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Merging man and maths

Chap # 02

Limit, Continuity and Derivative:

- Indeterminate Form (where function is undefined)

$\frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0, 0^0, 1^\infty, \infty^0, \infty+\infty, \infty-\infty$

Example.

$F(x) = \frac{x^2-9}{x-3}$

Put $x = 3$

$F(3) = \frac{9-9}{3-3} = \frac{0}{0}$ form

In these cases we use Limit.

$\lim_{x\to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3}$

$\lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$

$\lim_{x\to 3} x+3$

$3+3 = 6$

LIMIT

- $\lim_{x\to 3} 2x+1$, approaches

$= 2(3)+1 \cong 7$ (Approximate)

OR

Function

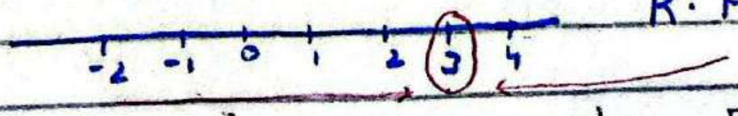
$f(x) = 2x+1$ at $x=3$

$F(3) = 2(3)+1$

$= 6+1 = 7$ (exact)

- Limit exist only If R.H.L = L.H.L

L.H.L R.H.L



$\lim_{x\to 3^-} F(x)$ $\quad 3 \quad$ $\lim_{x\to 3^+} F(x)$

3^- $\quad$ 3^+

L.H.L $\lim_{x\to 3^-} (2x+3)$
$2(3)+3 \cong 9$

R.H.L $\lim_{x\to 3^+} (2x+3)$
$2(3)+3 \cong 9$

So Limit exist
L.H.L = R.H.L

Exc 2.1

Q#1: Use Graph to Find Limit If it exist

i) $\lim_{x \to 5} \sqrt{x-1}$

Put $x - 1 \geq 0$

$x \geq 1$

x	1	2	3	4	5	
y	0	1	1.4	1.7	2	

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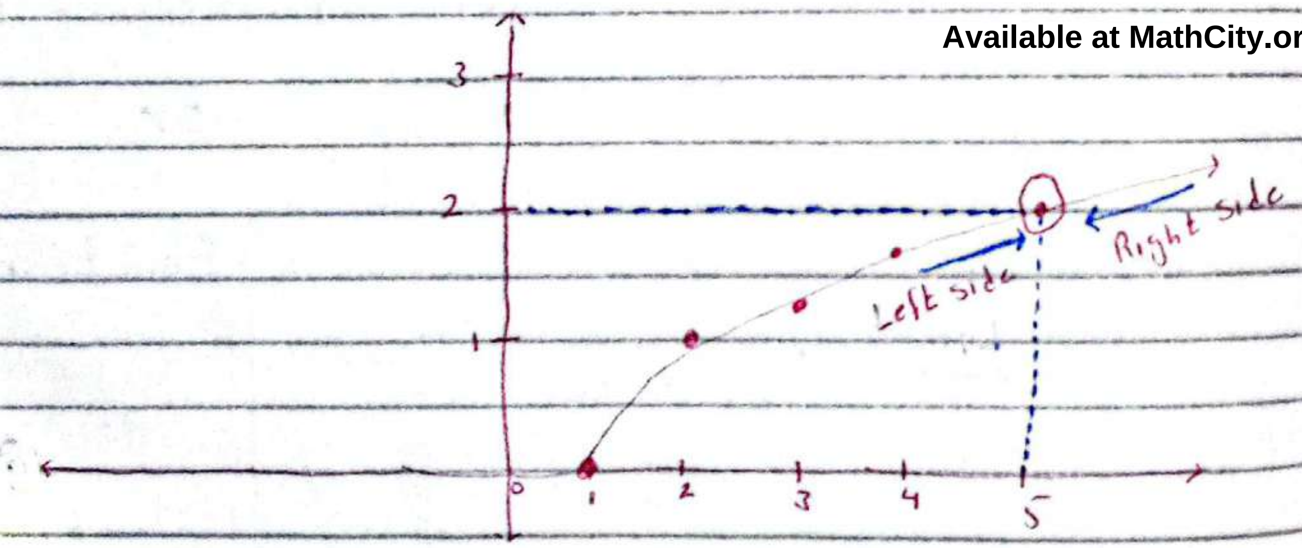

R.H.L

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} \sqrt{x-1}$

$= \sqrt{5-1}$

$= \sqrt{4}$

$= 2$

L.H.L

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} \sqrt{x-1}$

$= \sqrt{5-1}$

$= \sqrt{4}$

$= 2$

R.H.L = L.H.L

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So Limit exist

$$\lim_{x \to 5} f(x) = 2$$

(ii) $\lim_{x \to 1} \frac{x^2-1}{x-1}$

Sol

$$\lim_{x \to 1} \frac{x^2 - 1^2}{x-1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

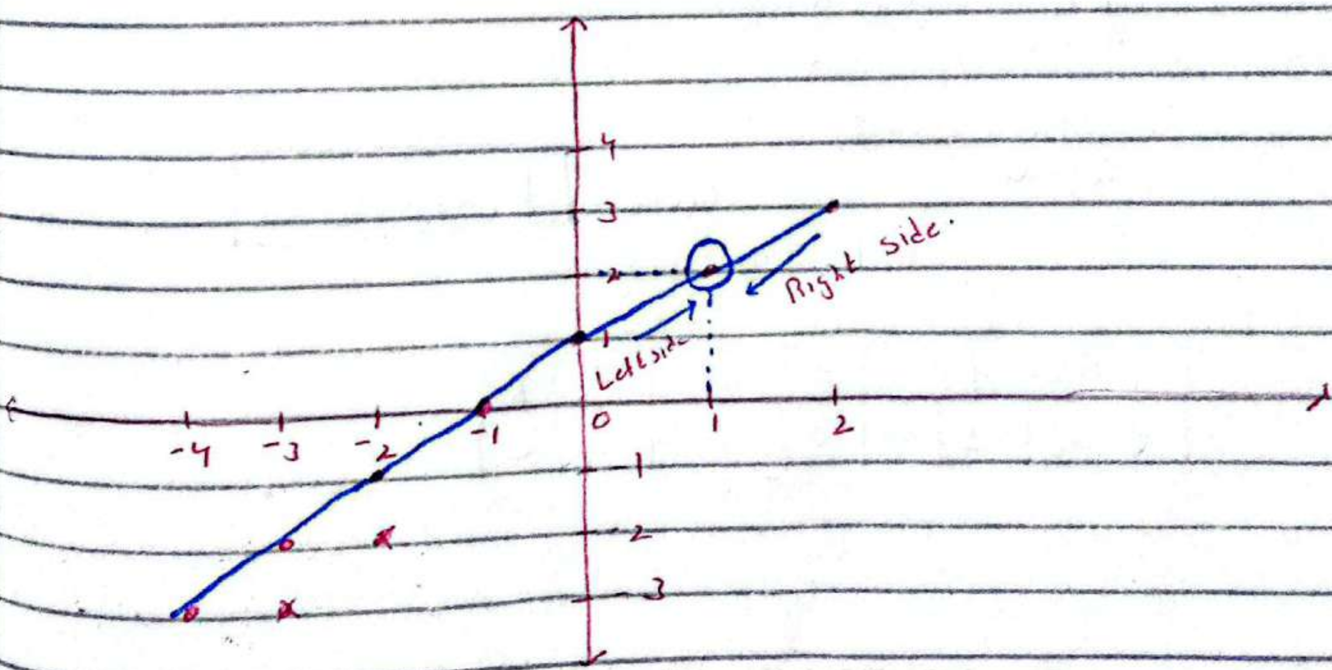
$$\lim_{x \to 1} x+1$$

Here $\boxed{f(x) = x+1}$

put $x+1=0 \quad x=-1$

Graph

x	-4	-3	-2	-1	0	1	2
0	-3	-2	-1	0	1	2	3



L.H.L	R.H.L
$\lim_{x\to1^+} F(x) = \lim_{x\to1^+} x+1$ $= 1+1$ $= 2$	$\lim_{x\to1^-} F(x) = \lim_{x\to1^-} x+1$ $= 1+1$ $= 2$

L.H.L = R.H.L

So Limit exist

$$\lim_{x\to1} F(x) = 2$$

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(iii) $\lim_{x\to0} \dfrac{x^2-3x}{x}$

Sol

$$\lim_{x\to0} \frac{\cancel{x}(x-3)}{\cancel{x}}$$

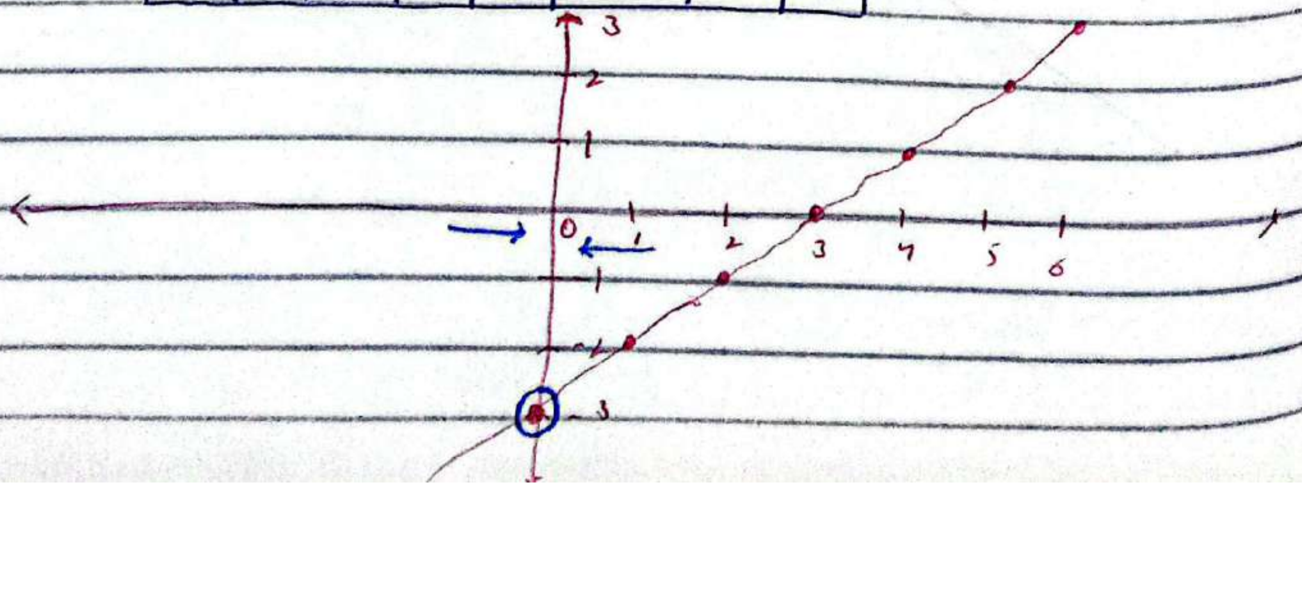
$$\lim_{x\to0} (x-3)$$

Here $\boxed{F(x) = x-3}$

Put $x-3=0$

$x = 3$

x	0	1	2	3	4	5	6
0	-3	-2	-1	0	1	2	3



L.H.L | R.H.L

$$\lim_{x\to 0^-} F(x) = \lim_{x\to 0^-} x-3 \qquad \lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} x-3$$

$$= 0-3 \qquad = 0-3$$

$$= -3 \qquad = -3$$

$$L.H.L = R.H.L$$

So Limit exist

$$\lim_{x\to 0} F(x) = -3$$

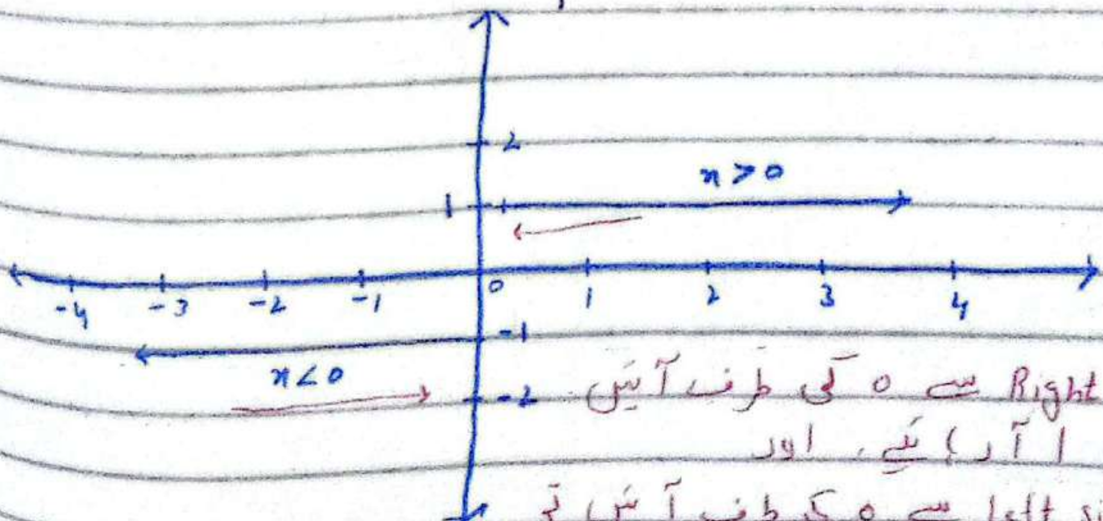
(IV) $\lim_{x\to 0} \frac{|x|}{x}$ $\qquad |x| = \pm x$

Sol:

$$\lim_{x\to 0} \frac{+x}{x} \quad \text{If} \quad x>0 \Rightarrow \lim_{x\to 0} 1 \quad \text{If } x>0$$

$$\lim_{x\to 0} \frac{-x}{x} \quad \text{If} \quad x<0 \Rightarrow \lim_{x\to 0} -1 \quad \text{If } x<0$$

$$\lim_{x\to 0} 1 \quad \text{If} \quad x>0 \qquad \Bigg| \qquad \lim_{x\to 0} -1 \quad \text{If} \quad x<0$$



اگر ہم Right side سے 0 کی طرف آئیں تو answer 1 آرہا ہے، اور اگر ہم left side سے 0 کی طرف آئیں تو جواب -1 آرہا ہے، Limit exist نہیں کرے گا۔

L.H.L

$$\lim_{x \to 0^-} F(x) = \lim_{x \to 0^-} -1$$

$$= -1$$

R.H.L

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} 1$$

$$= 1$$

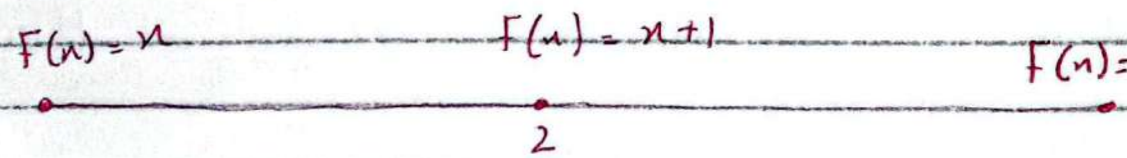
$$L.H.L \neq R.H.L$$

So Limit doesnot exist. Also graphically Limit doesnot exist.

(V) $\lim_{x \to 2} F(x)$ where $F(x) = \begin{cases} x & x < 2 \\ x+1 & x \geq 2 \end{cases}$

(Piece wise Function)

Sol



L.H.L

$$\lim_{x \to 2^-} F(x) = \lim_{x \to 2^-} x$$

$$= 2$$

R.H.L

$$\lim_{x \to 2^+} F(x) = \lim_{x \to 2^+} x+1$$

$$= 2+1$$

$$= 3$$

$$L.H.L \neq R.H.L$$

So Limit doesnot exist.

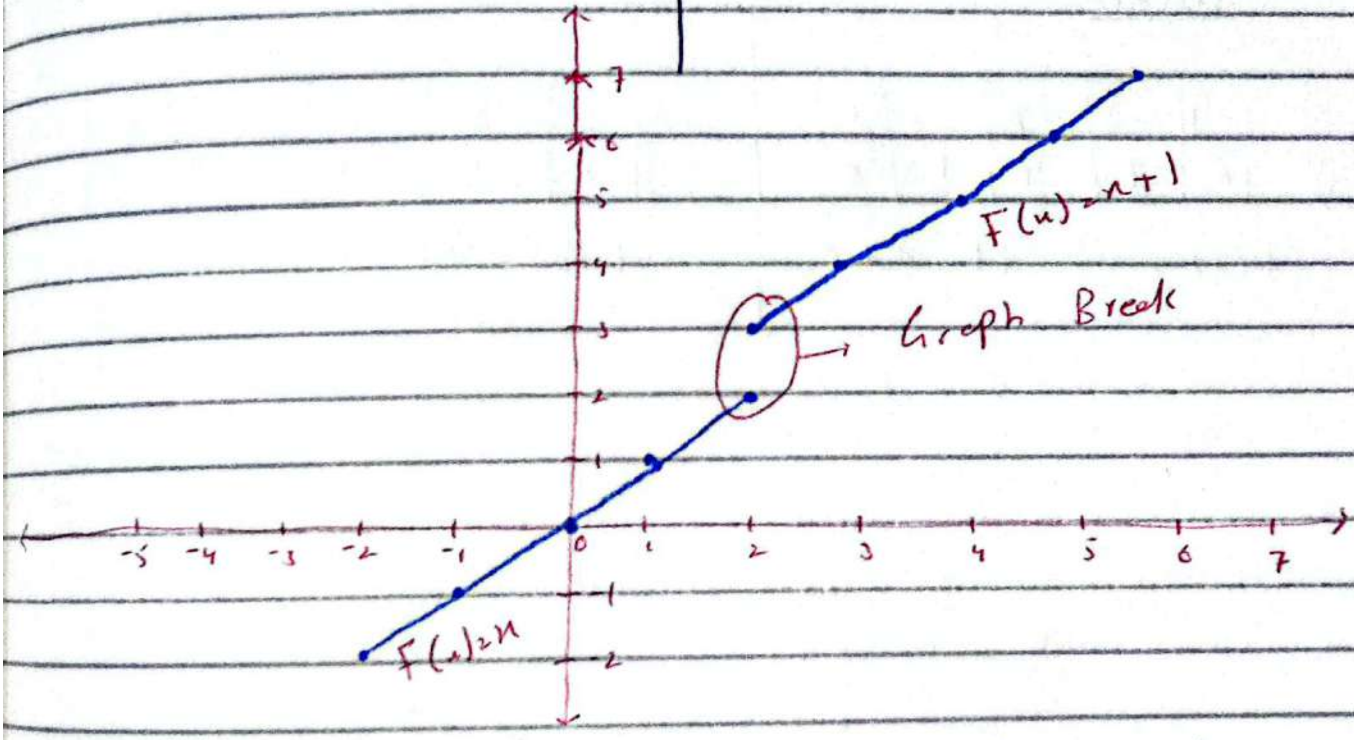
Graph

x	-2	-1	0	1	2
y	-2	-1	0	1	2

x	2	3	4	5	6
y	3	4	5	6	7

$F(x) = x$ If $x < 2$

$F(x) = x+1$ If $x \geq 2$



Graphically also Limit doesnot exist.

(vi) $\lim_{x \to 0} F(x)$ where $F(x) = \begin{cases} x^2 & \text{If } x < 0 \\ 2 & x = 0 \\ \sqrt{x} - 1 & x > 0 \end{cases}$

Sol

$F(x) = x^2$ $\quad$ 2 $\quad$ 0 $\quad$ $F(x) = \sqrt{x} - 1$

L.H.L

$\lim_{x \to 0^-} F(x) = \lim_{x \to 0^-} x^2$

$= 0$

R.H.L

$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \sqrt{x} - 1$

$= 0 - 1$

$= -1$

$$L.H.L \neq R.H.L$$

So Limit does not exist.

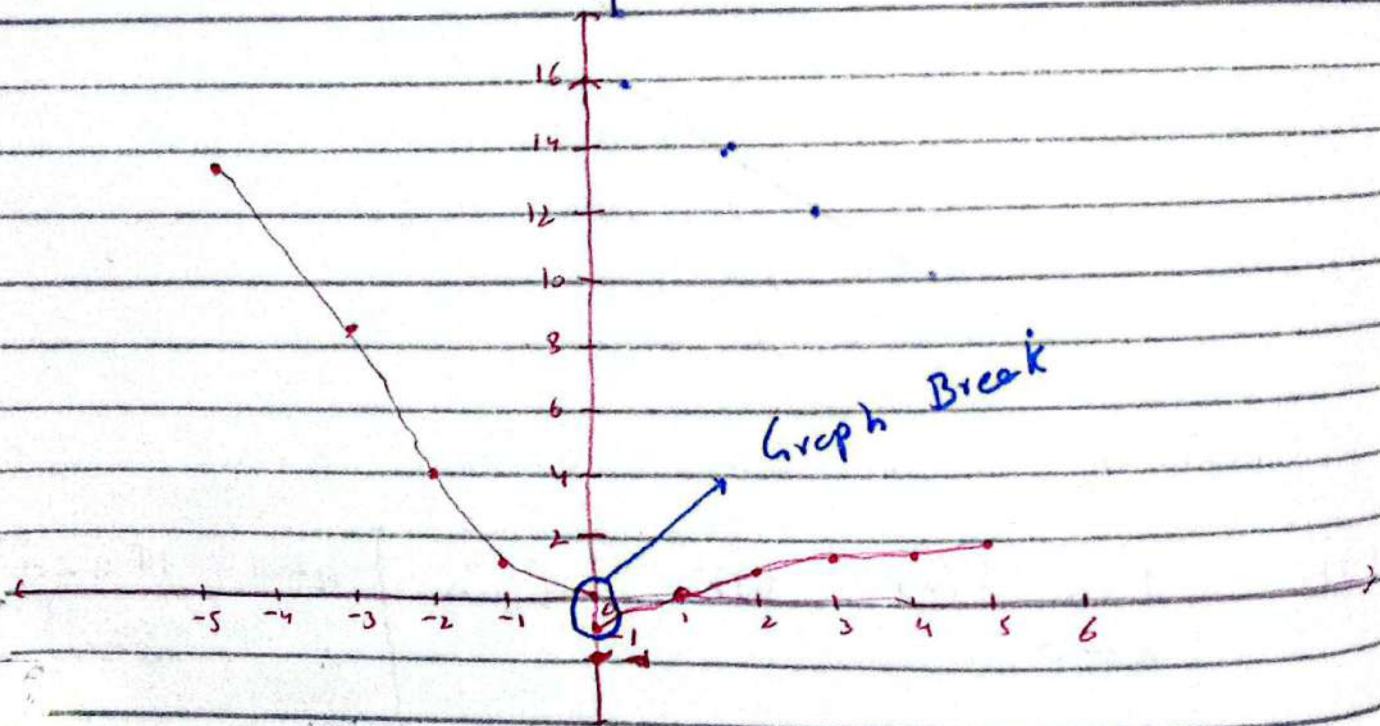
Graph

x	-4	-3	-2	-1	0
y	16	9	4	1	0

$F(x) = x^2$ IF $x < 0$

x	1	2	3	4	5	0
y	0	0.4	0.7	1	1.2	-1

$F(x) = \sqrt{x} - 1$ IF $x \geq 0$



Graphically also Limit does not exist.

(vii) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

Calculator should be in Radian Mode.

Sol

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} \times \frac{1 + \cos x}{1 + \cos x}$$

$$\lim_{x \to 0} \frac{1 - \cos^2 x}{x^2(1+\cos x)}$$

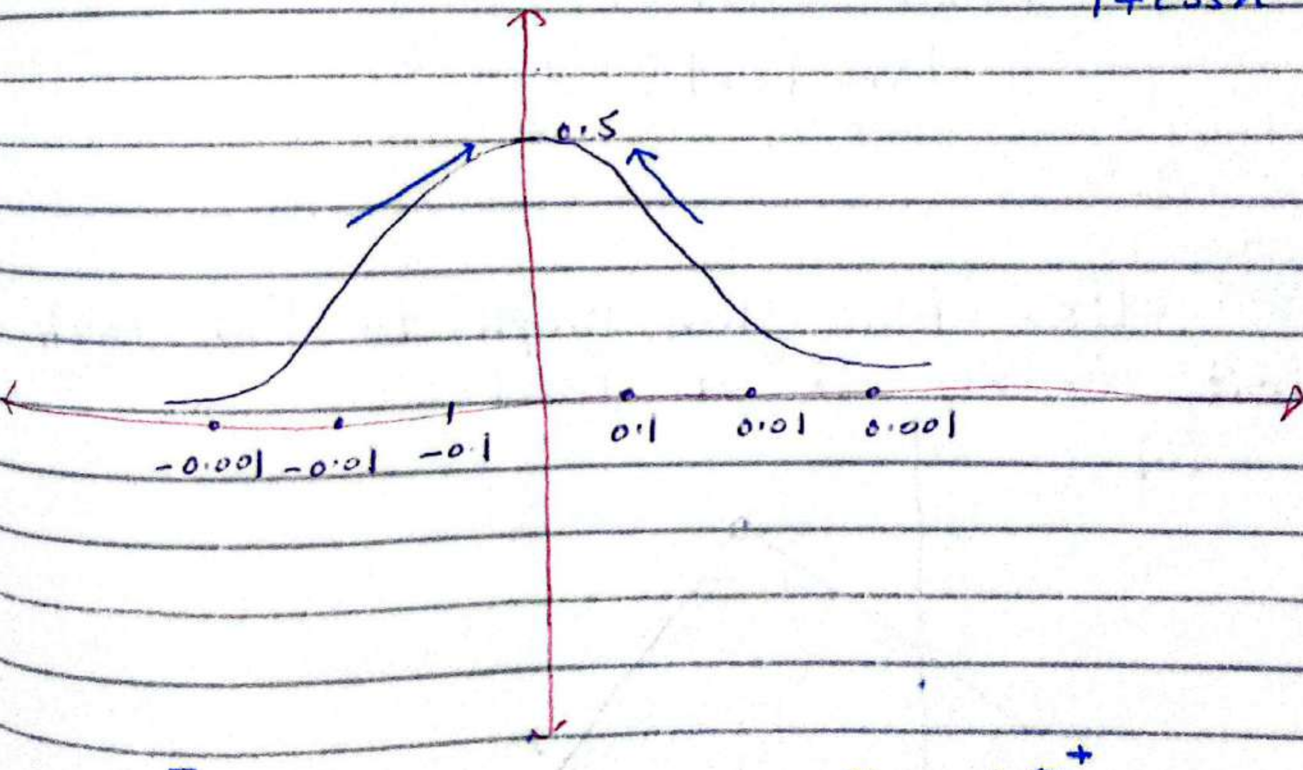
$$\lim_{x \to 0} \frac{\sin^2 x}{x^2(1+\cos x)}$$

$$\lim_{x \to 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x \to 0} \frac{1}{1+\cos x}$$

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

$$= 1 \cdot \frac{1}{1+\cos(0)} = \frac{1}{2} = 0.5$$

$$F(x) = \frac{1}{1+\cos x}$$



$x \to 0^-$

-0.1	$= 0.49958$
-0.01	$= 0.49995$
-0.001	$= 0.499999$
	$\cong 5$

$x \to 0^+$

0.1	$= 0.4999583$
0.01	$= 0.499995$
0.001	$= 0.499999$
	$\cong 5$

L.H.L	R.H.L
$\lim_{x \to 0^-} F(x) = \lim_{x \to 0^-} \frac{1}{1+\cos x}$	$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \frac{1}{1+\cos x}$
$= \frac{1}{1+1}$	$= \frac{1}{1+1}$
$= \frac{1}{2}$	$= \frac{1}{2}$

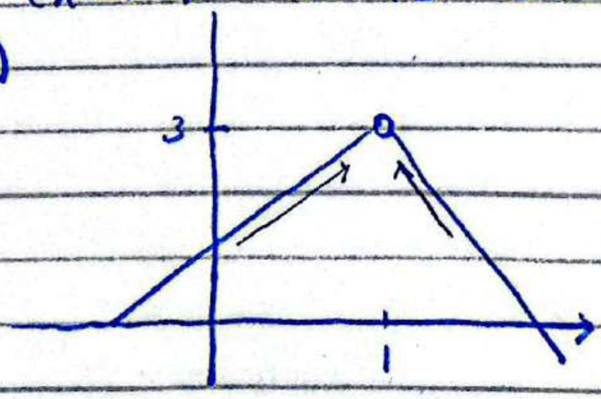
$$L.H.L = R.H.L$$

So Limit exist

$$\lim_{x \to 0} F(x) = \frac{1}{2}$$

Q#2 Use the given graph to find each Limit $(x \to 1)$ If it exists

(a)



SoL

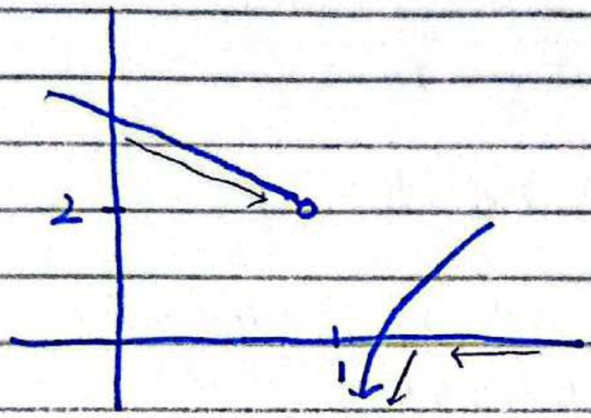
L.H.L	R.H.L
$\lim_{x \to 1^-} F(x) = 3$	$\lim_{x \to 1^+} F(x) = 3$

$L.H.L = R.H.L$

So Limit exist

$$\lim_{x \to 1} F(x) = 3$$

b)



L.H.L

R.H.L

$$\lim_{x \to 1^-} F(x) = 2$$

$$\lim_{x \to 1^+} F(x) = -\infty$$

$L.H.L \neq R.H.L$

Limit does not exist.

Q#3 Evaluate the Following Undefined Forms

1) $\frac{0}{0}$ 2) $\frac{x}{0}$ 3) $\frac{\infty}{\infty}$ 4) ∞

(i) 3 $\lim_{x \to -2} \frac{x^3+8}{x+2}$

Sol

$a^3+b^3 = (a+b)(a^2-ab+b^2)$

$= \lim_{x \to -2} \frac{(x)^3+(2)^3}{x+2}$

$= \lim_{x \to -2} \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}}$

$= \lim_{x \to -2} x^2-2x+4$

$= (-2)^2-2(-2)+4$

$= 4+4+4$

$= 12$

4 $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$

Sol

$= \lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} \times \frac{\sqrt{x}+2}{\sqrt{x}+2}$

$= \lim_{x \to 4} \frac{(\sqrt{x})^2-(2)^2}{(x-4)(\sqrt{x}+2)}$

$= \lim_{x \to 4} \frac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x}+2)}$

$= \lim_{x \to 4} \frac{1}{(\sqrt{x}+2)} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$

(5) $\lim_{x \to 7} \frac{x^2 - 21}{x + 2}$

Sol

Apply Limit

$= \frac{(7)^2 - 21}{7 + 2}$

$= \frac{49 - 21}{9}$

$= \frac{28}{9}$ Ans.

(6) $\lim_{x \to 0} \frac{x^2 - 6x}{x^2 - 7x + 6}$

Sol

Apply Limit

$= \frac{(0)^2 - 6(0)}{(0)^2 - 7(0) + 6}$

$= \frac{0}{6} = 0$ Ans

(7) $\lim_{y \to 1} \frac{y^3 - 1}{y - 1}$

Sol

$\lim_{y \to 1} \frac{y^3 - 1^3}{y - 1}$ $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$\lim_{y \to 1} \frac{\cancel{(y - 1)}(y^2 + y + 1)}{\cancel{(y - 1)}} = \lim_{y \to 1} y^2 + y + 1$

$= 1 + 1 + 1 = 3$

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8. $$\lim_{x \to 3^+} \frac{(x+3)^2}{\sqrt{x-3}}$$

Sol

$$= \frac{(3+3)^2}{\sqrt{3-3}}$$

$$= \frac{(6)^2}{0} = \frac{36}{0} = \infty$$

Limit does not exist.

9. $$\lim_{x \to 2} (x-4)^4 (x^2-3)^{10}$$

Sol

$$= (2-4)^4 (2^2-3)^{10}$$

$$= (-2)^4 (4-3)^{10}$$

$$= 16 (1)^{10}$$

$$= 16$$

10. $$\lim_{x \to 0} \left(x - \frac{1}{x-2} \right)$$

Sol

$$= \left(0 - \frac{1}{0-2} \right)$$

$$= -\frac{1}{-2}$$

$$= \frac{1}{2}.$$

(11) $\lim_{x \to -3} \dfrac{2x+6}{4x^2-36}$ $\left(\dfrac{0}{0}\right)$

Sol

$= \lim_{x \to -3} \dfrac{2x+6}{(2x)^2 - (6)^2}$

$= \lim_{x \to -3} \dfrac{\cancel{2x+6}}{(\cancel{2x+6})(2x-6)}$

$= \lim_{x \to -3} \dfrac{1}{2x-6}$

$= \dfrac{1}{2(-3)-6}$

$= \dfrac{1}{-6-6}$

$= \dfrac{1}{-12}$ Ans.

$\lim_{x \to -3} \dfrac{\cancel{2}^{1}(x+3)}{{}_{2}\cancel{4}(x^2-9)}$

$\lim_{x \to -3} \dfrac{x+3}{2\left[x^2-3^2\right]}$

$\lim_{x \to -3} \dfrac{\cancel{x+3}}{2(\cancel{x+3})(x-3)}$

$\lim_{x \to -3} \dfrac{1}{2(x-3)}$

$= \dfrac{1}{2(-3-3)}$

$= \dfrac{1}{-12}$ Ans.

SANDWICH Theorem

$\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

(12) $\lim_{x \to 0} \dfrac{\tan x}{x}$

Sol

$= \lim_{x \to 0} \dfrac{\sin x}{\cos x \cdot x}$

$= \lim_{x \to 0} \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x}$

$= \lim_{x \to 0} \dfrac{\sin x}{x} \cdot \lim_{x \to 0} \dfrac{1}{\cos x}$

$= 1 \cdot \dfrac{1}{\cos(0)} = 1 \cdot \dfrac{1}{1} = 1$

13) $$\lim_{x \to 0} \frac{x}{\sin 3x}$$

Sol

$$= \frac{1}{3} \lim_{x \to 0} \frac{3x}{\sin 3x}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{1}{\frac{\sin 3x}{3x}} \Rightarrow$$

Put $3x = t$
If $x \to 0$ $t \to 0$

$$= \frac{1}{3} \lim_{t \to 0} \frac{1}{\frac{\sin t}{t}}$$

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

$$= \frac{1}{3} \text{ Ans.}$$

Complete.

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