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Merging man and maths

Function: A Function f from a set A to a set B assigns to each element of A exactly one element of B. The set A is called the domain of the function and the set B is called the Codomain of the function.

Note:
A Function is like a machine that takes an input and produces a corresponding out put.

Indeterminate form:
(i) $\frac{1}{0} = \infty$ (ii) $\sqrt{-1} = i$ Complex
(Domain: Real Numbers)

Exc 1.1

Q#1 Find the domain of following functions

i) $f(x) = x^2 - 6$

Sol.

$-\infty \; -3 \; -2 \; -1 \; 0 \; 1 \; 2 \; 3 \; +\infty$

$\text{Dom}\, f(x) = \mathbb{R}$

or

$\text{Dom}\, f(x) = (-\infty, +\infty)$

or

$\text{Dom}\, f(x) =$ Set of Real numbers.

ii) $g(x) = \dfrac{x}{x+3}$

$\boxed{\begin{aligned} x+3 &= 0 \\ x &= -3 \end{aligned}}$

Sol. $\text{Dom}\, g(x) = \mathbb{R} - \{-3\}$ or $(-\infty, -3) \cup (-3, \infty)$ or Set of all real numbers except -3.

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(iii) $h(x) = \dfrac{x+4}{x^2-9}$

$x^2 - 9 = 0$
$x^2 = 9$
$x = \pm 3$

Sol Dom $h(x) = R - \{-3, 3\}$

or

Dom $h(x)$ = Set of real numbers except -3 and 3.

or

Dom $h(x) = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

(iv) $i(x) = \dfrac{x}{5x+2}$

$5x + 2 = 0$
$5x = -2$
$x = \dfrac{-2}{5}$

Sol Dom $i(x) = R - \{-2/5\}$

or

Dom $i(x)$ = Set of real numbers except $-2/5$

or (not included $-2/5$)

Dom $i(x) = \left(-\infty, \dfrac{-2}{5}\right) \cup \left(\dfrac{-2}{5}, \infty\right)$

v) $j(x) = \dfrac{x}{x^2+4}$

$x^2 + 4 = 0$
$\sqrt{x^2} = \sqrt{-4}$
$x = \pm 2i$
only put real numbers

Sol Dom $j(x) = R$

or

Dom $j(x)$ = Set of real numbers

or

Dom $j(x) = (-\infty, +\infty)$.

$x + 1 \geq 0$
$x \geq -1$

vi) $K(x) = \sqrt{x+1}$

Sol Dom $K(x) = x \geq -1$ or $[-1, +\infty)$

Set of real numbers greater than or equal to -1.

Domain = Input

Range = output

Q#2 Find the Domain and Range of the Functions:

i) $F(x) = x + 7$

Sol: Dom $F(x) = (-\infty, \infty)$ or R

Ran $f(x) = (-\infty, \infty)$ or R

ii) $F(x) = 2x^2 + 1$

Sol:

Dom $F(x) = (-\infty, +\infty)$ or R

Range $F(x) =$

$$x^2 \geq 0$$
$$2x^2 \geq 0$$
$$2x^2 + 1 \geq 0 + 1$$
$$2x^2 + 1 \geq 1$$

Range $F(x) = [1, \infty)$

Square always +ve

$x^2 \geq 0$

$y = 2x^2 \geq 0$

$y = 2x^2 + 1 \geq 1$

Range = $[1, \infty)$

iii) $F(x) = 2\sqrt{x - 5}$

Sol: Dom $F(x) = x \geq 5$ or $[5, \infty)$

Range $F(x) = [0, \infty)$

$x - 5 \geq 0$

$x \geq 5$

$\sqrt{x-5} \geq 0$

$2\sqrt{x-5} \geq 0$

iv) $F(x) = |x - 2| - 3$

Sol: Dom $F(x) = (-\infty, +\infty)$

Range $F(x) = [-3, \infty)$

Absolute always +ve

$|x - 2| \geq 0$

$|x - 2| - 3 \geq -3$

$\sin x$

Domain $= (-\infty, +\infty)$

Range $[-1, 1]$

$+1 \quad +1$

v) $F(x) = 1 + \sin x$

Sol

Dom $F(x) = (-\infty, +\infty)$

Range $F(x) = [0, 2]$

vi) $F(x) = 3 + \sqrt{x-2}$

$x - 2 \geq 0$

$x \geq 2$

Sol

Dom $F(x) = [2, +\infty)$ or $x \geq 2$

Range $F(x) =$

$\sqrt{x-2} \geq 0$

$3 + \sqrt{x-2} \geq 0 + 3$

$3 + \sqrt{x-2} \geq 3$

$[3, +\infty)$

vii) $F(x) = \dfrac{3e^x}{7}$

*exponential Function Cannot give undefined answer:

Sol

Dom $= (-\infty, +\infty)$

$e^x > 0$

$\dfrac{3e^x}{7} > 0$

Range $F(x) = (0, \infty)$

$e^0 = 1 \quad e = 2.73$

$e^{-1} = \dfrac{1}{e} = \dfrac{1}{2.73}$

All answers > 0

viii) $F(x) = \dfrac{x^2-16}{x+4}$

$x+4=0$
$x=-4$

Sol/ $\text{Dom } F(x) = R - \{-4\}$ or $(-\infty, -4) \cup (-4, +\infty)$

Range $F(x)$

$$y = \frac{x^2-4^2}{x+4} = \frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}}$$

$y = x-4$

$y = -4-4 \neq -8$

Range $= R - \{-8\}$

Ix) $F(x) = (x-1)^2+1$

Sol/ $\text{Dom } F(x) = (-\infty, +\infty)$ or R

Range $F(x) =$

$(x-1)^2 \geq 0$

$y = (x-1)^2+1 \geq 1$

Range $F(x) = [1, +\infty)$

$F(x)$ x^2-2x+2
Range $= ?$
$x^2-2x = -2$
$x^2-2x+(-1)^2 = -2+1$
$(x-1)^2 = -1$
$(x-1)^2+1 = 0$

x) $F(x) = \dfrac{1}{x-1}$

$x-1=0$
$x=1$

Sol/ $\text{Dom } F(x) = R - \{1\}$ or $(-\infty, 1) \cup (1, \infty)$

Range $F(x)$

$$y = \frac{1}{x-1}$$

$$y(x-1) = 1$$
$$xy - y = 1$$
$$xy = 1 + y$$
$$x = \frac{1+y}{y}$$

Range = y

$$y \neq 0$$

Range = $R - \{0\}$

xi) $F(x) = \dfrac{x-2}{x+3}$

Sol

Dom $F(x) = R - \{-3\}$

$x + 3 = 0$
$x = -3$

Range $F(x)$ =

$$y = \frac{x-2}{x+3}$$
$$(x+3)y = x-2$$
$$xy + 3y = x - 2$$
$$xy - x = -2 - 3y$$
$$x(y-1) = -2 - 3y$$
$$x = \frac{-2-3y}{y-1}$$

Range $F(x) = R - \{1\}$.

xii) $F(x) = \dfrac{x^2 - x - 6}{x-3}$

Sol

Dom $F(x) = R - \{3\}$

$x - 3 = 0$
$x = 3$

Rang $F(x) =$

$$y = \frac{x^2 - 3x + 2x - 6}{x - 3}$$

$$y = \frac{x(x-3) + 2(x-3)}{x-3}$$

$$y = \frac{(x+2)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$y = x + 2$$

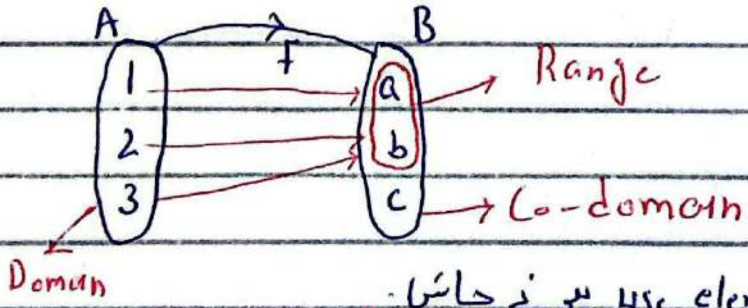
$$y = 3 + 2 \neq 5$$

$$\text{Range} = R - \{5\}.$$

Types of Function

i) Into Function: Let

$$A = \{1, 2, 3\} \qquad B = \{a, b, c\}$$



$$F = \{(1,a)\ (2,b)\ (3,b)\}$$

$$\text{Dom}F = \{1, 2, 3\}$$

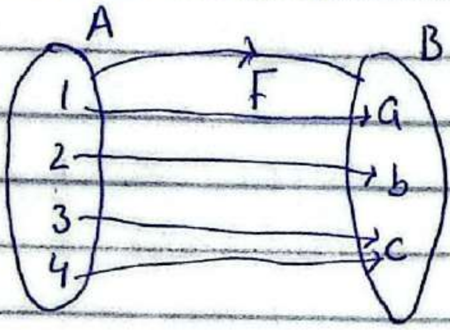
$$\text{Range} = \{a, b\} \subseteq \text{Co-domain}$$

A کے سارے element use ہونے چاہئیں۔ پورا Set B Codomain ہوتا ہے لیکن جو element use ہوتے ہیں وہ Range میں آتے ہیں، Range Subset ہونی چاہئے Codomain کے۔

ii) On to Function:- (Surjective)

Let

$A = \{1, 2, 3, 4\}$ $\quad B = \{a, b, c\}$



• Domain unique ; All are used

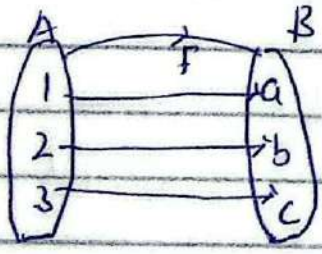
Range $= \{a, b, c\}$ = Codomain

اگر Range , Codomin کے برابر آ جائے تو وہ Function onto ہوگا۔

iii) one - one Function:

$A = \{1, 2, 3\}$ $\quad B = \{a, b, c\}$

• All Linear Functions are one to one.

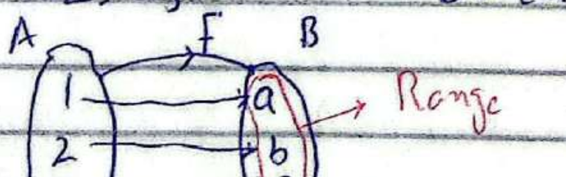


• Distinct element have distinct image.

$$\because F(x_1) = F(x_2)$$
$$\Rightarrow x_1 = x_2$$

iv) Into and one - one:

$A = \{1, 2, 3\}$ $\quad B = \{a, b, c, d\}$



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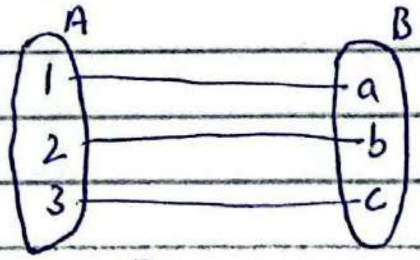
- Range = {a, b, c} ⊆ Codomain
- All element in Set A has unique images in Set B.

Into and one-one is called Injective.

v) onto and one - one :

$A = \{1, 2, 3\}$ $\qquad$ $B = \{a, b, c\}$



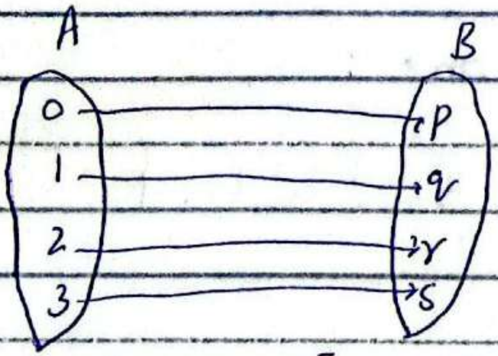
- Range = {a, b, c} = Codomain.
- It is also one - one

Onto and one - one is called bijective

Note. سب سے پہلے Function ہونا چاہیے. جالیت نہ ہو اور زیادتی نہ ہو
- Range ⊆ Codomain → Into Function
- Range = Codomain → on to Function.
- $F(x_1) = F(x_2) \Rightarrow x_1 = x_2$ → one - one Function.
  one to one correspondence

Q#3 Given that $A = \{0, 1, 2, 3\}$ $B = \{p, q, r, s\}$ and $f = \{(0,p)\ (1,q)\ (2,r)\ (3,s)\}$ Check whether the function is one to one onto and / or into.

Sol



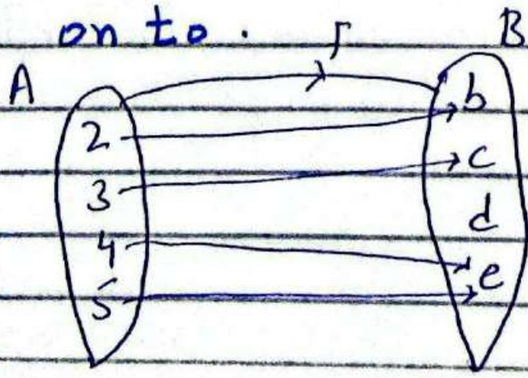
Range = $\{p, q, r, s\}$ = Codomain

one to one Correspondence

This Function is onto and one to one. Also called bijective Function.

Q #4 $A = \{2, 3, 4, 5\}$ $B = \{b, c, d, e\}$. The function is defined as $f = \{(2,b)\ (3,c)\ (4,e)\ (5,e)\}$. Check whether the function is one to one, into or onto.

Sol



Range = {b, c, e} $\subseteq$ Codomain

- This is into Function.
- Not unique image. This is not one to one.

Q#5 Check whether the function are one to one or not.

i) $F(x) = 4x - 7$

Sol Let

$F(x_1) = F(x_2)$

$4x_1 - 7 = 4x_2 - 7$

$4x_1 - \cancel{7} + \cancel{7} = 4x_2$

$\cancel{4}x_1 = \cancel{4}x_2$

$\boxed{x_1 = x_2}$

one to one function

(ii) $F(x) = 6x^2 + 2$

Sol Let $F(x_1) = F(x_2)$

$6x_1^2 + \cancel{2} = 6x_2^2 + \cancel{2}$

$\cancel{6}x_1^2 = \cancel{6}x_2^2$

$x_1^2 = x_2^2$

Taking square root

$\sqrt{x_1^2} = \sqrt{x_2^2}$

$x_1 = \pm x_2$

Not one-one function

i) $F(x) = 4x - 7$

Sol Let $y = 4x - 7$

x	1	2	3	4	5	6
y	-3	1	5	9	13	17

Distinct elements has distinct images

one to one function.

(ii) $F(x) = 6x^2 + 2$

Sol Let $y = 6x^2 + 2$

x	1	0	-1	2	3
y	8	2	8	26	56

Distinct elements has same images

$x = 1$

$F(1) = 8$

$x = -1$

$F(-1) = 8$

$F(1) = F(-1)$

$1 \neq -1$

(iii) $F(x) = \dfrac{x^3 - 1}{2}$

Sol Let $F(x_1) = F(x_2)$

$$\frac{x_1^3 - 1}{\cancel{2}} = \frac{x_2^3 - 1}{\cancel{2}}$$

$$x_1^3 - \cancel{1} = x_2^3 - \cancel{1}$$

$$x_1^3 = x_2^3$$

Taking cube root

$$\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3}$$

$$x_1 = x_2$$

This is one to one Function

(iii) $F(x) = \dfrac{x^3 - 1}{2}$

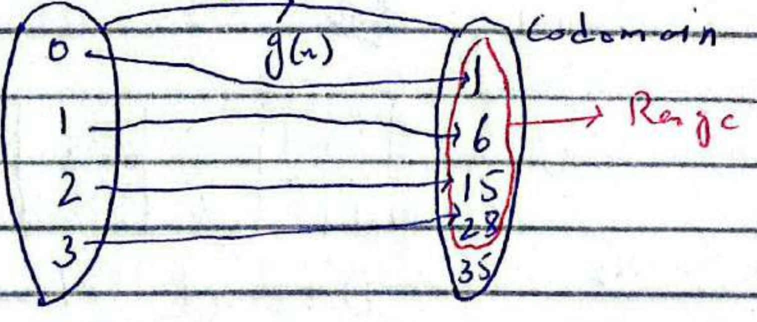
Sol Let $y = \dfrac{x^3 - 1}{2}$

x	-1	0	1	2
y	-1	-0.5	0	3.5

Here distinct elements has distinct images one to one Function

Q #6 Check the type of Function
$g(x) = 2x^2 + 3x + 1$ IF Dom $g(x) = \{0, 1, 2, 3\}$
and ~~Range~~ Codomain $g(x) = \{1, 6, 15, 28, 35\}$.

Sol



$g(x) = 2x^2 + 3x + 1$

$g(0) = 2(0)^2 + 3(0) + 1$

$= 1$

$g(1) = 2(1)^2 + 3(1) + 1$
$= 6$

$g(2) = 2(2)^2 + 3(2) + 1$
$= 8 + 6 + 1$
$= 15$

$g(3) = 2(3)^2 + 3(3) + 1$
$= 18 + 9 + 1$
$= 28$

Range $= \{1, 6, 15, 28\} \subseteq$ Codomain.

→ Into function
→ one to one Function
• This is also called Injective function

Q#7 Find the type of the function $h(x) = 2x + 1$ defined on $h: R \to R$

Sol
Domain $= R =$ real numbers
Codomain $= R =$ real numbers

Check one to one
$h(x) = 2x + 1$
$h(x_1) = h(x_2)$
$2x_1 + \cancel{1} = 2x_2 + \cancel{1}$
$\cancel{2}x_1 = \cancel{2}x_2$
$x_1 = x_2$

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Range = ?

Let $y = 2x + 1$

$y - 1 = 2x$

$\frac{y-1}{2} = x$ Range = y = All values input

Range = R = Co-domain

On-to Function

- one to one and on to Function. This Function is called bijective Function.

Q#8 If $f: A \to B$ is defined by $f(x) = \frac{x-2}{x-3}$ For all $x \in A$ where
$A = R - \{3\}$ and $B = R - \{1\}$.
Then show that the function f is bijective.

Sol:
$A \to$ Domain $\qquad$ $B \to$ Co-domain

$R - \{3\}$ $\qquad$ $R - \{1\}$

$$f(x) = \frac{x-2}{x-3}$$

Check one to one

$$f(x_1) = f(x_2)$$

$$\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

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$$(x_1-2)(x_2-3) = (x_2-2)(x_1-3)$$
$$\cancel{x_1x_2} + 3x_1 - 2x_2 + \cancel{6} = \cancel{x_1x_2} - 3x_2 - 2x_1 + \cancel{6}$$
$$-3x_1 - 2x_2 = -3x_2 - 2x_1$$
$$-3x_1 + 2x_1 = -3x_2 + 2x_2$$
$$-x_1 = -x_2$$
$$x_1 = x_2$$

This shows that Function is 1-1.

Range = ?

$$y = \frac{x-2}{x-3}$$
$$y(x-3) = x-2$$
$$xy - 3y = x - 2$$
$$xy - x = -2 + 3y$$
$$x(y-1) = -2 + 3y$$
$$x = \frac{-2+3y}{y-1} \qquad y \neq 1$$

Range = $R - \{1\}$ = Codomain

This Show on to Function.

Hence proved It is bijective because It is one to one and on to.

Q#9 Find the Domain and Range of inverse Functions. Also prove that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

i) $f(x) = 4x-3$

Sol

$$\boxed{\begin{aligned} \text{Dom } f &= \mathbb{R} = (-\infty, +\infty) \\ \text{Range } f &= \mathbb{R} = (-\infty, +\infty) \end{aligned}}\ \begin{aligned} &= \text{Range } f^{-1} \\ &= \text{Domain } f^{-1} \end{aligned}$$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$$y = 4x-3$$

$$y+3 = 4x$$

$$\frac{y+3}{4} = x$$

$$f^{-1}(y) = \frac{y+3}{4}$$

replace y by x.

$$\boxed{f^{-1}(x) = \frac{x+3}{4}}$$

$\text{Dom } f^{-1} = (-\infty, +\infty)$

$\text{Range } f^{-1} = (-\infty, +\infty)$

Proof $f(f^{-1}(x)) = f^{-1}(f(x))$.

$$f\left(\frac{x+3}{4}\right) = f^{-1}(4x-3)$$

$$4\left(\frac{x+3}{4}\right)-3 = \frac{(4x-3)+3}{4}$$

$$x+\cancel{3}-\cancel{3} = \frac{\cancel{4}x-\cancel{3}+\cancel{3}}{\cancel{4}}$$

$$x = x$$

Proved.

(ii) $F(x) = \dfrac{x}{x-5}$

Sol Dom $F(x) = R-\{5\} =$ Range F^{-1}

Range $F(x) = R-\{1\} =$ Domain F^{-1}

Let $y = \dfrac{x}{x-5} \Rightarrow (x-5)y = x$

$xy - 5y = x$

$xy - x = 5y$

$x(y-1) = 5y$

$x = \dfrac{5y}{y-1}$

As $F(x) = y$

$x = F^{-1}(y)$

$F^{-1}(y) = \dfrac{5y}{y-1}$

replace y by x

$F^{-1}(x) = \dfrac{5x}{x-1}$

Domain $F^{-1} = R-\{1\}$

Range $F^{-1} = R-\{5\}$.

Proof $F(F^{-1}(x)) = F^{-1}(F(x))$

$$F\left(\frac{5x}{x-1}\right) = F^{-1}\left(\frac{x}{x-5}\right)$$

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$$= \frac{\frac{5x}{x-1}}{\frac{5x}{x-1} - 5} = \frac{5\left(\frac{x}{x-5}\right)}{\frac{x}{x-5} - 1}$$

$$= \frac{5x/\cancel{x-1}}{\frac{5x-5(x-1)}{\cancel{x-1}}} = \frac{5x/\cancel{x-5}}{\frac{x-(x-5)}{\cancel{x-5}}}$$

$$\frac{5x}{\cancel{5x}-\cancel{5x}+5} = \frac{5x}{\cancel{x}-\cancel{x}+5}$$

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{\cancel{5}x}{\cancel{5}}$$

$$x = x$$ Proved

(iii) $F(x) = \frac{x+2}{x-1}$

Sol

Domain $F(x) = R - \{1\} = \text{Rang } F^{-1}$

Range $F(x) = R - \{1\} = \text{Dom } F^{-1}$

As $F(x) = y \quad x = F^{-1}(y)$

$$y = \frac{x+2}{x-1}$$

$$xy - y = x + 2$$

$$xy - x = 2 + y$$

$$x(y-1) = 2 + y$$

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$$x = \frac{2+y}{y-1}$$

$$f^{-1}(y) = \frac{2+y}{y-1}$$

replace y by x

$$f^{-1}(x) = \frac{2+x}{x-1}$$

Proof

$$f(f^{-1}(x)) = f^{-1}(f(x))$$

$$f\left(\frac{2+x}{x-1}\right) = f^{-1}\left(\frac{x+2}{x-1}\right)$$

$$= \frac{\frac{2+x}{x-1} + \frac{2}{1}}{\frac{2+x}{x-1} - \frac{1}{1}} = \frac{\frac{2}{1} + \frac{x+2}{x-1}}{\frac{x+2}{x-1} - \frac{1}{1}}$$

$$= \frac{\cancel{2}+x+2x-\cancel{2}}{2+\cancel{x}-\cancel{x}+1} = \frac{2x-\cancel{2}+x+\cancel{2}}{\cancel{x}+2-\cancel{x}+1}$$

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{\cancel{3}x}{\cancel{3}}$$

$$x = x \qquad \text{proved}$$

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(IV) $f(x) = \sqrt{x+2}$

Sol

Domain $f(x) = [-2, \infty) =$ Range f^{-1}
Range $f(x) = [0, \infty) =$ Domain f^{-1}

$x+2 \geq 0$	$\sqrt{x+2} \geq 0$
$x \geq -2$	$y \geq 0$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$y = \sqrt{x+2}$

$y^2 = x+2$

$y^2 - 2 = x$

$f^{-1}(y) = y^2 - 2$

replace y by x

$f^{-1}(x) = x^2 - 2$

Proof

$f(f^{-1}(x)) = f^{-1}(f(x))$

$\sqrt{x^2 - \cancel{2} + \cancel{2}} = (\sqrt{x+2})^{\cancel{2}} - 2$

$x = x + \cancel{2} - \cancel{2}$

$x = x$

Proved

(v) $F(x) = x^2 + 6$

Sol

$\text{Dom } f(x) = [0, \infty) = \text{Range } f^{-1}$

$\text{Rang } F(x) = [6, \infty) = \text{Domain } f^{-1}$

$x^2 \geq 0$

$x^2 + 6 \geq 6$

Let $y = F(x) \Rightarrow x = F^{-1}(y)$

$y = x^2 + 6$

$y - 6 = x^2$

$\pm\sqrt{y-6} = x$

$x = \sqrt{y-6} \qquad x^2 \geq 0$

$F^{-1}(y) = \sqrt{y-6}$

replace y by x

$F^{-1}(x) = \sqrt{x-6}$

Domain and Range of $f^{-1}(x)$

Domain

$x - 6 \geq 0$

$x \geq 6$

$[6, \infty)$

$y = x^2 + 6 \quad \sqrt{x-6} \geq 0$

$y \geq 0$

Range

$[0, \infty)$

Proof

$F(F^{-1}(x)) = f^{-1}(F(x))$

$(\cancel{\sqrt{x-6}})^{\cancel{2}} + 6 = \sqrt{x^2 + \cancel{6} - \cancel{6}}$

$x - \cancel{6} + \cancel{6} = \cancel{\sqrt{x^2}}$

$x = x.$

Proved

(vi) $F(x) = \dfrac{2x-1}{x+4}$

Sol

Domain $F(x) = R - \{-4\}$ Range F^{-1}
Range $F(x) = R - \{2\}$ Domain F^{-1}

Let $y = F(x)$ $x = F^{-1}(y)$

$$y = \frac{2x-1}{x+4}$$

$$xy + 4y = 2x - 1$$

$$xy - 2x = -1 - 4y$$

$$x(y-2) = -1 - 4y$$

$$x = \frac{-1-4y}{y-2} \qquad R - \{2\}$$

$$F^{-1}(y) = \frac{-1-4y}{y-2}$$

replace y by x

$$\boxed{F^{-1}(x) = \frac{-1-4x}{x-2}}$$

Proof

$$F(F^{-1}(x)) = F^{-1}(f(x)) = x$$

L.H.S

$$F(F^{-1}(x))$$

$$F\left(\frac{-1-4x}{x-2}\right)$$

$$= \frac{2\left(\frac{-1-4x}{x-2}\right) - \frac{1}{1}}{\frac{-1-4x}{x-2} + \frac{4}{1}}$$

$$= \frac{-\cancel{2} - 8x - x + \cancel{2}}{-1 - \cancel{4x} + \cancel{4x} - 8}$$

$$= \frac{-\cancel{9}x}{-\cancel{9}}$$

$$= x$$

R.H.S

$$F^{-1}(F(x))$$

$$F^{-1}\left(\frac{2x-1}{x+4}\right)$$

$$= \frac{-1 - 4\left(\frac{2x-1}{x+4}\right)}{\frac{2x-1}{x+4} - \frac{2}{1}}$$

$$\frac{-x - \cancel{4} - 8x + \cancel{4}}{\cancel{2x} - 1 - \cancel{2x} - 8} = \frac{-9x}{-9} = x$$

Proved

Complete

Available at MathCity.org

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