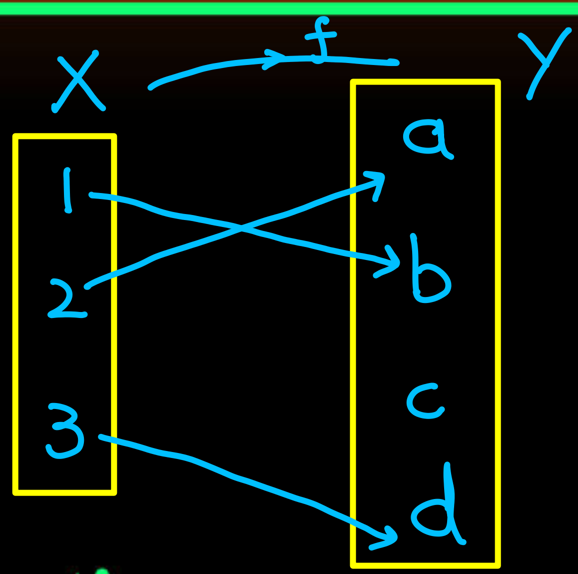
بِسْمِ اللهِ الرَّحْمٰنِ الرَّحِيْمِ

(In the Name of Allah, the Most Compassionate, the Most Merciful)

Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 2: Functions and Graphs
- Concept of Function
- Exercise 2.1: Concepts and Examples

Unit 2 Functions and Graphs



(1, b), (2, a), (3, d)

D = X

Codomain = Y

Range = {a, b, d}

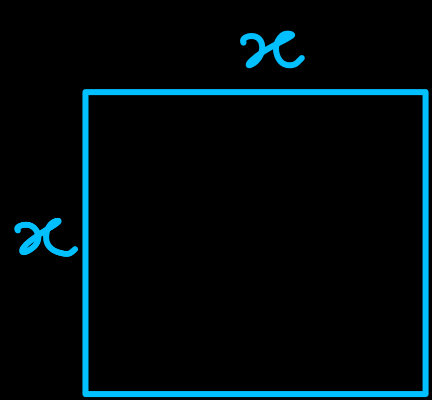
2.1 Concept of Function

The term function was recognized by a German Mathematician Leibniz (1646-1716) to describe the dependence of one quantity on another. The following examples illustrate how this term is used:

(i) The area "A" of a square depends on one of its sides "x" by the formula $A = x^2$, so we say that A is a function of x.

(ii) The volume "V" of a sphere depends on its radius "r" by the formula $V = \frac{4}{3}\pi r^3$, so we say that V is a function of r.

$A = x^2$



A **function** is a rule or correspondence, relating two sets in such a way that each element in the first set corresponds to one and only one element in the second set.

Thus in, (i) above, a square of a given side has only one area and in, (ii) above, a sphere of a given radius has only one volume.

Now we have a formal definition:

2.1.1 Definition (Function, Domain, Codomain, Range)

A **function** f from a set X to a set Y is a rule or a correspondence that assigns to each element x in X a unique element y in Y. The set X is called the **domain** of f.

The set of corresponding elements y in Y is called the **range** of f. While the **codomain** of a function is the set Y in which function's output values (range) lie.

Unless stated to the contrary, we shall assume hereafter that the set X and Y consist of real numbers.

Note: Co-domain is the set of all possible outputs but the range is the actual set of outputs produced by the function under the given domain that is range set is always a subset of co-domain.

2.1.2 Notation and Value of a Function

$y = f(x)$

dependent variable (output)

independent variable (input)

$x \to \boxed{f} \to f(x)$

$1 \to \boxed{f = x^2+1} \to 2$

$3 \to \boxed{f = x^2+1} \to 10$

Example 1: Given $f(x) = x^3 - 2x^2 + 4x - 1$, find: (i) $f(0)$ (ii) $f(1)$

(iii) $f(-2)$ (iv) $f(1 + x)$ (v) $f\left(\frac{1}{x}\right)$, $x \neq 0$

Solution: $f(x) = x^3 - 2x^2 + 4x - 1$

$\because (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(i) $f(0) = 0 - 0 + 0 - 1 = -1$

(ii) $f(1) = (1)^3 - 2(1)^2 + 4(1) - 1 = 1 - 2 + 4 - 1 = 2$

(iii) $f(-2) = (-2)^3 - 2(-2)^2 + 4(-2) - 1 = -8 - 8 - 8 - 1 = -25$

(iv) $f(1 + x) = (1 + x)^3 - 2(1 + x)^2 + 4(1 + x) - 1$

$= 1 + 3x + 3x^2 + x^3 - 2 - 4x - 2x^2 + 4 + 4x - 1$

$f(1+x) = x^3 + x^2 + 3x + 2$

(iv) $f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - 2\left(\frac{1}{x}\right)^2 + 4\left(\frac{1}{x}\right) - 1 = \frac{1}{x^3} - \frac{2}{x^2} + \frac{4}{x} - 1$, $x \neq 0$

Example 2: Find the domain and range of $f(x) = x^2$.

Solution: For every real number x, $f(x) = x^2$ is a non-negative real number. So,

Domain f = set of all real numbers ; Range f = set of all non-negative real numbers.

$\mathbb{R}$, $(-\infty, \infty)$

Domain: Set of inputs

Range: Set of outputs

Remember!

There are two types of intervals known as open interval and closed interval. In an open interval (a, b), the endpoints are not included. In a closed interval [a, b], the endpoints are included.

Closed ← $[3, 5]$

Open ← $(3, 5)$

Half-open ← $(3, 5]$

Example 3: Find the domain and range of $f(x) = \dfrac{x}{x^2 - 4}$.

Solution: At $x = 2$ and $x = -2$, $f(x) = \dfrac{x}{x^2 - 4}$ is not defined. So,

Domain f = set of all real numbers except -2 and 2 or $R - \{-2, 2\}$

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

Let $y = \dfrac{x}{x^2 - 4} \Rightarrow y(x^2 - 4) = x \Rightarrow x^2 y - 4y = x$

$f \longrightarrow f^{-1}$

$D(f) = R(f^{-1})$

$R(f) = D(f^{-1})$

$$x^2 y - x - 4y = 0$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(y)(-4y)}}{2y}$$

$$\longrightarrow x = \frac{1 \pm \sqrt{1 + 16y^2}}{2y}, \; y \neq 0$$

x is defined as $\forall \; y \neq 0$

For $y = 0$, we have $0 = \dfrac{x}{x^2 - 4} \Rightarrow x = 0$

$f(0) = 0$

So, range f = set of all real numbers or $(-\infty, \infty)$

Example 4: Find the domain and range of $f(x) = \sqrt{x^2 - 9}$.

Solution: $\sqrt{x^2 - 9} \geq 0 \Rightarrow x^2 - 9 \geq 0$ ...(i)

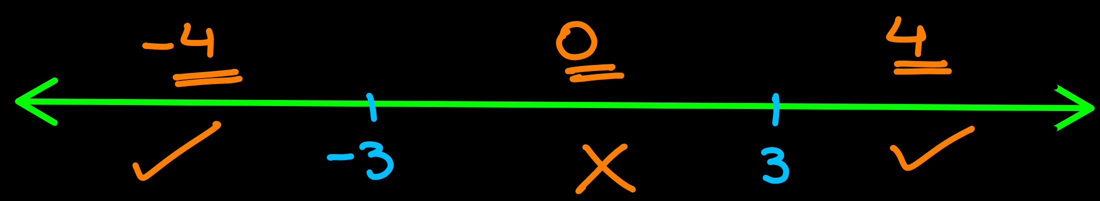
Let $x^2 - 9 = 0 \Rightarrow x = \pm 3$

Critical points divide the number line into three regions:

Put $x = -4$ in (i), $16 - 9 \geq 0$ (True)

Put $x = 0$ in (i), $0 - 9 \geq 0$ (False)

Put $x = 4$ in (i), $16 - 9 \geq 0$ (True)



So, domain $f = (-\infty, -3] \cup [3, \infty)$

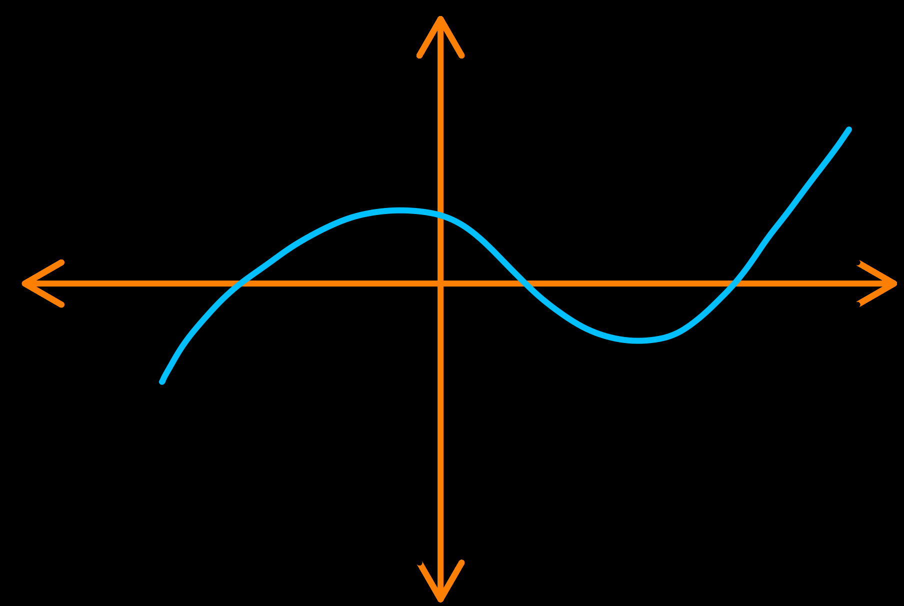
The smallest value of $x^2 - 9$ is 0 (when $x = \pm 3$).

$\Rightarrow \quad y = \sqrt{0} = 0$

As $|x|$ increases beyond 3, $x^2 - 9$ grows to $+\infty$, so y grows to $+\infty$.

So, range $f = [0, \infty)$

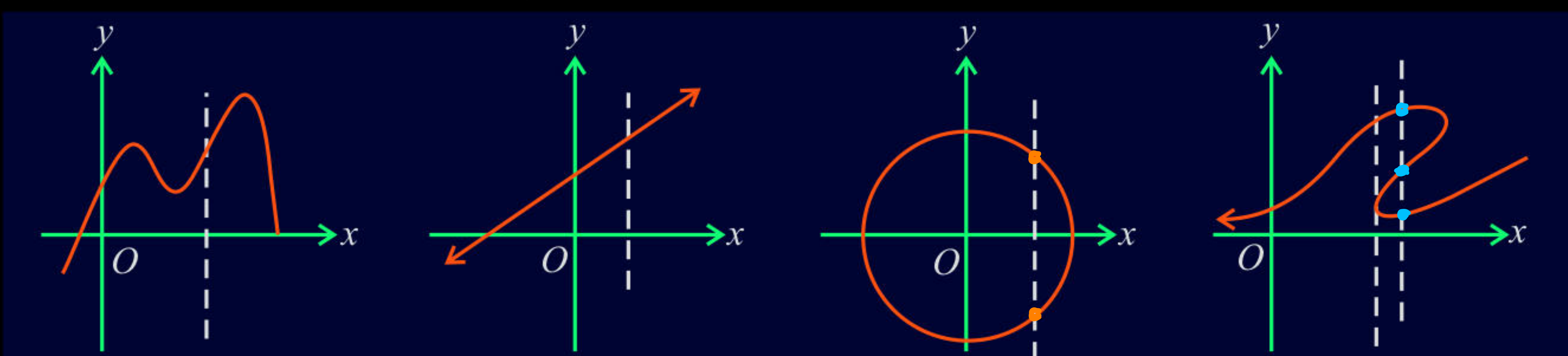
$y = f(x)$



2.1.3 Vertical Line Test

The vertical line test is a method used to determine whether a graph represents a function. A graph represents a function if and only if no vertical line intersects the graph more than once. If any vertical line passes through the graph more than once, it is not a function.

Explanation is given in the figure.



(a) a function (b) a function (c) not a function (d) not a function

2.1.4 Types of Function

(i) One-to-One (Injective) Function

A function f is one-to-one if different inputs produce different outputs, i.e., if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. This means that no two different elements of the domain map to the same element of the co-domain.

For example, $f(x) = 5x + 7$ is one-to-one because if $5x_1 + 7 = 5x_2 + 7$ implies $x_1 = x_2$.

$\rightarrow f(x_1) = f(x_2) \implies x_1 = x_2$

OR

$x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

$f(x_1) = 5x_1 + 7$

$f(x_2) = 5x_2 + 7$

$5x_1 + \cancel{7} = 5x_2 + \cancel{7}$

$\cancel{5}x_1 = \cancel{5}x_2$

$x_1 = x_2$

(ii) Onto (Surjective) Function

A function $f : X \rightarrow Y$ is called onto (or surjective) function if every element in the co-domain Y has at least one pre-image in the domain X. In other words, for every y in Y, there exists an x in X such that $f(x) = y$.

For example, $f(x) = 2x + 3$, where the domain and co-domain are both real numbers.

Here $y = 2x + 3 \Rightarrow x = \dfrac{y-3}{2}$. Here for each y in R, there exists $\dfrac{y-3}{2}$ in R such that $f\left(\dfrac{y-3}{2}\right) = y$. Hence f is an onto function.

Codomain = Range

$y = 2x + 3$

$x = \dfrac{y-3}{2}$

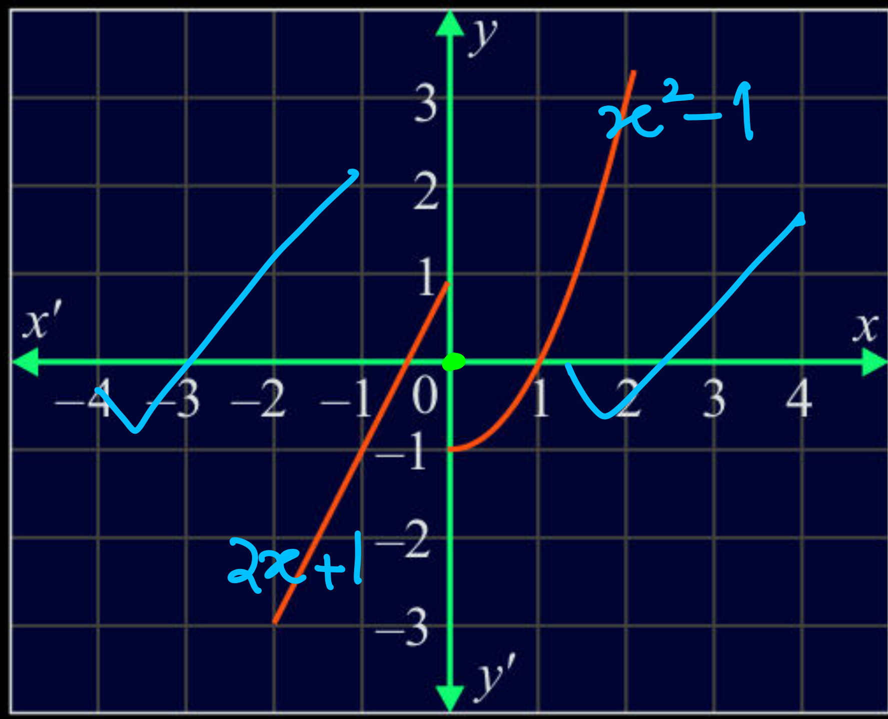
$f\left(\dfrac{y-3}{2}\right) = \cancel{2}\left(\dfrac{y-3}{\cancel{2}}\right) + 3 = y - \cancel{3} + \cancel{3} = y$

(iii) Bijective Function

A function $f: X \to Y$ is called bijective if it is both one-to-one and onto.

Piecewise Function

A piecewise function is a function that is defined by different expressions (or "pieces") over different intervals of its domain. Each piece applies to a specific part of the domain.



For example, $f(x) = \begin{cases} 2x+1 & \text{if } x<0 \\ x^2-1 & \text{if } x \geq 0 \end{cases}$

For $x<0$, the function behaves like $2x+1$ and for $x \geq 0$, it behaves like x^2-1

Example 5: Show that the function $f(x) = x+1$, where the domain and co-domain are all real numbers, is bijective.

Solution: A function is bijective if it is both one-to-one and onto.

A function is one-to-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for $f(x) = x+1$

Suppose $f(x_1) = f(x_2)$

$$x_1 + 1 = x_2 + 1$$

$$\Rightarrow \quad x_1 = x_2$$

So, the given function is one-to-one.

It is also onto because for every real number y, there is a real number x (specifically $x = y-1$) such that $f(y-1) = y - 1 + 1 = y$. Therefore, $f(x)$ is bijective.

$y = x+1$

$x = y-1$ Put in $f(x)$

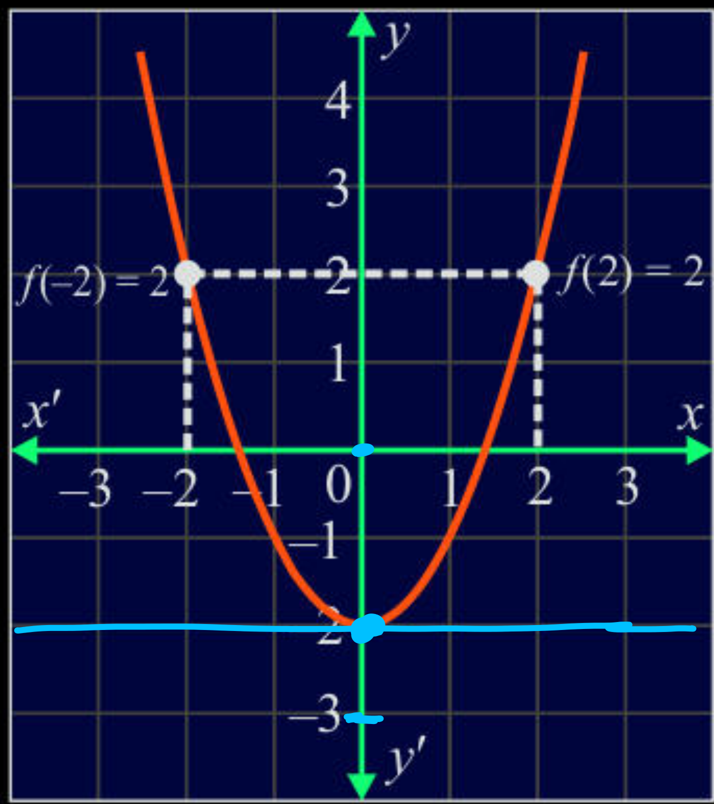
$f(y-1) = y - \cancel{1} + \cancel{1} = y$

Example 6: Show that the function $f(x) = x^2 - 2$, where the domain and co-domain are all real numbers, is neither one-to-one nor onto.

Solution: As $f(x_1) = f(x_2) \Rightarrow x_1^2 - 2 = x_2^2 - 2 \Rightarrow x_1^2 = x_2^2$

Taking square root, we get $x_1 = x_2$ or $x_1 = -x_2$

Codomain



This does not imply that $x_1 = x_2$, for example

$x_1 = 2, x_2 = -2 \Rightarrow x_1 \neq x_2$ and $f(2) = 2 = f(-2)$.

Thus, f is not one-to-one.

Also, the element –2 in the co-domain R is the smallest value that $f(x) = x^2 - 2$ can attain, and it is only achieved when $x = 0$. However, any number less than –2 (in co-domain R) is not the image of any real number x in domain R. For example, $f(x) = -3 \Rightarrow x^2 - 2 = -3$ has no real root.

$f(x) = x^2 - 2$

$f(x_1) = f(x_2)$

$x_1^2 + \cancel{2} = x_2^2 + \cancel{2}$

$x_1^2 = x_2^2$

$\sqrt{x_1^{\cancel{2}}} = \pm\sqrt{x_2^{\cancel{2}}}$

$x_1 = \pm x_2$

Not one-to-one

$y = x^2 - 2$

$y + 2 = x^2$

$x = \pm\sqrt{y+2}$

$\sqrt{-3+2} = \sqrt{-1}$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 2: Functions and Graphs
- Concept of Function
- Exercise 2.1: Q1 - Q5

EXERCISE 2.1

1. Given that: (a) $f(x) = x^2 - 1$ (b) $f(x) = \sqrt{2x+3}$ Find:

(i) $f(-3)$ (ii) $f(0)$ (iii) $f(x-2)$ (iv) $f(x^2+3)$

Sol (a)

i) $f(-3) = ?$

$f(-3) = (-3)^2 - 1$

$= 9 - 1$

$= 8$

ii) $f(0) = ?$

$f(0) = 0^2 - 1$

$= -1$

iii) $f(x-2) = ?$

$\because (a-b)^2 = a^2 - 2ab + b^2$

$f(x-2) = (x-2)^2 - 1$

$= x^2 - 4x + 4 - 1$

$= x^2 - 4x + 3$

iv) $f(x^2+3) = ?$

$\because (a+b)^2 = a^2 + 2ab + b^2$

$f(x^2+3) = (x^2+3)^2 - 1$

$= x^4 + 6x^2 + 9 - 1$

$= x^4 + 6x^2 + 8$

Sol (b)

$f(x) = \sqrt{2x+3}$

i) $f(-3) = ?$

$f(-3) = \sqrt{2(-3)+3}$

$= \sqrt{-6+3}$

$= \sqrt{-3}$

ii) $f(0) = ?$

$f(0) = \sqrt{2(0)+3}$

$= \sqrt{3}$

iii) $f(x-2) = ?$

$f(x-2) = \sqrt{2(x-2)+3}$

$= \sqrt{2x-4+3}$

$= \sqrt{2x-1}$

iv) $f(x^2+3) = ?$

$f(x^2+3) = \sqrt{2(x^2+3)+3}$

$= \sqrt{2x^2+6+3}$

$= \sqrt{2x^2+9}$

2. Find $\frac{f(a+h)-f(a)}{h}$ and simplify where,

(i) $f(x) = 4x + 7$

Sol

$$\frac{f(a+h)-f(a)}{h} \quad \text{———} \ (1)$$

$$f(a+h) = 4(a+h)+7$$
$$= 4a+4h+7$$

$$f(a) = 4a+7$$

From (1)

$$\frac{f(a+h)-f(a)}{h} = \frac{4a+4h+7-(4a+7)}{h}$$

$$= \frac{\cancel{4a}+4h+\cancel{7}-\cancel{4a}-\cancel{7}}{h}$$

$$= \frac{4\cancel{h}}{\cancel{h}}$$

$$\frac{f(a+h)-f(a)}{h} = 4$$

(ii) $f(x) = \sin x$

Sol

$$\frac{f(a+h)-f(a)}{h} = \frac{\sin(a+h)-\sin a}{h}$$

$$\because \sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

$$= \frac{2\cos\left(\frac{a+h+a}{2}\right)\cdot\sin\left(\frac{\cancel{a}+h-\cancel{a}}{2}\right)}{h}$$

$$= \frac{2\cos\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{2\cos\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

(iii) $f(x) = x^3 + x^2 - 1$

Sol

$$\frac{f(a+h) - f(a)}{h} \quad \text{——} \quad ①$$

$$f(a+h) = (a+h)^3 + (a+h)^2 - 1$$

$$f(a+h) = a^3 + 3a^2h + 3ah^2 + h^3 + a^2 + 2ah + h^2 - 1$$

&

$$f(a) = a^3 + a^2 - 1$$

Substitute in exp①,

$$\frac{f(a+h) - f(a)}{h} = \frac{a^3 + 3a^2h + 3ah^2 + h^3 + a^2 + 2ah + h^2 - 1 - (a^3 + a^2 - 1)}{h}$$

$$= \frac{\cancel{a^3} + 3a^2h + 3ah^2 + h^3 + \cancel{a^2} + 2ah + h^2 \cancel{-1} - \cancel{a^3} - \cancel{a^2} \cancel{+1}}{h}$$

$$= \frac{\cancel{h}(3a^2 + 3ah + h^2 + 2a + h)}{\cancel{h}}$$

$$\frac{f(a+h) - f(a)}{h} = 3a^2 + 3ah + h^2 + 2a + h$$

(iv) $f(x) = \tan x$

Sol

$$\frac{f(a+h)-f(a)}{h} = \frac{\tan(a+h) - \tan(a)}{h}$$

$$= \frac{\frac{\sin(a+h)}{\cos(a+h)} - \frac{\sin a}{\cos a}}{h}$$

$$= \frac{\frac{\sin(a+h)\cos a - \cos(a+h)\sin a}{\cos(a+h).\cos a}}{h}$$

$\because \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{\sin(a+h-a)}{h.\cos(a+h).\cos a}$$

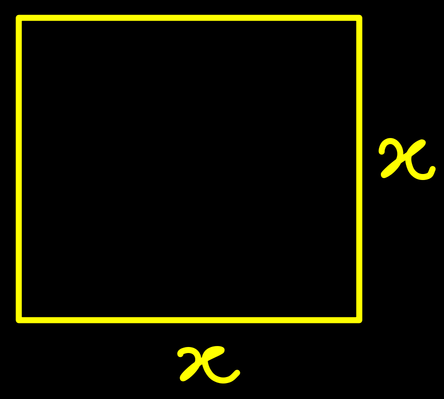
$$\frac{f(a+h)-f(a)}{h} = \frac{\sin h}{h\cdot\cos(a+h)\cdot\cos a}$$

3. Express the following:

(a) The area A of a square as a function of its perimeter P.

Sol

$A = l \times w$



$A = x \times x$

$A = x^2$ —— 1

As

$P = 4x$

$\frac{P}{4} = x$ Put in 1

$A = \left(\frac{P}{4}\right)^2$

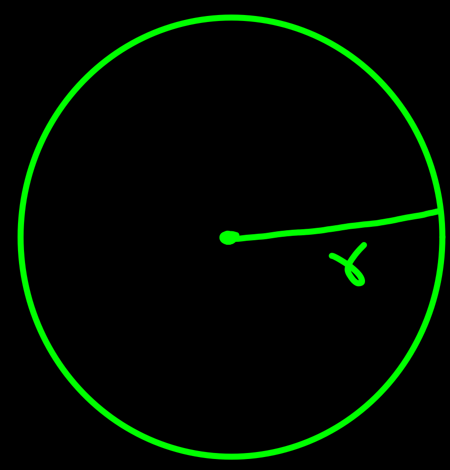
$A = \frac{P^2}{16}$

(b) The circumference C of a circle as a function of its area A.

sol

As

$$C = 2\pi r \quad \text{—— } ①$$



and

$$A = \pi r^2$$

$$\frac{A}{\pi} = r^2$$

$$\sqrt{\frac{A}{\pi}} = \sqrt{r^2}$$

$$r = \sqrt{\frac{A}{\pi}} \qquad \text{Put in } ①$$

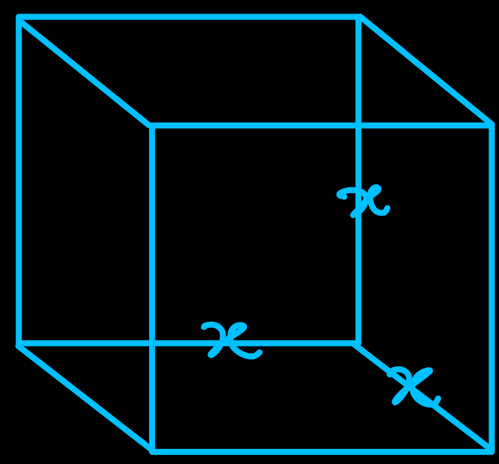
$$C = 2\pi\sqrt{\frac{A}{\pi}}$$

$$= 2\sqrt{\pi}\,\cancel{\sqrt{\pi}}\,\frac{\sqrt{A}}{\cancel{\sqrt{\pi}}}$$

$$C = 2\sqrt{\pi A}$$

(c) The surface area S of a cube as a function of its volume V.

Sol



$S = 6x^2$ —— ①

and

$V = l \times w \times h$

$V = x \cdot x \cdot x$

$V = x^3$

$V^{1/3} = (x^3)^{1/3}$

$x = V^{1/3}$ put in ①

$S = 6\left(V^{1/3}\right)^2$

$S = 6V^{2/3}$

4. Find the domain and the range of the function g defined below:

(i) $g(x) = 5 - x$

Sol

$g(x) = 5 - x$

Domain: $\mathbb{R}$

$(-\infty, \infty)$

Range: $\mathbb{R}$

$g(-2) = 5-(-2) = 7$

$g(0) = 5-0 = 5$

$g(6) = 5-6 = -1$

(ii) $g(x) = \sqrt{x+2}$

$\sqrt{-1}$

Sol

$x+2 \geq 0$

$x \geq -2$

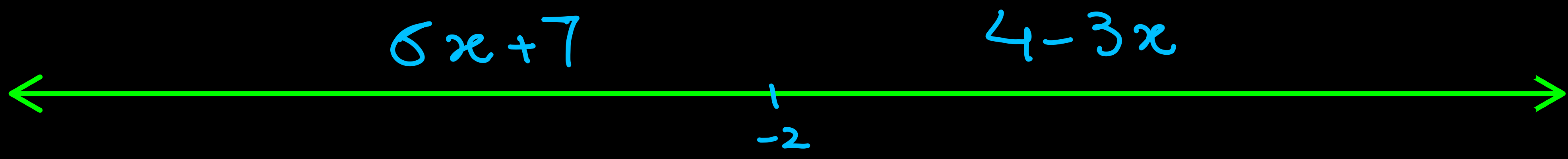
Domain: $[-2, \infty)$

Range: $[0, \infty)$ OR $\mathbb{R}^{+} \cup \{0\}$

(iii) $g(x) = \begin{cases} 6x+7, & x \le -2 \\ 4-3x, & x > -2 \end{cases}$

Sol

Domain : $(-\infty, -2] \cup (-2, \infty) = \mathbb{R}$



x	-6	-5	-4	-3	-2	-1	0	1	2
$f(x)$	-29	-23	-17	-11	-5	7	4	1	-2

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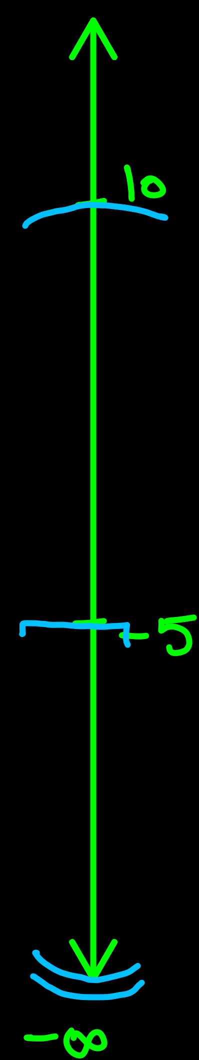
Range $(6x+7) = (-\infty, -5]$

Range $(4-3x) = (-\infty, 10)$



Range: $(-\infty, 10)$

(iv) $g(x) = |x - 5|$

Sol

$$|x-5| = \begin{cases} x-5 & \text{if } x \geqslant 5 \\ -(x-5) & \text{if } x < 5 \end{cases}$$

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

$|2| = 2$

$|-3| = -(-3) = 3$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

$\mathbb{R}^+ \cup \{0\}$

(v) $g(x) = \dfrac{x+2}{3-x}$

Sol

$$g(x) = \frac{x+2}{-x+3}$$

$$\frac{3+2}{3-3} = \frac{5}{0} = \infty$$

For domain:

$$3-x=0$$

$$3=x$$

Domain: $\mathbb{R}-\{3\}$ OR $(-\infty,3)\cup(3,\infty)$

$$f(x)=\frac{ax+b}{cx+d}$$

Range: $\mathbb{R}-\left\{\frac{1}{-1}\right\}$

$$\text{Range} = \mathbb{R}\setminus\left\{\frac{a}{c}\right\}$$

$$\mathbb{R}-\{-1\}$$

Alternatively

$$y=\frac{x+2}{3-x}$$

$$y(3-x)=x+2$$

$$3y-xy=x+2$$

$$3y-2=x+xy$$

$$3y-2=x(1+y)$$

$$\frac{3y-2}{y+1}=x$$

$$D(f)=R(f^{-1})$$

$$R(f)=D(f^{-1})$$

$$f \rightleftharpoons f^{-1}$$

$D(f^{-1})=\mathbb{R}-\{-1\}$ So, $R(f)=\mathbb{R}-\{-1\}$

5. Given $f(x) = x^3 - ax^2 + bx + 1$. If $f(2) = -3$ and $f(-1) = 0$. Find the values of a and b.

Sol

Since $f(2) = -3$

$$2^3 - a(2^2) + b(2) + 1 = -3$$

$$8 - 4a + 2b + 1 = -3$$

$$-4a + 2b + 9 = -3$$

$$-4a + 2b = -12$$

$$\cancel{-2}(2a - b) = \cancel{-2} \times 6$$

$$2a - b = 6 \quad \text{——— } (1)$$

Now,

$$f(-1) = 0$$

$$(-1)^3 - a(-1)^2 + b(-1) + 1 = 0$$

$$\cancel{-1} - a - b \cancel{+1} = 0$$

$$-a - b = 0$$

$$-1(a + b) = 0$$

$$a + b = 0 \quad \text{——— } (2)$$

Adding (1) & (2),

$$\begin{array}{r} 2a - \cancel{b} = 6 \\ a + \cancel{b} = 0 \\ \hline 3a = 6 \end{array}$$

$$a = \frac{\cancel{6}^{2}}{\cancel{3}}$$

$a = 2$ put in 2

$$2 + b = 0$$

$$b = -2$$

So,

$$\boxed{a = 2, \quad b = -2}$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 2: Functions and Graphs
- Concept of Function
- Exercise 2.1: Q6 - Q10

EXERCISE 2.1

6. A stone falls from a height of 60m on the ground, the height h after x seconds is approximately given by $h(x) = 40 - 10x^2$.

(i) What is the height of stone when:

(a) $x = 1$ sec ? (b) $x = 1.5$ sec ? (c) $x = 1.7$ sec ?

Sol

(a) At $x = 1$

$$h(1) = 40 - 10(1)^2$$
$$= 40 - 10$$
$$h(1) = 30\text{ m}$$

(b) At $x = 1.5$

$$h(1.5) = 40 - 10(1.5)^2$$
$$= 40 - 10(2.25)$$
$$= 40 - 22.5$$
$$h(1.5) = 17.5\text{ m}$$

(c) At $x = 1.7$

$$h(1.7) = 40 - 10(1.7)^2$$
$$= 40 - 10(2.89)$$
$$= 40 - 28.9$$
$$h(1.7) = 11.1\text{ m}$$

(ii) When does the stone strike the ground?

sol

When stone strikes the ground, $h=0$

$$0 = 40 - 10x^2$$

$$10x^2 = 40$$

$$x^2 = 4$$

$$\sqrt{x^2} = \pm\sqrt{4}$$

$$x = \pm 2$$

Hence $x = 2$ sec

7. Consider the function $f(x) = 3x - 5$.

(i) Determine the domain and range of $f(x)$.

Sol

$D_f = (-\infty, \infty)$

$R_f = (-\infty, \infty)$

(ii) Is the function f one-to-one? Justify your answer.

Sol

$$f(x) = 3x - 5$$

$$f(x_1) = f(x_2)$$

$$3x_1 - \cancel{5} = 3x_2 - \cancel{5}$$

$$\cancel{3}x_1 = \cancel{3}x_2$$

$$x_1 = x_2$$

Yes, $f(x)$ is one-to-one.

(iii) Is the function f onto if the co-domain is all real numbers? Explain.

Sol

$$y = 3x - 5$$

$$y + 5 = 3x$$

$$\frac{y+5}{3} = x$$

$\boxed{\exists \text{ (there exists)}}$

for each y $\exists$ $x = \frac{y+5}{3}$ such that

$$f(x) = f\left(\frac{y+5}{3}\right) = \cancel{3}\left(\frac{y+5}{\cancel{3}}\right) - 5 = y + 5 - 5 = y$$

Yes, $f(x)$ is onto.

8. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \dfrac{2x-3}{x+1}$

(i) Find the domain and range of $f(x)$.

Sol

For domain,

$$x+1=0 \implies x=-1$$

$$D_f = \mathbb{R} - \{-1\} \quad \text{OR} \quad (-\infty,-1) \cup (-1,\infty)$$

For range,

$$R_f = \mathbb{R} - \{2\}$$

Alternatively,

$$y = \frac{2x-3}{x+1}$$

$$y(x+1) = 2x-3$$

$$yx + y = 2x-3$$

$$y+3 = 2x-yx$$

$$y+3 = x(2-y)$$

$$\frac{y+3}{2-y} = x \quad \text{OR} \quad x = \frac{y+3}{2-y}$$

$$y \neq 2$$

Hence $R_f = (-\infty, 2) \cup (2, \infty)$

(ii) Determine whether $f(x)$ is onto.

$y = \frac{2x-3}{x+1}$, $\quad x = \frac{y+3}{2-y}$

for each y $\nexists$ $x = \frac{y+3}{2-y}$ such that $y = f(x)$

For $y = 2$, $\quad x = \frac{2+3}{2-2} = \frac{5}{0} = \infty$

Not onto.

(iii) Prove that $f(x)$ is one-to-one.

Sol

$$f(x) = \frac{2x-3}{x+1}$$

$$f(a) = f(b)$$

$$\frac{2a-3}{a+1} = \frac{2b-3}{b+1}$$

$$(2a-3)(b+1) = (a+1)(2b-3)$$

$$2\cancel{ab} + 2a - 3b - \cancel{3} = 2\cancel{ab} - 3a + 2b - \cancel{3}$$

$$2a + 3a = 2b + 3b$$

$$\cancel{5}a = \cancel{5}b$$

$$a = b$$

Hence One-to-one.

9. Consider the function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $f(x) = e^{-x}$. Show that $f(x)$ is a bijective.

Sol

For injective,

$$f(x_1) = f(x_2)$$

$$e^{-x_1} = e^{-x_2}$$

Take ln,

$$\ln e^{-x_1} = \ln e^{-x_2}$$

$$-x_1 \cdot \ln e = -x_2 \cdot \ln e$$

$$-x_1 (1) = -x_2 (1)$$

$$x_1 = x_2$$

For surjective,

$$y = e^{-x}$$

Take ln,

$$\ln y = \ln e^{-x}$$

$$= -x \cdot \ln e$$

$$\ln y = -x (1)$$

$$-\ln y = x$$

$$\boxed{f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+}$$

for each y $\exists$ $x = -\ln y$ such that

$$f(x) = f(-\ln y) = e^{-(-\ln y)} = e^{\ln y} = y$$

Hence, $f(x)$ is bijective.

10. Let $g: \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^3 - 3x$. Determine if $g(x)$ is injective and/or surjective.

<u>Sol</u>

For injective,

$$g(a) = g(b)$$

$$a^3 - 3a = b^3 - 3b$$

$$a^3 - b^3 = 3a - 3b$$

$$(a-b)(a^2 + ab + b^2) = 3(a-b)$$

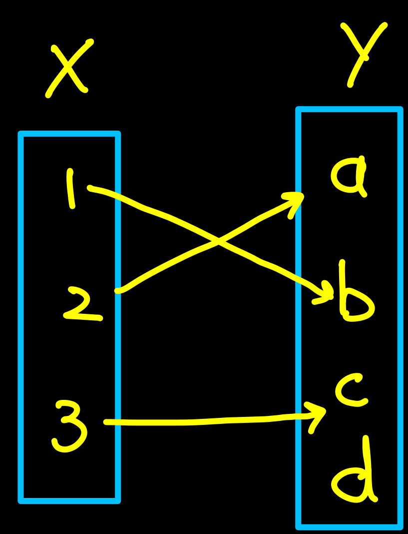
Not One to one.

For surjective,

$$y = x^3 - 3x$$

$f(x)$ is surjective, because

$$\text{Codomain} = \text{Range} = (-\infty, \infty)$$



Codomain = Range

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 2: Functions and Graphs
- Finding the Inersection Point(s) Graphically
- Exercise 2.2: Concepts and Examples

2.2 Finding the Intersecting Point(s) Graphically

The point of intersection is a point where two or more graphs meet on the coordinate plane. This point represents the solution(s) to the equations of the given functions.

2.2.1 Intersection of a Linear Function and Coordinate Axes

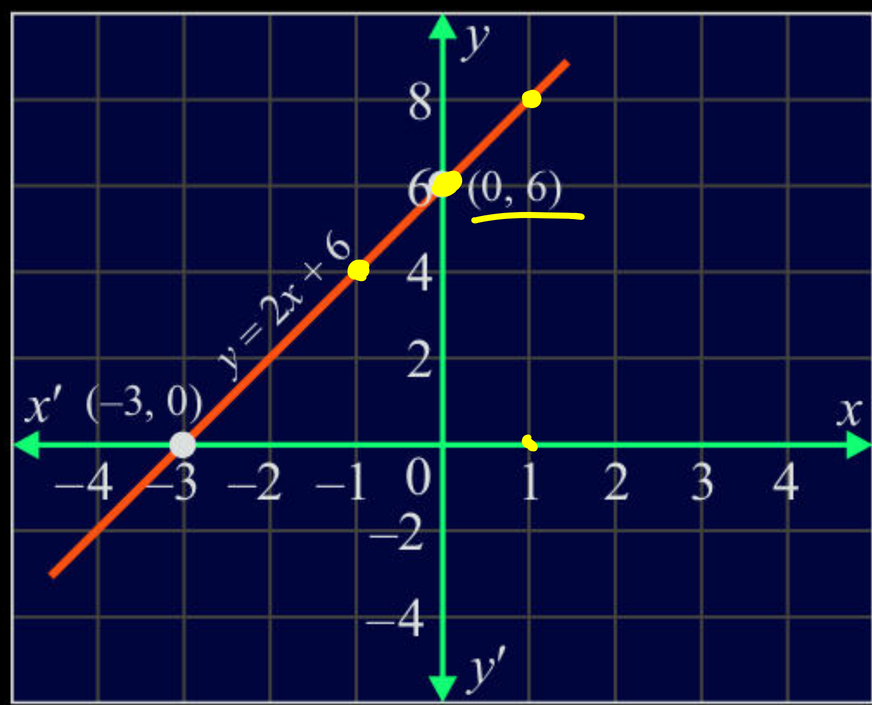
As we know that linear function is a function in which the highest power of the variable is one. While the coordinate axes refers to x-axis and y-axis in the Cartesian coordinate system.

Example 7: Find the points of intersection of a linear function $y = 2x + 6$ and coordinate axes.

Solution: Table values and the graph of $y = 2x + 6$ is given below:

	x	$y = 2x + 6$
(-1,4)	–1	4
(0,6)	0	6
(1,8)	1	8

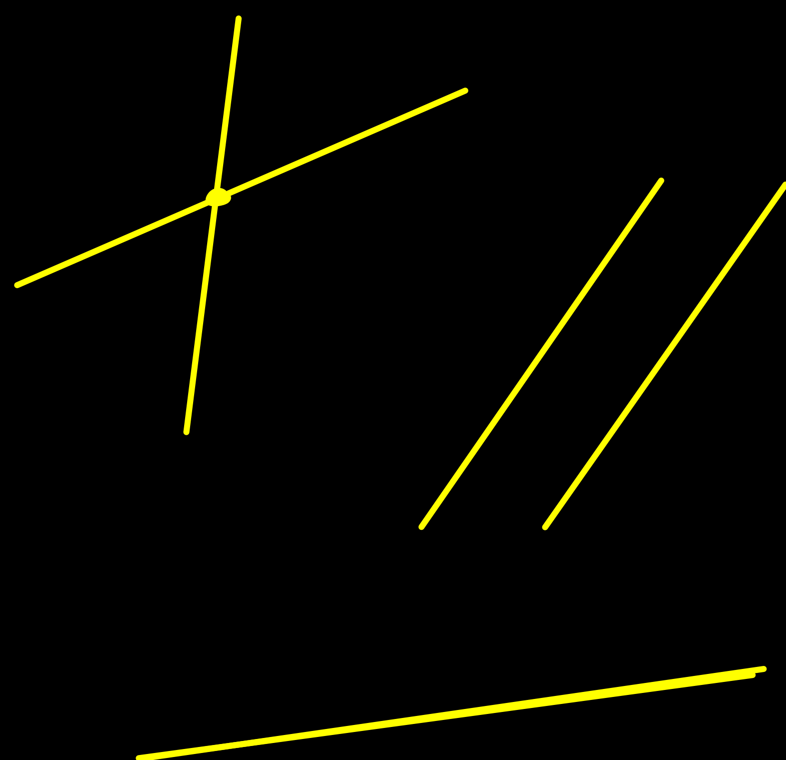
$2(-1) + 6 = 4$



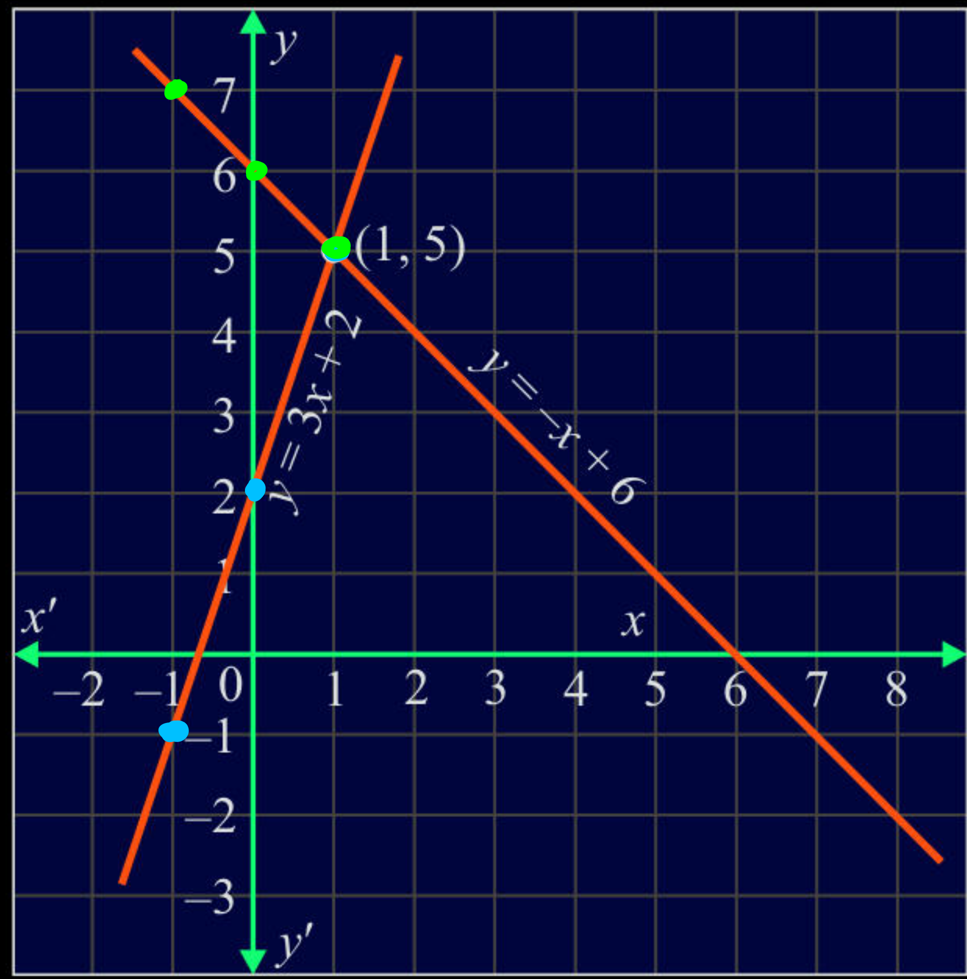
Hence, from the above graph, the points (–3, 0) and $(0, 6)$ are the points of intersections of $y = 2x + 6$ and coordinate axes.

2.2.2 Intersection of Two Linear Functions

The point of intersection of two linear functions is the point where their graphs cross each other. This means the two functions have the same x and y values at that point.



Example 8: Find the point of intersection of $y = 3x + 2$ and $y = -x + 6$.

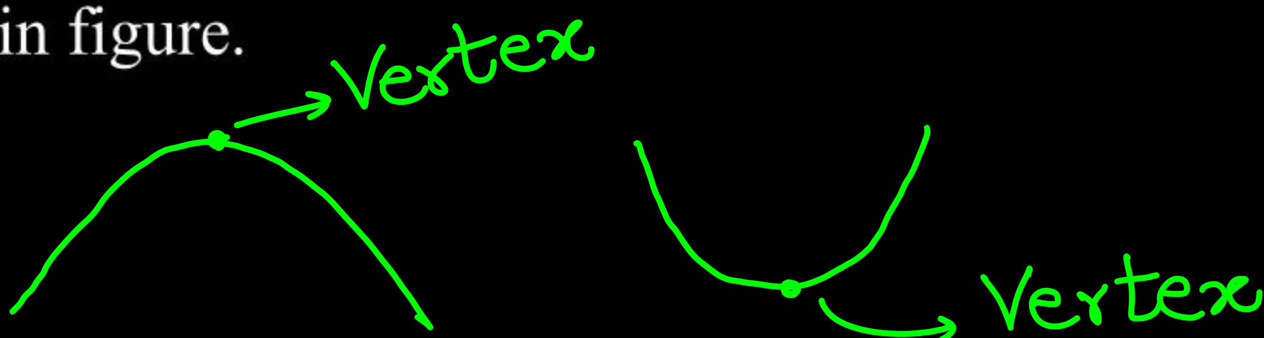


Solution: Table of different values of x and y is given below:

x	$y = 3x + 2$	$y = -x + 6$
-1	-1	7
0	2	6
1	5	5

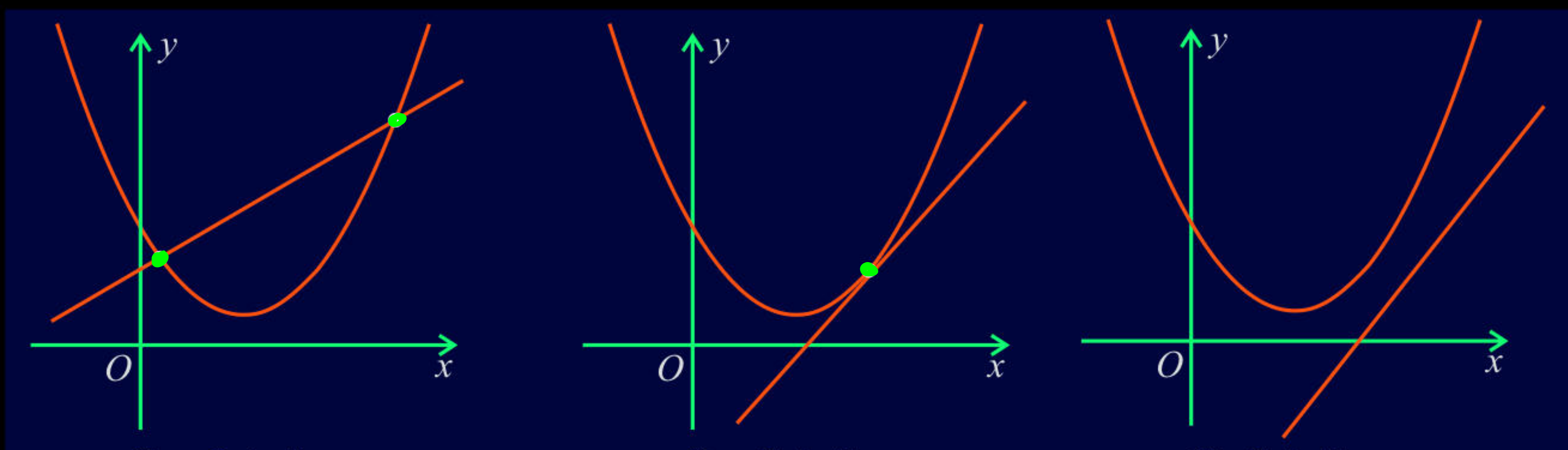
$-(-1)+6$

By plotting the above points, we see that (1, 5) is the point of intersection of both the straight lines as shown in figure.



2.2.3 Intersection of a Linear Function and a Quadratic Function

A line and a parabola can either intersect at two points, one point or not intersect at all. If there are two solutions, the system has two points of intersection. A single solution indicates that there is only one intersection point, suggesting that the line may be tangent to the parabola. If no solution exists, it means the line and the parabola do not intersect.



Two Solutions One Solutions No Solutions

Example 9: Solve the linear function $y=-x+3$ and quadratic function $y=x^2-6x+3$ graphically.

Solution: Clearly (3, 0) and (0, 3) are the x-intercept and y-intercept respectively of $y=-x+3$.

$y=x^2-6x+3$

$a=1, \quad b=-6, \quad c=3$

$$y=x^2-6x+3 \qquad \ldots(\text{i})$$

Put $x=0$ in (i), so (0, 3) is the y-intercept.

Put $y=0$ in (i), we have

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$0=x^2-6x+3$$

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(3)}}{2(1)}$$

$$x=\frac{6\pm\sqrt{36-12}}{2}$$

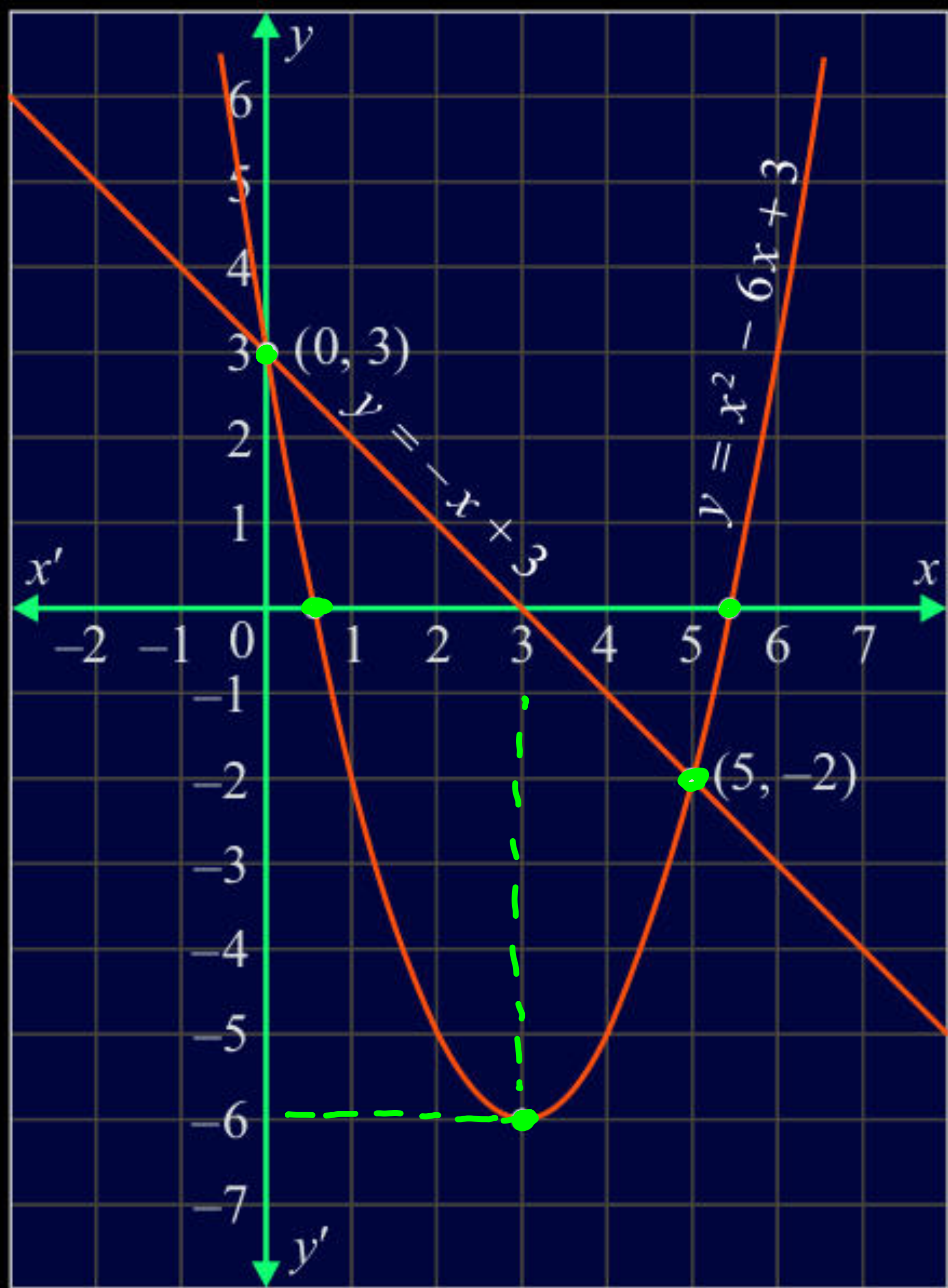
$$x=\frac{6\pm\sqrt{24}}{2}$$

$$x=\frac{6\pm2\sqrt{6}}{2}$$

$$x=3\pm\sqrt{6}$$

$$x=3-\sqrt{6},\ 3+\sqrt{6}$$

$$x=0.6,\ 5.4$$



So (0.6, 0) and (5.4, 0) are the x-intercepts.

Now we find vertex $(h,\ k)$ of the parabola

$$h=-\frac{b}{2a}=-\frac{-6}{2(1)}=3$$

$K=f(h)=f(3)=3^2-6(3)+3=-6$

$$k=(3)^2-6(3)+3=-6$$

$(h,K)=(3,-6)$

So, the vertex is $(3,-6)$

Hence (0, 3) and $(5,-2)$ are the solutions (points of intersection) of the given functions.

2.3 Graph of the Square Root Function

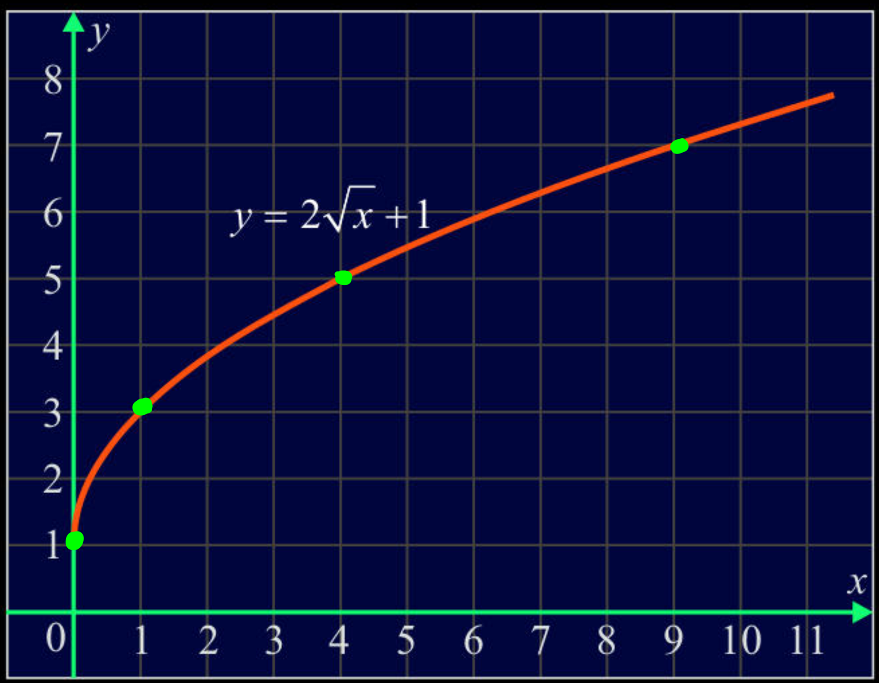
$\sqrt{x} = x^{1/2}$

Example 10: Graph the square root function $y = 2\sqrt{x} + 1$

Solution: Clearly the domain of $y = 2\sqrt{x} + 1$ is $x \geq 0$, as the square root of a negative number is not a real number. The range of $y = 2\sqrt{x} + 1$ is $y \geq 0$, as the square root of a non-negative number is also non-negative.

Table values and the graph of the function are given below:

x	$y = 2\sqrt{x} + 1$
0	1
1	3
2	3.8
3	4.5
4	5
5	5.5
6	5.9
7	6.3
8	6.7
9	7
10	7.3



2.4 Graph of the Cube Root Function

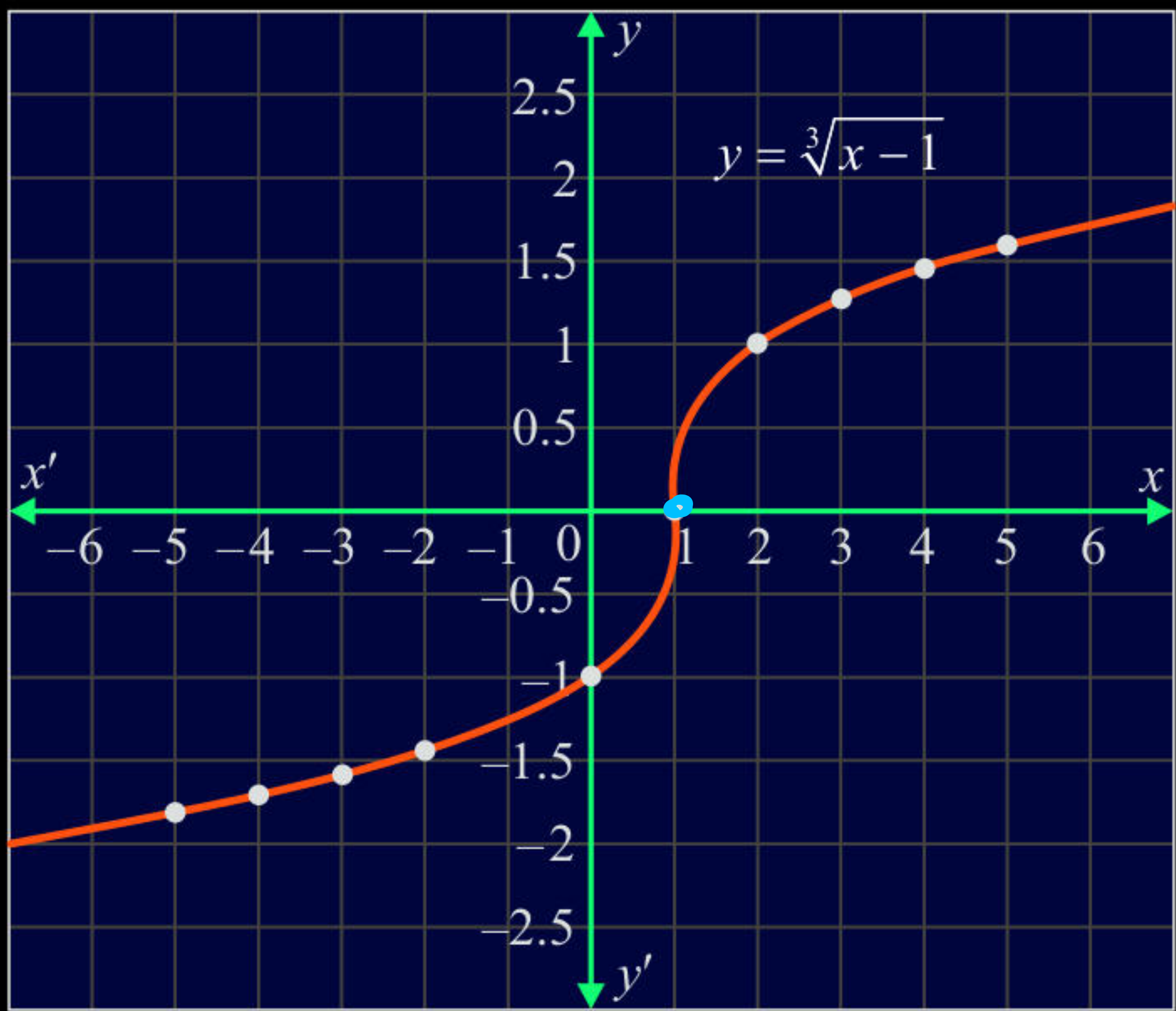
$\sqrt[3]{x} = x^{1/3}$

Example 11: Graph the cube root function $y = \sqrt[3]{x-1}$

Solution: As we know that cube root function is defined for all real numbers because the cube root of any number (positive, negative or zero) is always real. Therefore, the domain of the given cube root function is all real numbers. The range of the given function is also the set of real numbers.

Table values and the graph of the function are given below:

x	$y = \sqrt[3]{x-1}$
–5	–1.8
–4	–1.7
–3	–1.6
–2	–1.4
–1	–1.3
0	–1
1	0
2	1
3	1.3
4	1.4
5	1.6



2.5 Real Life Applications

Growth and Decay in Finance (Predicting Long-Term Stock Prices)

When something increases in quantity or size over time, it is called **growth**. For example, money in a bank account earning interest (it grows larger), a population of rabbits is increasing over months.

When something decreases in quantity or size over time, it is called decay. For example, a radioactive substance is losing its strength over years, a cup of hot coffee is cooling down over time.

Example 12: The value of a stock follows the exponential growth model $P(t) = P_0 e^{rt}$, where P_0 is the initial stock price, r is the growth rate per year and t is the time in years. Suppose a stock is currently valued at Rs. 5,000, and it is expected to grow at a rate of 5% per year.

(i) Find the value of the stock after 10 years.

(ii) After how many years will the stock double in value?

Solution: (i) The formula for the exponential growth is:

$$P(t) = P_0 e^{rt}$$

Given $P_0 = 5{,}000$, $r = 0.05$ (5% growth rate), and $t = 10$ years.

$$\longrightarrow P(10) = 5{,}000\, e^{0.05\times 10} = 5{,}000\, e^{0.5}$$

Using $e^{0.5} \approx 1.6487$

$$P(10) = 5{,}000 \times 1.6487 = 8244$$

So, the value of the stock after 10 years is approximately Rs. 8244.

(ii) We want to find t when the stock doubles, i.e., when $P(t) = 2P_0$. Using the equation:

$$2\cancel{P}_0 = \cancel{P}_0 e^{rt}$$

Dividing both sides by P_0, we have $2 = e^{rt}$ $\longrightarrow$ $\ln 2 = \ln e^{rt}$

Taking the natural logarithm on both sides: $\ln 2 = rt$

and $t = \ln 2 / r = 0.69310.05 = 13.86$

So, the stock will double in value in approximately 13.86 years.

Example 13: The concentration of a pollutant in a lake, in parts per million (ppm), decays over time according to the function

$$C(t) = \frac{100}{\sqrt{t+1}}$$

where t is the time in days since the pollutant was introduced.

(i) What is the concentration of the pollutant after 4 days?

(ii) After how many days will the concentration drop below 10 ppm?

Solution: (i) The pollutant concentration function is $C(t) = \frac{100}{\sqrt{t+1}}$, where t is the time in days.

Concentration after 4 days:

$$C(4) = \frac{100}{\sqrt{4+1}} = \frac{100}{\sqrt{5}} \approx 44.72\,\text{ppm}$$

The concentration after 4 days is about 44.72 ppm.

(ii) When will the concentration drop below 10 ppm? Set $C(t) = 10$:

$$10 = \frac{100}{\sqrt{t+1}} \Rightarrow \sqrt{t+1} = 10 \Rightarrow t+1 = 100 \Rightarrow t = 99$$

After 99 days, the concentration will drop below 10 ppm.

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 2: Functions and Graphs
- Finding the Inersection Point(s) Graphically
- Exercise 2.2: Q1

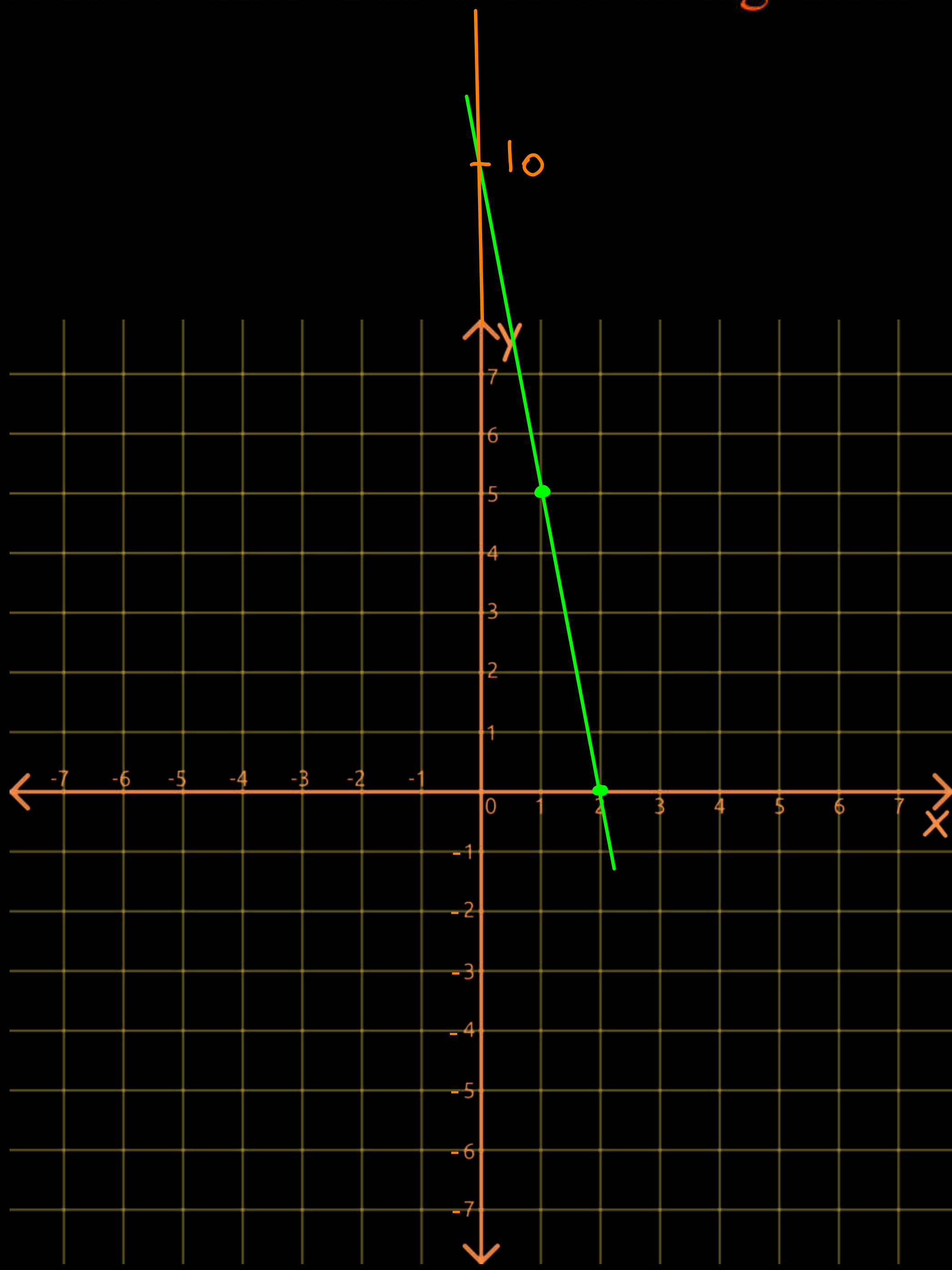
EXERCISE 2.2

1. Find the point of intersection of the coordinate axes and the following linear functions graphically:

(i) $y=-5x+10$

Sol

x	$y=-5x+10$
0	$y=-5(0)+10=10$
1	$y=-5(1)+10=5$
2	$y=-5(2)+10=0$



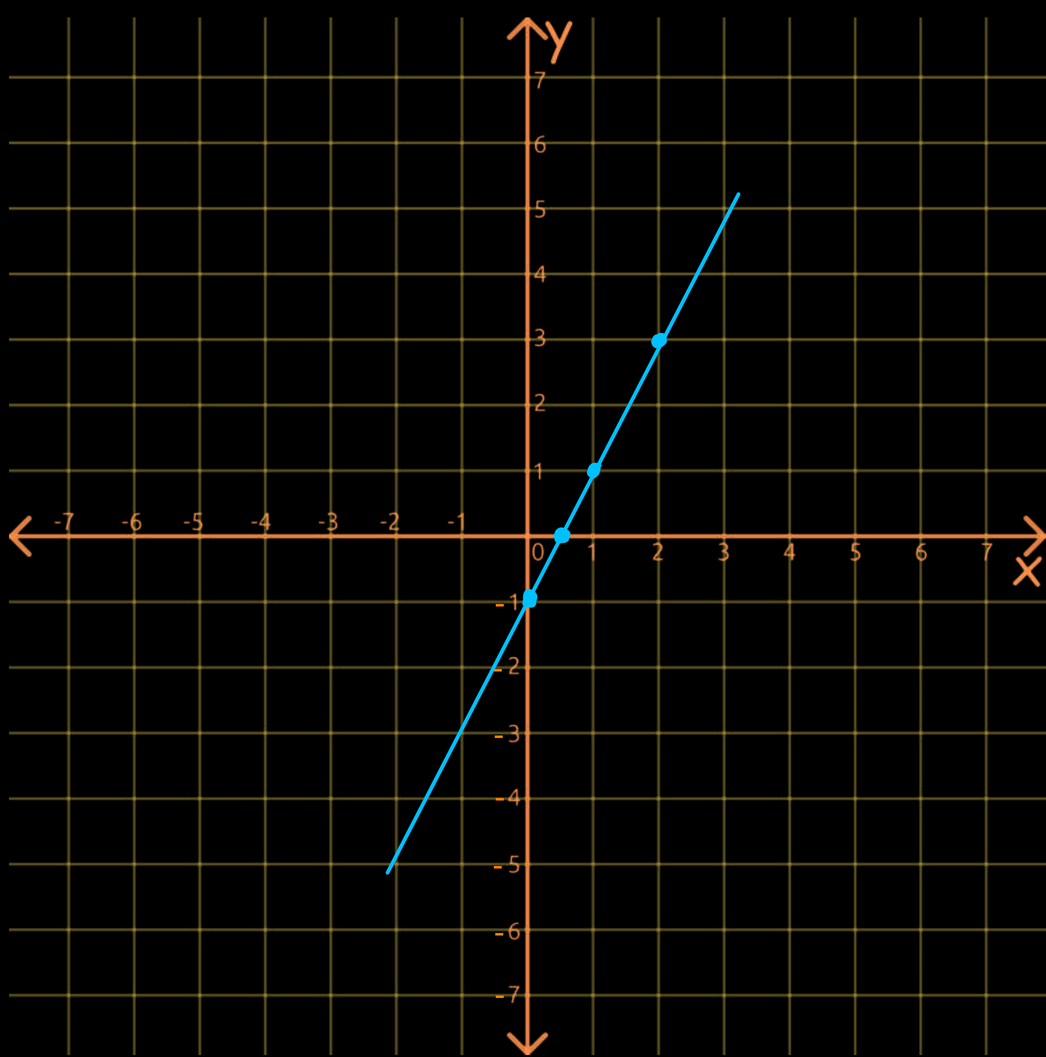
From the graph, the points of intersections are:

$(0,10)$ & $(2,0)$

(ii) $y=2x-1$

Sol

x	$y=2x-1$
0	$y=2(0)-1=-1$
1	$y=2(1)-1=1$
2	$y=2(2)-1=3$

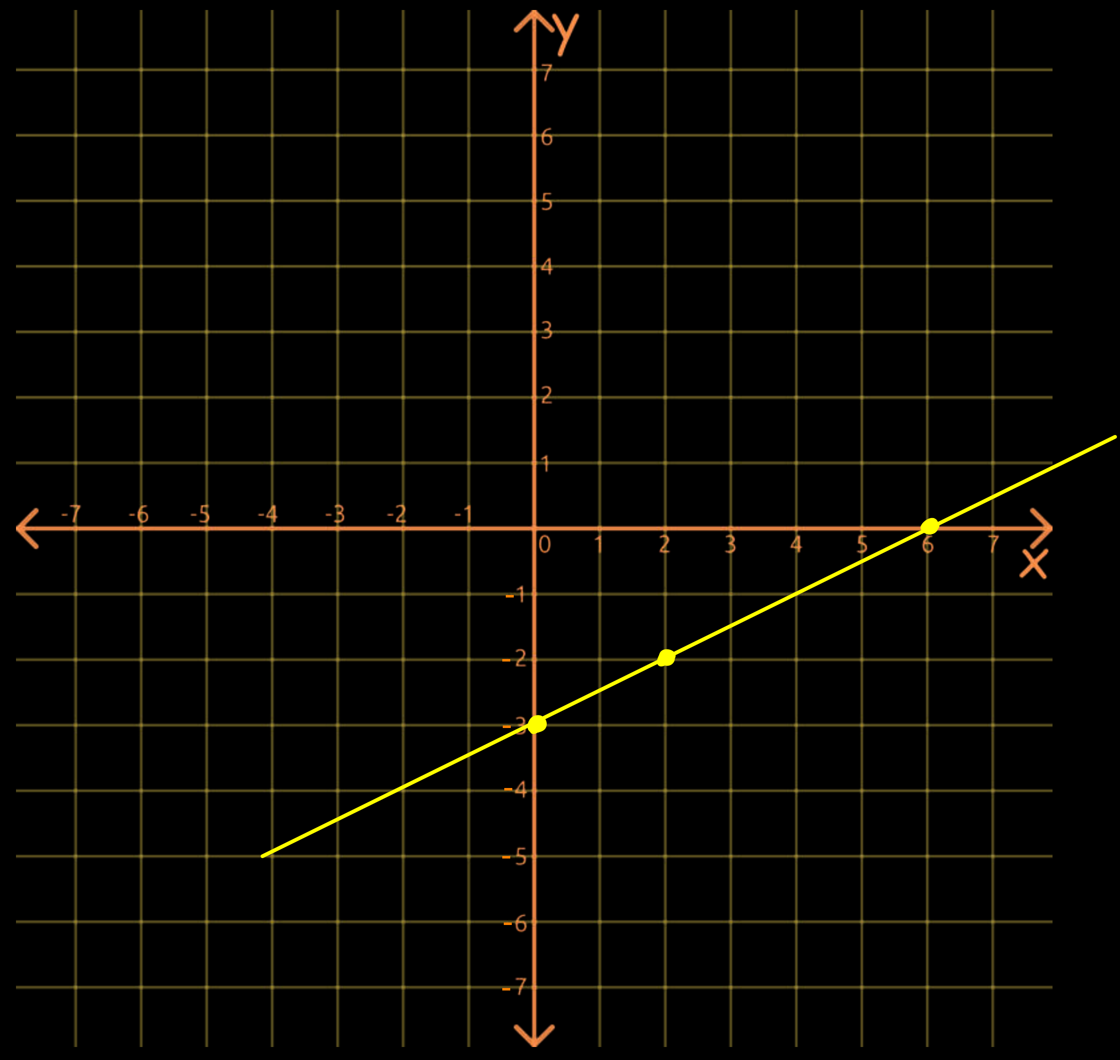


From the graph, the points of intersections are:

$\left(\frac{1}{2},0\right)$ & $(0,-1)$

(iii) $y = \frac{1}{2}x - 3$

x	$y = \frac{1}{2}x - 3$
0	$y = \frac{1}{2}(0) - 3 = -3$
2	$y = \frac{1}{\cancel{2}}(\cancel{2}) - 3 = -2$
6	$y = \frac{1}{\cancel{2}}(\overset{3}{\cancel{6}}) - 3 = 0$



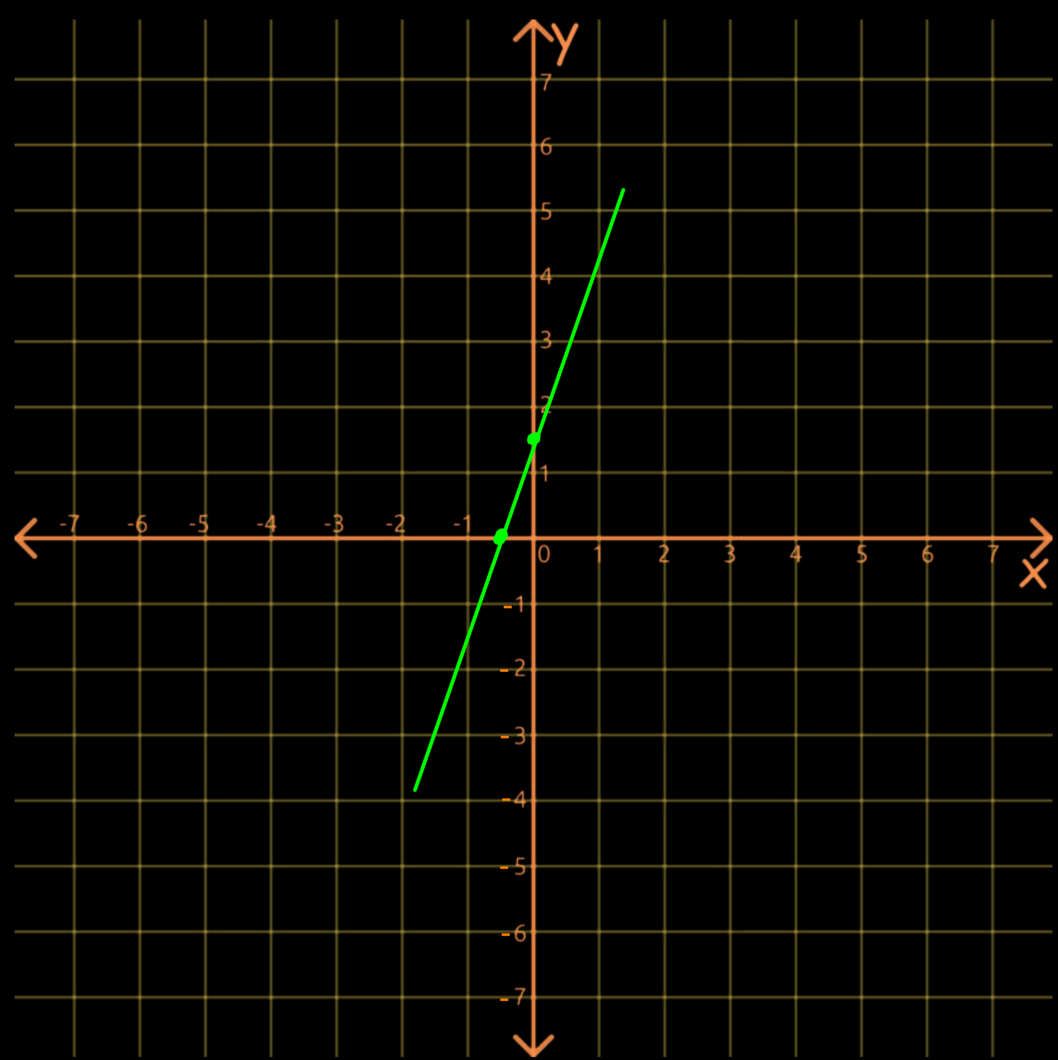
From the graph, the points of intersections are:

$(6, 0)$ & $(0, -3)$

(iv) $y = 3x + \frac{3}{2}$

Sol

x	$y = 3x + \frac{3}{2}$
$-\frac{1}{2}$	$y = 3\left(-\frac{1}{2}\right) + \frac{3}{2} = -\frac{3}{2} + \frac{3}{2} = 0$
0	$y = 3(0) + \frac{3}{2} = \frac{3}{2}$



From the graph, the points of intersections are:

$\left(-\frac{1}{2}, 0\right)$ & $\left(0, \frac{3}{2}\right)$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 2: Functions and Graphs
- Finding the Inersection Point(s) Graphically
- Exercise 2.2: Q2

YouTube Channel: The Mathematics Outlet

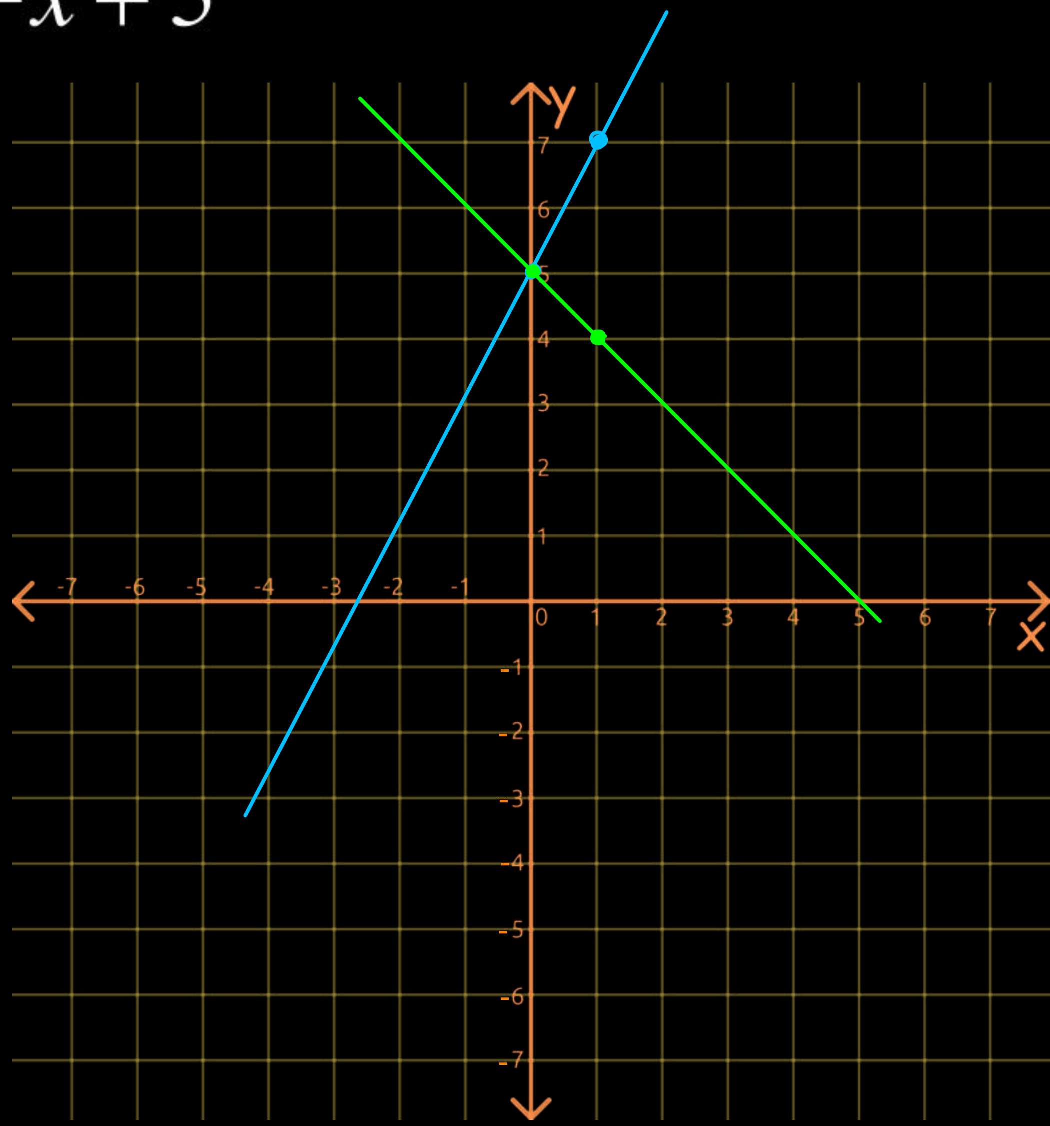
EXERCISE 2.2

2. Find the point(s) of intersection of the following functions graphically:

(i) $f(x) = 2x + 5$, $g(x) = -x + 5$

Sol

x	$f(x) = 2x+5$	$g(x) = -x+5$
0	$f(0) = 2(0)+5 = 5$	$g(0) = -0+5 = 5$
1	$f(1) = 2(1)+5 = 7$	$g(1) = -1+5 = 4$



From the graph, the point of intersection is $(0, 5)$.

Algebraically,

$$f(x) = g(x)$$

$$2x+5 = -x+5$$

$$2x+x = 5-5$$

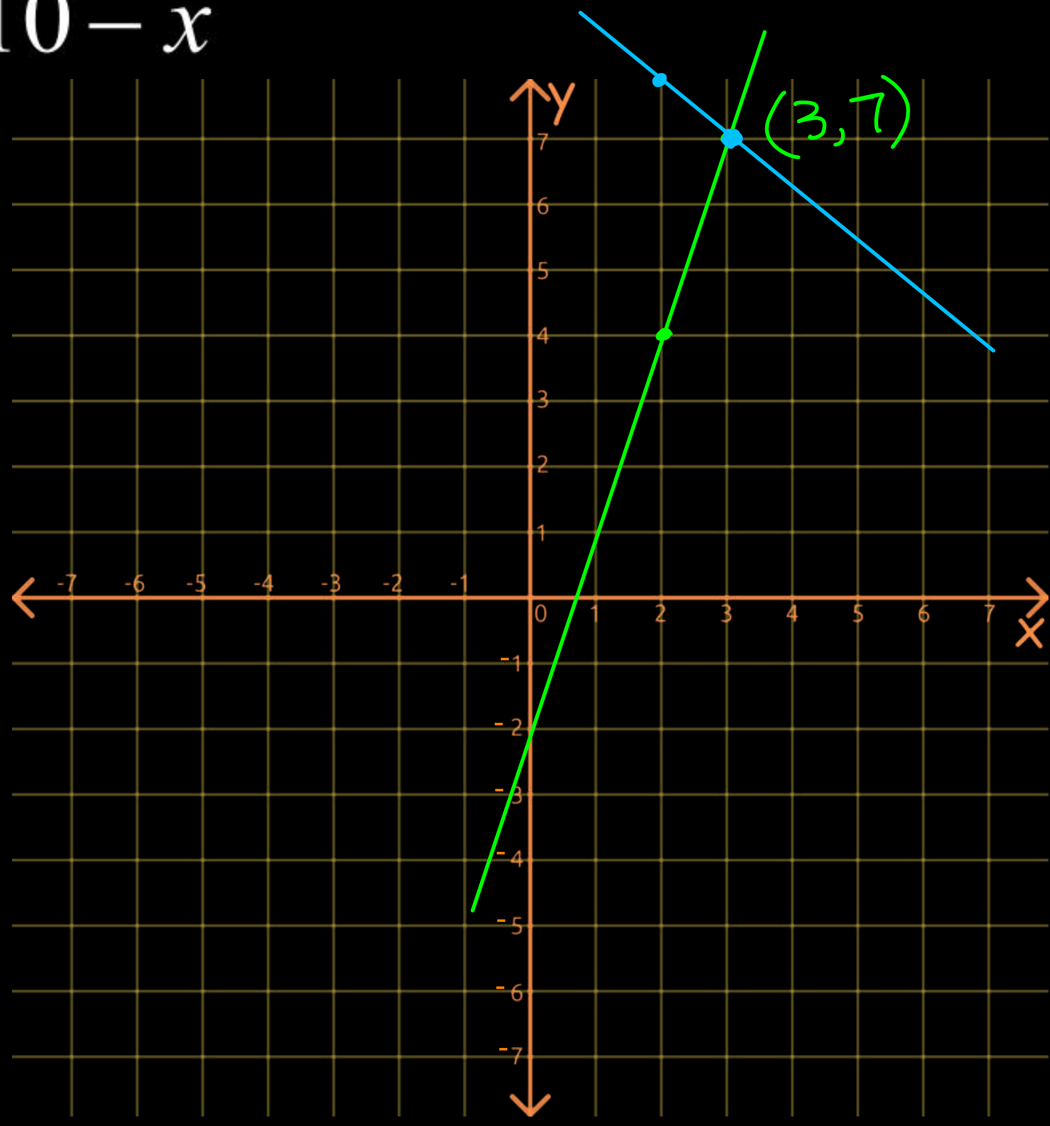
$$3x = 0$$

$x = 0$ and $g(0) = -0+5 = 5$

(ii) $f(x) = 3x - 2$, $g(x) = 10 - x$

Sol

x	$f(x) = 3x - 2$	$g(x) = 10 - x$
2	4	8
3	7	7



Algebraically,

$$f(x) = g(x)$$

$$3x - 2 = 10 - x$$

$$3x + x = 10 + 2$$

$$4x = 12$$

$$x = \frac{12}{4} = 3$$

Substitute $x = 3$ into $f(x)$ or $g(x)$:

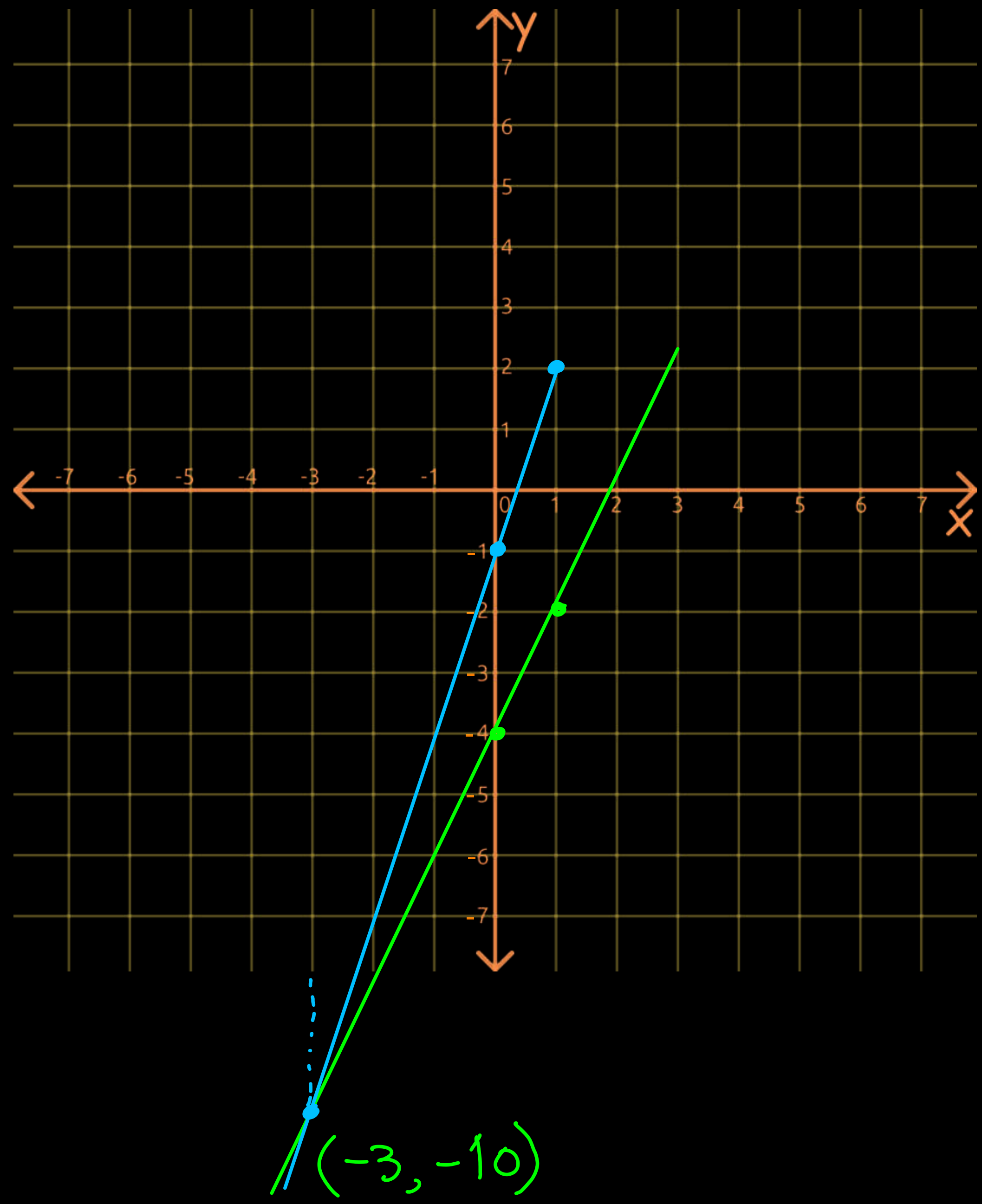
$$g(3) = 10 - 3 = 7$$

Hence, the point of intersection is $(3, 7)$.

(iii) $f(x) = 2x - 4$, $g(x) = 3x - 1$

Sol

x	$f(x)$	$g(x)$
0	-4	-1
1	-2	2
-3	-10	



$f(x) = g(x)$

$2x - 4 = 3x - 1$

$1 - 4 = 3x - 2x$

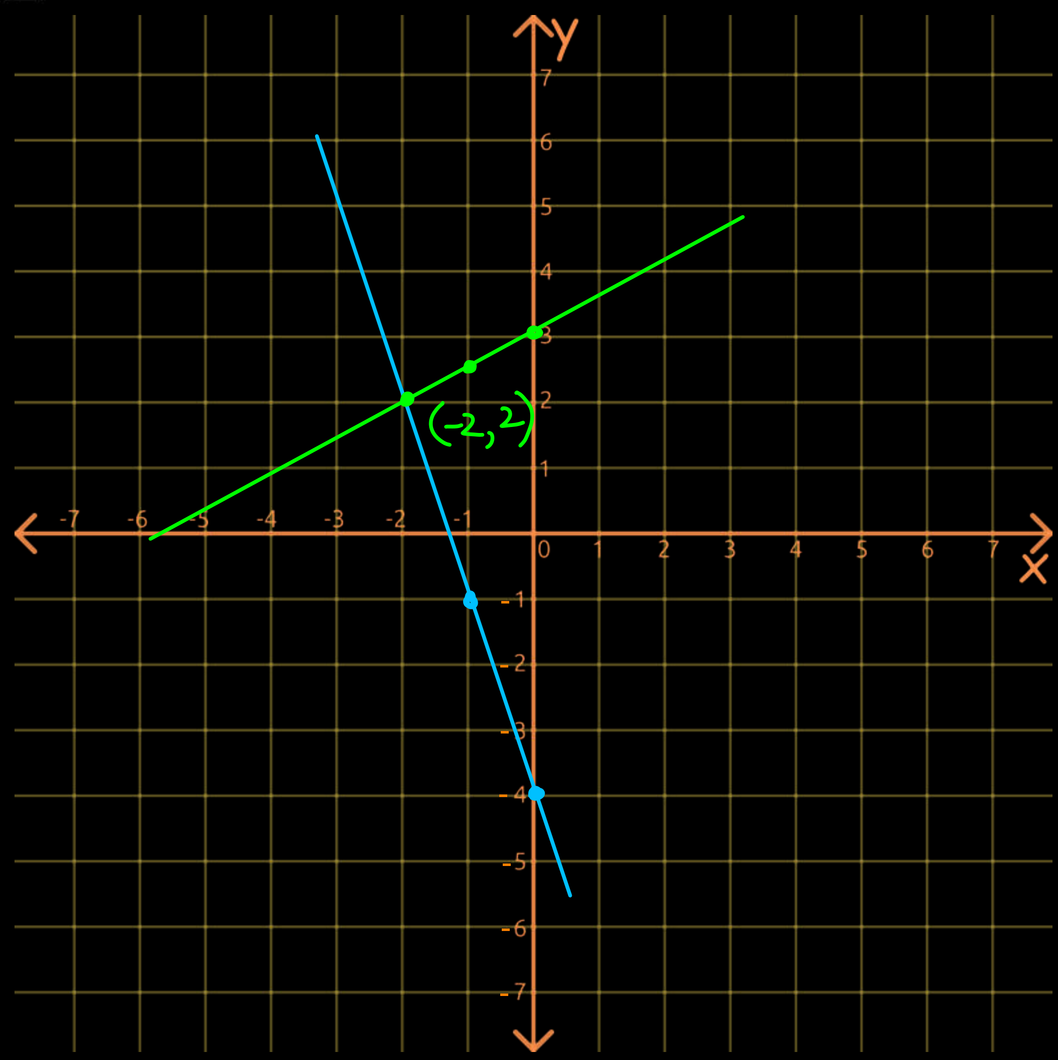
$-3 = x$

Hence, the point of intersection is $(-3, -10)$.

(iv) $f(x) = -3x - 4$, $g(x) = \frac{1}{2}x + 3$

Sol

x	$f(x)$	$g(x)$
-1	$-3(-1)-4=-1$	$\frac{1}{2}(-1)+3=2.5$
0	$-3(0)-4=-4$	$\frac{1}{2}(0)+3=3$

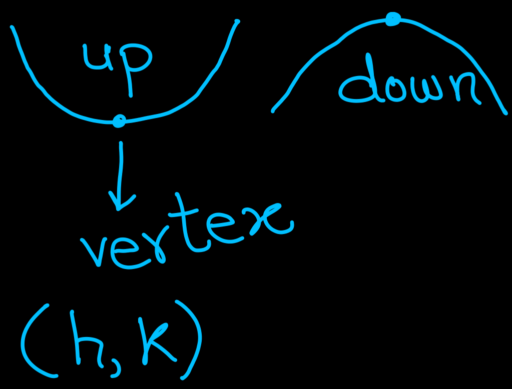


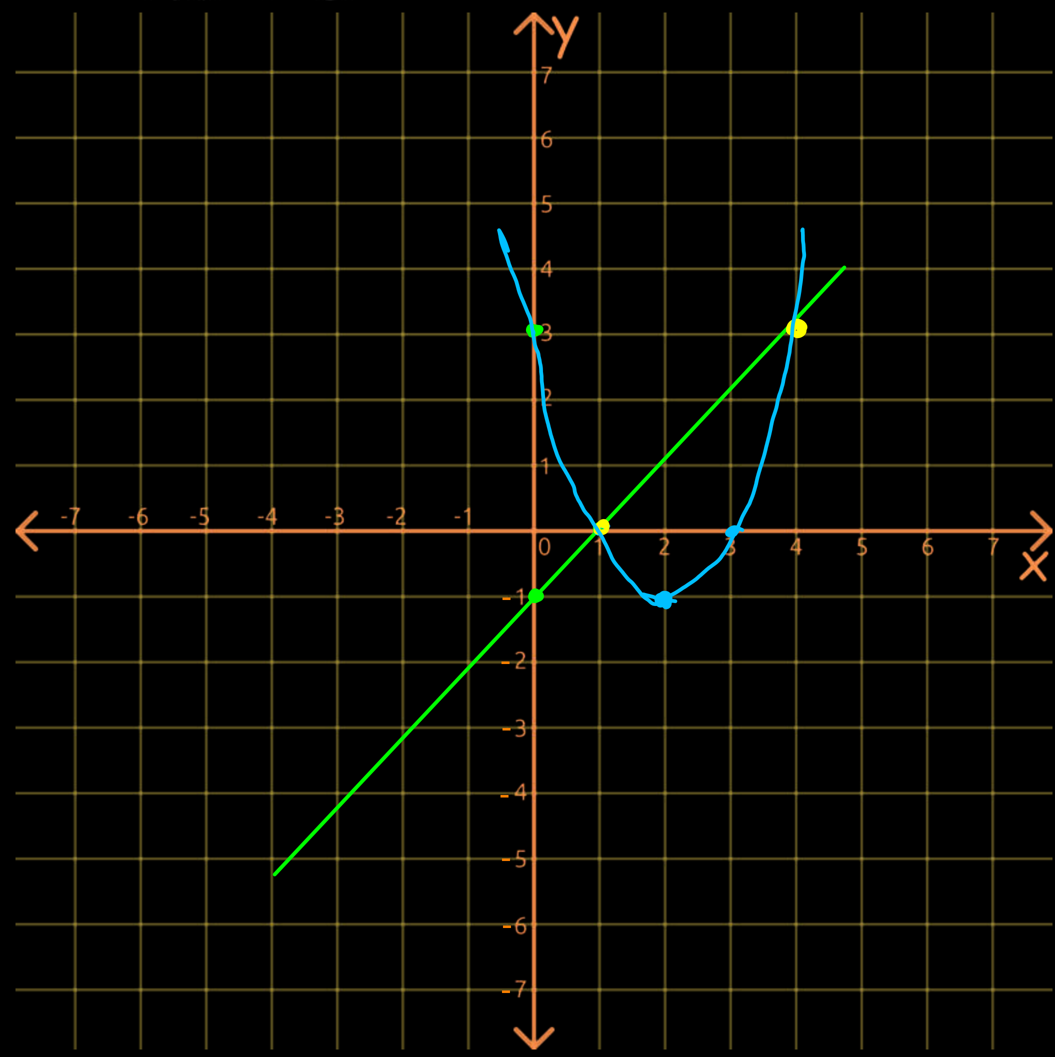
From the graph, the point of intersection is $(-2,2)$.

(v) $f(x) = x - 1, \ g(x) = x^2 - 4x + 3$

Sol

x	$f(x)$
0	-1
1	0





$h = \frac{-b}{2a}$, $K = f(h)$

$h = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$

$K = g(2) = 2^2 - 4(2) + 3 = -1$

To find x-intercepts, $y = 0$

$0 = x^2 - 4x + 3$

$x^2 - 4x + 3 = 0$

$x^2 - x - 3x + 3 = 0$

$x(x-1) - 3(x-1) = 0$

$(x-1)(x-3) = 0$

$x - 1 = 0$ or $x - 3 = 0$

$x = 1$ $\quad$ $x = 3$

To find y-intercepts, $x = 0$

$g(0) = 0^2 - 4(0) + 3 = 3$

Algebraically,

$f(x) = g(x)$

$x - 1 = x^2 - 4x + 3$

$0 = x^2 - 4x + 3 - x + 1$

$x^2 - 5x + 4 = 0$

$x^2 - x - 4x + 4 = 0$

$x(x-1) - 4(x-1) = 0$

$(x-1).(x-4) = 0$

$x - 1 = 0$ or $x - 4 = 0$

$x = 1$ $\quad$ $x = 4$

Hence

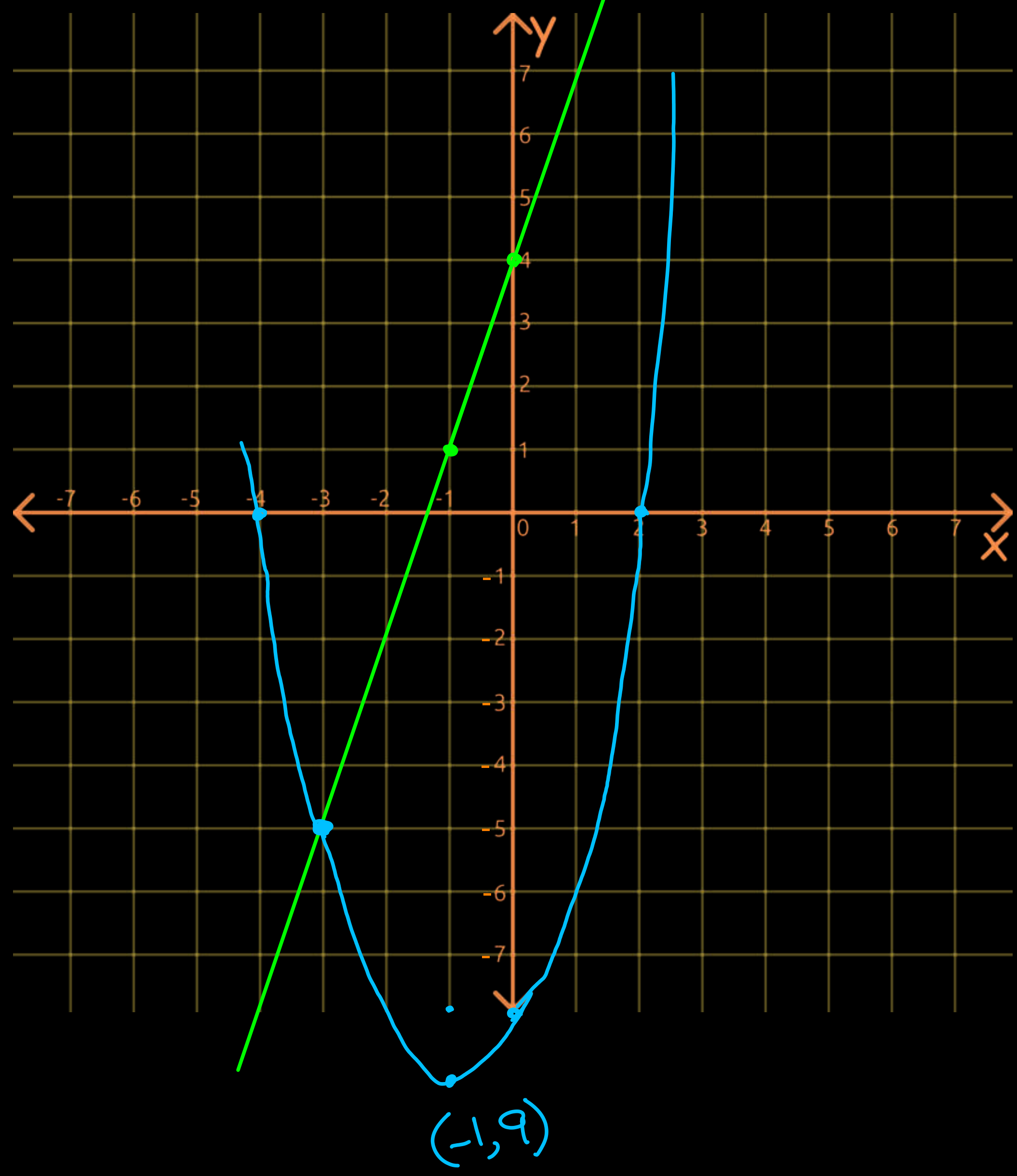
Points of intersections:

$(1, 0)$ & $(4, 3)$

(vi) $f(x)=3x+4$, $g(x)=x^2+2x-8$

Sol

x	$f(x)$
-1	1
0	4
4	16



$g(x)=x^2+2x-8$

$h=\frac{-b}{2a}=\frac{-2}{2(1)}=-1$

$k=g(-1)=(-1)^2+2(-1)-8=-9$

To find x-intercepts, $y=0$

$x^2+2x-8=0$

$x=-4, 2$

To find y-intercepts, $x=0$

$g(0)=0^2+2(0)-8=-8$

To find points of intersections algebraically,

$f(x)=g(x)$

$3x+4=x^2+2x-8$

$0=x^2+2x-8-3x-4$

$x^2-x-12=0$

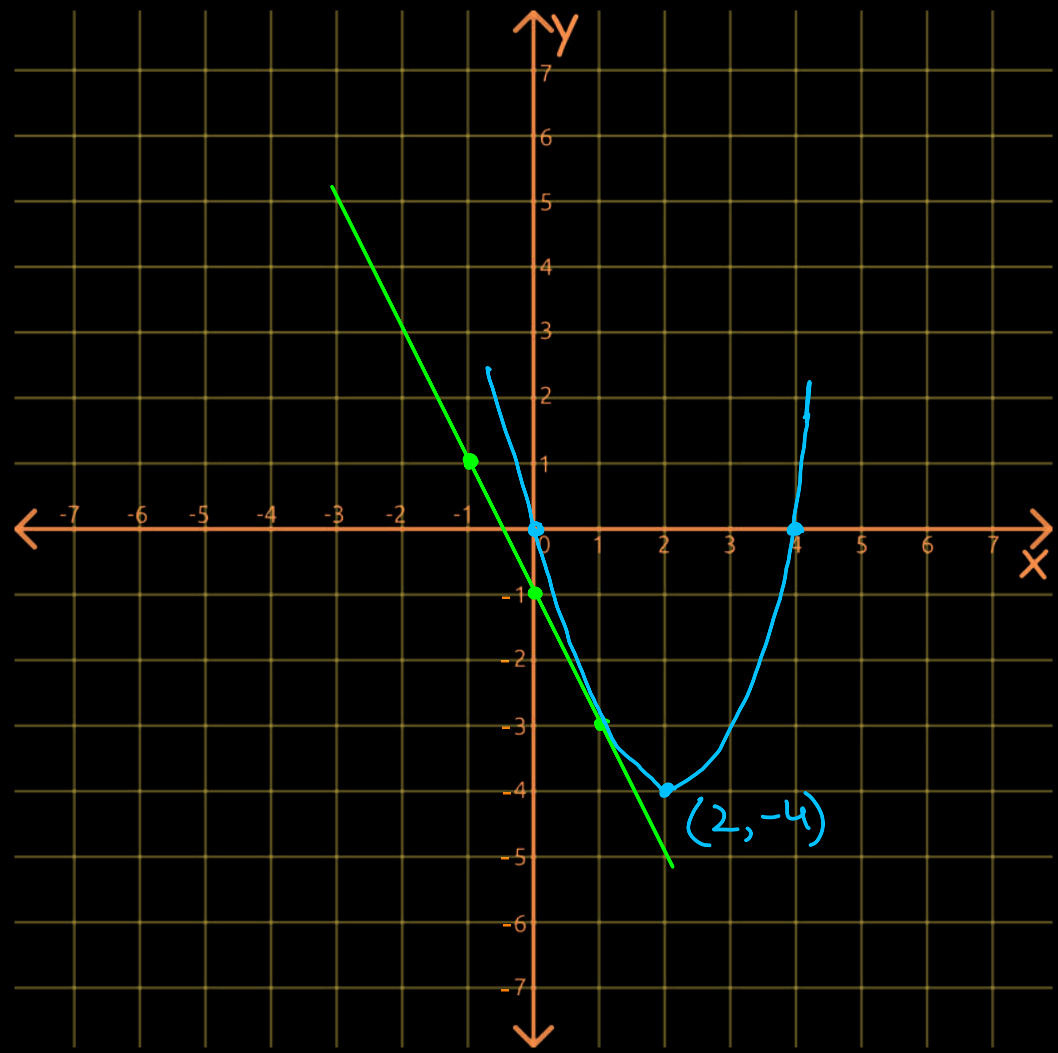
$x=-3$ or $x=4$

Hence, the points of intersections are $(-3,-5)$ & $(4,16)$.

(vii) $f(x) = -2x - 1$, $g(x) = x^2 - 4x$

Sol

x	$f(x)$
-1	1
0	-1
1	-3



For vertex:

$$g(x) = x^2 - 4x$$

$$h = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$$

$$k = g(2) = 2^2 - 4(2) = -4$$

For x-intercepts, put $y = 0$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0, \quad x = 4$$

Alternatively,

$$f(x) = g(x)$$

$$-2x - 1 = x^2 - 4x$$

$$0 = x^2 - 4x + 2x + 1$$

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

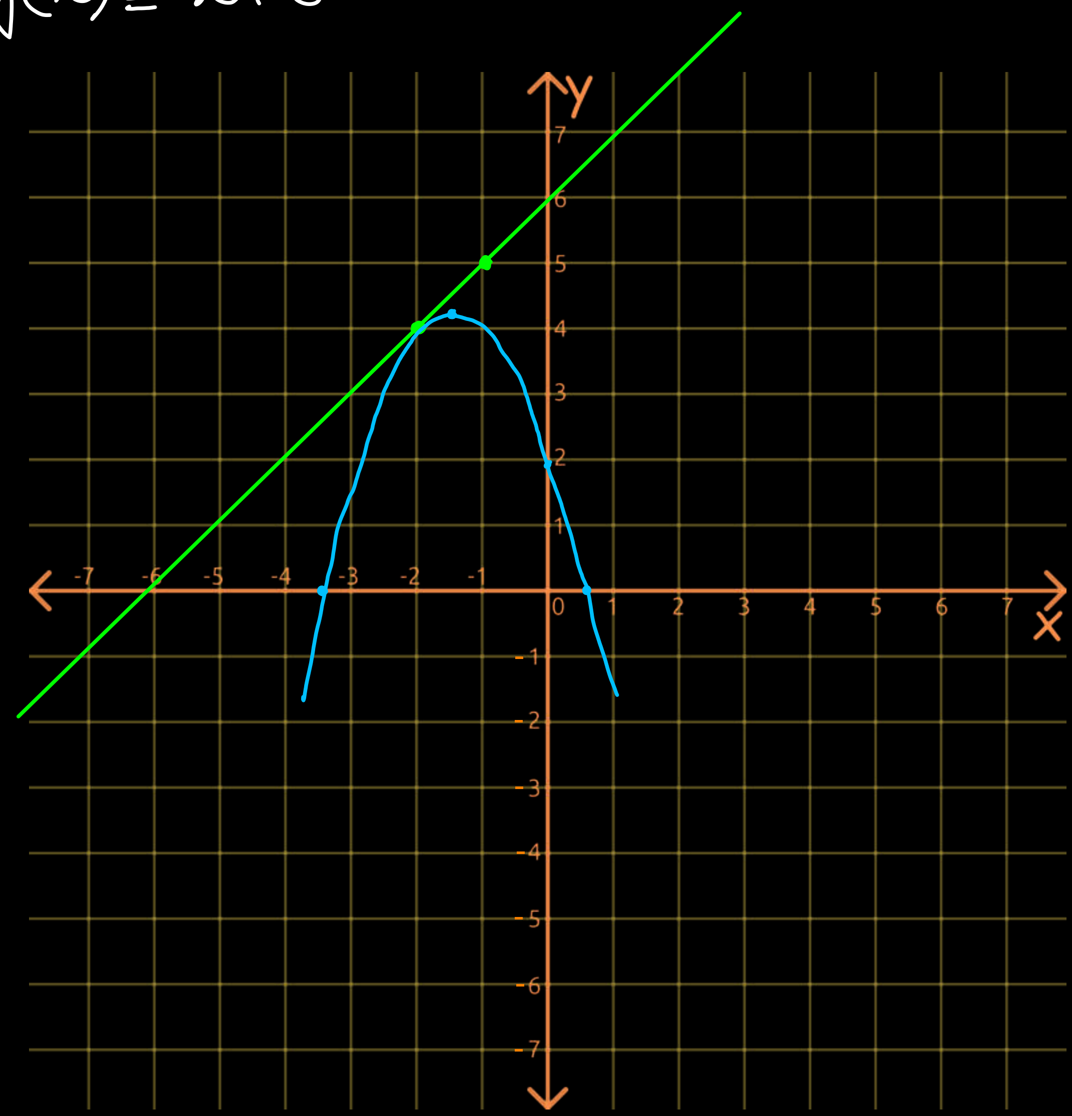
$$x - 1 = 0$$

$$x = 1 \quad \text{and} \quad g(1) = 1^2 - 4(1) = -3$$

Hence, the point of intersection is $(1, -3)$.

(viii) $f(x) = -x^2 - 3x + 2$, $g(x) = x + 6$

x	$g(x)$
-2	4
-1	5



For vertex:

$$f(x) = -x^2 - 3x + 2$$

$$h = \frac{-(-3)}{2(-1)} = \frac{-3}{2}$$

$$K = f\left(\frac{-3}{2}\right) = -\left(\frac{-3}{2}\right)^2 - 3\left(\frac{-3}{2}\right) + 2 = \frac{17}{4}$$

For x-intercepts, put $y = 0$

$$-x^2 - 3x + 2 = 0 \implies x^2 + 3x - 2 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{17}}{2}$$

$$x = \frac{-3 - \sqrt{17}}{2} = -3.56$$

$$x = \frac{-3 + \sqrt{17}}{2} = 0.56$$

Alternatively,

$$f(x) = g(x)$$

$$-x^2 - 3x + 2 = x + 6$$

$$0 = x^2 + 3x - 2 + x + 6$$

$$x^2 + 4x + 4 = 0$$

$$(x+2)^2 = 0$$

$$x + 2 = 0$$

$x = -2$ and so $f(-2) = -(-2)^2 - 3(-2) + 2 = 4$

Hence, the point of intersection is $(-2, 4)$.

بِسْمِ اللهِ الرَّحْمٰنِ الرَّحِيْمِ

(In the Name of Allah, the Most Compassionate, the Most Merciful)

Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 2: Functions and Graphs
- Finding the Inersection Point(s) Graphically
- Exercise 2.2: Q3 - Q5

EXERCISE 2.2

3. Graph the following functions:

(i) $y = \sqrt{3x}$

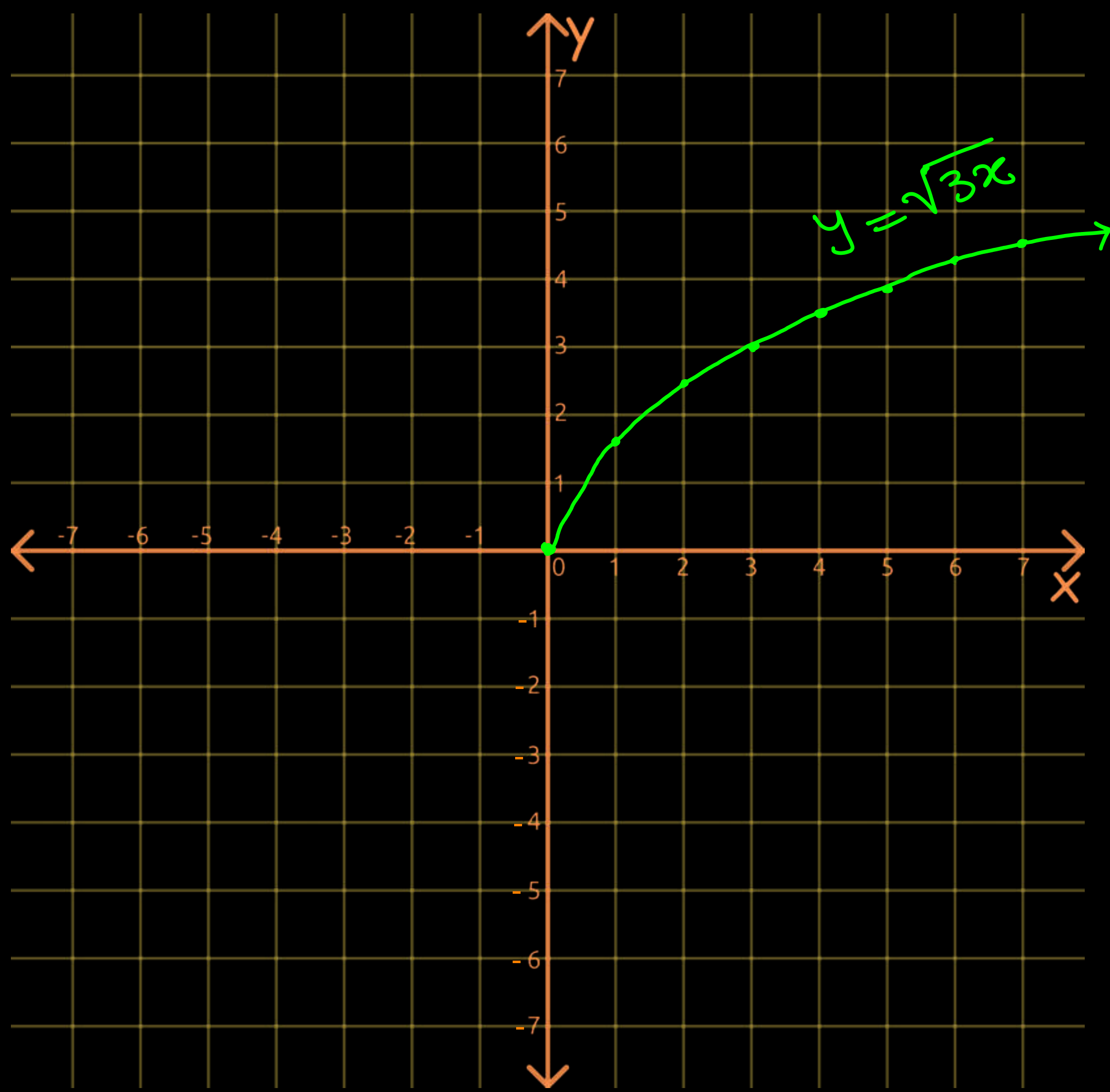
Sol

Given $y = \sqrt{3x}$

Domain: $3x \geq 0$

$\implies x \geq 0$

x	$y = \sqrt{3x}$
0	$y = \sqrt{3(0)} = 0$
1	$y = \sqrt{3(1)} = \sqrt{3} = 1.73$
2	$\sqrt{3(2)} = 2.45$
3	$\sqrt{3(3)} = \sqrt{9} = 3$
4	3.46
5	3.87



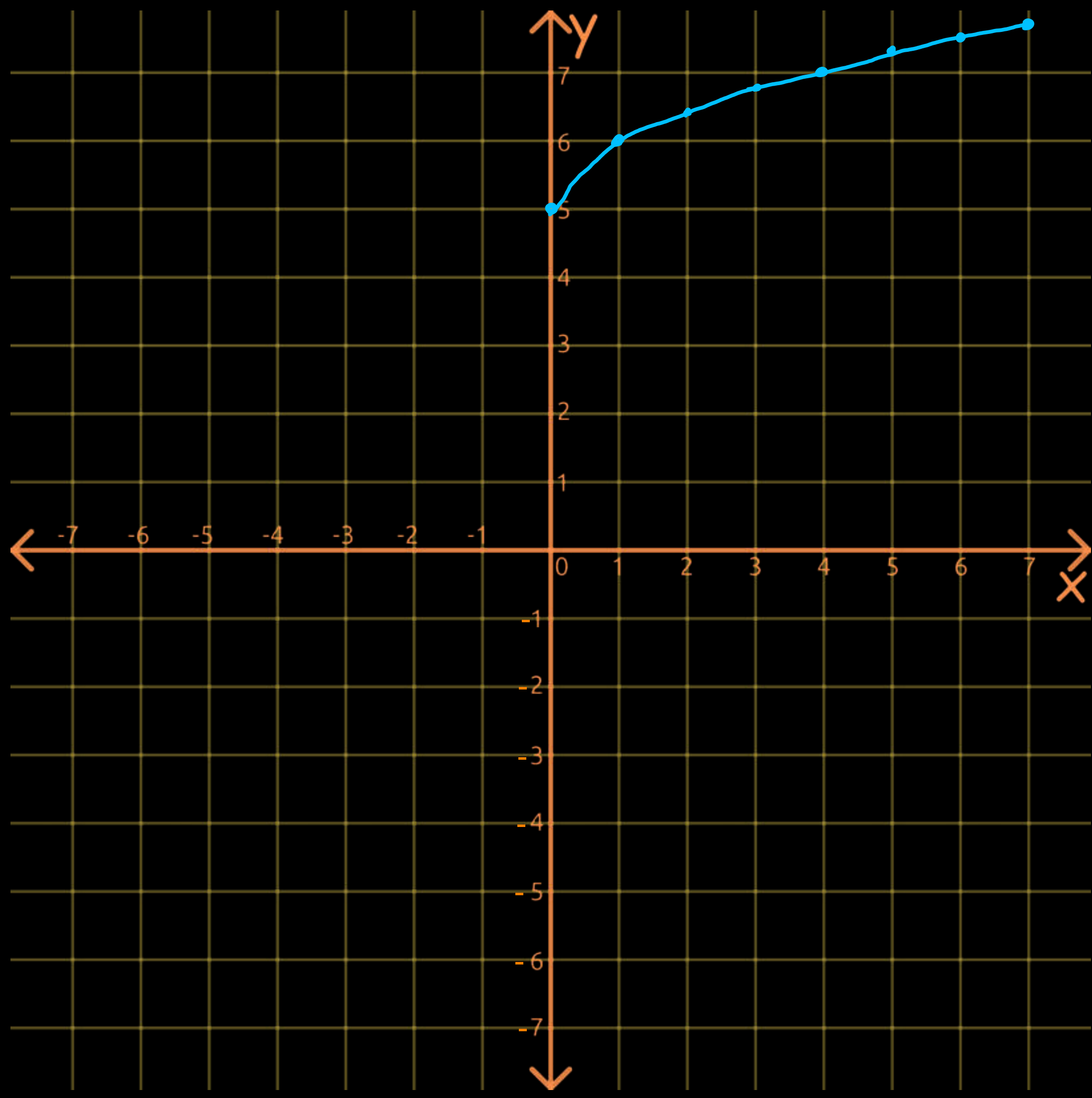
(ii) $y = \sqrt{x} + 5$

Sol

Given $y = \sqrt{x} + 5$

Domain: $x \geq 0$

x	$y = \sqrt{x} + 5$
0	$y = \sqrt{0} + 5 = 5$
1	$\sqrt{1} + 5 = 6$
2	6.41
3	6.73
4	7
5	7.24
6	7.45



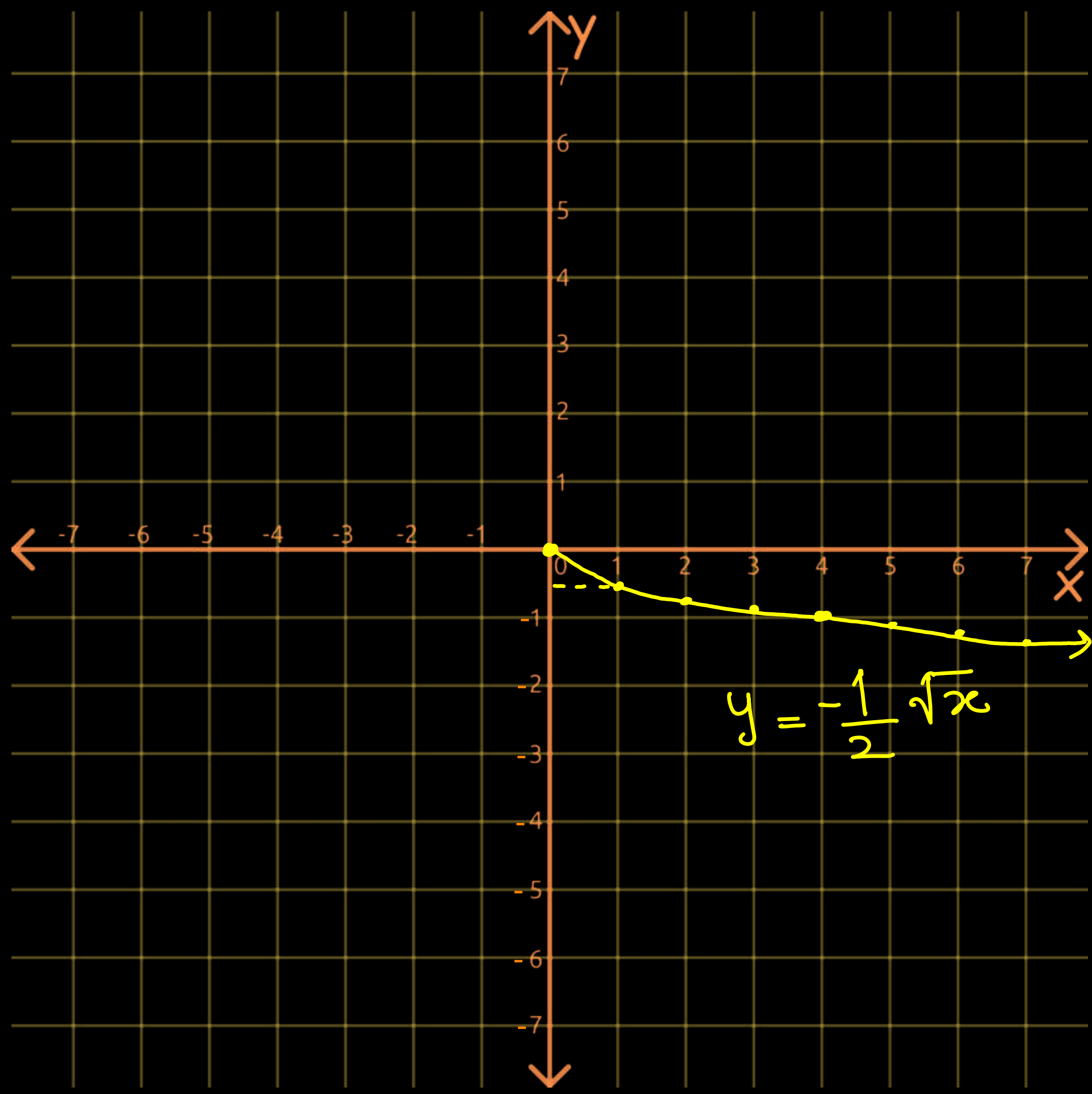
(iii) $y = -\frac{1}{2}\sqrt{x}$

Sol

Given $y = -\frac{1}{2}\sqrt{x}$

Domain: $x \geq 0$

x	$y = -\frac{1}{2}\sqrt{x}$
0	0
1	$-1/2$
2	-0.71
3	-0.87
4	-1
5	-1.12
6	-1.22



(iv) $y = -\sqrt{x+1} + 2$

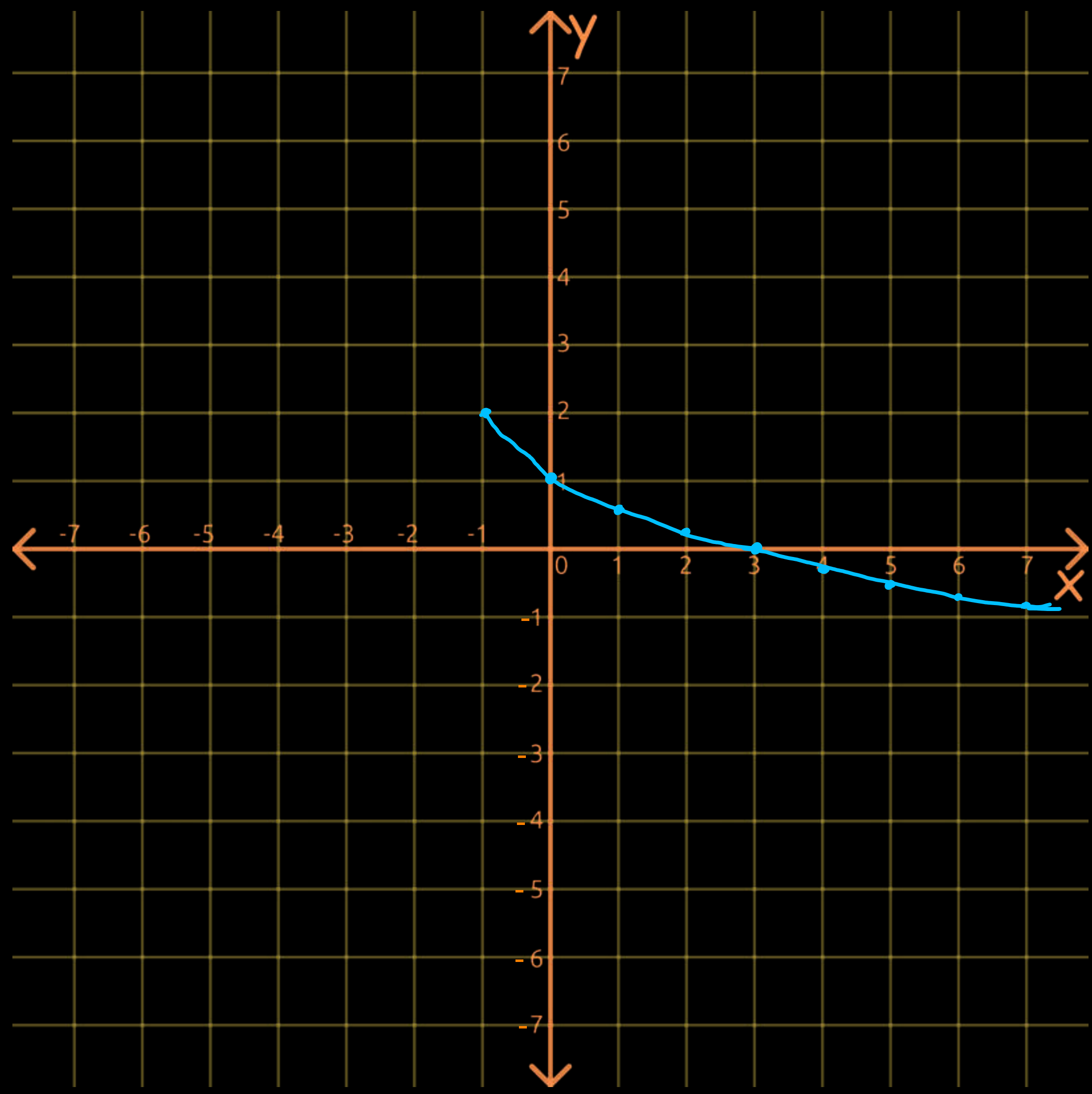
Sol

Given $y = -\sqrt{x+1} + 2$

Domain: $x + 1 \geq 0$

$x \geq -1$

x	$y = -\sqrt{x+1} + 2$
-1	$-\sqrt{-1+1} + 2 = 2$
0	$-\sqrt{0+1} + 2 = 1$
1	$-\sqrt{1+1} + 2 = 0.58$
2	0.27
3	0
4	-0.24
5	-0.45



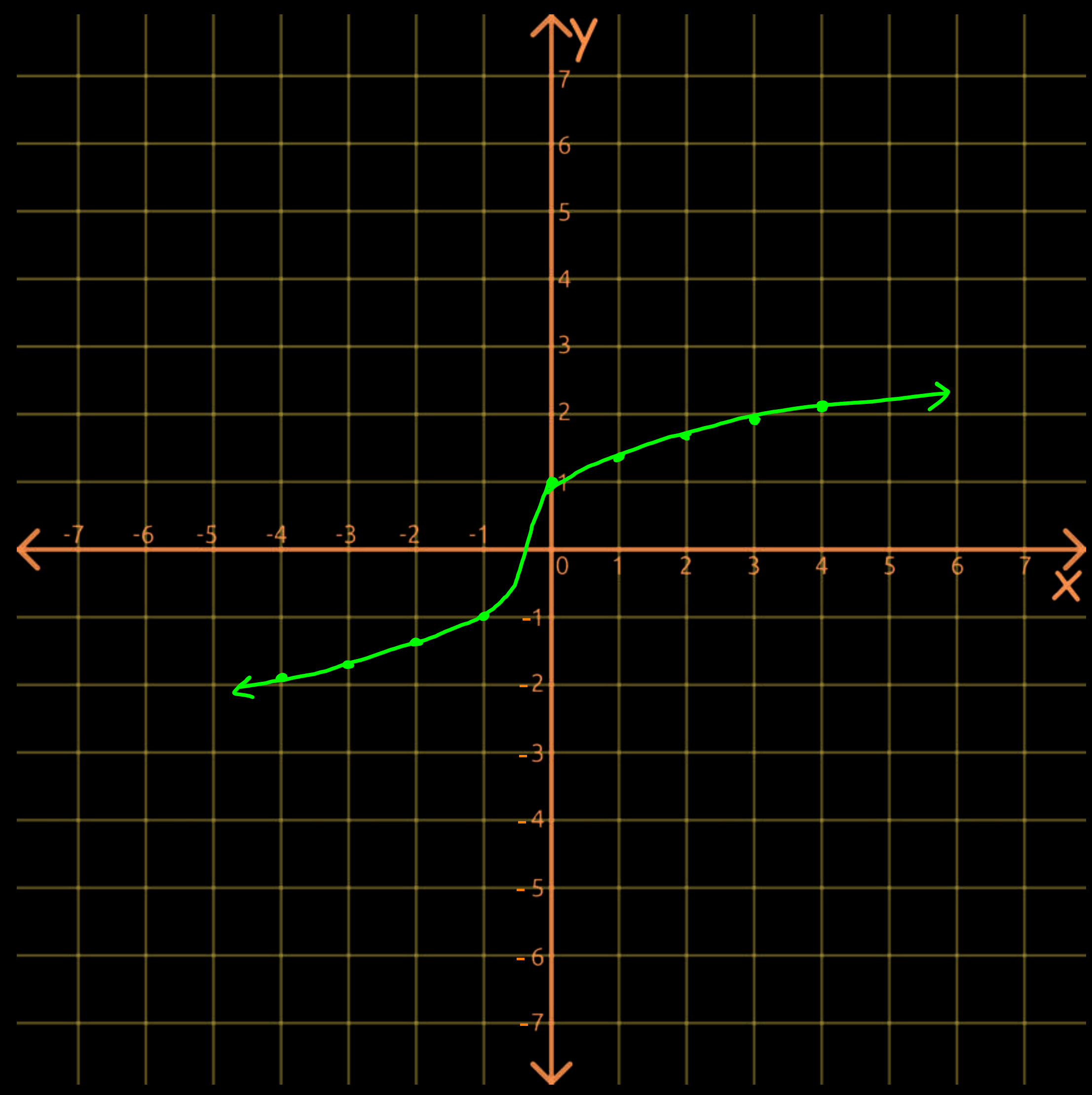
(v) $y=\sqrt[3]{2x+1}$

Sol

Given $y=\sqrt[3]{2x+1}$

Domain: All real numbers

x	$y=\sqrt[3]{2x+1}$
-4	$\sqrt[3]{2(-4)+1} = (-7)^{1/3} = -1.91$
-3	-1.71
-2	-1.44
-1	-1
0	1
1	1.44
2	1.71
3	1.91
4	2.08



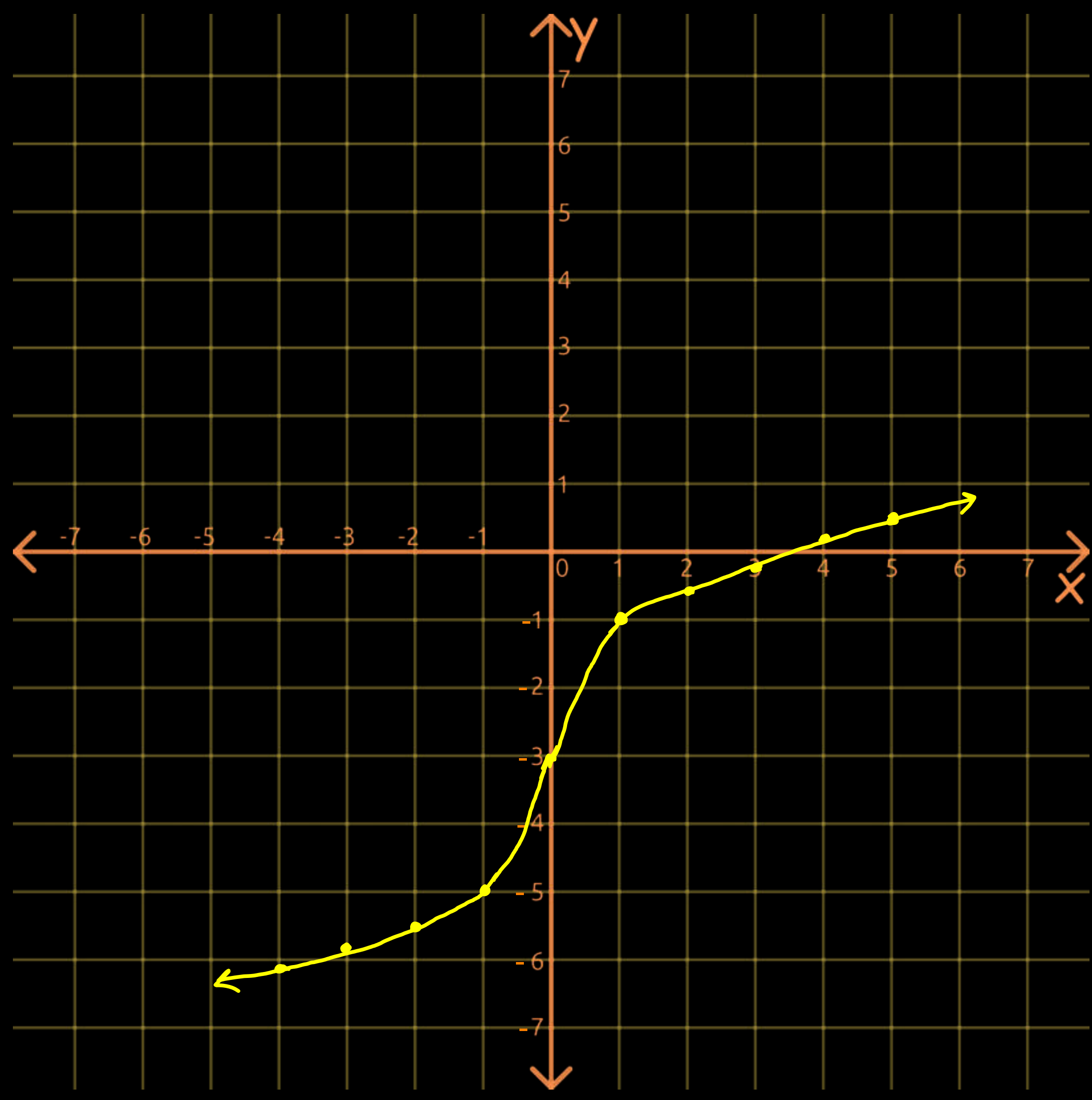
(vi) $y = 2\sqrt[3]{x} - 3$

Sol

Given $y = 2\sqrt[3]{x} - 3$

Domain: All real numbers

x	$y = 2\sqrt[3]{x} - 3$
-4	$2\sqrt[3]{-4} - 3 = -6.18$
-3	-5.88
-2	-5.52
-1	-5
0	-3
1	-1
2	-0.48
3	-0.12
4	0.18



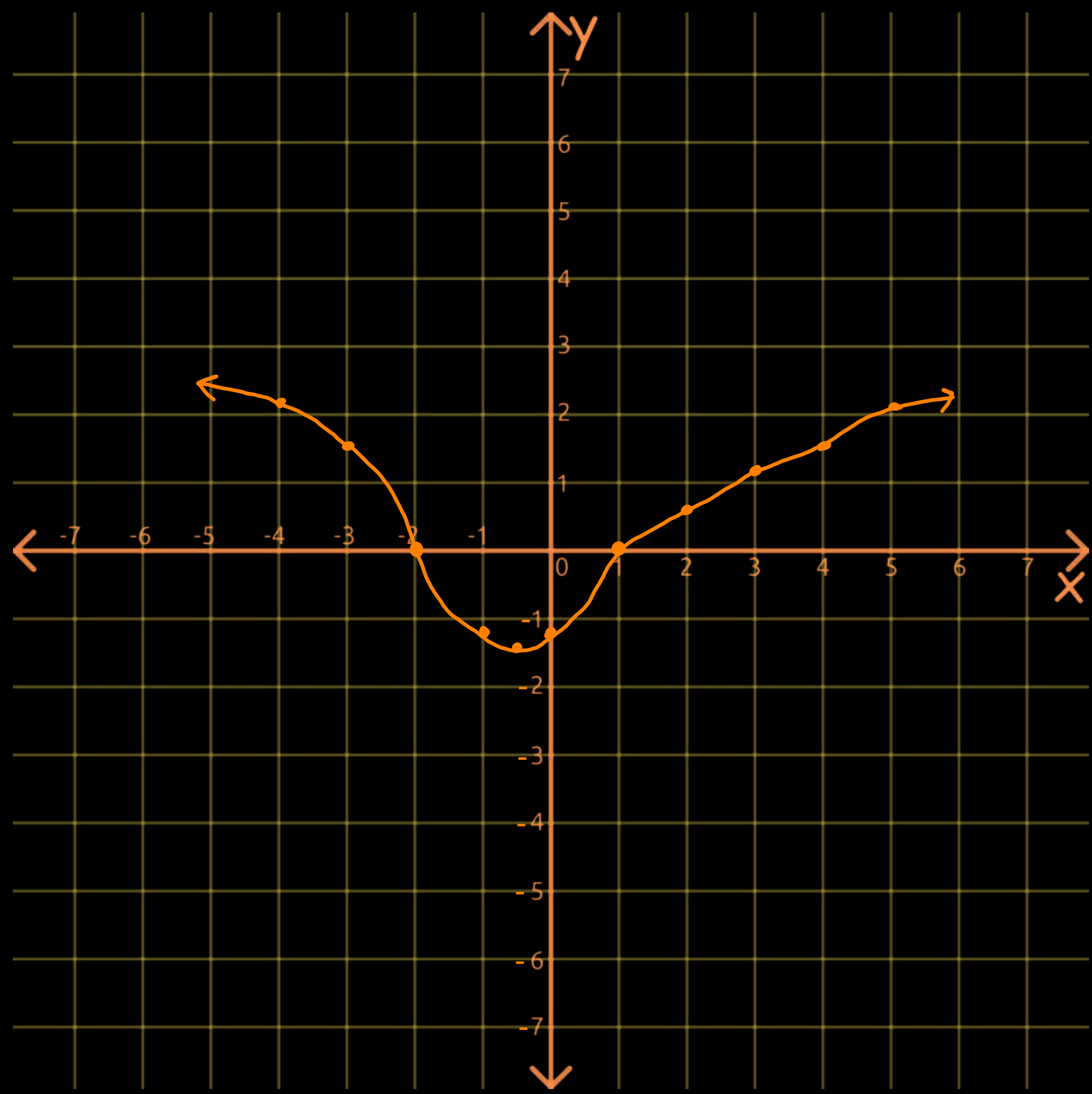
(vii) $y = \sqrt[3]{x^2 + x - 2}$

Sol

Given $y = \sqrt[3]{x^2+x-2}$

Domain: All real numbers

x	$y = \sqrt[3]{x^2+x-2}$
-4	$\sqrt[3]{(-4)^2+(-4)-2} = 2.15$
-3	1.59
-2	0
-1	-1.26
0	-1.26
1	0
2	1.59
3	2.15
4	2.62



4. A building's height over time is modeled by $H(t) = 100 + 20t$ which is in metres and t is the time in months. The height of a growing tree nearby is given by $T(t) = 50 + 10t + t^2$.

(i) At what time will the building and tree have the same height?

(ii) What will that height be?

Sketch the graphs of both functions and determine the time when the tree will overtake the height of the building.

Sol (i)

If the tree & building have the same height, then

$$T(t) = H(t)$$

$$50 + 10t + t^2 = 100 + 20t$$

$$50 + 10t + t^2 - 100 - 20t = 0$$

$$t^2 - 10t - 50 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-50)}}{2(1)}$$

$$t = \frac{10 \pm \sqrt{100 + 200}}{2}$$

$$\sqrt{3 \times 100} = \sqrt{3} \times \sqrt{100}$$

$$= 10\sqrt{3}$$

$$= \frac{10 \pm \sqrt{300}}{2}$$

$$t = \frac{10 \pm 10\sqrt{3}}{2}$$

$$= \frac{\cancel{10}^{5}(1 \pm \sqrt{3})}{\cancel{2}}$$

$$t = 5(1 \pm \sqrt{3})$$

$t = 5(1+\sqrt{3})$ $\qquad$ $t = 5(1-\sqrt{3}) < 0$

$t = 13.66$ $\quad$ ($t \approx 14$ months)

Sol (ii)

Substitute $t = 13.66$ into $H(t)$ to find the height:

$$H(13.66) = 100 + 20(13.66)$$

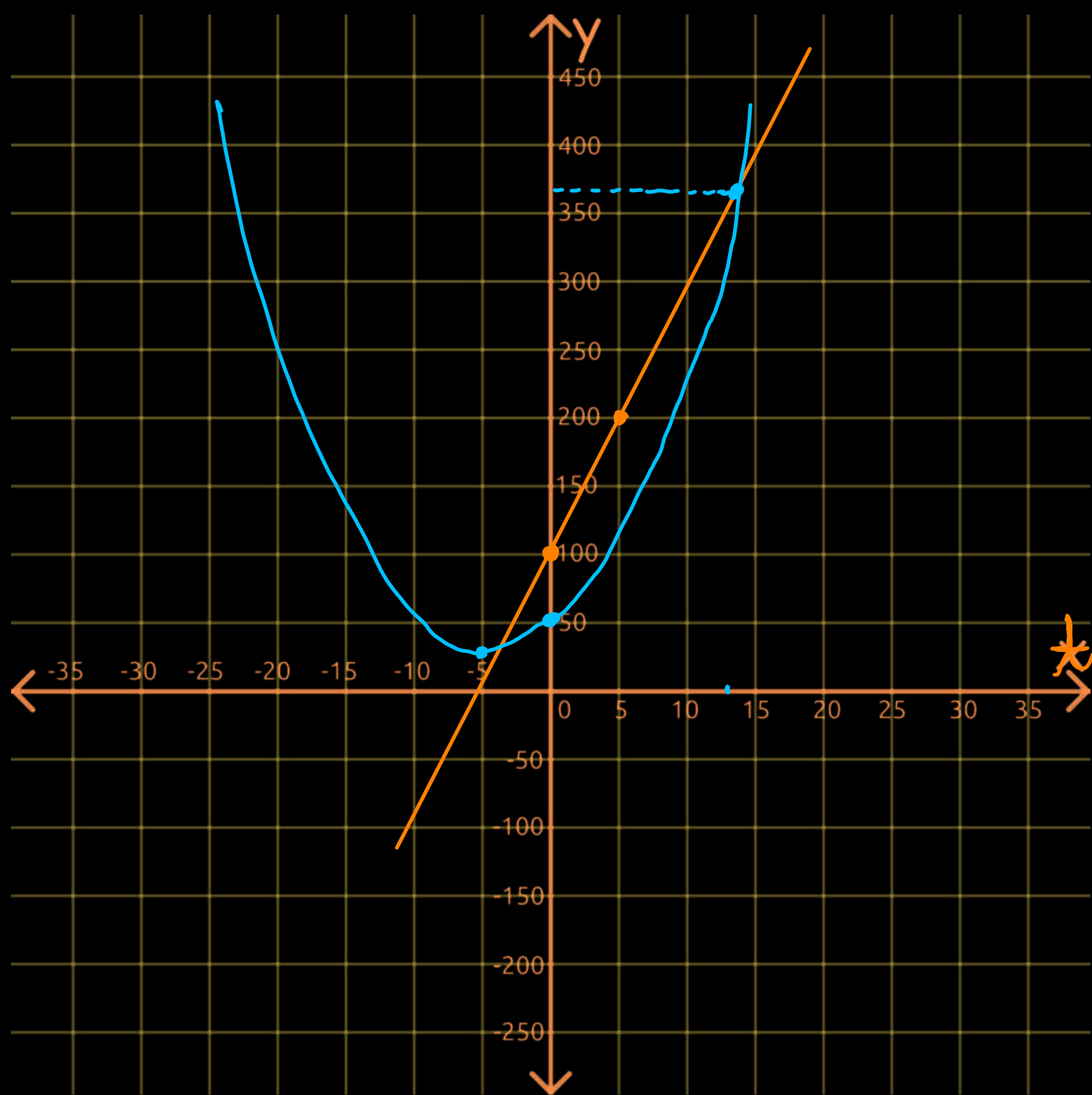
$$H(13.66) = 373.2 \text{ m}$$

Let's sketch $H(t)$,

$H(t) = 100 + 20t$

$t = 0 \longrightarrow H(0) = 100 + 20(0) = 100$

$t = 5 \longrightarrow H(5) = 100 + 20(5) = 200$



$T(t) = t^2 + 10t + 50$

Vertex: (h, k)

$$h = \frac{-b}{2a} = \frac{-10}{2(1)} = -5$$

$$k = T(-5) = (-5)^2 + 10(-5) + 50$$

$$= 25 - 50 + 50$$

$$k = 25$$

$(h, k) = (-5, 25)$

To find y-intercept, substitute $t=0$ into $T(t)$:

$$T(0) = 0^2 + 10(0) + 50 = 50$$

5. A radioactive substance has a half-life of 2 years. If the initial quantity is 200 grams and the exponential decay function is $Q(t) = Q\left(\frac{1}{2}\right)^{\frac{t}{h}}$, then find the remaining quantity after 6 years graphically?

Sol

$$Q(t) = Q_0\left(\frac{1}{2}\right)^{t/h} \quad \text{———} \; ①$$

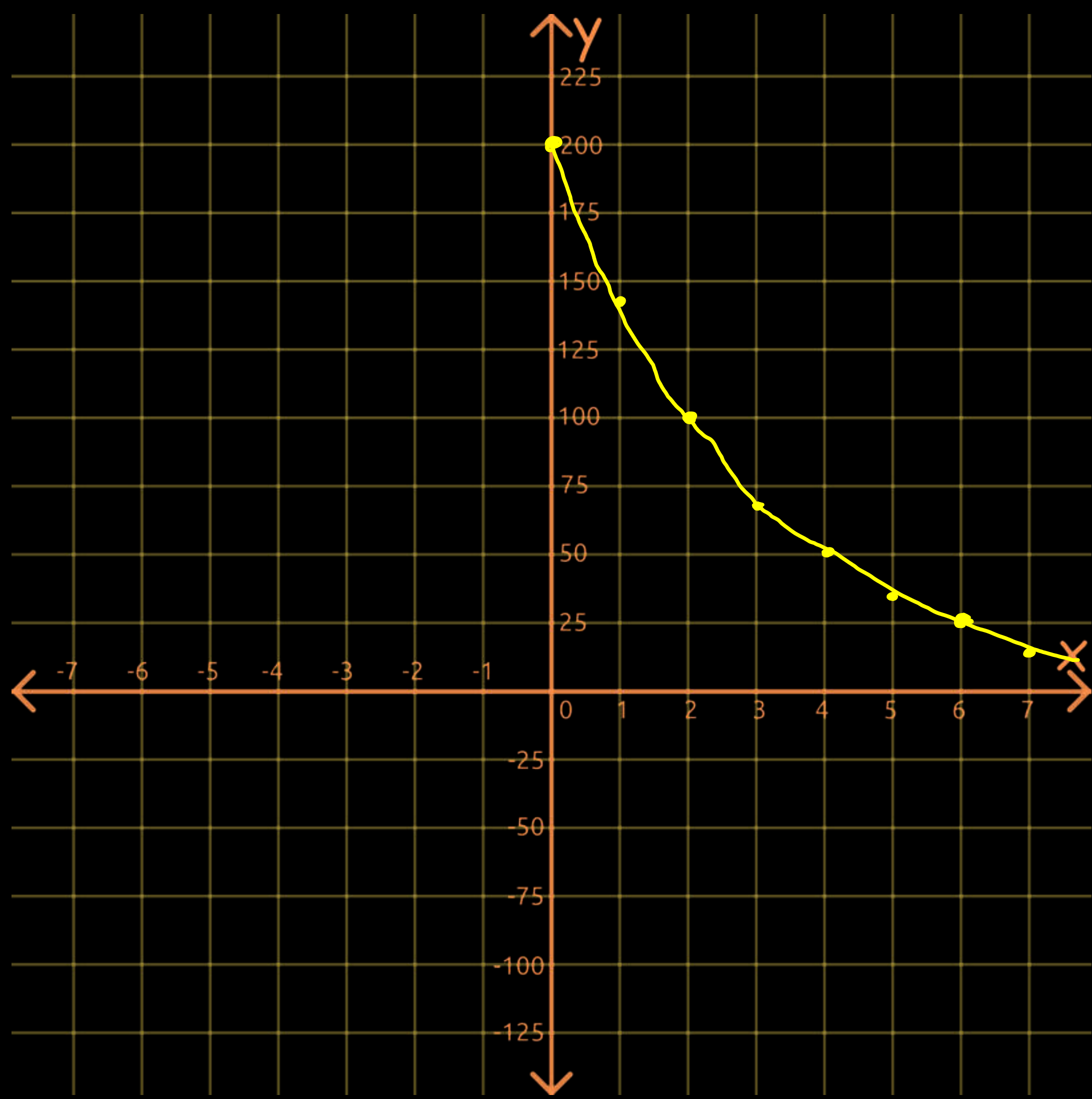
$h = 2$ years

$Q_0 = 200$ grams

Put in ①

$$Q(t) = 200\left(\frac{1}{2}\right)^{t/2}$$

t	$Q(t)$
0	$200\left(\frac{1}{2}\right)^{0/2} = 200$
1	141.4
2	100
3	70.7
4	50
5	35.35



From the graph, the remaining quantity after 6 years is 25g.