Exc 2.2

Definition of Limit

$$\lim_{x \to a} F(x) = L$$

Where L is a real number.

i) Left Hand Limit:

$\lim_{x \to c^-} F(x) = L$ is read as the Limit of F(x) is equal to L as x approches c from the Left.

ii) Right hand Limit:

$\lim_{x \to c^+} F(x) = M$ is read as the Limit of F(x) is equal to M as x approaches c from the right.

- Existence of Limit of a Function:

$$\lim_{x \to c^-} F(x) = \lim_{x \to c^+} F(x) = L$$

Explanation of Phrase

$x \to 0$	$x \to \infty$
x tends to Zero	x tends to infinity
x approaches to Zero	x approaches to Infinity

Continuity of a Function at a number

a) Continuous Function

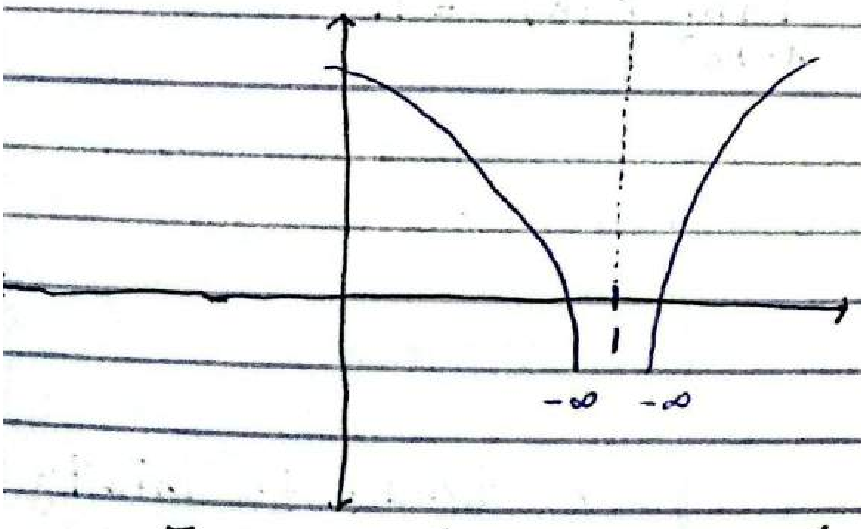
A Function F is said to be continuous at a number 'c' If and only If the following three conditions are satisfied.

a) $F(c)$ is defined

b) $\lim_{x \to c} F(x)$ exists $\left[\lim_{x \to c^-} F(x) = \lim_{x \to c^+} F(x)\right]$

c) $\lim_{x \to c} F(x) = F(c)$

Graphical Definition



Discontinuous.

اگر کسی Graph کے Point پر بریک ہو رہا ہو تو function og
Continuous نہیں ہو گا۔

- $\lim_{x\to 1} F(x)$ does not exist

$\lim_{x\to 1^-} F(x) = \infty$ $\lim_{x\to 1^+} F(x) = \infty$ (Not a real number)

- $F(1)$ is not defined.

Determine the numbers (If any) at which the given Function is discontinuous:

(i) $F(x) = x^2 - 5x + 6$

Sol Given Function is a Polynomial

- $F(x)$ is well defined for all Real numbers.
- Polynomial Functions (Quadratic) are Continuous every where.
- So No discontinuity in this Function.

(ii) $F(x) = \dfrac{2x}{x^2+5}$

Put $x^2 + 5 = 0$
$x^2 = -5$
$x = \pm\sqrt{5}i$
No real answer

- $F(x)$ is well defined for all real numbers.
- Denominator Cannot be Zero in given Function.
- Given Function is Continuous every where.
- No discontinuity in this Function For any real number.

③ $F(x) = \dfrac{1}{x^2 - 9x + 8}$

Sol

Put $x^2 - 9x + 8 = 0$

$x^2 - 8x - x + 8 = 0$

$x(x-8) - 1(x-8) = 0$

$(x-1)(x-8) = 0$

$x - 1 = 0$ $\quad\quad$ $x - 8 = 0$

$x = 1$ $\quad\quad$ $x = 8$

$= \dfrac{1}{(1)^2 - 9(1) + 8}$ $\quad\quad$ $\dfrac{1}{(8)^2 - 9(8) + 8}$

$= \dfrac{1}{9-9}$ $\quad\quad$ $\dfrac{1}{64 - 72 + 8}$

$= \dfrac{1}{0}$ $\quad\quad$ $\dfrac{1}{72-72} = \dfrac{1}{0}$

$= \infty$ $\quad\quad$ $= \infty$

• $f(x)$ is well defined for all real numbers except 1 and 8. Hence $x = 1$ and $x = 8$ are the numbers where function is discontinuous.

(4) $f(x) = \dfrac{x^2 - 1}{x^4 - 1}$

Sol

At $x = 1$ and $x = -1$ given function is discontinuous.

• $f(x)$ is well defined for all real numbers except 1 and -1. Point of discontinuity $\{-1, +1\}$.

(5) $F(x) = \dfrac{x-1}{\sin 2x}$

Sol

$F(x) = \dfrac{x-1}{\sin 2x}$ $\sin(0) = 0$

F(x) is well defined for all real num except 0.

Point of discontinuity = 0

Sol

Put $\sin 2x = 0$

$2x = \sin^{-1}(0)$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi, —$

$2x = n\pi \quad \forall n \in Z$

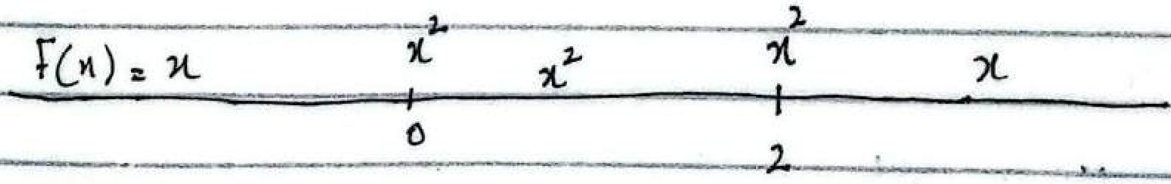
$\left\{x = \dfrac{n\pi}{2}\right\}$

Hence $x = \dfrac{n\pi}{2}$, $n \in Z$ Given Function is Discontinuous.

(6)

$$F(x) = \begin{cases} x & x < 0 \\ x^2 & 0 \leq x \leq 2 \\ x & x > 2 \end{cases}$$

Sol



For $x = 0$

1) F(c) is defined

$F(x) = x^2$

$F(0) = (0)^2$

$F(0) = 0$

Condition 1 is satisfied.

(ii) $\lim_{x \to c} f(x)$ exist

L.H.L	R.H.L
$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x$	$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2$
$= 0$	$= (0)^2$
	$= 0$

L.H.L = R.H.L

So $\lim_{x \to 0} f(x) = 0$ exist.

(iii) $f(c) = \lim_{x \to c} f(x)$

$f(0) = \lim_{x \to 0} f(x)$

$0 = 0$

Hence Function is Continuous at $x = 0$

At $x = 2$

i) $f(c)$ is defined

$f(x) = x^2$

$f(2) = (2)^2$

$f(2) = 4$

$\therefore$ Condition 1 is satisfied.

(iii) $\lim_{x \to c} F(x)$ exist.

L.H.L	R.H.L
$\lim_{x \to 2^-} F(x) = \lim_{x \to 2^-} x^2$	$\lim_{x \to 2^+} F(x) = \lim_{x \to 2^+} x$
$= (2)^2$	$= 2$
$= 4$	$= 2$

$$L.H.L \neq R.H.L$$

Therefore $\lim_{x \to 2} F(x)$ does not exist.

Hence given function is discontinuous at $x = 2$.

(7) $F(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$

Sol

At $x = 0$

i) $f(c)$ is defined

$F(0) = \frac{1}{2}$

(ii) $\lim_{x \to c} F(x)$ exist.

L.H.L	R.H.L
$\lim_{x \to 0^-} F(x) = \lim_{x \to 0^-} \frac{\sin x}{x}$	$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \frac{\sin x}{x}$
$= 1$	$= 1$

L.H.L = R.H.L

So $\lim_{x \to 0} f(x)$ exists

(iii) $f(0) = \lim_{x \to 0} f(x)$

$\frac{1}{2} \neq 1$

Condition III is not satisfied.

Hence given function is discontinuous at $x=0$

8. $F(x) = \begin{cases} \frac{x^2-36}{x-6}, & x \neq 6 \\ 12, & x = 6 \end{cases}$

Sol: At $x = 6$

i) $F(c)$ is defined

$f(6) = 12$

ii) $\lim_{x \to c} F(x)$ exists

L.H.L	R.H.L
$\lim_{x \to 6^-} F(x) = \lim_{x \to 6^-} \frac{x^2-36}{x-6}$	$\lim_{x \to 6^+} F(x) = \lim_{x \to 6^+} \frac{x^2-36}{x-6}$

$\lim_{x\to 6^-} \frac{(\cancel{x-6})(x+6)}{(\cancel{x-6})}$	$\lim_{x\to 6^+} \frac{(\cancel{x-6})(x+6)}{(\cancel{x-6})}$
$\lim_{x\to 6^-} x+6$	$\lim_{x\to 6^+} x+6$
$= 6+6$	$= 6+6$
$= 12$	$= 12$

L.H.L = R.H.L

So $\lim_{x\to 6} F(x)$ exist.

(iii) $F(6) = \lim_{x\to 6} F(x)$

$12 = 12$

All 3 conditions are satisfied so given function is continuous at $x=6$.

No discontinuity in this given function.

(9) Determine whether the given function is continuous in the indicated intervals.

$F(x) = x^2 + 1$

(a) $[-1, 3]$ $\quad -1, 0, 1, 2, 3$

The given function $f(x)$ is continuous every where in the interval $[-1, 3]$

b) $[3, \infty)$ 3, 4, 5, 6 ---

The given function $F(x)$ is continous at every where in the interval $[3, \infty)$

Q #10 $F(x) = \frac{1}{x}$

a) $(-3, 3)$ -2, -1, 0, 1, 2

$F(x)$ is discontinous at $x = 0$ in Interval $(-3, 3)$

b) $(0, 10]$ 1, 2, 3, --- 10

$F(x)$ is continous at $(0, 10]$.

Q #11 $F(x) = \frac{1}{\sqrt{x}}$

a) $[1, 4)$ 1, 2, 3, — 3.9

Clearly $F(x)$ is continous at $[1, 4)$

b) $[1, 9]$ 1, 2, 3, 4, --- 9

Clearly $F(x)$ is continous at interval $[1, 9]$

Q#12 $f(x) = \sqrt{x^2-9}$

a) $[-3, 3]$ $-3, -2, -1, 0, 1, 2, 3$

Clearly $f(x)$ is not defined at points $-2, -1, 0, 1, 2, 3$.

So $f(x)$ is discontinuous at $[-3, 3]$.

b) $[3, \infty)$ $3, 4, 5, 6, \ldots$

Clearly $f(x)$ is continuous at interval $[3, \infty)$

Q#13

$f(x) = \dfrac{x}{x^3+8}$

At $x = -2$ the function is undefined

Sol

a) $[-4, -3]$ $\cdot -3, -4.$

Given function is continuous in the interval $[-4, -3]$

b) $[-10, 10]$ $-10, -9, -8, -7, \ldots 8, 9, 10$

Given function is discontinuous at $x = -2$ which lie in this interval.

So function is discontinuous on $[-10, 10]$.

(14) $F(x) = \sin\frac{1}{x}$

Given Function is discontinuous at $x=0$

a) $\left[\frac{1}{\pi}, 5\right)$

$F(x) = \sin\frac{1}{x}$ is continuous at $\left[\frac{1}{\pi}, 5\right)$.

b) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$F(x)$ is continuous at every where in interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

VERY IMPORTANT QUESTIONS

In Problems 15 – 18 Find the values of m and n So that given Function is continuous

Q #15 $F(x) = \begin{cases} mx, & x < 4 \\ x^2, & x \geq 4 \end{cases}$

$F(x) = mx$ $\quad$ $F(x) = x^2$ $\quad$ $F(x) = x^2$

4

L.H.L	R.H.L
$\lim\limits_{x\to 4^-} F(x) = \lim\limits_{x\to 4^-} mx$	$\lim\limits_{x\to 4^+} F(x) = \lim\limits_{x\to 4^+} x^2$

$= m \cdot 4$	$= (4)^2$
$= 4m$	$= 16$

Given Function is continuous at $x = 4$
So Limit must exists.

$$L.H.L = R.H.L$$

$$4m = 16$$

$$m = \frac{16}{4}$$

$$m = 4 \quad \text{Ans.}$$

Q#16

$$F(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ m & x = 2 \end{cases}$$

$F(x) = \dfrac{x^2-4}{x-2}$ | 2 | $F(x) = \dfrac{x^2-4}{x-2}$

L.H.L	R.H.L
$\lim\limits_{x \to 2^-} F(x) = \lim\limits_{x \to 2^-} \dfrac{x^2-4}{x-2}$	$\lim\limits_{x \to 2^+} F(x) = \lim\limits_{x \to 2^+} \dfrac{x^2-4}{x-2}$
$\lim\limits_{x \to 2^-} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$	$\lim\limits_{x \to 2^+} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$
$= 2+2$	$= 2+2$
$= 4$	$= 4$

As L.H.L = R.H.L So Limit exists.
It means $\lim_{n \to 4} F(n) = 4$
Since Given function is Continuous
So
$$\lim_{n \to 2} F(n) = F(2) \quad [\text{3rd Condition}]$$
$$\boxed{4 = m}$$

Q#17
$$F(n) = \begin{cases} mn, & n < 3 \\ n, & n = 3 \\ -2n+9, & n > 3 \end{cases}$$

Sol
$F(n) = mn$ | $F(n) = n$ | $F(n) = -2n+9$
3

L.H.L
$$\lim_{n \to 3^-} F(n) = \lim_{n \to 3^-} mn$$
$$= 3m$$

R.H.L
$$\lim_{n \to 3^+} F(n) = \lim_{n \to 3^+} -2n+9$$
$$= -2(3)+9$$
$$= -6+9$$
$$= 3$$

Since given Function is continuous at $n=3$
So Limit exists It means

L.H.L = R.H.L

$3m = 3$ —— (A)

ii) Condition ii) $F(c)$ is defined.

$F(3) = n$

Since given function is continuous at $x = 3$

$$\lim_{x \to c} F(x) = F(c)$$

$$\lim_{x \to 3} F(x) = F(3)$$

$3m = 3 \quad = n$

$3m = 3 \qquad\qquad 3 = n$

$\boxed{m = 1} \qquad\qquad \boxed{n = 3}$

Q #18

$$F(x) = \begin{cases} mx - n & x < 1 \\ 5 & x = 1 \\ 2mx + n & x > 1 \end{cases}$$

Sol

$F(x) = mx - n \qquad 5 \qquad F(x) = 2mx + n$

1

i) $F(c)$ is defined

$c = 1$

$F(1) = 5$

ii) $\lim_{x \to c} F(x)$ exists

L.H.L	R.H.L
$\lim_{x \to c^-} F(x) = \lim_{x \to 1^-} mx - n$	$\lim_{x \to c^+} F(x) = \lim_{x \to 1^+} 2mx + n$
$= m(1) - n$	$= 2m(1) + n$
$= m - n$	$= 2m + n$

Since given Function is Continuous So

L.H.L = R.H.L

$m - n = 2m + n$

(iii) $\lim_{x \to c} F(x) = F(c)$

$\lim_{x \to 1} F(x) = F(1)$

$m - n = 2m + n = 5$

$m - n = 5$ — ① $2m + n = 5$ — ②

Adding

$$\begin{array}{r} 2m + n = 5 \\ m - n = 5 \\ \hline 3m = 10 \end{array}$$

$$\boxed{m = \frac{10}{3}}$$

Put in (i)

$$m - n = 5$$

$$-n = 5 - m$$

$$-n = 5 - \frac{10}{3}$$

$$-n = \frac{15 - 10}{3}$$

$$-n = \frac{5}{3} \qquad \boxed{n = -\frac{5}{3}}$$

Intermediate Value Theorem

If a function $f(x)$ is

i) Continuous on a closed interval $[a, b]$

ii) If $f(a)$ and $f(b)$ have opposite signs then there is atleast one number $c \in (a, b)$ such that $f(c) = 0$

19 Prove that the equation $\frac{x^2+1}{x+3} + \frac{x^4+1}{x-4} = 0$ has a solution in the interval $(-3,4)$

Sol: (i) Given Function

$$F(x) = \frac{x^2+1}{x+3} + \frac{x^4+1}{x-4}$$

is continuous in the interval $(-3,4)$.

ii) Let $a = -2$, $b = 3$

• $$F(a) = F(-2) = \frac{(-2)^2+1}{-2+3} + \frac{(-2)^4+1}{-2-4}$$

$$= \frac{4+1}{1} + \frac{16+1}{-6}$$

$$= \frac{5}{1} + \frac{17}{-6}$$

$$= \frac{5}{1} - \frac{17}{6}$$

$$= \frac{30-17}{6} = \frac{13}{6} > 0$$

• $$F(b) = F(3) = \frac{(3)^2+1}{3+3} + \frac{(3)^4+1}{3-4}$$

$$= \frac{10}{6} - \frac{85}{1} = \frac{-250}{3} < 0$$

$\Rightarrow F(a) > 0$ and $F(b) < 0$
have opposite Signs

By I.V.T there is atleast one number $c \in (a,b)$ Such that $F(c) = 0$

Proved.

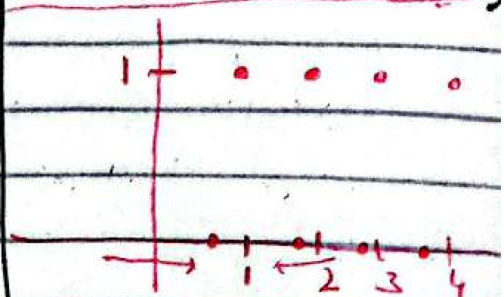
(20) Prove that $F(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$ is discontinuous at every real value.

Proof

$$F(x) = \begin{cases} 1 ; & x \text{ rational} \quad x = \frac{2}{5}, \frac{3}{5}, 0.6 \\ 0 ; & x \text{ irrational} \quad \sqrt{2}, \sqrt{3}, \sqrt{5} \end{cases}$$

We want to calculate Limit of Function at $x = 1$ (اس '1' کی جگہ کوئی بھی نمبر لے سکتے ہیں۔)

L.H.L	R.H.L
$\lim\limits_{x \to 1^-} F(x) = 0$ (If x is irrational)	$\lim\limits_{x \to 1^+} F(x) = 1$ (If x is rational)



$$\lim_{x \to 1^-} F(x) \neq \lim_{x \to 1^+} F(x)$$
$$0 \neq 1$$

So function is discontinuous for all real values.