 + y^2$

Sol

Diff w.r. to x.

$$\frac{d}{dx}(x^3y^2) = \frac{d}{dx}2x^2 + \frac{d}{dx}y^2$$

$$x^3\frac{d}{dx}y^2 + y^2\frac{d}{dx}x^3 = 2\frac{d}{dx}x^2 + \frac{d}{dx}y^2$$

$$x^3\left[2y\frac{dy}{dx}\right] + y^2\left[3x^2\right] = 2\left[2x\right] + \left[2y\frac{dy}{dx}\right]$$

$$2x^3y\frac{dy}{dx} + 3y^2x^2 = 4x + 2y\frac{dy}{dx}$$

$$2x^3y\frac{dy}{dx} - 2y\frac{dy}{dx} = 4x - 3x^2y^2$$

$$\frac{dy}{dx}\left(2x^3y - 2y\right) = 4x - 3x^2y^2$$

$$\frac{dy}{dx} = \frac{4x - 3x^2y^2}{2x^3y - 2y} \quad \text{Ans.}$$

(13) $xy = \sin x + y$

Sol

Diff w.r. to x

$$\frac{d}{dx}(xy) = \frac{d}{dx}(\sin x + y)$$

$$\Rightarrow x \cdot \frac{dy}{dx} + y\frac{d}{dx}x = \frac{d}{dx}\sin x + \frac{dy}{dx}$$

$$\Rightarrow x\frac{dy}{dx} + y \cdot 1 = \cos x + \frac{dy}{dx}.$$

$$x\frac{dy}{dx} - \frac{dy}{dx} = \cos x - y$$

$$\frac{dy}{dx}(x-1) = \cos x - y$$

$$\frac{dy}{dx} = \frac{\cos x - y}{x-1} \quad \text{Ans.}$$

(14) $x + y = \cos xy.$

Sol Diff w.r.to x.

$$\frac{d}{dx}(x+y) = \frac{d}{dx}(\cos xy)$$

$$\frac{d}{dx}x + \frac{d}{dx}y = \frac{d}{dx}\cos xy$$

$$1 + \frac{dy}{dx} = -\sin xy\frac{d}{dx}(xy)$$

$$1 + \frac{dy}{dx} = -\sin xy\left[x\frac{dy}{dx} + y\frac{d}{dx}x\right]$$

$$1 + \frac{dy}{dx} = -\sin xy\left[x\frac{dy}{dx} + y\right]$$

$$1 + \frac{dy}{dx} = -x \sin xy \frac{dy}{dx} - y \sin xy$$

$$\frac{dy}{dx} + x \sin xy \frac{dy}{dx} = -1 - y \sin xy$$

$$\frac{dy}{dx}(1 + x \sin xy) = -1 - y \sin xy$$

$$\frac{dy}{dx} = \frac{-1 - y \sin xy}{1 + x \sin xy}$$

(15) $x \sin y - y \cos x = 1$

$$\frac{d}{dx}(x \sin y - y \cos x) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x \sin y) - \frac{d}{dx}(y \cos x) = 0$$

$$\left[x \frac{d}{dx} \sin y + \sin y \frac{d}{dx} x\right] - \left[y \frac{d}{dx} \cos x + \cos x \frac{d}{dx} y\right] = 0$$

$$x \cdot \cos y \cdot \frac{dy}{dx} + \sin y - y(-\sin x) - \cos x \frac{dy}{dx} = 0$$

$$x \cos y \frac{dy}{dx} + \sin y + y \sin x - \cos x \frac{dy}{dx} = 0$$

$$x \cos y \frac{dy}{dx} - \cos x \frac{dy}{dx} = -\sin y - y \sin x$$

$$\frac{dy}{dx}(x \cos y - \cos x) = -\sin y - y \sin x$$

$$\frac{dy}{dx} = \frac{-\sin y - y\sin x}{x\cos y - \cos x}$$ Ans.

(16) $\sin y = y\cos 2x$

Sol Diff w.r.to x.

$$\frac{d}{dx}\sin y = \frac{d}{dx}(y\cos 2x)$$

$$\frac{d}{dx}\sin y = y\frac{d}{dx}\cos 2x + \cos 2x\frac{d}{dx}y$$

$$\cos y\frac{dy}{dx} = y[-\sin 2x \cdot 2] + \cos 2x\frac{dy}{dx}$$

$$\cos y\frac{dy}{dx} - \cos 2x\frac{dy}{dx} = -2y\sin 2x$$

$$\frac{dy}{dx}(\cos y - \cos 2x) = -2y\sin 2x$$

$$\frac{dy}{dx} = \frac{-2y\sin 2x}{\cos y - \cos 2x}$$ Ans.

Derivative of Exponential Function.

$$\frac{d}{dx} e^x = e^x$$

When ever we take derivative of exponential function then we always take derivative of its Power.

Example:

$$\frac{d}{dx} e^{5x} = e^{5x} \frac{d}{dx} 5x$$

$$= e^{5x} \cdot 5 \frac{d}{dx} x$$

$$= 5e^{5x} \quad \text{Ans}$$

Derivative of Logarithmic function:

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

When ever we take derivative of Logarithmic function then we always take derivative of its angle.

Example:

$$\frac{d}{dx} \ln 3x = \frac{1}{3x} \frac{d}{dx} 3x$$

$$= \frac{1}{\cancel{3}x} \cdot \cancel{3}(1)$$

$$= \frac{1}{x} \quad \text{Ans}$$

Q #17 $\quad y = x^3 e^{5x}$

Sol. Diff w.r.to x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 \cdot e^{5x}\right)$$

$$= x^3 \frac{d}{dx} e^{5x} + e^{5x} \frac{d}{dx} x^3$$

$$= x^3 \left[e^{5x} \frac{d}{dx} 5x\right] + e^{5x}\left[3x^{3-1}\right]$$

$$= x^3\left[e^{5x} \cdot 5\right] + e^{5x}\left[3x^2\right]$$

$$= 5x^3 e^{5x} + 3x^2 e^{5x}$$

$$= x^2 e^{5x}\left(5x + 3\right) \quad \text{Ans.}$$

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(18) $\quad y = e^{4x}(1 + \ln x)$

Sol.

Diff w.r.to x

$$\frac{dy}{dx} = \frac{d}{dx} e^{4x} \cdot (1 + \ln x)$$

Using product rule.

$$\frac{dy}{dx} = e^{4x} \frac{d}{dx}(1 + \ln x) + (1 + \ln x)\frac{d}{dx} e^{4x}$$

$$= e^{4x}\left[\frac{d}{dx} 1 + \frac{d}{dx} \ln x\right] + (1 + \ln x)\left[e^{4x} \frac{d}{dx} 4x\right]$$

$$= e^{4x}\left[0+\frac{1}{x}\right]+(1+\ln x)\left[e^{4x}\cdot 4\right]$$

$$= \frac{e^{4x}}{x}+4e^{4x}(1+\ln x)$$

$$= e^{4x}\left[\frac{1}{x}+4(1+\ln x)\right]$$

$$= e^{4x}\left[\frac{1+4x(1+\ln x)}{x}\right] \quad \text{Ans}$$

(19)

$$y = \frac{e^{2x}}{e^{-2x}+1}$$

Sol: Diff w.r.to x

$$\frac{dy}{dx} = \frac{d}{dx}\frac{e^{2x}}{e^{-2x}+1} \quad \text{(Apply Quotient rule)}$$

$$= \frac{(e^{-2x}+1)\frac{d}{dx}e^{2x} - e^{2x}\frac{d}{dx}(e^{-2x}+1)}{(e^{-2x}+1)^2}$$

$$= \frac{(e^{-2x}+1)\left[e^{2x}\frac{d}{dx}2x\right] - e^{+2x}\left[e^{-2x}\frac{d}{dx}-2x+\frac{d}{dx}1\right]}{(e^{-2x}+1)^2}$$

$$= \frac{(e^{-2x}+1)\left[2e^{2x}\right] - e^{+2x}\left[-2e^{-2x}\right]}{(e^{-2x}+1)^2}$$

$$= \frac{2e^{2x}e^{-2x}+2e^{2x}+2e^{+2x}e^{-2x}}{(e^{-2x}+1)^2}$$

$$= \frac{2e^{0} + 2e^{2x} + 2e^{0}}{(e^{-2x}+1)^2}$$

$$= \frac{4 + 2e^{2x}}{(e^{-2x}+1)^2} = \frac{2(2 + e^{2x})}{(e^{-2x}+1)^2} \quad \text{Ans.}$$

(2e) $y = \ln(e^{x} + e^{-x})$

Sol

Diff w.r.t. to x

$$\frac{dy}{dx} = \frac{d}{dx}\ln(e^{x} + e^{-x})$$

$$= \frac{1}{e^{x}+e^{-x}} \frac{d}{dx}(e^{x} + e^{-x})$$

$$= \frac{1}{e^{x}+e^{-x}} \cdot \left[\frac{d}{dx}e^{x} + \frac{d}{dx}e^{-x}\right]$$

$$= \frac{1}{(e^{x}+e^{-x})}\left[e^{x} + e^{-x}\frac{d}{dx}(-x)\right]$$

$$= \frac{1}{(e^{x}+e^{-x})}\left[e^{x} + e^{-x}(-1)\right]$$

$$= \frac{1}{(e^{x}+e^{-x})}\left[e^{x} - e^{-x}\right]$$

$$= \frac{e^{x} - e^{-x}}{e^{x}+e^{-x}} \quad \text{Ans}$$

(21) $y = \ln\left(x + \sqrt{x^2+1}\right)$

Sol Diff w.r. to x.

$$\frac{dy}{dx} = \frac{d}{dx} \ln\left(x + \sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \frac{d}{dx}\left(x + \sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \cdot \left[\frac{d}{dx} x + \frac{d}{dx}(x^2+1)^{1/2}\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}} \cdot \left[1 + \frac{1}{2}(x^2+1)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(x^2+1)\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}} \left[1 + \frac{1}{2}(x^2+1)^{-1/2} \cdot (2x+0)\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}} \left[1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right]$$

$$= \frac{1}{x+\sqrt{x^2+1}} \left[1 + \frac{x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{\cancel{x+\sqrt{x^2+1}}} \left[\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{\sqrt{x^2+1}} \quad \text{Ans.}$$

22. $y = e^{-3x} \cos x.$

Sol Diff w.r.to x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{-3x} \cdot \cos x\right).$$

$$= e^{-3x} \cdot \frac{d}{dx} \cos x + \cos x \frac{d}{dx} e^{-3x}$$

$$= e^{-3x}\left[-\sin x\right] + \cos x \left[e^{-3x} \frac{d}{dx} -3x\right]$$

$$= -e^{-3x} \sin x + \cos x \left[e^{-3x} \cdot -3\right]$$

$$= -e^{-3x} \sin x - 3 e^{-3x} \cos x.$$

$$= -e^{-3x}\left[\sin x + 3 \cos x\right] \quad \text{Ans.}$$

CHAIN RULE

This rule is used when atleast three or more than three variable involved.

$x, y, t \Rightarrow$ Find $\frac{dy}{dx} \Rightarrow \frac{dy}{dt} \frac{dt}{dx}$

$x, y, \theta \Rightarrow$ Find $\frac{dy}{dx} \Rightarrow \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$x, y, z \Rightarrow$ Find $\frac{dy}{dx} \Rightarrow \frac{dy}{dz} \cdot \frac{dz}{dx}$

Variables

x, y, z
t, θ

Find $\frac{dy}{dx}$.

(23) $x = t + \frac{1}{t}$ — (1) $\qquad y = t + 1$ — (2)

Diff (i) w.r.to t

$$\frac{dx}{dt} = \frac{d}{dt}(t + t^{-1})$$

$$= \frac{d}{dt}t + \frac{d}{dt}t^{-1}$$

$$= 1 + [-1\, t^{-1-1}]$$

$$= 1 - \frac{1}{t^2}$$

$$\boxed{\frac{dx}{dt} = \frac{t^2 - 1}{t^2}} \qquad \boxed{\frac{dt}{dx} = \frac{t^2}{t^2-1}}$$

Diff eq (ii) w.r.to t

$$\frac{dy}{dt} = \frac{d}{dt}(t+1)$$

$$= \frac{d}{dt}t + \frac{d}{dt}1$$

$$\frac{dy}{dt} = 1 + 0$$

$$\boxed{\frac{dy}{dt} = 1}$$

Now by chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= 1 \cdot \frac{t^2}{t^2-1}$$

$$\frac{dy}{dx} = \frac{t^2}{t^2-1} \quad \text{Ans.}$$

24) $x = t^2 + \frac{1}{t^2}$ —(i) $\quad y = t - \frac{1}{t}$ — (ii)

Diff (i) w.r.to t

$$\frac{dx}{dt} = \frac{d}{dt}\left(t^2 + t^{-2}\right)$$

$$= \frac{d}{dt}t^2 + \frac{d}{dt}t^{-2}$$

$$= 2t^{2-1} + \left[-2t^{-2-1}\right]$$

$$= 2t - 2t^{-3}$$

$$= 2t - \frac{2}{t^3}$$

$$\boxed{\frac{dx}{dt} = \frac{2t^4 - 2}{t^3}} \text{ or } \boxed{\frac{dt}{dx} = \frac{t^3}{2t^4 - 2}}$$

Diff (ii) w.r.to t

$$\frac{dy}{dt} = \frac{d}{dt}\left(t - t^{-1}\right)$$

$$= \frac{d}{dt}t - \frac{d}{dt}t^{-1}$$

$$= 1 - \left[-1\, t^{-1-1}\right]$$

$$= 1 + t^{-2}$$

$$= 1 + \frac{1}{t^2}$$

$$\frac{dy}{dt} = \frac{t^2 + 1}{t^2}$$

Now by Chain Rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{t^2+1}{\cancel{t^2}} \cdot \frac{t^{\cancel{3}}}{2t^4 - 2}$$

$$= \frac{t(t^2+1)}{2(t^4-1)} = \frac{t\,\cancel{(t^2+1)}}{2\cancel{(t^2+1)}(t^2-1)}$$

$$\frac{dy}{dx} = \frac{t}{2(t^2-1)} \quad \text{Ans.}$$

25 $x = \dfrac{\theta^2-1}{\theta^2+1}$ — (1)

Sol

Diff (i) w.r.to θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\,\frac{\theta^2-1}{\theta^2+1}$$

$$= \frac{(\theta^2+1)\frac{d}{d\theta}(\theta^2-1) - (\theta^2-1)\frac{d}{d\theta}(\theta^2+1)}{(\theta^2+1)^2}$$

$$= \frac{(\theta^2+1)(2\theta-0) - (\theta^2-1)(2\theta+0)}{(\theta^2+1)^2}$$

$$= \frac{2\theta(\theta^2+1) - 2\theta(\theta^2-1)}{(\theta^2+1)^2}$$

$$= \frac{2\theta[\cancel{\theta^2}+1-\cancel{\theta^2}+1]}{(\theta^2+1)^2}$$

$$\frac{dx}{d\theta} = \frac{4\theta}{(\theta^2+1)^2} \quad \text{or} \quad \frac{d\theta}{dx} = \frac{(\theta^2+1)^2}{4\theta}$$

$y = \dfrac{\theta-1}{\theta+1}$ — (2)

Diff (ii) w.r.to θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\,\frac{\theta-1}{\theta+1}$$

$$= \frac{(\theta+1)\frac{d}{d\theta}(\theta-1) - (\theta-1)\frac{d}{d\theta}(\theta+1)}{(\theta+1)^2}$$

$$= \frac{(\theta+1)[1-0] - (\theta-1)[1+0]}{(\theta+1)^2}$$

$$= \frac{\cancel{\theta}+1-\cancel{\theta}+1}{(\theta+1)^2}$$

$$\frac{dy}{d\theta} = \frac{2}{(\theta+1)^2}$$

Now by chain Rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= \frac{2}{(\theta+1)^2}\cdot\frac{(\theta^2+1)^2}{4\theta} = \frac{(\theta^2+1)^2}{2\theta(\theta+1)^2} \quad \text{Ans}$$

26 $x = \sin 2\theta$ —(i)

$y = \cos 4\theta$ —(ii)

Sol

Diff (i) w.r.to θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin 2\theta)$$

$$= \cos 2\theta \cdot \frac{d}{d\theta} 2\theta$$

$$= \cos 2\theta \cdot 2$$

$$\frac{dx}{d\theta} = 2\cos 2\theta$$

$$\boxed{\frac{d\theta}{dx} = \frac{1}{2\cos 2\theta}}$$

Diff (ii) w.r.to θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta} \cos 4\theta$$

$$= -\sin 4\theta \frac{d}{d\theta} 4\theta$$

$$= -\sin 4\theta \cdot 4$$

$$\boxed{\frac{dy}{d\theta} = -4\sin 4\theta}$$

By Chain Rule.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= -4\sin 4\theta \cdot \frac{1}{2\cos 2\theta}$$

$$= \frac{-2\sin 4\theta}{\cos 2\theta}$$ Ans.

Derivative (d/dx)

$y = x^2 + 1$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{d}{dx}1$$

$$= 2x + 0$$

$$\frac{dy}{dx} = 2x$$

Differential (d)

$y = x^2 + 1$

Taking Differential.

$$dy = d(x^2 + 1)$$

$$= d(x^2) + d(1)$$

$$= 2x\,dx + 0$$

$$dy = 2x\,dx$$

In General

$$dx \simeq \Delta x$$

$$dy \neq \Delta y.$$

Find Δy and dy.

27) $y = x^2 + 1$

Sol $y = F(x) = x^2 + 1$

$$\Delta y = F(x + \Delta x) - F(x)$$

$$F(x) = x^2 + 1$$

$$F(x + \Delta x) = (x + \Delta x)^2 + 1$$

$$F(x + \Delta x) = x^2 + \Delta x^2 + 2x\Delta x + 1$$

$$\Delta y = \cancel{x^2} + \Delta x^2 + 2x\Delta x + \cancel{1} - \cancel{x^2} - \cancel{1}$$

$$\Delta y = 2x\Delta x + (\Delta x)^2 \quad \text{Ans.}$$

Differentials.

i) $\Delta y = F(x + \Delta x) - F(x)$

2) $dy = F'(x)\Delta x$

$dy = F'(x)\,dx$

$dy = ?$

$dy = F'(x)\,dx$ $\quad\quad \Delta x \cong dx$

$F(x) = x^2 + 1$

$F'(x) = 2x$

$dy = 2x\,dx$

Q #28 $\quad y = \sin x$

Sol

$y = F(x)$

$y = F(x) = \sin x$

$\boxed{\Delta y = F(x + \Delta x) - F(x)}$

$\Delta y = \sin(x + \Delta x) - \sin x$

$dy = F'(x)\,dx$

$\sin P - \sin Q = 2\cos\frac{P+Q}{2} \cdot \sin\frac{P-Q}{2}$

~~Ans~~

$F(x) = \sin x \Rightarrow F'(x) = \cos x$

$dy = \cos x\,dx$ Ans.

Formula of Approximation.

$$F(x+\Delta x) = F(x) + F'(x)\,\Delta x.$$

29 $(1.8)^5$

Sol $F(x+\Delta x) = (1.8)^5$

$F(x+\Delta x) = (2-0.2)^5$

$F(x) = x^5$ $\quad x = 2, \quad \Delta x = -0.2$

$F'(x) = 5x^4$

$$F(x+\Delta x) = F(x) + F'(x)\,\Delta x$$

$$F(2-0.2) = x^5 + 5x^4\,\Delta x$$

$$F(1.8) = (2)^5 + 5(2)^4(-0.2)$$

$$(1.8)^5 = 32 - 16$$

$(1.8)^5 = 16$ Ans.

30 $\sqrt{37}$

Sol $F(x) = x^{1/2}$; $F'(x) = \frac{1}{2} x^{-1/2}$

$F(x+\Delta x) = \sqrt{37}$

$= (37)^{1/2}$

$F(x+\Delta x) = (36+1)^{1/2}$

$x = 36 \qquad \Delta x = 1$

$F(x+\Delta x) = F(x) + F'(x)\Delta x$

$\sqrt{37} = x^{1/2} + \frac{1}{2} x^{-1/2}(1)$

$= (36)^{1/2} + \frac{1}{2(36)^{1/2}}$

$= \frac{6}{1} + \frac{1}{12}$

$\sqrt{37} = \frac{73}{12}$

$\sqrt{37} = 6.0833$ Ans.

31) $\sin 31$

Sol

$F(x) = \sin x \quad ; \quad F'(x) = \cos x$

$F(x+\Delta x) = \sin 31$

$F(x+\Delta x) = \sin(30+1)$

$x = 30^\circ \qquad \Delta x = 1^\circ$

$= 1 \times \frac{\pi}{180}$

$\Delta x = 0.017$

$F(x+\Delta x) = F(x) + F'(x)\,\Delta x$

$\sin 31 = \sin x + \cos x\,(0.017)$

$= \sin 30 + \cos 30\,(0.017)$

$\sin 31 = \frac{1}{2} + \frac{\sqrt{3}}{2}(0.017)$

$= \frac{1}{2} + 0.01472$

$\sin 31^\circ = 0.51472$

32 $\tan\left(\frac{\pi}{4} + 0.1\right)$

Sol $F(x) = \tan x \qquad F'(x) = \sec^2 x$

$$F(x+\Delta x) = \tan\left(\frac{\pi}{4} + 0.1\right)$$

$$x = \frac{\pi}{4} \qquad \Delta x = 0.1$$

$$F(x+\Delta x) = F(x) + F'(x)\,\Delta x$$

$$\tan\left(\frac{\pi}{4}+0.1\right) = \tan x + \sec^2 x \; 0.1$$

$$= \tan\frac{\pi}{4} + \left(\sec\frac{\pi}{4}\right)^2 \cdot 0.1$$

$$= 1 + 0.2$$

$$\tan\left(\frac{\pi}{4}+0.1\right) = 1.2$$

Complete.