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(In the Name of Allah, the Most Compassionate, the Most Merciful)

Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 5: Partial Fractions
- Partial Fractions
- Exercise 5.1: Q1 - Q6

Unit 5 Partial Fractions

INTRODUCTION

We have learnt in the previous classes how to add two or more rational fractions into a single rational fraction. For example,

(i) $$\frac{1}{x-1}+\frac{2}{x+2}=\frac{3x}{(x-1)(x+2)}$$

and (ii) $$\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x-2}=\frac{5x^2+5x-3}{(x+1)^2(x-2)}$$

In this unit we shall learn how to reverse the order in (i) and (ii) that is to express a single rational function as a sum of two or more single rational functions which are called **Partial Fractions**.

Expressing a rational function as a sum of partial fractions is called **Partial Fraction Resolution**. It is an extremely valuable tool in the study of calculus to decompose a complex rational function into a sum of simpler fractions.

An open sentence formed by using the sign of equality '=' is called an equation. The equations can be divided into the following two kinds:

Conditional equation: It is an equation in which two algebraic expressions are equal for particular values of the variable e.g.,

(a) $2x = 3$ is a conditional equation and it is true only if $x=\frac{3}{2}$.

(b) $x^2+x-6=0$ is a conditional equation and it is true for $x = 2, -3$ only.

Note:

For simplicity, a conditional equation is called an equation.

Identity: It is an equation which holds good for all values of the variable e.g.,

(a) $(a+b)x \equiv ax+bx$ is an **identity** and its two sides are equal for all values of x.

(b) $(x+3)(x+4) \equiv x^2+7x+12$ is also an identity which is true for all values of x.

For convenience, the symbol "=" shall be used both for equation and identity.

5.1 Rational Fraction

An expression of the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x with real coefficients and $Q(x) \neq 0$, is called a rational fraction. A rational fraction is of two types.

5.1.1 Proper Rational Fraction

A rational function $\dfrac{P(x)}{Q(x)}$ is called a **Proper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator. For example, $\dfrac{3}{x+1}, \dfrac{2x-5}{x^2+4}$ and $\dfrac{9x^2}{x^3-1}$ are proper rational fractions or proper fractions.

5.1.2 Improper Rational Fraction

A rational fraction $\dfrac{P(x)}{Q(x)}$ is called an **Improper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator.

For example, $\dfrac{x}{2x-3}, \dfrac{(x-2)(x+1)}{(x-1)(x+4)}, \dfrac{x^2-3}{3x+1}$ and $\dfrac{x^3-x^2+x+1}{x^2+5}$

are improper rational fractions or improper fractions.

Any improper rational fraction can be reduced by division to a mixed form, consisting of the sum of a polynomial and a proper rational fraction.

For example, $\frac{3x^2+1}{x-2}$ is an improper rational fraction. By long division we obtain $\frac{3x^2+1}{x-2} = 3x+6+\frac{13}{x-2}$

Quotient

Divisor
$$\begin{array}{r} 3x+6 \\ x-2\,\overline{)\,\cancel{3x^2}+1} \\ \pm\cancel{3x^2}\mp 6x \\ \hline 6\cancel{x}+1 \\ \pm\cancel{6x}\mp 12 \\ \hline 13 \end{array}$$
Rem.

i.e., an improper rational fraction has $\frac{3x^2-1}{x-2}$ been reduced to the sum of a polynomial $3x + 6$ and a proper rational fraction $\frac{13}{x-2}$.

$Q + \frac{R}{D} = 3x+6+\frac{13}{x-2}$

When a rational fraction is separated into partial fractions, the result is an identity; i.e., it is true for all values of the variable in the domain of identity.

The evaluation of the coefficients of the partial fractions is based on the following theorem:

> "*If two polynomials are equal for all values of the variable, then the polynomials have same degree and the coefficients of like powers of the variable in both the polynomials must be equal*".

For example,

If $px^3+qx^2-ax+b = 2x^3-3x^2-4x+5,\ \forall\, x$ then $p = 2$, $q = -3$, $a = 4$ and $b = 5$.

5.1.3 Resolution of a Rational Fraction $\frac{P(x)}{Q(x)}$ into Partial Fractions

Following are the main points of resolving a rational fraction $\frac{P(x)}{Q(x)}$ into partial fractions:

(i) The degree of $P(x)$ must be less than that of $Q(x)$. If not, divide and work with the remainder theorem.

(ii) Factor the denominator $Q(x)$ into its irreducible factor, write the rational fraction into partial fractions.

(iii) Multiply the identity with the denominator of left hand side.

(iv) Equate the coefficients of like terms (powers of x).

(v) Solve the resulting equations for the coefficients.

We now discuss the following cases of partial fractions resolution.

Case I: Resolution of $\frac{P(x)}{Q(x)}$ into partial fractions when $Q(x)$ has only non-repeated linear factors:

$(x-1)(x-2)$

The polynomial $Q(x)$ may be written as:

$$Q(x) = (x - a_1)(x - a_2) \dots (x - a_n), \text{ where } a_1 \neq a_2 \neq \dots \neq a_n$$

$$\therefore \quad \frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n} \text{ is an identity.}$$

Where $A_1, A_2, \dots, A_n$ are numbers to be found.

The method is explained by the following examples:

Example 1: Resolve $\frac{7x+25}{(x+3)(x+4)}$ into partial fractions.

Solution: Suppose $\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

Multiplying both sides by $(x + 3)(x + 4)$, we get

$$7x + 25 = A(x + 4) + B(x + 3)$$

$$\Rightarrow \quad 7x + 25 = Ax + 4A + Bx + 3B$$

$$\Rightarrow \quad 7x + 25 = (A + B)x + 4A + 3B$$

this is an identity in x.

So, equating the coefficients of like powers of x we have

$$7 = A + B \quad \text{and} \quad 25 = 4A + 3B$$

Solving these equations, we get $\boxed{A = 4}$ and $\boxed{B = 3}$.

Hence, $\frac{7x+25}{(x+3)(x+4)} = \frac{4}{x+3} + \frac{3}{x+4}$.

Alternative method

Suppose $\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

$$\Rightarrow \quad 7x + 25 = A(x + 4) + B(x + 3)$$

As two sides of the identity are equal for all values of x,

Let us put $x = -3$ and $x = -4$ in it.

For A, putting $x + 3 = 0$ i.e., $x = -3$ we get,

$$-21 + 25 = A(-3 + 4)$$

$$\Rightarrow \quad \boxed{A = 4}$$

For B, putting $x + 4 = 0$ i.e., $x = -4$ we get,

$$-28 + 25 = B(-4 + 3)$$

$$\Rightarrow \quad \boxed{B = 3}$$

Hence, $\dfrac{7x+25}{(x+3)(x+4)} = \dfrac{4}{x+3} + \dfrac{3}{x+4}$.

Example 2: Resolve $\dfrac{x^2-10x+13}{(x-1)(x^2-5x+6)}$ into Partial Fractions.

Solution: The polynomial $x^2 - 5x + 6$ in the denominator can be factorized and its factors are $x - 3$ and $x - 2$.

$$\therefore \quad \frac{x^2-10x+13}{(x-1)(x^2-5x+6)} = \frac{x^2-10x+13}{(x-1)(x-2)(x-3)}$$

Suppose $\dfrac{x^2-10x+13}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$

$$\Rightarrow \quad x^2 - 10x + 13 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

which is an identity in x.

For A, putting $x - 3 = 0$ i.e., $x = 1$, we get

$$(1)^2 - 10(1) + 13 = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$$

$$\Rightarrow \quad 1 - 10 + 13 = A(-1)(-2) + B(0)(-2) + C(0)(-1)$$

$$4 = 2A$$

$$\therefore \quad \boxed{A = 2}$$

For B, putting $x-2 = 0$ i.e., $x = 2$, we get

$$(2)^2 - 10(2) + 13 = A(0)(2-3) + B(2-1)(2-3) + C(2-1)(0)$$

$$\Rightarrow \quad 4 - 20 + 13 = B(1)(-1)$$

$$\Rightarrow \quad -3 = -B$$

$$\therefore \quad \boxed{B = 3}$$

For C, putting $x-3 = 0$ i.e., $x = 3$, we get

$$(3)^2 - 10(3) + 13 = A(3-2)(0) + B(3-1)(0) + C(3-1)(3-2)$$

$$\Rightarrow \quad 9 - 30 + 13 = C(2)(1)$$

$$\Rightarrow \quad -8 = 2C$$

$$\therefore \quad \boxed{C = -4}$$

Hence partial fractions are: $\dfrac{2}{x-1} + \dfrac{3}{x-2} - \dfrac{4}{x-3}$

Example 3: Resolve $\dfrac{2x^3 + x^2 - x - 3}{x(2x+3)(x-1)}$ into Partial Fractions.

Solution: $\therefore$ $\dfrac{2x^3 + x^2 - x - 3}{x(2x+3)(x-1)}$ is an improper fraction so, first transform it into mixed form.

$$\begin{array}{r|l} & \quad 1 \\ 2x^3 + x^2 - 3x & 2x^3 + x^2 - x - 3 \\ & \pm 2x^3 \pm x^2 \mp 3x \\ \hline & \qquad\qquad 2x - 3 \end{array}$$

Denominator $= x(2x+3)(x-1) = 2x^3 + x^2 - 3x$

$\therefore$ Dividing $2x^3 + x^2 - x - 3$ by $2x^3 + x^2 - 3x$, we have

Quotient $= 1$ and Remainder $= 2x - 3$

$$\therefore \quad \frac{2x^3 + x^2 - x - 3}{x(2x+3)(x-1)} = 1 + \frac{2x-3}{x(2x+3)(x-1)}$$

Suppose $\dfrac{2x-3}{x(2x+3)(x-1)} = \dfrac{A}{x} + \dfrac{B}{2x+3} + \dfrac{C}{x-1}$

$$\Rightarrow \quad 2x - 3 = A(2x+3)(x-1) + B(x)(x-1) + C(x)(2x+3)$$

which is an identity in x.

For A, putting $x = 0$ in the identity, we get $\boxed{A = 1}$

For B, putting $2x + 3 = 0 \Rightarrow x = -\dfrac{3}{2}$ in the identity, we get $\boxed{B = -\dfrac{8}{5}}$

For C, putting $x - 1 = 0 \Rightarrow x = 1$ in the identity, we get $\boxed{C = -\dfrac{1}{5}}$

Hence partial fractions are: $1 + \dfrac{1}{x} - \dfrac{8}{5(2x+3)} - \dfrac{1}{5(x-1)}$

EXERCISE 5.1

Resolve the following into partial fractions:

1. $\dfrac{2}{x^2-1}$

Sol

$$\frac{2}{x^2-1}=\frac{2}{(x-1)(x+1)}$$

$$\frac{2}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1} \quad \text{——— ①}$$

Multiply by $(x-1)(x+1)$,

$$(x-1)(x+1)\times\frac{2}{(x-1)(x+1)}=(x-1)(x+1)\left[\frac{A}{x-1}+\frac{B}{x+1}\right]$$

$$2=A(x+1)+B(x-1) \quad \text{——— ②}$$

Put $x=1$ in ②,

$$2=A(1+1)+B(1-1)$$

$$2=2A+B(0)$$

$$2=2A$$

$$\Rightarrow \quad A=1$$

Put $x=-1$ in ②,

$$2=A(-1+1)+B(-1-1)$$

$$2=A(0)+B(-2)$$

$$\frac{2}{-2}=\frac{-2B}{-2} \Rightarrow \quad B=-1$$

Hence, ① $\Rightarrow \dfrac{2}{(x-1)(x+1)}=\dfrac{1}{x-1}-\dfrac{1}{x+1}$

2. $\dfrac{a-b}{(x-a)(x-b)}$

Sol

$$\frac{a-b}{(x-a)(x-b)} = \frac{A_1}{x-a} + \frac{A_2}{x-b} \quad \text{——} \ (1)$$

Multiply by $(x-a)(x-b)$,

$$(x-a)(x-b) \times \frac{a-b}{(x-a)(x-b)} = (x-a)(x-b)\left[\frac{A_1}{x-a} + \frac{A_2}{x-b}\right]$$

$$a-b = A_1(x-b) + A_2(x-a) \quad \text{——} \ (2)$$

Put $x=a$ in (2),

$$a-b = A_1(a-b) + A_2(a-a)$$

$$a-b = A_1(a-b) + 0$$

$$\Longrightarrow \quad A_1 = 1$$

Put $x=b$ in (2),

$$a-b = A_1(b-b) + A_2(b-a)$$

$$a-b = A_1(0) - A_2(a-b)$$

$$a-b = -A_2(a-b)$$

$$\frac{(a-b)}{(a-b)} = -A_2$$

$$\Longrightarrow \quad A_2 = -1$$

Hence,

$$(1) \Longrightarrow \quad \frac{a-b}{(x-a)(x-b)} = \frac{1}{x-a} - \frac{1}{x-b}$$

3. $\frac{x^2+1}{(x+1)(x-1)}$

Sol

$$\frac{x^2+1}{(x+1)(x-1)} = \frac{x^2+1}{x^2-1}$$

$$\frac{x^2+1}{(x+1)(x-1)} = 1 + \frac{2}{x^2-1} \quad \text{---} \ (1)$$

$$x^2-1 \overline{\smash{\big)}\,x^2+1\,}^{\;1}$$
$$\underline{-x^2 \mp 1}$$
$$2$$

Take

$$\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \quad \text{---} \ (2)$$

Multiply by $(x-1)(x+1)$,

$$2 = A(x+1) + B(x-1) \quad \text{---} \ (3)$$

Put $x=1$ in (3),

$$2 = A(1+1) + B(1-1)$$

$$2 = 2A + 0$$

$$\Rightarrow A = 1$$

Put $x=-1$ in (3)

$$2 = A(-1+1) + B(-1-1)$$

$$2 = A(0) + B(-2)$$

$$2 = -2B \quad \Rightarrow \quad B = -1$$

Now,

(2) $\Rightarrow$ $\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$

Hence $\frac{x^2+1}{x^2-1} = 1 + \frac{1}{x-1} - \frac{1}{x+1}$

4. $$\frac{2x+3}{(x+1)(x+2)(x+3)}$$

Sol

$$\frac{2x+3}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \quad \text{—— } 1$$

Multiply by $(x+1)(x+2)(x+3)$

$$(x+1)\cancel{(x+2)}(x+3)\frac{2x+3}{(x+1)\cancel{(x+2)(x+3)}} = (x+1)(x+2)(x+3)\left(\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}\right)$$

$$2x+3 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2) \quad 2$$

Put $x=-1$ in 2

$$2(-1)+3 = A(-1+2)(-1+3) + B(-1+1)(-1+3) + C(-1+1)(-1+2)$$

$$-2+3 = A(1)(2) + B(0)(2) + C(0)(1)$$

$$1 = 2A + 0 + 0$$

$$\frac{1}{2} = A$$

Put $x=-2$ in 2

$$2(-2)+3 = A(-2+2)(-2+3) + B(-2+1)(-2+3) + C(-2+1)(-2+2)$$

$$-4+3 = A(0)(1) + B(-1)(1) + C(-1)(0)$$

$$-1 = 0 - B + 0$$

$$1 = B$$

Put $x=-3$ in 2,

$$2(-3)+3 = A(-3+2)(-3+3) + B(-3+1)(-3+3) + C(-3+1)(-3+2)$$

$$-6+3 = A(-1)(0) + B(-2)(0) + C(-2)(-1)$$

$$-3 = 0 + 0 + 2C$$

$$-\frac{3}{2} = C$$

Hence

$$① \implies \frac{2x+3}{(x+1)(x+2)(x+3)} = \frac{\frac{1}{2}}{x+1} + \frac{1}{x+2} + \frac{-3/2}{x+3}$$

$$= \frac{1}{2(x+1)} + \frac{1}{x+2} - \frac{3}{2(x+3)}$$

5. $\dfrac{x^2+4x+5}{(x+1)(x^2+5x+6)}$

Sol

$$\frac{x^2+4x+5}{(x+1)(x^2+5x+6)} = \frac{x^2+4x+5}{(x+1)(x+2)(x+3)}$$

$$\because \; x^2+5x+6 = x^2+2x+3x+6$$
$$= x(x+2)+3(x+2)$$
$$= (x+2)(x+3)$$

$$\frac{x^2+4x+5}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \quad \text{--- } (1)$$

Multiply by $(x+1)(x+2)(x+3)$

$$(x+1)(x+2)(x+3)\frac{x^2+4x+5}{(x+1)(x+2)(x+3)} = (x+1)(x+2)(x+3)\left(\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}\right)$$

$$x^2+4x+5 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2) \quad (2)$$

Put $x=-1$ in (2),

$$(-1)^2+4(-1)+5 = A(-1+2)(-1+3) + B(-1+1)(-1+3) + C(-1+1)(-1+2)$$

$$1-4+5 = A(1)(2) + B(0)(2) + C(0)(1)$$

$$2 = 2A + 0 + 0$$

$$\Rightarrow A = 1$$

Put $x=-2$ in (2),

$$(-2)^2+4(-2)+5 = A(-2+2)(-2+3) + B(-2+1)(-2+3) + C(-2+1)(-2+2)$$

$$4-8+5 = A(0)(1) + B(-1)(1) + C(-1)(0)$$

$$1 = 0 - B - 0$$

$$\Rightarrow B = -1$$

Put $x = -3$ in ②,

$$(-3)^2 + 4(-3) + 5 = A(-3+2)(-3+3) + B(-3+1)(-3+3) + C(-3+1)(-3+2)$$

$$9 - 12 + 5 = A(-1)(0) + B(-2)(0) + C(-2)(-1)$$

$$2 = 0 + 0 + 2C$$

$$\Rightarrow C = 1$$

Hence,

$$① \Rightarrow \quad \frac{x^2+4x+5}{(x+1)(x+2)(x+3)} = \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3}$$

6. $\dfrac{4x^3+5x^2-3x-2}{x^2-1}$

Sol

$$\frac{4x^3+5x^2-3x-2}{x^2-1}=4x+5+\frac{x+3}{x^2-1} \quad \text{— ①}$$

$$\begin{array}{r|l} & 4x+5 \\ \hline x^2-1 & 4x^3+5x^2-3x-2 \\ & \underline{\cancel{4x^3} \qquad \mp 4x} \\ & \cancel{5x^2}+x-2 \\ & \underline{\cancel{5x^2} \qquad \mp 5} \\ & x+3 \end{array}$$

$$Q+\frac{R}{D}$$

Take

$$\frac{x+3}{x^2-1}=\frac{x+3}{(x-1)(x+1)}$$

$$\frac{x+3}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1} \quad \text{— ②}$$

Multiply by $(x-1)(x+1)$,

$$x+3=A(x+1)+B(x-1) \quad \text{— ③}$$

Put $x=1$ in ③,

$$1+3=A(1+1)+B(1-1)$$

$$4=A(2)+B(0)$$

$$4=2A$$

$$\Rightarrow A=2$$

Put $x=-1$ in ③

$$-1+3=A(-1+1)+B(-1-1)$$

$$2=A(0)+B(-2)$$

$$2=-2B \quad \Rightarrow \quad B=-1$$

From ②,

$$\frac{x+3}{(x-1)(x+1)}=\frac{2}{x-1}-\frac{1}{x+1}$$

Hence, ① $\Rightarrow$ $\dfrac{4x^3+5x^2-3x-2}{x^2-1}=4x+5+\dfrac{2}{x-1}-\dfrac{1}{x+1}$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 5: Partial Fractions
- Partial Fractions
- Exercise 5.1: Q7 - Q11

Case II: When $Q(x)$ has repeated linear factors:

If the polynomial $Q(x)$ has a repeated linear factors $(x-a)^n$, $n \geq 2$ and n is a positive integer, then $\dfrac{P(x)}{Q(x)}$ may be written as the following identity:

$\frac{1}{2} - \frac{3}{4} = \frac{2-3}{4} = \frac{-1}{4}$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_n}{(x-a)^n}$$

$(x-2).(x-3)$

where $A_1, A_2, \ldots, A_n$ are numbers to be found.

The method is explained by the following examples:

Example 4: Resolve $\dfrac{x^2+x-1}{(x+2)^3}$ into partial fractions.

Solution: Suppose $\dfrac{x^2+x-1}{(x+2)^3} = \dfrac{A}{(x+2)} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}$

$$\Rightarrow \quad x^2 + x - 1 = A(x+2)^2 + B(x+2) + C \qquad \text{(i)}$$

$$\Rightarrow \quad x^2 + x - 1 = A(x^2 + 4x + 4) + B(x+2) + C \qquad \text{(ii)}$$

For C, putting $x + 2 = 0$, i.e., $x = -2$ in (i), we get

$$(-2)^2 + (-2) - 1 = A(0) + B(0) + C$$

$$\Rightarrow \qquad \boxed{1 = C}$$

Equating the coefficients of x^2 and x in (ii), we get $\boxed{A = 1}$

and $\quad 1 = 4A + B$

$$\Rightarrow \qquad 1 = 4 + B \quad \Rightarrow \quad \boxed{B = -3}$$

Hence the partial fractions are: $\dfrac{1}{x+2} - \dfrac{3}{(x+2)^2} + \dfrac{1}{(x+2)^3}$

Example 5: Resolve $\dfrac{1}{(x+1)^2(x^2-1)}$ into partial fractions.

Solution: Here denominator $= (x+1)^2(x^2-1)$

$$= (x+1)^2(x+1)(x-1) = (x+1)^3(x-1)$$

$$\therefore \quad \frac{1}{(x+1)^2(x^2-1)} = \frac{1}{(x+1)^3(x-1)}$$

Suppose $\dfrac{1}{(x-1)(x+1)^3} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2} + \dfrac{D}{(x+1)^3}$

$$\Rightarrow \quad 1 = A(x+1)^3 + B(x+1)^2(x-1) + C(x-1)(x+1) + D(x-1) \quad \ldots\text{(i)}$$

$$\Rightarrow \quad 1 = A(x^3 + 3x^2 + 3x + 1) + B(x^3 + x^2 - x - 1) + C(x^2 - 1) + D(x - 1)$$

$$\Rightarrow \quad 1 = (A+B)x^3 + (3A+B+C)x^2 + (3A - B + D)x + (A - B - C - D) \quad \ldots\text{(ii)}$$

For A, putting $x - 1 = 0 \Rightarrow x = 1$ in (i), we get

$$1 = A(2)^3 \quad \Rightarrow \quad \boxed{A = \frac{1}{8}}$$

For D, putting $x + 1 = 0 \Rightarrow x = -1$ in (i), we get

$$1 = D(-1-1) \quad \Rightarrow \quad \boxed{D = -\frac{1}{2}}$$

Equating the coefficients of x^3 and x^2 in (ii), we get

$$0 = A + B \quad \Rightarrow \quad B = -A \quad \Rightarrow \quad \boxed{B = -\frac{1}{8}}$$

and

$$0 = 3A + B + C \quad \Rightarrow \quad 0 = \frac{3}{8} - \frac{1}{8} + C \quad \Rightarrow \quad \boxed{C = -\frac{1}{4}}$$

Hence the partial fractions are:

$$\frac{\frac{1}{8}}{x-1} + \frac{-\frac{1}{8}}{x+1} + \frac{-\frac{1}{4}}{(x+1)^2} + \frac{-\frac{1}{2}}{(x+1)^3} = \frac{1}{8(x-1)} - \frac{1}{8(x+1)} - \frac{1}{4(x+1)^2} - \frac{1}{2(x+1)^3}$$

EXERCISE 5.1

Resolve the following into partial fractions:

7. $\dfrac{3x^2-12x+11}{(x-1)(x-2)(x-3)}$

Sol

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \quad \text{—— } ①$$

Multiply by $(x-1)(x-2)(x-3)$,

$$(x-1)\cancel{(x-2)}(x-3)\frac{3x^2-12x+11}{(x-1)\cancel{(x-2)}(x-3)} = (x-1)(x-2)(x-3)\left(\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}\right)$$

$$3x^2-12x+11 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \text{—— } ②$$

Put $x=1$ in ②,

$$3(1)^2-12(1)+11 = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$$

$$3-12+11 = A(-1)(-2) + B(0)(-2) + C(0)(-1)$$

$$2 = A(2) + 0 + 0$$

$$\Rightarrow A = 1$$

Put $x=2$ in equ ②,

$$3(2)^2-12(2)+11 = A(2-2)(2-3) + B(2-1)(2-3) + C(2-1)(2-2)$$

$$12-24+11 = A(0)(-1) + B(1)(-1) + C(-1)(0)$$

$$-1 = -B$$

$$\Rightarrow B = 1$$

Put $x=3$ in ②,

$$3(3)^2-12(3)+11 = A(3-2)(3-3) + B(3-1)(3-3) + C(3-1)(3-2)$$

$$27-36+11 = A(1)(0)+B(2)(0)+C(2)(1)$$

$$2 = 0+0+2C$$

$$\Rightarrow C=1$$

Hence,

$$① \Rightarrow \quad \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$$

8. $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$

Sol

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = \frac{(x-1)(x^2-5x+6)}{(x-4)(x^2-11x+30)}$$

$$= \frac{x^3-5x^2+6x-x^2+5x-6}{x^3-11x^2+30x-4x^2+44x-120}$$

$$= \frac{x^3-6x^2+11x-6}{x^3-15x^2+74x-120}$$

$$\begin{array}{r|l} & 1 \\ x^3-15x^2+74x-120 & x^3-6x^2+11x-6 \\ & \underline{-x^3 \mp 15x^2 \pm 74x \mp 120} \\ & 9x^2-63x+114 \end{array}$$

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{9x^2-63x+114}{(x-4)(x-5)(x-6)} \quad \text{——} ①$$

Take

$$\frac{9x^2-63x+114}{(x-4)(x-5)(x-6)} = \frac{A}{x-4} + \frac{B}{x-5} + \frac{C}{x-6} \quad \text{——} ②$$

Multiply by $(x-4)(x-5)(x-6)$

$$\cancel{(x-4)(x-5)(x-6)}\frac{9x^2-63x+114}{\cancel{(x-4)(x-5)(x-6)}} = (x-4)(x-5)(x-6)\left(\frac{A}{x-4} + \frac{B}{x-5} + \frac{C}{x-6}\right)$$

$$9x^2-63x+114 = A(x-5)(x-6) + B(x-4)(x-6) + C(x-4)(x-5) \quad ③$$

Put $x=4$ in ③,

$$9(4)^2-63(4)+114 = A(4-5)(4-6) + B(4-4)(4-6) + C(4-4)(4-5)$$

$$144-252+114 = A(-1)(-2) + B(0)(-2) + C(0)(-1)$$

$$258-252 = 2A+0+0$$

$$6 = 2A$$

$$\Longrightarrow \quad A = 3$$

Put $x = 5$ in ③,

$$9(5)^2 - 63(5) + 114 = A(5-5)(5-6) + B(5-4)(5-6) + C(5-4)(5-5)$$

$$225 - 315 + 114 = A(0)(-1) + B(1)(-1) + C(1)(0)$$

$$24 = 0 - B + 0$$

$$\Longrightarrow \quad B = -24$$

Put $x = 6$ in ③,

$$9(6)^2 - 63(6) + 114 = A(6-5)(6-6) + B(6-4)(6-6) + C(6-4)(6-5)$$

$$9(36) - 378 + 114 = A(1)(0) + B(2)(0) + C(2)(1)$$

$$324 - 378 + 114 = 0 + 0 + 2C$$

$$60 = 2C$$

$$\Longrightarrow \quad C = 30$$

From ②

$$\frac{9x^2 - 63x + 114}{(x-4)(x-5)(x-6)} = \frac{3}{x-4} - \frac{24}{x-5} + \frac{30}{x-6}$$

Hence

$$① \Longrightarrow \quad \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{3}{x-4} - \frac{24}{x-5} + \frac{30}{x-6}$$

9. $$\frac{x^2}{(x^2+a)(x^2+b)(x^2+c)}$$

Sol

Let $y = x^2$

$$\frac{y}{(y+a)(y+b)(y+c)} = \frac{A_1}{y+a} + \frac{A_2}{y+b} + \frac{A_3}{y+c} \quad \text{—— ①}$$

Multiply by $(y+a)(y+b)(y+c)$,

$$(y+a)(y+b)(y+c)\frac{y}{(y+a)(y+b)(y+c)} = (y+a)(y+b)(y+c)\left[\frac{A_1}{y+a} + \frac{A_2}{y+b} + \frac{A_3}{y+c}\right]$$

$$y = A_1(y+b)(y+c) + A_2(y+a)(y+c) + A_3(y+a)(y+b) \quad \text{—— ②}$$

Put $y = -a$ in ②,

$$-a = A_1(-a+b)(-a+c) + A_2(-a+a)(-a+c) + A_3(-a+a)(-a+b)$$

$$-a = A_1(a-b)(a-c) + A_2(0) + A_3(0)$$

$$\frac{-a}{(a-b)(a-c)} = A_1$$

Put $y = -$ in ②,

$$-b = A_1(-b+b)(-b+c) + A_2(-b+a)(-b+c) + A_3(-b+a)(-b+b)$$

$$-b = A_1(0) + A_2(a-b)(-b+c) + A_3(0)$$

$$-b = -A_2(a-b)(b-c)$$

$$\Rightarrow \frac{b}{(a-b)(b-c)} = A_2$$

Put $y = -c$ in ②,

$$-c = A_1(-c+b)(-c+c) + A_2(-c+a)(-c+c) + A_3(-c+a)(-c+b)$$

$$-c = A_1(0) + A_2(0) + A_3(a-c)(b-c)$$

$$\Rightarrow \quad \frac{-c}{(a-c)(b-c)} = A_3$$

Hence

$$① \Rightarrow \quad \frac{y}{(y+a)(y+b)(y+c)} = \frac{\frac{-a}{(a-b)(a-c)}}{y+a} + \frac{\frac{b}{(a-b)(b-c)}}{y+b} + \frac{\frac{-c}{(a-c)(b-c)}}{y+c}$$

$$= \frac{a}{(a-b)(c-a)(x^2+a)} + \frac{b}{(a-b)(b-c)(x^2+b)} + \frac{c}{(b-c)(c-a)(x^2+c)}$$

10. $\dfrac{x+1}{(x-1)^2}$

Sol

$$\frac{x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \quad \text{——} \ 1$$

Multiply by $(x-1)^2$,

$$\cancel{(x-1)^2} \times \frac{x+1}{\cancel{(x-1)^2}} = (x-1)^2\left[\frac{A}{x-1} + \frac{B}{(x-1)^2}\right]$$

$$x+1 = A(x-1) + B$$

$$x+1 = Ax - A + B$$

By comparing coefficients,

$$x^1: \quad A = 1$$

$$x^0: \quad -A+B = 1 \quad \text{——} \ 2$$

Put $A=1$ in 2,

$$-1 + B = 1$$

$$B = 2$$

Hence,

$$1 \Rightarrow \quad \frac{x+1}{(x-1)^2} = \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

11. $\dfrac{x^2+x}{(x^2-1)^2}$

Sol

$$\frac{x^2+x}{(x^2-1)^2} = \frac{x(x+1)}{[(x-1)(x+1)]^2}$$

$$= \frac{x(x+1)}{(x-1)^2(x+1)^2}$$

$$= \frac{x}{(x-1)^2(x+1)}$$

So,

$$\frac{x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \quad \text{——} \ (1)$$

Multiply by $(x+1)(x-1)^2$,

$$\cancel{(x+1)}\cancel{(x-1)^2} \times \frac{x}{\cancel{(x+1)}\cancel{(x-1)^2}} = (x+1)(x-1)^2\left[\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right]$$

$$x = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$$= A(x^2-2x+1) + B(x^2-1) + Cx + C$$

$$x = Ax^2 - 2Ax + A + Bx^2 - B + Cx + C$$

$$x = (A+B)x^2 + (-2A+C)x + A - B + C$$

Comparing Coefficients,

$$A+B=0 \quad \text{——} \ (2)$$

$$-2A+C=1 \quad \text{——} \ (3)$$

$$A-B+C=0 \quad \text{——} \ (4)$$

Adding (2) & (4),

$$\begin{array}{r} A+B \quad = 0 \\ A-B+C=0 \\ \hline 2A+C=0 \end{array}$$

$C = -2A$ ——— ⑤ Put in ③

$$-2A - 2A = 1$$

$$-4A = 1$$

$$A = -\frac{1}{4}$$ Put in ② & then ⑤

$$-\frac{1}{4} + B = 0$$

$$C = -2\left(-\frac{1}{4}\right) = \frac{1}{2}$$

$$B = \frac{1}{4}$$

Hence

① $\Longrightarrow$

$$= \frac{-1/4}{x+1} + \frac{1/4}{x-1} + \frac{1/2}{(x-1)^2}$$

$$= \frac{-1}{-4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 5: Partial Fractions
- Partial Fractions
- Exercise 5.1: Q12 - Q15

EXERCISE 5.1

Resolve the following into partial fractions:

12. $\dfrac{3x^2+4x-5}{(x-1)^3}$

Sol

$$\frac{3x^2+4x-5}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \quad \text{—— } (1)$$

Multiply by $(x-1)^3$,

$$\cancel{(x-1)^3}\,\frac{3x^2+4x-5}{\cancel{(x-1)^3}} = (x-1)^3\left[\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}\right]$$

$$3x^2+4x-5 = A(x-1)^2 + B(x-1) + C \quad \text{—— } (2)$$

Put $x=1$ in equ. (2)

$$3(1)^2+4(1)-5 = A(1-1)^2 + B(1-1) + C$$

$$2 = A(0) + B(0) + C$$

$$\Longrightarrow \quad C = 2$$

From equ. (2)

$$3x^2+4x-5 = A(x^2-2x+1) + Bx - B + C$$

$$= Ax^2 - 2Ax + A + Bx - B + C$$

$$3x^2+4x-5 = Ax^2 + (-2A+B)x + A - B + C$$

By comparing coefficients

$$x^2: \quad A = 3$$

$$x^1: \quad -2A+B=4 \quad \text{———} \quad ③$$

Put $A=3$ in ③,

$$-2(3)+B=4$$

$$-6+B=4$$

$$B=10$$

Hence from equ. ①,

$$\frac{3x^2+4x-5}{(x-1)^3} = \frac{3}{x-1} + \frac{10}{(x-1)^2} + \frac{2}{(x-1)^3}$$

13. $\dfrac{1}{x(x+1)^3}$

Sol

$$\frac{1}{x(x+1)^3}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3} \quad \text{——} \quad ①$$

Multiply by $x(x+1)^3$,

$$x(x+1)^3\frac{1}{x(x+1)^3}=x(x+1)^3\left(\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}\right)$$

$$1=A(x+1)^3+Bx(x+1)^2+Cx(x+1)+Dx \quad \text{——} \quad ②$$

Put $x=0$ in equ.②,

$$1=A(0+1)^3+B(0)(0+1)^2+C(0)(0+1)+D(0)$$

$$\boxed{1=A}$$

Put $x=-1$ in equ.②,

$$1=A(-1+1)^3+B(-1)(-1+1)^2+C(-1)(-1+1)+D(-1)$$

$$1=A(0)+B(-1)(0)+C(-1)(0)-D$$

$$1=-D$$

$$\boxed{-1=D}$$

From equ.②,

$$1=A(x^3+3x^2+3x+1)+Bx(x^2+2x+1)+C(x^2+x)+Dx$$

$$1=Ax^3+3Ax^2+3Ax+A+Bx^3+2Bx^2+Bx+Cx^2+Cx+Dx$$

$$1=(A+B)x^3+(3A+2B+C)x^2+(3A+B+C+D)x+A$$

By comparing coefficients,

x^3: $\quad A+B=0 \quad$ —— ③

$x^2: \quad 3A+2B+C=0 \quad \text{———} \quad (4)$

$x: \quad 3A+B+C+D=0 \quad \text{———} \quad (5)$

Put $A=1$ in equ. (3),

$$1+B=0 \quad \Longrightarrow \quad \boxed{B=-1}$$

Now,

$$(4) \Longrightarrow \quad 3(1)+2(-1)+C=0$$

$$3-2+C=0$$

$$1+C=0$$

$$\boxed{C=-1}$$

Hence from equ. (1),

$$\frac{1}{x(x+1)^3} = \frac{1}{x} + \frac{-1}{x+1} + \frac{-1}{(x+1)^2} + \frac{-1}{(x+1)^3}$$

$$\frac{1}{x(x+1)^3} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}$$

14. $\dfrac{4x^2-3x+1}{(x+1)(x-1)^2}$

Sol

$$\frac{4x^2-3x+1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \quad \text{——— (1)}$$

Multiply by $(x+1)(x-1)^2$,

$$(x+1)\cancel{(x-1)^2}\,\frac{4x^2-3x+1}{\cancel{(x+1)(x-1)^2}} = (x+1)(x-1)^2\left[\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right]$$

$$4x^2-3x+1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1) \quad \text{——— (2)}$$

Put $x=-1$ in equ.(2),

$$4(-1)^2-3(-1)+1 = A(-1-1)^2 + B(-1+1)(-1-1) + C(-1+1)$$

$$4(1)+3+1 = A(-2)^2 + B(0)(-2) + C(0)$$

$$8 = 4A$$

$$\Longrightarrow \boxed{A=2}$$

Put $x=1$ in equ.(2),

$$4(1)^2-3(1)+1 = A(1-1)^2 + B(1+1)(1-1) + C(1+1)$$

$$4-3+1 = A(0)^2 + B(2)(0) + C(2)$$

$$2 = 0+0+2C$$

$$\Longrightarrow \boxed{C=1}$$

From equ.(2),

$$4x^2-3x+1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$$= A(x^2-2x+1) + B(x^2-1^2) + Cx + C$$

$$4x^2-3x+1 = Ax^2-2Ax+A+Bx^2-B+Cx+C$$

$$4x^2-3x+1 = (A+B)x^2+(-2A+C)x+A-B+C$$

By comparing coefficients,

$$x^2: \quad A+B=4 \quad \text{——— } ③$$

Put $A=2$ in equ. ③,

$$2+B=4$$

$$B=4-2$$

$$\boxed{B=2}$$

Hence from equ. ①,

$$\frac{4x^2-3x+1}{(x+1)(x-1)^2} = \frac{2}{x+1} + \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

15. $$\frac{12x^2-48}{(x-2)^2(x+2)^2}$$

Sol

$$\frac{12x^2-48}{(x-2)^2(x+2)^2} = \frac{12(x^2-4)}{(x-2)^2(x+2)^2}$$

$$\because x^2-4 = x^2-2^2$$
$$= (x-2)(x+2)$$

$$= \frac{12(x-2)(x+2)}{(x-2)^2(x+2)^2}$$

$$\frac{12x^2-48}{(x-2)^2(x+2)^2} = \frac{12}{(x-2)(x+2)}$$

Now,

$$\frac{12}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \quad \text{——} \quad ①$$

Multiply by $(x-2)(x+2)$,

$$(x-2)(x+2)\frac{12}{(x-2)(x+2)} = (x-2)(x+2)\left(\frac{A}{x-2} + \frac{B}{x+2}\right)$$

$$12 = A(x+2) + B(x-2) \quad \text{——} \quad ②$$

Put $x=2$ in equ. ②,

$$12 = A(2+2) + B(2-2)$$

$$12 = 4A + B(0)$$

$$12 = 4A$$

$$\Rightarrow A = 3$$

Put $x=-2$ in equ. ②,

$$12 = A(-2+2) + B(-2-2)$$

$$12 = A(0) + B(-4)$$

$$12 = -4B$$

$$\implies B = -3$$

Hence, from equ. 1,

$$\frac{12}{(x-2)(x+2)} = \frac{3}{x-2} + \frac{-3}{x+2}$$

$$\frac{12}{(x-2)(x+2)} = \frac{3}{x-2} - \frac{3}{x+2}$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 5: Partial Fractions
- Partial Fractions
- Exercise 5.2

Case III: When $Q(x)$ contains non-repeated irreducible quadratic factors

Definition: A quadratic factor is irreducible if it cannot be written as the product of two linear factors with real coefficients. For example, $x^2 + x + 1$ and $x^2 + 3$ are irreducible quadratic factors.

If the polynomial $Q(x)$ contains non-repeated irreducible quadratic factors then $\dfrac{P(x)}{Q(x)}$ may be written as the identity having partial fractions of the form:

A

$$\frac{Ax+B}{ax^2+bx+c} \quad \text{where } A \text{ and } B \text{ are the numbers to be found.}$$

The method is explained by the following examples:

Example 6: Resolve $\dfrac{3x-11}{(x^2+1)(x+3)}$ into partial fractions.

Solution: Suppose $\dfrac{3x-11}{(x^2+1)(x+3)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x+3}$

$$\Rightarrow \quad 3x - 11 = (Ax+B)(x+3) + C(x^2+1) \qquad \text{(i)}$$

$$\Rightarrow \quad 3x - 11 = (A+C)x^2 + (3A+B)x + (3B+C) \qquad \text{(ii)}$$

For C, putting $x + 3 = 0 \quad \Rightarrow \quad x = -3$ in (i), we get

$$-9 - 11 = C(9+1) \quad \Rightarrow \quad \boxed{C = -2}$$

Equating the coefficients of x^2 and x in (ii), we get

$$0 = A + C \;\Rightarrow\; A = -C \quad \Rightarrow \boxed{A = 2}$$

and $\quad 3 = 3A + B \Rightarrow B = 3 - 3A \;\Rightarrow\; B = 3 - 6 \;\Rightarrow\; B = -3$

Hence, the partial fractions are: $\dfrac{2x-3}{x^2+1} - \dfrac{2}{x+3}$

Example 7: Resolve $\dfrac{4x^2+8x}{x^4+2x^2+9}$ into partial fractions.

Solution: Here, denominator $= x^4 + 2x^2 + 9 = (x^2 + 2x + 3)(x^2 - 2x + 3)$

$$\therefore \quad \frac{4x^2+8x}{x^4+2x^2+9} = \frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)}$$

Suppose $\quad \dfrac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)} = \dfrac{Ax+B}{x^2+2x+3} + \dfrac{Cx+D}{x^2-2x+3}$

$$\Rightarrow \quad 4x^2 + 8x = (Ax + B)(x^2 - 2x + 3) + (Cx + D)(x^2 + 2x + 3)$$

$$\Rightarrow \quad 4x^2 + 8x = (A + C)x^3 + (-2A + B + 2C + D)x^2$$

$$+ (3A - 2B + 3C + 2D)x + 3B + 3D \qquad \text{(i)}$$

which is an identity in x.

Equating the coefficients of x^3, x^2, x, x^0 in (i), we have

$$0 = A + C \qquad \text{(ii)}$$

$$4 = -2A + B + 2C + D \qquad \text{(iii)}$$

$$8 = 3A - 2B + 3C + 2D \qquad \text{(iv)}$$

$$0 = 3B + 3D \qquad \text{(v)}$$

Solving (ii), (iii), (iv) and (v), we get

$$\boxed{A = 1},\ \boxed{B = 2},\ \boxed{C = -1} \text{ and } \boxed{D = -2}$$

Hence the partial fractions are: $\dfrac{x+2}{x^2+2x+3} + \dfrac{-x-2}{x^2-2x+3}$

Case IV: When $Q(x)$ has repeated irreducible quadratic factors

If the polynomial $Q(x)$ contains a repeated irreducible quadratic factors $(ax^2 + bx + c)^n$, $n \geq 2$ and n is a positive integer, then $\dfrac{P(x)}{Q(x)}$ may be written as the following identity:

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ are numbers to be found. The method is explained through the following example:

Example 8: Resolve $\dfrac{4x^2}{(x^2+1)^2(x-1)}$ into partial fractions.

Solution: Let $\dfrac{4x^2}{(x^2+1)^2(x-1)} = \dfrac{Ax+B}{(x^2+1)} + \dfrac{Cx+D}{(x^2+1)^2} + \dfrac{E}{x-1}$

$$\Rightarrow 4x^2 = (Ax + B)(x^2 + 1)(x - 1) + (Cx + D)(x - 1) + E(x^2 + 1)^2 \qquad \text{(i)}$$

$$\Rightarrow \quad 4x^2 = (A + E)x^4 + (-A + B)x^3 + (A - B + C + 2E)x^2$$

$$+ (-A + B - C + D)x + (-B - D + E) \qquad \text{(ii)}$$

For E, putting $x-1=0 \quad \Rightarrow x=1$ in (i), we get

$$4=E(1+1)^2 \quad \Rightarrow \quad \boxed{E=1}$$

Equating the coefficients of x^4, x^3, x^2, x, in (ii), we get

$$0=A+E \quad \Rightarrow \quad A=-E \quad \Rightarrow \quad \boxed{A=-1}$$

$$0=-A+B \quad \Rightarrow \quad B=A \quad \Rightarrow \quad \boxed{B=-1}$$

$$4=A-B+C+2E$$

$$\Rightarrow \quad C=4-A+B-2E=4+1-1-2 \quad \Rightarrow \quad \boxed{C=2}$$

and $\quad 0=-A+B-C+D$

$$\Rightarrow \quad D=A-B+C=-1+1+2=2 \quad \Rightarrow \quad \boxed{D=2}$$

Hence partial fractions are: $\dfrac{-x-1}{x^2+1}+\dfrac{2x+2}{(x^2+1)^2}+\dfrac{1}{x-1}$

EXERCISE 5.2

Resolve into partial fractions:

1. $$\frac{2x^2+3x+3}{(x+1)(x^2+1)}$$

Sol

$$\frac{2x^2+3x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \quad \text{——} \quad (1)$$

Multiply by $(x+1)(x^2+1)$,

$$(x+1)(x^2+1)\frac{2x^2+3x+3}{(x+1)(x^2+1)} = (x+1)(x^2+1)\left[\frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right]$$

$$2x^2+3x+3 = A(x^2+1) + (Bx+C)(x+1) \quad \text{——} \quad (2)$$

Put $x=-1$ in equ.(2),

$$2(-1)^2+3(-1)+3 = A((-1)^2+1) + (B(-1)+C)(-1+1)$$

$$2-3+3 = A(2) + (-B+C)(0)$$

$$2 = 2A$$

$$\Rightarrow A = 1$$

From equ.(2),

$$2x^2+3x+3 = A(x^2+1) + (Bx+C)(x+1)$$

$$= Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$2x^2+3x+3 = (A+B)x^2 + (B+C)x + A + C$$

By comparing coefficients:

x^2: $\quad A+B=2 \quad \text{——} \quad (3)$

$x^1: \quad B+C=3$

$x^0: \quad A+C=3 \quad \text{———} \quad (4)$

Put $A=1$ in equ.(3) & equ.(4),

$(3) \Rightarrow 1+B=2$

$B=2-1$

$B=1$

$(4) \Rightarrow 1+C=3$

$C=3-1$

$C=2$

Hence from equ.(1),

$$\frac{2x^2+3x+3}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{1.x+2}{x^2+1}$$

$$\frac{2x^2+3x+3}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{x+2}{x^2+1}$$

2. $\dfrac{2x+1}{(x-2)(x^2+3x+5)}$

Sol

Let $\dfrac{2x+1}{(x-2)(x^2+3x+5)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+3x+5}$ —— ①

Multiply each term by LCM $(x-2)(x^2+3x+5)$,

$$(x-2)(x^2+3x+5)\frac{2x+1}{(x-2)(x^2+3x+5)} = (x-2)(x^2+3x+5)\left(\frac{A}{x-2} + \frac{Bx+C}{x^2+3x+5}\right)$$

$$2x+1 = A(x^2+3x+5) + (Bx+C)(x-2) \quad \text{—— ②}$$

Put $x=2$ in equ.②,

$$2(2)+1 = A[(2)^2+3(2)+5] + (B(2)+C)(2-2)$$

$$4+1 = A(4+6+5) + (2B+C)(0)$$

$$5 = A(15)$$

$$\frac{5}{15} = A$$

$$\Rightarrow \boxed{A = \frac{1}{3}}$$

From equ.②,

$$2x+1 = Ax^2+3Ax+5A+Bx^2-2Bx+Cx-2C$$

$$2x+1 = (A+B)x^2+(3A-2B+C)x+5A-2C$$

By comparing coefficients,

x^2: $\quad A+B=0$ —— ③

x^1: $\quad 3A-2B+C=2$

$x^0: \quad 5A-2C=1$ —— ④

Put $A=\frac{1}{3}$ in equ.③ & equ.④,

③ $\Rightarrow \frac{1}{3}+B=0$

$$\boxed{B=\frac{-1}{3}}$$

④ $\Rightarrow 5\left(\frac{1}{3}\right)-2C=1$

$$\frac{5}{3}-1=2C$$

$$\frac{5-3}{3}=2C$$

$$\frac{2}{3}=2C$$

$$\Rightarrow \boxed{C=\frac{1}{3}}$$

Hence, from equ. ①

$$\frac{2x+1}{(x-2)(x^2+3x+5)}=\frac{\frac{1}{3}}{x-2}+\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+3x+5}$$

$$=\frac{1}{3(x-2)}+\frac{\frac{-x+1}{3}}{x^2+3x+5}$$

$$\frac{2x+1}{(x-2)(x^2+3x+5)}=\frac{1}{3(x-2)}-\frac{x-1}{3(x^2+3x+5)}$$

3. $\dfrac{2x+32}{(x-2)(x^2+2)^2}$

Sol

$$\frac{2x+32}{(x-2)(x^2+2)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} \quad \text{———} \quad ①$$

Multiply each term by LCM $(x-2)(x^2+2)^2$

$$(x-2)(x^2+2)^2 \frac{2x+32}{(x-2)(x^2+2)^2} = (x-2)(x^2+2)^2 \left(\frac{A}{x-2} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} \right)$$

$$2x+32 = A(x^2+2)^2 + (Bx+C)(x-2)(x^2+2) + (Dx+E)(x-2) \quad ②$$

Put $x=2$ in equ. ②,

$$2(2)+32 = A(2^2+2)^2 + (B(2)+C)(2-2)(2^2+2) + (D(2)+E)(2-2)$$

$$4+32 = A(6)^2 + (2B+C)(0)(6) + (2D+E)(0)$$

$$36 = 36A + 0 + 0$$

$$\frac{36}{36} = A$$

$$\Rightarrow \boxed{A=1}$$

From equ. ②,

$$2x+32 = A(x^2+2)^2 + (Bx+C)(x-2)(x^2+2) + (Dx+E)(x-2)$$

$$2x+32 = A(x^4+4x^2+4) + (Bx+C)(x^3+2x-2x^2-4) + Dx^2-2Dx+Ex-2E$$

$$= Ax^4+4Ax^2+4A+Bx^4+2Bx^2-2Bx^3-4Bx+Cx^3+2Cx$$
$$-2Cx^2-4C+Dx^2-2Dx+Ex-2E$$

$$2x+32 = (A+B)x^4 + (-2B+C)x^3 + (4A+2B-2C+D)x^2$$
$$+(-4B+2C-2D+E)x + (4A-4C-2E)$$

By comparing coefficients,

x^4: $A+B=0$ —— ③

x^3: $-2B+C=0$ —— ④

x^2: $4A+2B-2C+D=0$ —— ⑤

x: $-4B+2C-2D+E=2$

x^0: $4A-4C-2E=32$ —— ⑥

Put $A=1$ in equ.③,

$$1+B=0 \implies \boxed{B=-1}$$

Put in ④

$$④ \implies -2(-1)+C=0$$

$$2+C=0$$

$$\boxed{C=-2}$$

Put $A=1$, $B=-1$, $C=-2$ in equ.⑤ & equ.⑥,

$$⑤ \implies 4(1)+2(-1)-2(-2)+D=0$$

$$4-2+4+D=0$$

$$6+D=0$$

$$\boxed{D=-6}$$

$$⑥ \implies 4(1)-4(-2)-2E=32$$

$$4+8-2E=32$$

$$12-32=2E$$

$$\frac{-20}{2}=E$$

$$\implies \boxed{E=-10}$$

Hence from equ.①,

$$\frac{2x+32}{(x-2)(x^2+2)^2}=\frac{1}{x-2}+\frac{(-1)x+(-2)}{x^2+2}+\frac{-6x+(-10)}{(x^2+2)^2}$$

$$\frac{2x+32}{(x-2)(x^2+2)^2}=\frac{1}{x-2}-\frac{x+2}{x^2+2}-2\frac{(3x+5)}{(x^2+2)^2}$$ ✓

4. $\dfrac{3x^2+3}{x^3+1}$

Sol

$$\frac{3x^2+3}{x^3+1} = \frac{3x^2+3}{(x+1)(x^2-x+1)} \qquad \because a^3+b^3 = (a+b)(a^2-ab+b^2)$$

$$\frac{3x^2+3}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \quad \text{——— } ①$$

Multiply each term by LCM $(x+1)(x^2-x+1)$,

$$(x+1)(x^2-x+1)\frac{3x^2+3}{(x+1)(x^2-x+1)} = (x+1)(x^2-x+1)\left(\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\right)$$

$$3x^2+3 = A(x^2-x+1) + (Bx+C)(x+1) \quad \text{——— } ②$$

Put $x=-1$ in equ. ②

$$3(-1)^2+3 = A((-1)^2-(-1)+1) + (Bx+C)(-1+1)$$

$$3(1)+3 = A(1+1+1) + (Bx+C)(0)$$

$$6 = A(3) + 0$$

$$\Longrightarrow \boxed{A=2}$$

From equ. ②,

$$3x^2+3 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$3x^2+3 = (A+B)x^2 + (-A+B+C)x + A+C$$

By comparing coefficients,

x^2: $\quad A+B=3 \quad$ ——— ③

x^1: $\quad -A+B+C=0$

x^0: $\quad A+C=3 \quad$ ——— ④

Put $A=2$ in equ③ and equ④,

③ $\Longrightarrow$ $2+B=3$

$B=3-2$

$\boxed{B=1}$

④ $\Longrightarrow$ $2+C=3$

$C=3-2$

$\boxed{C=1}$

Hence from equ①

$$\frac{3x^2+3}{(x+1)(x^2-x+1)}=\frac{2}{x+1}+\frac{(1)x+1}{x^2-x+1}$$

$$\frac{3x^2+3}{(x+1)(x^2-x+1)}=\frac{2}{x+1}+\frac{x+1}{x^2-x+1}$$

5. $$\frac{x^4}{x^4+2x^2+1}$$

Sol

$$\frac{x^4}{x^4+2x^2+1} \longrightarrow \text{Improper fraction}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^4+2x^2+1 & x^4 \\ & \underline{-x^4 - 2x^2 - 1} \\ & -2x^2-1 \end{array}$$

$$Q + \frac{R}{D}$$

$$\frac{x^4}{x^4+2x^2+1} = 1 + \frac{-2x^2-1}{x^4+2x^2+1}$$

$$(a+b)^2 = a^2+2ab+1$$

$$\frac{x^4}{x^4+2x^2+1} = 1 - \frac{2x^2+1}{(x^2)^2+2(x^2)(1)+1^2}$$

$$\frac{x^4}{x^4+2x^2+1} = 1 - \frac{2x^2+1}{(x^2+1)^2} \quad \text{------} \quad (1)$$

Take

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \quad \text{------} \quad (2)$$

Multiply each term by LCM $(x^2+1)^2$,

$$(x^2+1)^2\frac{2x^2+1}{(x^2+1)^2} = (x^2+1)^2\left[\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}\right]$$

$$2x^2+1 = (Ax+B)(x^2+1) + (Cx+D)$$

$$= Ax^3+Ax+Bx^2+B+Cx+D$$

$$2x^2+1 = Ax^3+Bx^2+(A+C)x+B+D$$

By comparing coefficients,

$$x^3: \quad A = 0$$

$$x^2: \quad B = 2$$

$$x: \quad A + C = 0 \quad \text{———} \quad ③$$

$$x^0: \quad B + D = 1 \quad \text{———} \quad ④$$

Put $A = 0$ in equ.③ and $B = 2$ in equ.④,

$$③ \implies 0 + C = 0$$

$$C = 0$$

$$④ \implies 2 + D = 1$$

$$D = 1 - 2$$

$$D = -1$$

Put $A = 0$, $B = 2$, $C = 0$, and $D = -1$ in equ.②,

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{0.x+2}{x^2+1} + \frac{0.x+(-1)}{(x^2+1)^2}$$

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{2}{x^2+1} - \frac{1}{(x^2+1)^2}$$

Hence from equ.①,

$$\frac{x^4}{x^4+2x^2+1} = 1 - \left(\frac{2}{x^2+1} - \frac{1}{(x^2+1)^2}\right)$$

$$\frac{x^4}{x^4+2x^2+1} = 1 - \frac{2}{x^2+1} + \frac{1}{(x^2+1)^2}$$

6. $\dfrac{6x^4+40x^2}{(4-x^2)(x^2+4)^2}$

Sol

$$\frac{6x^4+40x^2}{(4-x^2)(x^2+4)^2}=\frac{6x^4+40x^2}{(2^2-x^2)(x^2+4)^2}$$

$$\frac{6x^4+40x^2}{(4-x^2)(x^2+4)^2}=\frac{6x^4+40x^2}{(2-x)(2+x)(x^2+4)^2}$$

$$\frac{6x^4+40x^2}{(4-x^2)(x^2+4)^2}=\frac{A}{2-x}+\frac{B}{2+x}+\frac{Cx+D}{x^2+4}+\frac{Ex+F}{(x^2+4)^2} \quad \text{——— (1)}$$

Multiply each term by LCM $(2-x)(2+x)(x^2+4)^2$,

$$(2-x)(2+x)(x^2+4)^2\frac{6x^4+40x^2}{(4-x^2)(x^2+4)^2}=(2-x)(2+x)(x^2+4)^2\left(\frac{A}{2-x}+\frac{B}{2+x}+\frac{Cx+D}{x^2+4}+\frac{Ex+F}{(x^2+4)^2}\right)$$

$$6x^4+40x^2=A(2+x)(x^2+4)^2+B(2-x)(x^2+4)^2+(Cx+D)(2-x)(2+x)(x^2+4)$$
$$+(Ex+F)(2-x)(2+x) \quad \text{——— (2)}$$

Put $x=2$ in (2),

$$6(2)^4+40(2)^2=A(2+2)(2^2+4)^2+B(2-2)(2^2+4)^2+(C(2)+D)(2-2)(2+2)(2^2+4)$$
$$+(E(2)+F)(2-2)(2+2)$$

$$6(16)+40(4)=A(4)(4+4)^2+B(0)(4+4)^2+(2C+D)(0)(4)(4+4)$$
$$+(2E+F)(0)(4)$$

$$96+160=A(4)(8)^2+0+0+0$$

$$256=A(256)$$

$$\Rightarrow \quad A=1$$

Put $x=-2$ in ②,

$$6(-2)^4+40(-2)^2=A(2+(-2))((-2)^2+4)^2+B(2-(-2))((-2)^2+4)^2+(C(-2)+D)(2-(-2))(2+(-2))((-2)^2+4)$$
$$+(E(-2)+F)(2-(-2))(2+(-2))$$

$$6(16)+40(4)=A(0)(4+4)^2+B(4)(4+4)^2+(-2C+D)(4)(0)(4+4)+(-2E+F)(4)(0)$$

$$96+160=0+B(4)(8)^2+0+0$$

$$256=B(256)$$

$$\Rightarrow B=1$$

From equ. ②,

$$6x^4+40x^2=A(2+x)(x^2+4)^2+B(2-x)(x^2+4)^2+(Cx+D)(2-x)(2+x)(x^2+4)$$
$$+(Ex+F)(2-x)(2+x)$$

$$=A(2+x)(x^4+8x^2+16)+B(2-x)(x^4+8x^2+16)$$
$$+(Cx+D)(4-x^2)(4+x^2)+(Ex+F)(4-x^2)$$

$$=A(2x^4+16x^2+32+x^5+8x^3+16x)+B(2x^4+16x^2+32$$
$$-x^5-8x^3-16x)+(Cx+D)(16-x^4)+4Ex-Ex^3+4F-Fx^2$$

$$=2Ax^4+16Ax^2+32A+Ax^5+8Ax^3+16Ax+2Bx^4+16Bx^2$$
$$+32B-Bx^5-8Bx^3-16Bx+16Cx-Cx^5+16D-Dx^4$$
$$+4Ex-Ex^3+4F-Fx^2$$

$$6x^4+40x^2=(A-B-C)x^5+(2A+2B-D)x^4+(8A-8B-E)x^3$$
$$+(16A+16B-F)x^2+(16A-16B+16C+4E)x+(32A+32B$$

$+16D+4F)$

By comparing coefficients:

x^5: $A-B-C=0$ —— ③

x^4: $2A+2B-D=6$ —— ④

x^3: $8A-8B-E=0$ —— ⑤

x^2: $16A+16B-F=40$ —— ⑥

x^1: $16A-16B+16C+4E=0$

x^0: $32A+32B+16D+4F=0$

Put $A=1$, $B=1$ in equ. ③, equ. ④, equ. ⑤ & equ. ⑥,

③ $\Rightarrow$ $1-1-C=0$

$-C=0$

$C=0$

④ $\Rightarrow$ $2(1)+2(1)-D=6$

$4-D=6$

$-D=6-4$

$-D=2$

$D=-2$

⑤ $\Rightarrow$ $8(1)-8(1)-E=0$

$8-8-E=0$

$E=0$

⑥ $\Rightarrow$ $16(1)+16(1)-F=40$

$$32-F=40$$

$$32-40=F$$

$$-8=F$$

Hence from equ. ①,

$$\frac{6x^4+40x^2}{(4-x^2)(x^2+4)^2}=\frac{1}{2-x}+\frac{1}{2+x}+\frac{0.x-2}{x^2+4}+\frac{0.x-8}{(x^2+4)^2}$$

$$\frac{6x^4+40x^2}{(4-x^2)(x^2+4)^2}=\frac{1}{2-x}+\frac{1}{2+x}-\frac{2}{x^2+4}-\frac{8}{(x^2+4)^2}$$

بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

(In the Name of Allah, the Most Compassionate, the Most Merciful)

Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 6: Sequences and Series
- Sequence
- Exercise 6.1

Unit 6 Sequences and Series

INTRODUCTION

In this unit, students will learn to analyze and solve problems involving arithmetic, geometric, and harmonic sequences and series, including their real-world applications. Learners will identify various sequence types, compute finite and infinite sums, and utilize sigma notation. Additionally, they will explore practical scenarios such as motor vehicle leasing, investment planning, and financial calculations. This unit also emphasizes applying these concepts to diverse fields, including healthcare, finance, and traffic modeling. Finally, Students will be able to solve both theoretical and real-life problems using sequences and series effectively.

Let us observe the following pattern of numbers.

(i) $5, 11, 17, 23, \ldots$ (ii) $6, 12, 24, 48, \ldots$

+6 +6

(iii) $4, 2, 0, -2, -4, \ldots$ (iv) $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$

$\times\frac{2}{3}$

In example (i), every number (except 5) is formed by adding 6 to the previous numbers. Hence a specific pattern is followed in the arrangement of these numbers. Similarly, in example (ii), every number is obtained by multiplying the previous number by 2. Similar cases are followed in example (iii) and (iv). When a set of numbers follows a pattern and there is a clear rule for finding next number in the pattern, then we have sequence as in above examples.

$a_1, a_2, a_3, \ldots, a_n, \ldots$

6.1 Sequence

A systematic arrangement of numbers according to a given rule is called a sequence. The numbers in a sequence are called its **terms.** We refer the first term of a sequence as a_1, second term as a_2 and so on. The n^{th} term of a sequence is denoted by a_n, which may also be referred to as the general term of the sequence, and the terms immediately preceding it are called the $(n-1)$st term, the $(n-2)$nd term and so on.

6.1.2 Finite and Infinite Sequences

1. A sequence which consists of a finite number of terms is called a finite sequence. For example, 2, 5, 8, 11, 14, 17, 20, 23 is a finite sequence of 8 terms.
2. A sequence which consists of an infinite number of terms is called an infinite sequence. For example, 3, 10, 17, 24, … is an infinite sequence, or more generally as 3, 10, 17, 24, … , $7n-4$, … to show how each term was generated.

$a_1=3,\quad a_2=10,\quad a_3=17,\ \ldots,\ a_n=7n-4,\ \ldots$

Note: If a sequence is given, then we can find its n term and if the n term of a sequence is given then we can find the terms of the sequence.

Example 1: Find the first four terms of the sequences whose n terms are given.

(i) $a_n = 3n + 1$

Substituting $n = 1$, we have

$$a_1 = 3(1) + 1 = 4$$

Similarly, $a_2 = 3(2) + 1 = 7$

$$a_3 = 3(3) + 1 = 10$$

$$a_4 = 3(4) + 1 = 13$$

The first four terms of the sequence are 4, 7, 10, 13

(ii) $a_n = 3n^2 - 3$

Substituting $n = 1$, we have

$$a_1 = 3(1)^2 - 3 = 0$$

$\{2, 3, 5, 7, 11, \ldots\}$

Similarly, $a_2 = 3(2)^2 - 3 = 9$

$$a_3 = 3(3)^2 - 3 = 24$$

$$a_4 = 3(4)^2 - 3 = 45$$

The first four terms of the sequence are 0, 9, 24, 45

Sequences of numbers which follow specific patterns are called progression. Depending on the pattern, the progression is classified as follows.

(i) Arithmetic progression (ii) Geometric progression

(iii) Harmonic progression

EXERCISE 6.1

1. Find the next four terms of each sequence.

(i) 12, 16, 20, …

Sol

12, 16, 20, ...

Here

$a_1 = 12$

$a_2 = 16 \quad (a_1 + 4)$

$a_3 = 20 \quad (a_2 + 4)$

$a_4 = a_3 + 4 = 24, \quad a_5 = a_4 + 4 = 28, \quad a_6 = a_5 + 4 = 32$

$a_7 = a_6 + 4 = 36$

(ii) 3, 1, –1, …

Sol

3, 1, –1, ...

Here

$a_1 = 3$

$a_2 = 1 \quad (3-2)$

$a_3 = -1 \quad (1-2)$

$a_4 = a_3 - 2 = -1 - 2 = -3$

$a_5 = a_4 - 2 = -3 - 2 = -5$

$a_6 = a_5 - 2 = -5 - 2 = -7$

$a_7 = a_6 - 2 = -7 - 2 = -9$

2. Write down the first three terms of each sequence.

(i) $a_n = 3n + 5$

Sol

$$a_n = 3n+5$$

Put $n=1, 2, 3$

$a_1 = 3(1)+5$ $\quad$ $a_2 = 3(2)+5$ $\quad$ $a_3 = 3(3)+5$

$a_1 = 8$ $\quad$ $a_2 = 6+5 = 11$ $\quad$ $a_3 = 9+5 = 14$

Hence, the first three terms are:

8, 11, 14

(ii) $a_{n+1} = 4a_n - 7$ and $a_1 = 3$

Sol

$$a_{n+1} = 4 \cdot a_n - 7 \quad , \quad a_1 = 3$$

Put $n=1, 2$

$a_{1+1} = 4a_1 - 7$ $\quad$ $a_{2+1} = 4a_2 - 7$

$a_2 = 4(3) - 7 = 5$ $\quad$ $a_3 = 4(5) - 7 = 13$

Hence, the first three terms are:

3, 5, 13

(iii) $a_n = (n-3)(n+1)$

Sol

$$a_n = (n-3)(n+1)$$

Put $n = 1, 2, 3$

$a_1 = (1-3)(1+1)$

$= (-2)(2)$

$a_1 = -4$

$a_2 = (2-3)(2+1)$

$= (-1)(3)$

$a_2 = -3$

$a_3 = (3-3)(3+1)$

$= (0)(4)$

$a_3 = 0$

Hence, the first three terms are:

$$-4, -3, 0$$

(iv) $a_1 = -1,\ a_{n+1} = \dfrac{3}{a_n + 2}$

Sol

$$a_{n+1} = \frac{3}{a_n + 2}, \quad a_1 = -1$$

Put $n = 1, 2$

$a_{1+1} = \dfrac{3}{a_1 + 2}$

$a_2 = \dfrac{3}{-1+2}$

$a_2 = \dfrac{3}{1} = 3$

$a_{2+1} = \dfrac{3}{a_2 + 2}$

$a_3 = \dfrac{3}{3+2}$

$a_3 = \dfrac{3}{5}$

Hence, the first three terms are:

$$-1, 3, \frac{3}{5}$$

(v) $a_n = 8 - \dfrac{20}{3+n}$

Sol

$$a_n = 8 - \frac{20}{3+n}$$

Put $n = 1, 2, 3$

$a_1 = 8 - \dfrac{20}{3+1}$

$a_1 = 8 - \dfrac{20}{4} = 3$

$a_2 = 8 - \dfrac{20}{3+2}$

$a_2 = 8 - 4 = 4$

$a_3 = 8 - \dfrac{20}{3+3}$

$= 8 - \dfrac{20}{6}$

$= \dfrac{8}{1} - \dfrac{10}{3} = \dfrac{24-10}{3}$

$a_3 = \dfrac{14}{3}$

Hence, the first three terms are:

$3, 4, \dfrac{14}{3}$

(vi) $a_1 = 1,\ a_{n+1} = (3a_n + 2)^2$

Sol

$$a_{n+1} = (3a_n+2)^2 \quad \& \quad a_1 = 1$$

Put $n = 1, 2$

$a_{1+1} = (3a_1+2)^2$

$= (3(1)+2)^2$

$a_2 = 5^2 = 25$

$a_{2+1} = (3a_2+2)^2$

$= (3(25)+2)^2$

$a_3 = 77^2 = 5929$

Hence, the first three terms are:

$1, 25, 5929$

(vii) $a_n = (-2n)^2$

Sol

$$a_n = (-2n)^2$$

Put $n = 1, 2, 3$

$a_1 = [-2(1)]^2$ | $a_2 = [-2(2)]^2$ | $a_3 = [-2(3)]^2$
$= (-2)^2$ | $= (-4)^2$ | $= (-6)^2$
$a_1 = 4$ | $a_2 = 16$ | $a_3 = 36$

Hence, the first three terms are: 4, 16, 36

(viii) $a_n = (-1)^n 7n^2$

Sol

$$a_n = (-1)^n 7n^2$$

Put $n = 1, 2, 3$

$a_1 = (-1)^1 7(1)^2$ | $a_2 = (-1)^2 7(2)^2$ | $a_3 = (-1)^3 7(3^2)$
$a_1 = -7$ | $a_2 = 28$ | $a_3 = -63$

Hence, the first three terms are:

$-7, 28, -63$

3. An expression for the n^{th} triangular number is $\frac{n(n+1)}{2}$. Write down the 15^{th} triangular number. Make a triangle of dots by taking $n=5$.

Sol

The formula for n^{th} triangular number is:

$$T_n = \frac{n(n+1)}{2} \qquad \text{———} \quad (1)$$

Substitute $n=15$,

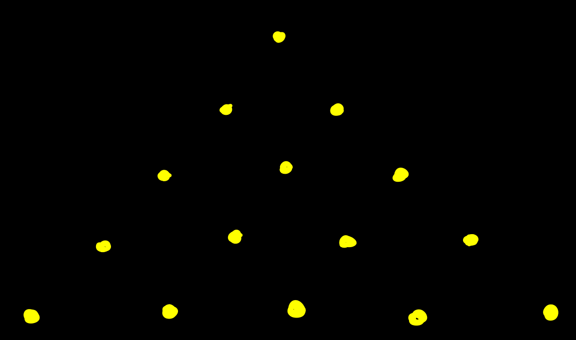
$$T_{15} = \frac{15(15+1)}{2} = \frac{15(16)}{2}$$

$$T_{15} = 120$$

Substitute $n=5$ in (1),

$$T_5 = \frac{5(5+1)}{2} = \frac{5(6)}{2} = 15$$

So, we arrange 15 dots in rows of 1, 2, 3, 4, 5.



4. Write down the n^{th} term of each sequence.

(i) $1, 4, 9, \dots$

Sol

$$1,\ 4,\ 9,\ \dots,\ a_n,\ \dots$$

Here

$a_1 = 1 = 1^2$ $\qquad a_2 = 4 = 2^2$ $\qquad a_3 = 9 = 3^2$

Hence

$$a_n = n^2$$

(ii) $1,\ 1+2,\ 1+2+3,\ \dots$

Sol

$$1,\ 1+2,\ 1+2+3,\ \dots,\ a_n,\ \dots$$

Here

$a_1 = 1,$ $\qquad a_2 = 1+2,$ $\qquad a_3 = 1+2+3$

Hence the n^{th} term would be:

$$a_n = 1+2+3+\dots+n$$

(iii) $a_1b_1,\ a_2b_2,\ a_3b_3, \ldots$

Sol

$$a_1b_1,\ a_2b_2,\ a_3b_3, \ldots, x_n, \ldots$$

Here

$$x_1 = a_1b_1, \qquad x_2 = a_2b_2, \qquad x_3 = a_3b_3$$

Hence the n^{th} term would be:

$$x_n = a_nb_n$$

(iv) $x,\ 2x^2,\ 3x^3, \ldots$

Sol

$$x,\ 2x^2,\ 3x^3, \ldots, a_n, \ldots$$

Here

$$a_1 = x (= 1 \cdot x^1) \qquad a_2 = 2x^2, \qquad a_3 = 3x^3$$

Hence the n^{th} term would be:

$$a_n = nx^n$$

(v) $a_1, \quad a_1+d, \quad a_1+2d, \ldots$

Sol

$$a_1, \quad a_1+d, \quad a_1+2d, \ldots, a_n, \ldots$$

Here

$$a_1 = a_1 + 0 \cdot d$$

$$a_2 = a_1 + 1 \cdot d$$

$$a_3 = a_1 + 2d$$

Hence the n^{th} term would be:

$$a_n = a_1 + (n-1)d$$

(vi) $a_1, \quad a_1 r, \quad a_1 r^2, \ldots$

Sol

$$a_1, \quad a_1 r, \quad a_1 r^2, \ldots, a_n, \ldots$$

Here

$$a_1 = a_1 r^0 \qquad a_2 = a_1 r^1 \qquad a_3 = a_1 r^2$$

Hence the n^{th} term would be:

$$a_n = a_1 r^{n-1}$$

(vii) $\frac{a_1}{b_1+c_1}, \frac{2a_2}{b_2+c_2}, \frac{3a_3}{b_3+c_3}, \ldots$

Sol

$$\frac{a_1}{b_1+c_1}, \frac{2a_2}{b_2+c_2}, \frac{3a_3}{b_3+c_3}, \ldots, x_n, \ldots$$

Here

$$x_1 = \frac{1a_1}{b_1+c_1}, \quad x_2 = \frac{2a_2}{b_2+c_2}, \quad x_3 = \frac{3a_3}{b_3+c_3}$$

Hence the n^{th} term would be:

$$x_n = \frac{na_n}{b_n+c_n}$$

(viii) $\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \ldots$

Sol

$$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \ldots, x_n, \ldots$$

Here

$$x_1 = \frac{1}{a_1}, \quad x_2 = \frac{1}{a_1+d}, \quad x_3 = \frac{1}{a_1+2d}$$

Hence the n^{th} term would be:

$$x_n = \frac{1}{a_1+(n-1)d}$$