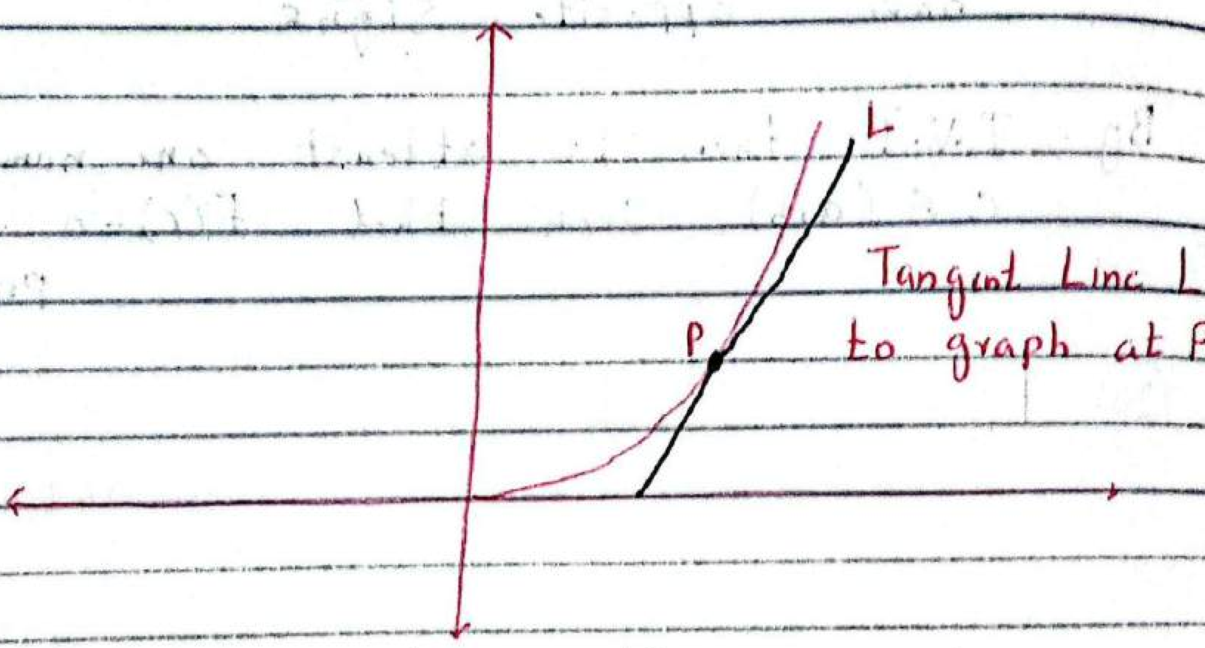
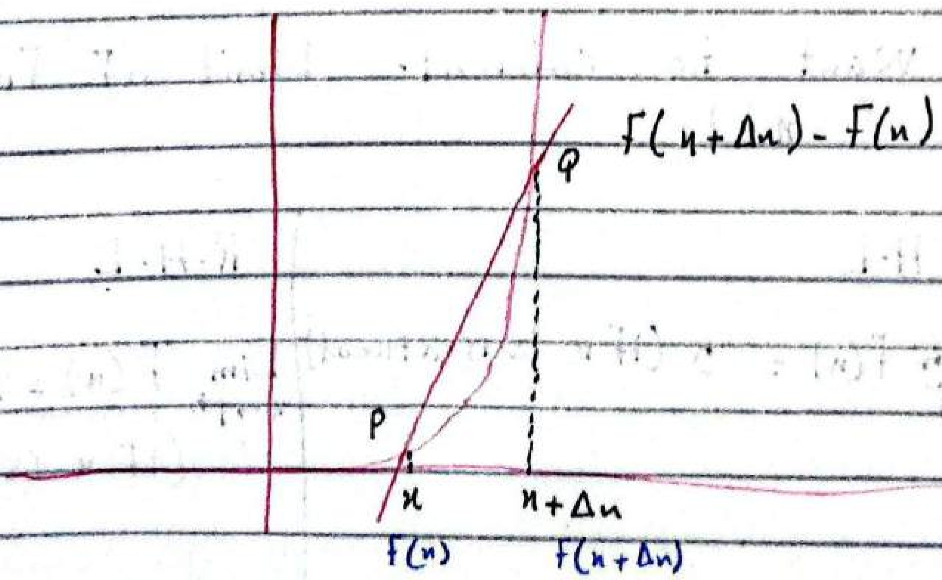
Exc 2.3

Tangent of a Graph



A Line which touches the curve at one point is called tangent to the curve

Slope of Tangent Line.



$$m_{sec} = \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

Def: Tangent Line.

Let $y = F(x)$ be continuous function at a point $(x, F(x))$ the tangent Line to the graph is the Line that passes through the point with slope.

$$\text{Slope} = m_{tan} = \boxed{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}}$$

Find slope of tangent Line to the graph.

1) $F(x) = 2x - 1$; $(x, F(x)) = (4, 7)$ (x, y)

Sol

We know that slope of tangent Line is

$$m_{tan} = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

Given, $F(x) = 2x - 1$ $\quad F(x+\Delta x) = 2(x+\Delta x) - 1$

$$= \lim_{\Delta x \to 0} \frac{[2(x+\Delta x) - 1] - [2x - 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{2x} + 2\Delta x \cancel{-1} - \cancel{2x} \cancel{+1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\cancel{\Delta x}}{\cancel{\Delta x}}$$

$\therefore$ $m_{tan} = 2$

At point $(x, y) = (4, 7)$

$\boxed{m_{tan} = 2}$ Ans.

Q #1
(II) $F(x) = -\frac{1}{2}x + 3$ $\quad (a, F(a))$

Sol. We know that slope of tangent line is

$$m_{tan} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

Here $F(x) = -\frac{1}{2}x + 3$ $\quad F(x + \Delta x) = -\frac{1}{2}(x + \Delta x) + 3$

$$= \lim_{\Delta x \to 0} \frac{\left[-\frac{1}{2}(x + \Delta x) + 3\right] - \left[-\frac{1}{2}x + 3\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{-\frac{1}{2}x} - \frac{1}{2}\Delta x + \cancel{3} + \cancel{\frac{1}{2}x} - \cancel{3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\frac{1}{2}\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$m_{tan} = \frac{-1}{2}$$

At point $(a, F(a))$ [Same result]

$$\boxed{m_{tan} = \frac{-1}{2}}$$ Ans.

(iii) $F(x) = x^2 + 4$

$\begin{matrix} x & y \\ (-1, & 5) \end{matrix}$

Sol: We know that slope of tangent line is

$$m_{tan} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

Here $F(x) = x^2 + 4$ $F(x + \Delta x) = (x + \Delta x)^2 + 4$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 4] - [x^2 + 4]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + (\Delta x)^2 + 2x\Delta x \cancel{+4} - \cancel{x^2} \cancel{-4}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{\Delta x} + \frac{2x\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \Delta x + 2x$$

$$= 0 + 2x$$

$$m_{tan} = 2x$$

At point $(-1, 5)$; $x = -1$ $y = 5$

$$m_{tan} = 2(-1)$$

$$\boxed{m_{tan} = -2}$$

(iv)

$F(x) = x^2 - 5x + 4$ $\quad (\overset{x}{2}, \overset{y}{-2})$

Sol

We know that slope of tangent line is

$$m_{tan} = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

Here $F(x) = x^2 - 5x + 4$ $\quad F(x+\Delta x) = (x+\Delta x)^2 - 5(x+\Delta x) + 4$

$$= \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 - 5(x+\Delta x) + 4] - [x^2 - 5x + 4]}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{x^2} + (\Delta x)^2 + 2x\Delta x - \cancel{5x} - 5\Delta x + \cancel{4} - \cancel{x^2} + \cancel{5x} - \cancel{4}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2x\,\Delta x \mp 5\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{\Delta x} + \frac{2x\,\Delta x}{\Delta x} \mp \frac{5\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x + 2x \mp 5$$

$$= 0 + 2x \mp 5$$

$$m_{tan} = 2x \mp 5$$

At Point $(2, -2)$; $x = 2$, $y = -2$

$$m_{tan} = 2(2) \mp 5$$
$$= 4 \mp 5$$
$$\boxed{m_{tan} = -1}$$ Ans.

(V)

$F(x) = x^3$ $\quad\quad$ $\begin{array}{cc} x & y \\ (1, & F(1)) \end{array}$

Sol

We know that Slope of tangent Line is

$$m_{tan} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

$F(x) = x^3$ $\quad\quad$ $F(x + \Delta x) = (x + \Delta x)^3$

$$\lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^3} + (\Delta x)^3 + 3x^2 \Delta x + 3x(\Delta x)^2 - \cancel{x^3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^3 + 3x^2 \Delta x + 3x(\Delta x)^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{(\Delta x)^3}{\Delta x} + \frac{3x^2 \Delta x}{\Delta x} + \frac{3x(\Delta x)^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} (\Delta x)^2 + 3x^2 + 3x\,\Delta x$$

$$= (0)^2 + 3x^2 + 3x(0)$$

$$m_{tan} = 3x^2$$

At point $(1, F(1))$; $x = 1$, $y = F(1)$

$$m_{tan} = 3(1)^2$$

$$\boxed{m_{tan} = 3}$$

(vi) $F(x) = \frac{1}{x}$; $\left(\frac{1}{3}, F(1/3)\right)$

Sol

We know that slope of tangent line is

$$m_{tan} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

Here $F(x) = \frac{1}{x}$; $F(x+\Delta x) = \frac{1}{x+\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{x\cdot(x+\Delta x)}}{\frac{\Delta x}{1}}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x}-\cancel{x}-\Delta x}{x\cdot(x+\Delta x)} \times \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\cancel{\Delta x}}{x\cdot(x+\Delta x)\cdot\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)}$$

$$= \frac{-1}{x(x+0)} = \frac{-1}{x^2}$$

$m_{tan} = -1/x^2$

At Point $(1/3, F(1/3))$ $x = 1/3$

$$m_{tan} = \frac{-1}{(1/3)^2} = \frac{-1}{1/9} = \frac{-9}{1}$$

$$\boxed{m_{tan} = -9}$$

Average rate of change

If $y = F(x)$, $[a, b]$

$$m_{sec} = \frac{F(b) - F(a)}{b - a} = \frac{\Delta y}{\Delta x}$$

Q#7 Find average rate of change

$F(x) = x^3 + 2x^2 - 4x$ $[-1, 2]$

Sol

$a = -1$ $b = 2$

we know that

$$\text{Average rate of change} = \frac{F(b) - F(a)}{b - a}$$

$$= \frac{F(2) - F(-1)}{2 - (-1)}$$

$$= \frac{F(2) - F(-1)}{3}$$

$$= \frac{[(2)^3 + 2(2)^2 - 4(2)] - [(-1)^3 + 2(-1)^2 - 4(-1)]}{3}$$

$$= \frac{(8 + \cancel{8} - \cancel{8}) - (-1 + 2 + 4)}{3}$$

$$= \frac{8-5}{3} = \frac{3}{3} = 1$$

Q#8. $F(x) = \cos x \qquad [-\pi, \pi]$

Sol. $a = -\pi, \quad b = \pi$

We know that

$$\text{Average rate of change} = \frac{F(b) - F(a)}{b-a}$$

$$= \frac{F(\pi) - F(-\pi)}{\pi - (-\pi)}$$

$$= \frac{\cos\pi - \cos(-\pi)}{\pi + \pi}$$

$$= \frac{\cos\pi - \cos\pi}{2\pi}$$

$$= 0 \quad \text{Ans.}$$

Instantaneous Velocity.

$$V_{inst} = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t}$$

Q#9 $F(t) = -4t^2 + 10t + 6$, $t = 3$

Sol. We Know that

$$V_{inst} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{[-4(t+\Delta t)^2 + 10(t+\Delta t) + 6] - [-4t^2 + 10t + 6]}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{[-4(t^2 + \Delta t^2 + 2t\Delta t) + 10t + 10\Delta t + 6] + 4t^2 - 10t - 6}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\cancel{-4t^2} - 4\Delta t^2 - 8t\Delta t + \cancel{10t} + 10\Delta t \cancel{+6} + \cancel{4t^2} - \cancel{10t} - \cancel{6}}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\cancel{\Delta t}[-4\Delta t - 8t + 10]}{\cancel{\Delta t}}$$

$$\lim_{\Delta t \to 0} -4\Delta t - 8t + 10$$

$$= -4(0) - 8t + 10$$

$$V_{inst} = -8t + 10$$

At $t = 3$

$$V_{Inst} = -8(3) + 10$$

$$= -14 \quad \text{Ans}$$

Q#12 $F(t) = t^2 + \frac{1}{5t+1}$, $t = 0$

Sol. We know that

$$V_{inst} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\left[(t+\Delta t)^2 + \frac{1}{5(t+\Delta t)+1}\right] - \left[t^2 + \frac{1}{5t+1}\right]}{\Delta t}$$

As $t = 0$ Given.

$$\lim_{\Delta t \to 0} \frac{\left[(\Delta t)^2 + \frac{1}{5\Delta t+1}\right] - \left[0 + \frac{1}{0+1}\right]}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{(\Delta t)^2 + \frac{1}{5\Delta t+1} - 1}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\frac{(\Delta t^2)(5\Delta t+1) + 1 - 5\Delta t - 1}{5\Delta t+1}}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\frac{5\Delta t^3 + \Delta t^2 \cancel{+1} - 5\Delta t \cancel{-1}}{5\Delta t+1}}{\frac{\Delta t}{1}}$$

$$\lim_{\Delta t \to 0} \frac{5\Delta t^3 + \Delta t^2 - 5\Delta t}{(5\Delta t+1) \cdot \Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\cancel{\Delta t}(5\Delta t^2 + \Delta t - 5)}{\cancel{\Delta t}(5\Delta t+1)}$$

$$= \frac{5(0) + 0 - 5}{5(0)+1} = \frac{-5}{1} = -5 \text{ Ans}$$

Q#11 The Height above ground of a ball droped from an initial altitude of 122.5m given by $S(t) = 122.5 - 4.9t^2$, where S is measured in metres.

Sol

i) What is the instantaneous velocity at $t = \frac{1}{2}$.

$$V_{INST} = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{F(\frac{1}{2} + \Delta t) - F(\frac{1}{2})}{\Delta t}$$

$$F(\tfrac{1}{2}) = 122.5 - 4.9(\tfrac{1}{2})^2$$
$$= 122.5 - 4.9(\tfrac{1}{4})$$

$$F(\tfrac{1}{2} + \Delta t) = 122.5 - 4.9(\tfrac{1}{2} + \Delta t)^2$$

$$\lim_{\Delta t \to 0} \left[122.5 - 4.9(\tfrac{1}{2} + \Delta t)^2\right] - \left[122.5 - 4.9(\tfrac{1}{4})\right]$$

$$F(t) = 122.5 - 4.9t^2$$

$$F(t + \Delta t) = 122.5 - 4.9(t + \Delta t)^2$$

$$= \lim_{\Delta t \to 0} \frac{[122.5 - 4.9(t+\Delta t)^2] - [122.5 - 4.9t^2]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[122.5 - 4.9(t^2 + \Delta t^2 + 2t\Delta t)] - [122.5 - 4.9t^2]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{122.5} - \cancel{4.9t^2} - 4.9\Delta t^2 - 9.8t\Delta t - \cancel{122.5} + \cancel{4.9t^2}}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\cancel{\Delta t}[-4.9\Delta t - 9.8t]}{\cancel{\Delta t}}$$

$$\lim_{\Delta t \to 0} -4.9\Delta t - 9.8t$$

$$= 0 - 9.8t$$

$$= -9.8t$$

At $t = 1/2$

$$V_{INST} = -9.8\left(\frac{1}{2}\right) = -4.9\,m/s \quad \text{Ans.}$$

(iii) At what time does the ball hit the ground?

$t = ?$ $\quad S(t) = 0$

When ball hits the ground then $S(t) = 0$

$$122.5 - 4.9t^2 = 0$$

$$t^2 = \frac{122.5}{4.9}$$

$$t^2 = 25$$

$$\boxed{t = 5\ sec}$$

(iii) What is the $\underbrace{\text{Impact velocity}}_{t=5\,\text{second}}$

$$F(t) = S(t) = 122.5 - 4.9t^2$$

$$V_{INST} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[122.5 - 4.9(t+\Delta t)^2\right] - \left[122.5 - 4.9t^2\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[122.5 - 4.9(t^2 + \Delta t^2 + 2t\Delta t)\right] - \left[122.5 - 4.9t^2\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[\cancel{122.5} - \cancel{4.9t^2} - 4.9\Delta t^2 - 9.8t\,\Delta t - \cancel{122.5} + \cancel{4.9t^2}\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{\Delta t}\,(-4.9\Delta t - 9.8t)}{\cancel{\Delta t}}$$

$$\lim_{\Delta t \to 0} -4.9\Delta t - 9.8t$$

$$= -9.8t$$

At $t = 5$

$$= -9.8(5)$$

$$= -49\,\text{m/sec}$$

(12) The height of a Projectile shot from ground Level is given by

$$S(t) = -16t^2 + 256t$$

Sol

$$S(t) = -16t^2 + 256t$$

i) Determine the Height of the Projectile at $t = 2$, $t = 6$, $t = 9$ and $t = 10$

$h = ?$

$t = 2$

$$S(2) = -16(2)^2 + 256(2)$$
$$S(2) = 448\text{ ft}$$

$t = 6$

$$S(6) = -16(6)^2 + 256(6)$$
$$S(6) = 960\text{ Ft}$$

$t = 9$

$$S(9) = -16(9)^2 + 256(9)$$
$$S(9) = 1008\text{ ft.}$$

$t = 10$

$$S(10) = -16(10)^2 + 256(10)$$
$$S(10) = 960\text{ ft}$$

(ii) Average velocity of Projectile between $t=2$ and $t=5$.

$$S(t) = -16t^2 + 256t$$

Average velocity $= \dfrac{S(b)-S(a)}{b-a}$ or $\dfrac{f(b)-f(a)}{b-a}$

$$= \frac{S(5)-S(2)}{5-2}$$

$$= \frac{[-16(5)^2+256(5)]-[-16(2)^2+256(2)]}{3}$$

$$= \frac{432}{2} = 144 \text{ ft/sec}$$

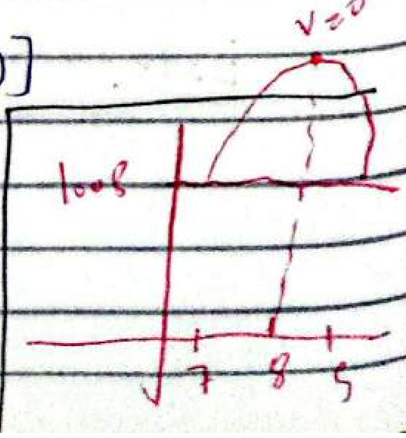
(iii) Show that Average velocity b/w $t=7$ and $t=9$ is Zero also interpret

Average velocity $= \dfrac{S(b)-S(a)}{b-a}$

$$= \frac{S(9)-S(7)}{9-7}$$

$$= \frac{[-16(9)^2+256(9)]-[-16(7)^2+256(7)]}{2}$$

$$= \frac{1008-1008}{2} = \frac{0}{2} = 0$$



~~Inter~~ (iv) At what time does the Projectile hit the ground.

When Projectile hits the ground then

$$S(t) = 0$$

$$-16t^2 + 256t = 0$$

$$t(-16t + 256) = 0$$

$$t = 0 \qquad -16t + 256 = 0$$

$$t = \frac{-256}{-16}$$

$$t = 16 \text{ Second.}$$

(V) Determine Instaneous velocity at time t=8

$$V_{INST} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{S(t+\Delta t) - S(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{[-16(t+\Delta t)^2 + 256(t+\Delta t)] - [-16t^2 + 256t]}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{[\cancel{-16t^2} - 16(\Delta t)^2 - 32t\Delta t \cancel{+256t} + 256\Delta t \cancel{+16t^2} \cancel{-256t}]}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\cancel{\Delta t}(-16\Delta t - 32t + 256)}{\cancel{\Delta t}}$$

$$= -16(0) - 32t + 256$$

$V_{INST} = 256 - 32t$

$t = 8$

$= 256 - 32(8)$

$= 256 - 256$

$V_{INST} = 0$

(VI) What is the Maximum height that the Projectile attains?

Given $S(t) = -16t^2 + 256t$ —— (1)

For maximum height put $t = 8$

$= -16(8)^2 + 256(8)$

$= -1024 + 2048$

$S(t) = 1024$

Complete