بِسْمِ اللّٰهِ الرَّحْمٰنِ الرَّحِيْمِ

(In the Name of Allah, the Most Compassionate, the Most Merciful)

Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Complex Numbers
- Exercise 1.1: Concepts and Examples

TABLE OF CONTENTS

Unit 1 Complex Numbers

$N = \{1, 2, 3, \dots\}$

W

$\mathbb{Z}$

$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$

1.1 Complex Numbers

$x^2 + 1 = 0$

$x^2 = -1$

$x = \sqrt{-1}$

$\therefore i = \sqrt{-1}$

$z = \underline{a} + i\underline{b}$

$a, b \in \mathbb{R},$

$i = \sqrt{-1}$

$i^2 = -1$

$2 + 3i,$ $1 - i,$ $0 + 4i$

$\underline{3 + 0i}$

$Re(z) = a$

$Im(z) = b$

Conjugate Complex Numbers

$z = a + bi,$

$\bar{z} = a - bi$

$z_1 = 2 - 3i$

$z_2 = 1 + i$

$\bar{z}_2 = 1 - i$

$\bar{z}_1 = 2 + 3i$

Operations on Complex Numbers

With a view to develop algebra of **complex numbers**, we state a few definitions.

The symbols *a, b, c, d, k,* where used, represent real numbers.

(i) Addition: $(a+ib)+(c+id)=(a+c)+i(b+d)$

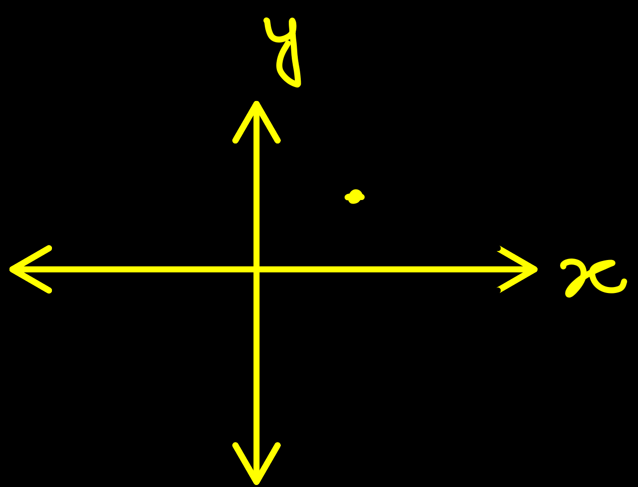
(ii) $k(a+ib)=ka+ikb$ $\quad$ $2(3-4i)=6-8i$

(iii) Subtraction: $(a+ib)-(c+id) = (a+ib)+[-(c+id)]$

$$= a+ib+(-c-id) = (a-c)+i(b-d)$$

(iv) Multiplication: $(a+ib)(c+id) = ac+iad+ibc+i^2bd = (ac-bd)+i(ad+bc)$

Complex Numbers as Ordered Pairs of Real Numbers



$a+ib = (a,b)$ $\qquad$ $\mathbb{R}^2$

We can define complex numbers also by using ordered pairs.

Let C be the set of ordered pairs belonging to $\mathcal{R} \times \mathcal{R}$ which are subject to the following properties:

(i) $(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d$

(ii) $(a, b) + (c, d) = (a+c, b+d)$

(iii) $(a, b)(c, d) = (ac-bd, ad+bc)$

Then C is called the set of **complex numbers**. It is easy to see that

$$(a, b) - (c, d) = (a-c, b-d)$$

(iv) If k is any real number, then $k(a,b) = (ka, kb)$

$$\mathbb{C} = \{a+ib : \quad a, b \in \mathbb{R}, \quad i = \sqrt{-1}\}$$

$$\mathbb{C} = \{(a,b) : \quad a, b \in \mathbb{R}\}$$

Example 1: Find the sum, difference and product of the complex numbers (8, 9) and (5, –6)

Sol

Sum: $(8,9)+(5,-6) = (8+5,\ 9+(-6))$

$= (13, 3)$

Difference: $(8,9)-(5,-6) = (8-5,\ 9-(-6))$

$= (3, 15)$

Product: $(8,9).(5,-6) = ((8\times5)-9\times(-6),\ -48+45)$

$= (94, -3)$

Properties of the Fundamental Operations on Complex Numbers

It can be easily verified that the set C satisfies all the field axioms i.e., it possesses the properties of real numbers.

By way of explanation of some points we observe as follows:

(i) The additive identity in C is $(0, 0)$.

(ii) Every complex number (a, b) has the additive inverse $(-a, -b)$ i.e.,

$$(a, b) + (-a, -b) = (0, 0)$$

(iii) The multiplicative identity is $(1, 0)$ i.e.,

$$(a, b)\cdot(1, 0) = (a\cdot 1 - b\cdot 0,\ b\cdot 1 + a\cdot 0) = (a, b).$$
$$= (1, 0)\cdot(a, b)$$

(iv) Every non-zero complex number {i.e., number not equal to (0,0)} has a multiplicative inverse.
The multiplicative inverse of (a, b) is

$$\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$\because \quad (a, b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1, 0), \text{ the identity element}$$

$$= \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)(a, b)$$

(v) $(a, b)\,[(c, d) \pm (e, f)] = (a, b)(c, d) \pm (a, b)(e, f)$

Note:

The set C of complex numbers does not satisfy the order axioms. In fact, there is no sense in saying that one complex number is greater or less than the other.

Example 2: If $z_1 = (4, 2)$ and $z_2 = (3, -1)$, then find $\frac{z_1}{z_2}$.

$$\frac{z_1}{z_2} = \frac{(4,2)}{(3,-1)} = \frac{4+2i}{3-i}$$

$$= \frac{4+2i}{3-i} \times \frac{3+i}{3+i}$$

$$= \frac{(4+2i)(3+i)}{(3-i)(3+i)}$$

$\because (a-b)(a+b) = a^2 - b^2$

$$= \frac{12+4i+6i+2i^2}{3^2 - i^2}$$

$$= \frac{12+10i+2(-1)}{9-(-1)}$$

$$= \frac{10+10i}{9+1}$$

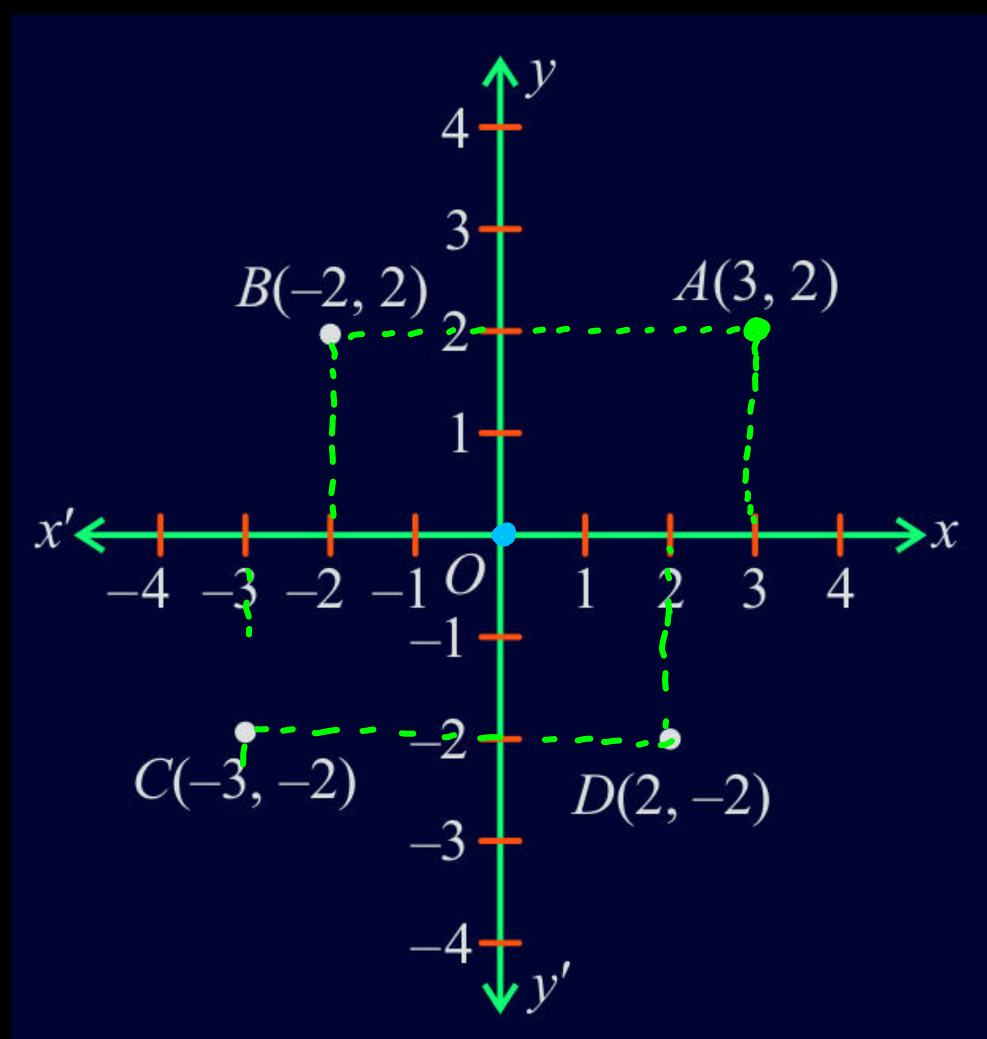
$$= \frac{10+10i}{10}$$

$$= \frac{\cancel{10}}{\cancel{10}} + \frac{\cancel{10}i}{\cancel{10}}$$

$$= 1+i$$

Argand Diagram

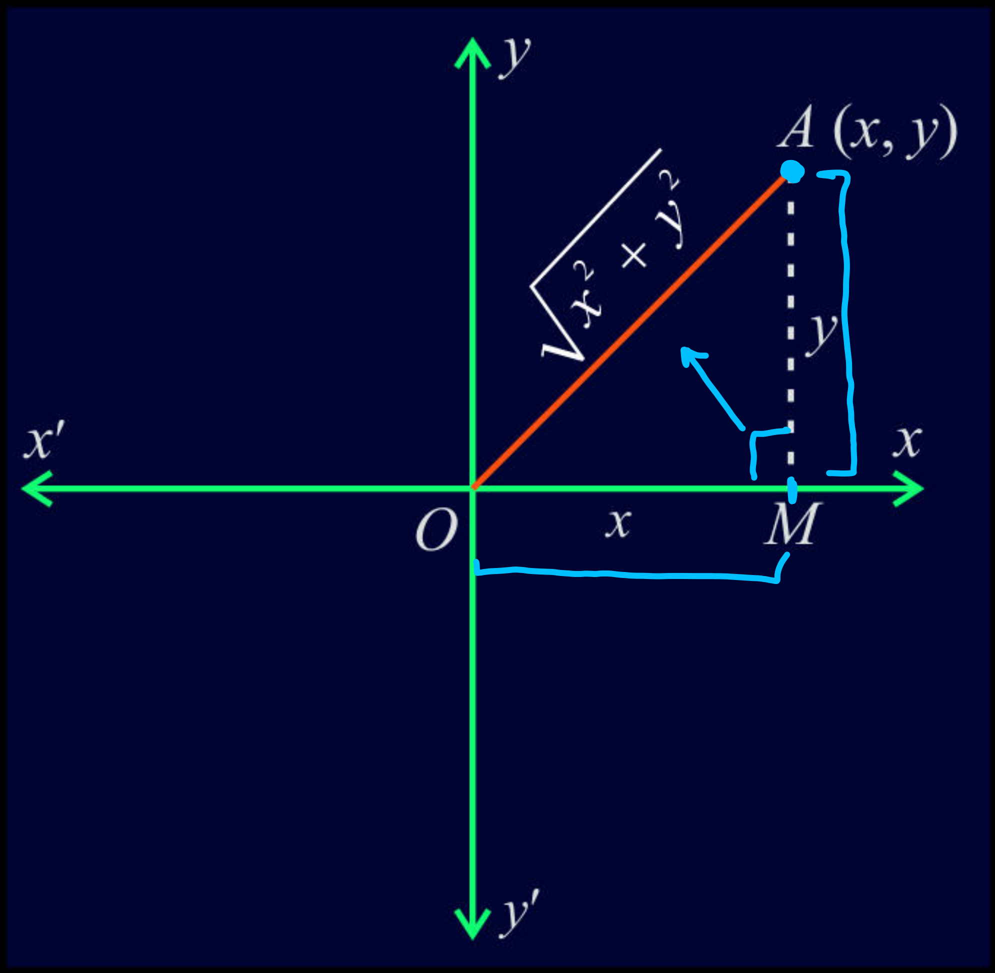
$$\mathbb{C} = \mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$$



Modulus of Complex Number

$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$



Example 3: If $z = \dfrac{(1+2i)^2}{2-i}$ then evaluate $|\bar{z}|$

$$z = \frac{1^2 + 2(1)(2i) + (2i)^2}{2-i}$$

$$= \frac{1 + 4i + 4(-1)}{2-i}$$

$$z = \frac{-3+4i}{2-i} \times \frac{2+i}{2+i}$$

$$= \frac{-6 - 3i + 8i + 4i^2}{2^2 - i^2}$$

$$= \frac{-6 + 5i - 4}{4 - (-1)}$$

$$= -\frac{10+5i}{5}$$

$$z = -2+i$$

$$\bar{z} = -2-i$$

$$|\bar{z}| = \sqrt{(-2)^2+(-1)^2}$$

$$|\bar{z}| = \sqrt{5}$$

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EXERCISE 1.1

Find the multiplicative inverse of each of the following numbers:

(i) $(-4, 7)$ $\quad a=-4, \quad b=7$

$$(a,b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$= \left(\frac{-4}{(-4)^2+7^2}, \frac{-7}{(-4)^2+7^2}\right)$$

$$= \left(\frac{-4}{16+49}, \frac{-7}{16+49}\right)$$

$$= \left(\frac{-4}{65}, \frac{-7}{65}\right) \checkmark$$

(ii) $(\sqrt{2}, -\sqrt{5})$ $\quad a=\sqrt{2}, \quad b=-\sqrt{5}$

Sol

$$= \left(\frac{\sqrt{2}}{(\sqrt{2})^2+(\sqrt{5})^2}, \frac{-(-\sqrt{5})}{(\sqrt{2})^2+(\sqrt{5})^2}\right)$$

$$= \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$$

(iii) $(1,0)$

Sol

$$= \left(\frac{1}{1^2+0^2}, \frac{-0}{1^2+0^2}\right)$$

$$= (1,0)$$

Separate into real and imaginary parts (write as a simple complex number):

(i) $\dfrac{2-7i}{4+5i}$

$a+ib$

Sol

$$\frac{2-7i}{4+5i} = \frac{2-7i}{(4+5i)} \times \frac{4-5i}{(4-5i)}$$

$(a+b)(a-b) = a^2-b^2$

$$= \frac{8-10i-28i+35i^2}{4^2-(5i)^2}$$

$$= \frac{8-38i+35(-1)}{16-25(-1)}$$

$$= \frac{-27-38i}{41}$$

$$= \frac{-27}{41} - \frac{38}{41}i \qquad a = \frac{-27}{41},\ b = \frac{-38}{41}$$

(ii) $\dfrac{(-2+3i)^2}{1+i}$

$\because (a-b)^2 = a^2 - 2ab + b^2$

Sol

$$\frac{(-2+3i)^2}{1+i} = \frac{(3i-2)^2}{1+i}$$

$$= \frac{(3i)^2 - 2(3i)(2) + 2^2}{1+i}$$

$$= \frac{9(-1) - 12i + 4}{1+i}$$

$$= \left(\frac{-5-12i}{1+i}\right) \times \frac{1-i}{1-i}$$

$$= \frac{-5+5i-12i+12i^2}{1^2 - i^2}$$

$$= \frac{-5-7i+12(-1)}{1-(-1)}$$

$$= \frac{-17-7i}{2}$$

$$\frac{(-2+3i)^2}{1+i} = -\frac{17}{2} - \frac{7}{2}i$$

(iii) $\dfrac{i}{1+i}$

Sol

$$\frac{i}{1+i} = \frac{i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{i-i^2}{1^2-i^2}$$

$$= \frac{i-(-1)}{1-(-1)}$$

$$= \frac{1+i}{2}$$

$$\frac{i}{1+i} = \frac{1}{2} + \frac{1}{2}i$$

(iv) $\dfrac{(4+3i)^2}{4-3i}$

Sol

$$\frac{(4+3i)^2}{4-3i} = \frac{4^2+2(4)(3i)+(3i)^2}{4-3i}$$

$$= \frac{16+24i+9(-1)}{4-3i}$$

$$= \left(\frac{7+24i}{4-3i}\right) \times \frac{4+3i}{4+3i}$$

$$= \frac{28+21i+96i+72i^2}{4^2-(3i)^2}$$

$$= \frac{28+117i+72(-1)}{16-9(-1)}$$

$$= \frac{-44+117i}{25}$$

$$= -\frac{44}{25}+\frac{117}{25}i$$

Prove that $\bar{z} = z$ iff z is real. if & only if

Sol

Let $\bar{z} = z$

If $z = x+iy$ then $\bar{z} = x-iy$

$$\cancel{x}-iy = \cancel{x}+iy$$

$$-iy = iy$$

$$0 = iy+iy$$

$$2iy = 0$$

$\Rightarrow \boxed{y=0}$ Hence $z = x+0i = x$

$\rightarrow$ So, z is real.

Conversely,

Let z be a real, then

$\rightarrow z = x + 0i$

$\bar{z} = x - 0i$

Hence $z = \bar{z}$

For $z \in C$, show that:

(i) $\frac{z+\bar{z}}{2} = \text{Re}(z)$

Sol

Let $z = x + iy$, $\bar{z} = x - iy$

$$\text{L.H.S.} = \frac{z+\bar{z}}{2}$$

$$= \frac{x + \cancel{iy} + x - \cancel{iy}}{2}$$

$$= \frac{\cancel{2}x}{\cancel{2}}$$

$$= x$$

$$= \text{Re}(z)$$

(ii) $\dfrac{z-\bar{z}}{2i} = \operatorname{Im}(z)$

Sol

Let $z = x + iy$ then $\bar{z} = x - iy$

$$\text{L.H.S.} = \frac{z-\bar{z}}{2i}$$

$$= \frac{x+iy-(x-iy)}{2i}$$

$$= \frac{\cancel{x}+iy-\cancel{x}+iy}{2i}$$

$$= \frac{\cancel{2i}y}{\cancel{2i}}$$

$$= y$$

$$= \operatorname{Im}(z)$$

(iii) $|z|^2 = z \cdot \bar{z}$

Sol

Let $z = x + iy$

$$\text{L.H.S.} = |z|^2$$

$$= \left(\cancel{\sqrt{x^2+y^2}}\right)^{\cancel{2}}$$

$|z|^2 = x^2 + y^2$ —— ①

R.H.S. $= z.\bar{z}$

$= (x+iy).(x-iy)$

$= x^2 - \cancel{xiy} + \cancel{xiy} - i^2y^2$

$= x^2 - (-1)y^2$

$z.\bar{z} = x^2 + y^2$ —— ②

From ① & ②, $|z|^2 = z.\bar{z}$ ✓

If $z_1 = 2+i, z_2 = 3-2i, z_3 = 1+3i$ then express $\frac{\overline{z_1}.\overline{z_3}}{z_2}$ in the form of $a+ib$.

Sol

$\bar{z}_1 = 2-i,$ $\bar{z}_3 = 1-3i$

$\frac{\bar{z}_1\bar{z}_3}{z_2} = \frac{(2-i)(1-3i)}{3-2i}$

$= \frac{2-6i-i+3i^2}{3-2i}$

$= \frac{2-7i+3(-1)}{3-2i} = \frac{-1-7i}{3-2i}$

$$= \frac{-1-7i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{-3-2i-21i-14i^2}{3^2-(2i)^2}$$

$$= \frac{-3-23i-14(-1)}{9-4(-1)}$$

$$= \frac{11-23i}{13}$$

$$\frac{\bar{z}_1 \bar{z}_3}{z_2} = \frac{11}{13} - \frac{23}{13}i \quad \checkmark \qquad a = \frac{11}{13}, \quad b = \frac{-23}{13}$$

If $z_1 = 2+7i$ and $z_2 = -5+3i$, the evaluate the following:

(i) $|2z_1 - 4z_2|$

Sol

$$|2z_1 - 4z_2| = |2(2+7i) - 4(-5+3i)|$$

$$= |4 + 14i + 20 - 12i|$$

$$= |24 + 2i| \qquad \because |x+iy| = \sqrt{x^2+y^2}$$

$$= \sqrt{24^2 + 2^2}$$

$= \sqrt{576+4}$

$= \sqrt{580}$

$580 = 4 \times 145$

$= \sqrt{2^2 \times 145}$

$= 2.\sqrt{145}$

(ii) $\left|3z_1 + 2\overline{z_1}\right|$

Sol

$|3z_1 + 2\overline{z}_1| = |3(2+7i) + 2(2-7i)|$

$= |6 + 21i + 4 - 14i|$

$= |10 + 7i|$

$= \sqrt{10^2 + 7^2}$

$= \sqrt{100+49}$

$= \sqrt{149}$

(iii) $\left|-7z_2 + 2\overline{z_2}\right|$

Sol

$|-7z_2 + 2\overline{z}_2| = |-7(-5+3i) + 2(-5-3i)|$

$= |35 - 21i - 10 - 6i|$

$$= |25 - 27i|$$

$$= \sqrt{25^2 + (-27)^2}$$

$$= \sqrt{625 + 729}$$

$$= \sqrt{1354}$$

(iv) $\left|(z_1 + z_2)^3\right|$

Sol

$$|(z_1+z_2)^3| = |(\underline{2+7i} - \underline{5+3i})^3|$$

$a = -3$

$b = 10$

$$= |(-3+10i)^3|$$

$$\because (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$= |(-3)^3 + 3(-3)^2 10i + 3(-3)(10i)^2 + (10i)^3|$$

$$= |-27 + 270i - 9(100i^2) + 1000i^3|$$

$$= |-27 + 270i - 900(-1) + 1000i^2 i|$$

$$= |873 + 270i - 1000i|$$

$$= |873 - 730i|$$

$$= \sqrt{(873)^2 + (-730)^2}$$

$$= \sqrt{762129 + 532900}$$

$$= \sqrt{1295029} = \sqrt{109^2 \times 109}$$

$$= 109\sqrt{109}$$

7) Show that: $i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} = 0$ for all $n \in \mathbb{N}$.

Sol.

$$\text{L.H.S.} = i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4}$$

$$= i^n \left(i + i^2 + i^3 + i^4\right)$$

$$= i^n \left(i + (-1) + (-i) + 1\right)$$

$$= i^n \left(\cancel{i} - \cancel{1} - \cancel{i} + \cancel{1}\right)$$

$$= i^n (0) = 0 = \text{R.H.S.}$$

$\because i^2 = -1$

$\because i^3 = i^2 \cdot i = -i$

$\because i^4 = i^2 . i^2 = (-1)(-1) = 1$

8) Find the least positive value of 'n', if

$$\left(\frac{1+i}{1-i}\right)^{2n} = 1$$

Sol.

Take $\frac{1+i}{1-i}$

$$= \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{(1+i)^2}{1^2 - i^2}$$

$$= \frac{1^2 + 2(1)(i) + i^2}{1 - (-1)}$$

$$= \frac{\cancel{1} + 2i - \cancel{1}}{1+1}$$

$$= \frac{2i}{2}$$

$$\frac{1+i}{1-i} = i$$

Now,

$$\left(\frac{1+i}{1-i}\right)^{2n} = i^{2n} = 1$$

$$i^{2n} = i^4$$

$$2n = 4$$

$$\boxed{n = 2}$$

9) Show that, the value of i^n for $n \in \mathbb{N}$ and $n > 4$ is i^r, where 'r' is the remainder when 'n' is divided by 4.

$$\underline{i^n = i^r} \qquad n > 4$$

Sol

The imaginary unit i has a cyclic pattern in its powers:

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$i^5 = i^4\, i = i$$

$$i^6 = i^4\, i^2 = -1$$

$$\vdots$$

$$\begin{array}{r} q \\ 4\overline{)\,n} \\ \underline{4q} \\ r \end{array}$$

$$n = 4q + r$$

(e.g).

$$\begin{array}{r} 1 \\ 4\overline{)\,6} \\ \underline{4} \\ 2 \end{array}$$

$$6 = 1 \times 4 + 2$$

Given that $n > 4$,

$n = 4q + r$, where $0 \leq r < 4$

So,

$$i^n = i^{4q+r}$$

$$= i^{4q} \cdot i^r$$

$$= (i^4)^q \cdot i^r$$

$$= (1)^q \cdot i^r \qquad \because r^4 = 1$$

$$\Longrightarrow \quad i^n = i^r$$

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- Unit 1: Complex Numbers
- Square Root of a Complex Number
- Exercise 1.2: Concepts and Examples

1.2 Equality of Two Complex Numbers

The two complex numbers $z_1 = a+bi$ and $z_2 = c+di$ are said to be equal iff their real and imaginary parts are equal i.e., $a+bi = c+di \Leftrightarrow a=c$ and $b=d$.

iff

Example 4: If $(3+2i)(x+iy) = 5+12i$, where $x, y \in R$, then find the values of x and y.

Solution: Given that $(3+2i)(x+iy) = 5+12i$

$$\Rightarrow \quad 3x + 3iy + 2ix + 2yi^2 = 5 + 12i$$

$$\Rightarrow \quad (3x - 2y) + (2x + 3y)i = 5 + 12i$$

Comparing real and imaginary part, we have

$$3x - 2y = 5 \qquad \dots(\text{i})$$

$$2x + 3y = 12 \qquad \dots(\text{ii})$$

Multiplying equation (i) by 3 and equation (ii) by 2, we have

$$9x - 6y = 15$$

$$4x + 6y = 24$$

Add the equations

$$9x - \cancel{6y} + 4x + \cancel{6y} = 15 + 24$$

$$13x = 39$$

$$x = 3$$

Substitute $x = 3$ in equation (i), we have

$$9(3) - 6y = 15 \quad \rightarrow \quad -6y = 15 - 27$$

$$\cancel{-}6y = \cancel{-}12$$

$$\boxed{y = 2}$$

Thus, $\boxed{x = 3, y = 2}$

Square Root of a Complex Number

The square root of a complex number is another complex number that, when squared, give the original complex number.

Let $w = p + qi$ is a square root of a complex number $z = x + iy$, where $p, q, x, y \in R$, then $w = \sqrt{z}$ ...(i), taking square on both sides, we get

$P = ?$

$q = ?$

$$\rightarrow w^2 = z$$

$$(p+qi)^2 = x+iy$$

$$p^2 + 2(p)(qi) + q^2 i^2 = x+iy$$

$$(p^2 - q^2) + 2pq\,i = x+iy$$

Comparing

$$x = p^2 - q^2 \quad \text{———} \quad ②$$

$$y = 2pq \quad \text{———} \quad ③$$

Squaring & then adding ② & ③

$$x^2 + y^2 = (p^2 - q^2)^2 + (2pq)^2$$

$$x^2 + y^2 = (p^2)^2 - 2(p^2)(q^2) + (q^2)^2 + 4p^2q^2$$

$$= (p^2)^2 + 2p^2q^2 + (q^2)^2$$

$$x^2 + y^2 = (p^2 + q^2)^2$$

$$\Rightarrow \sqrt{x^2+y^2} = p^2 + q^2$$

$$|z| = p^2 + q^2 \quad \text{———} \quad ④$$

Adding ② & ④,

$$p^2 - \cancel{q^2} = x$$
$$p^2 + \cancel{q^2} = |z|$$
$$2p^2 = |z| + x$$

$$p^2 = \frac{|z|+x}{2}$$

$$p = \pm\sqrt{\frac{|z|+x}{2}}$$

Subtract ② from ④,

$$\cancel{p^2} + q^2 = |z|$$
$$\cancel{p^2} - q^2 = x$$
$$- \quad + \quad -$$
$$2q^2 = |z| - x$$

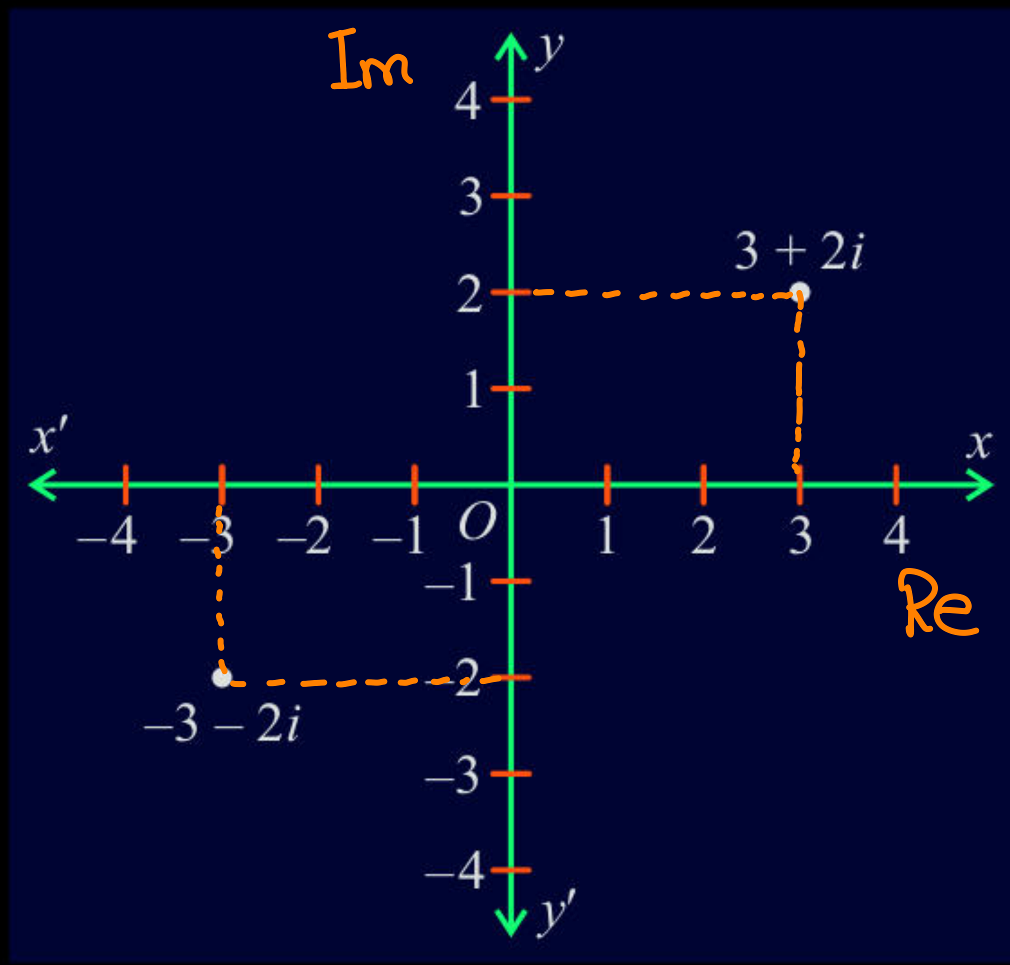
$$q^2 = \frac{|z|-x}{2}$$

$$q = \pm\sqrt{\frac{|z|-x}{2}}$$

$\underline{y = 2pq}$

$$\sqrt{z} = \sqrt{x+iy} = \pm\left(\sqrt{\frac{|z|+x}{2}} + i\frac{y}{|y|}\sqrt{\frac{|z|-x}{2}}\right) *$$

Example 5: Find the square root of complex number $5 + 12i$ and also represent the square roots on an Argand diagram.



Solution: Let $x + yi = 5 + 12i$

$$\Rightarrow x = 5 \text{ and } y = 12 > 0$$

$$|z| = |5+12i| = \sqrt{\underbrace{5^2 + 12^2}_{25+144}} = 13,$$

Applying the square root formula for complex numbers, we get

$$\sqrt{5+12i} = \pm\left(\sqrt{\frac{13+5}{2}} + \frac{i\cancel{12}}{|\cancel{12}|}\sqrt{\frac{13-5}{2}}\right)$$

$$= \pm\left(\sqrt{9} + i\sqrt{4}\right) = \pm(3 + 2i)$$

Thus, the square root of the complex number $5 + 12i$ are $3 + 2i$ and $-3 - 2i$ are shown in adjacent figure.

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Square Root of a Complex Number
- Exercise 1.2

EXERCISE 1.2

1. Find the values of x and y in each of the following:

(i) $x + iy + 2 - 3i = i(5 - i)(3 + 4i)$

$a + bi$

Sol

$$x + iy + 2 - 3i = i(5-i)(3+4i)$$

$$\underline{x+2} + iy - 3i = i(15 + 20i - 3i - 4i^2)$$

$$x + 2 + i(y-3) = i(15 + 17i - 4(-1))$$

$$\text{"} \quad = i(19 + 17i)$$

$$\text{"} \quad = 19i + 17i^2$$

$$x + 2 + i(y-3) = -17 + 19i$$

$x + 2 = -17$

$x = -17 - 2$

$x = -19$

$y - 3 = 19$

$y = 19 + 3$

$y = 22$

(ii) $(x + iy)(1 - i) = (2 - 3i)(-5 + 5i)(-i3/5)$

Sol

$$x - xi + yi - yi^2 = (-10 + 10i + 15i - 15i^2)\left(-\frac{3}{5}i\right)$$

$$x + i(y - x) - y(-1) = (-10 + 25i - 15(-1))\left(-\frac{3}{5}i\right)$$

$$x + i(y - x) + y = (5 + 25i)\left(-\frac{3}{5}i\right)$$

$$(x+y)+i(y-x) = -3i-3(5)i^2$$
$$= -3i-15(-1)$$

$$(x+y)+i(y-x) = 15-3i$$

Now,

$$x+y=15 \quad — ①$$
$$-x+y=-3 \quad — ②$$

Adding ① & ②

$$\cancel{x}+y=15$$
$$\cancel{-x}+y=-3$$
$$2y=12$$
$$y=6$$

$$① \Rightarrow \quad x+6=15$$
$$x=9$$

(iii) $\dfrac{x}{2+i}+\dfrac{y}{3-i}=4+5i$

Sol

$$\frac{x}{(2+i)} \times \frac{2-i}{(2-i)} + \frac{y}{(3-i)} \times \frac{3+i}{(3+i)} = 4+5i$$

$$\frac{x(2-i)}{2^2-i^2} + \frac{y(3+i)}{3^2-i^2} = 4+5i$$

$\because a^2-b^2=(a-b)(a+b)$

$$\frac{2x - xi}{4-(-1)} + \frac{3y+yi}{9-(-1)} = 4+5i$$

$$\frac{2x-xi}{5} + \frac{3y+yi}{10} = 4+5i$$

$$10\left(\frac{2x-xi}{5} + \frac{3y+yi}{10}\right) = 10(4+5i)$$

$$2(2x-xi)+3y+yi = 40+50i$$

$$4x-2xi+3y+yi = 40+50i$$

$$4x+3y+(-2x+y)i = 40+50i$$

Now,

$$4x+3y = 40 \quad \text{— ①}$$

$$-2x+y = 50 \quad \text{— ②}$$

$$y = 2x+50 \quad \text{— ③}$$

Put in ①

$$4x+3(2x+50) = 40$$

$$4x+6x+150 = 40$$

$$10x = 40-150$$

$$1\cancel{0}x = -11\cancel{0}$$

$$\boxed{x = -11}$$

Put in ③

$$y = 2(-11)+50$$

$$= -22+50$$

$$\boxed{y = 28}$$

2. If $z_1 = -13 + 24i$ and $z_2 = x + yi$, find the values of x and y such that $z_1 - z_2 = -27 + 15i$

Sol

$$z_1 - z_2 = -27 + 15i \quad \text{—— (1)}$$

$$z_1 - z_2 = -13 + 24i - (x + yi)$$

$$= -13 + 24i - x - yi$$

$$z_1 - z_2 = -13 - x + i(24 - y) \qquad \text{put in (1)}$$

$$-13 - x + i(24 - y) = -27 + 15i$$

$$-13 - x = -27$$

$$-13 + 27 = x$$

$$\boxed{x = 14}$$

$$24 - y = 15$$

$$24 - 15 = y$$

$$\boxed{y = 9}$$

3. Find the value of x and y if:

(i) $(x+iy)^2 = 25+60i$

Sol

$$(x+iy)^2 = 25+60i$$

$$x+iy = \sqrt{25+60i}$$

$z = a+bi$

$$\because \sqrt{a+bi} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$$

Let

$$z = 25+60i$$

$a = 25$, $\quad b = 60 > 0$

$$|z| = \sqrt{25^2+60^2} = \sqrt{625+3600}$$

$$= \sqrt{4225} = 65$$

Now,

$$\sqrt{25+60i} = \pm\left(\sqrt{\frac{65+25}{2}} + \frac{i\,\cancel{60}}{\cancel{|60|}}\sqrt{\frac{65-25}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{90}{2}} + i\sqrt{\frac{40}{2}}\right)$$

$$\sqrt{45} = \sqrt{3^2\times 5}$$

$$20 = 2^2\times 5$$

$$= \pm\left(\sqrt{45} + i\sqrt{20}\right)$$

$$x+iy = \pm\left(3\sqrt{5} + i\,2\sqrt{5}\right)$$

$3\sqrt{5}+i2\sqrt{5}$, $\quad -3\sqrt{5}-i2\sqrt{5}$

(ii) $(x+iy)^2 = 64+48i$

$x+iy = \sqrt{64+48i}$, $a = 64$

Let $z = 64+48i$

$|z| = \sqrt{64^2+48^2} = \sqrt{4096+2304}$

$= \sqrt{6400} = 80$

Now,

$$\sqrt{64+48i} = \pm\left(\sqrt{\frac{80+64}{2}} + i\,\frac{\cancel{48}}{\cancel{|48|}}\sqrt{\frac{80-64}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{144}{2}} + i\sqrt{\frac{16}{2}}\right)$$

$72 = 36\times 2$

$= \sqrt{6^2\times 2}$

$= 6\sqrt{2}$

$$= \pm\left(\sqrt{72} + i\sqrt{8}\right)$$

$$x+iy = \pm\left(6\sqrt{2} + i\,2\sqrt{2}\right)$$

$6\sqrt{2}+i2\sqrt{2}$, $-6\sqrt{2}-i2\sqrt{2}$

iii) $(x+iy)^2 = \frac{2i-3}{3+i}$

Sol

$$(x+iy)^2 = \frac{2i-3}{3+i} \quad \text{—} ①$$

$$\frac{2i-3}{3+i} = \frac{2i-3}{3+i} \times \frac{3-i}{3-i}$$

$$= \frac{(2i-3)(3-i)}{3^2 - i^2}$$

$$= \frac{6i-2i^2-9+3i}{9-(-1)}$$

$$= \frac{9i-2(-1)-9}{9+1}$$

$$= \frac{-7+9i}{10}$$

$$\frac{2i-3}{3+i} = \frac{-7}{10} + \frac{9}{10}i$$

From ①,

$$(x+iy)^2 = \frac{-7}{10} + \frac{9}{10}i$$

$$x + iy = \sqrt{\frac{-7}{10} + \frac{9}{10}i}$$ ✓

Let

$$z = \frac{-7}{10} + \frac{9}{10}i \quad \text{or} \quad z = -0.7 + 0.9i$$

$$|z| = \sqrt{\left(\frac{-7}{10}\right)^2 + \left(\frac{9}{10}\right)^2}$$

$$= \sqrt{\frac{49}{100} + \frac{81}{100}}$$

$$= \sqrt{\frac{130}{100}}$$

$$|z| = \sqrt{1.3} = 1.14$$

$$a = -0.7$$

$$\because \sqrt{a+bi} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$$

$$\sqrt{-0.7+0.9i} = \pm\left(\sqrt{\frac{1.14+(-0.7)}{2}} + \frac{\cancel{0.9}}{\cancel{|0.9|}}i\sqrt{\frac{1.14-(-0.7)}{2}}\right)$$

$$= \pm\left(\sqrt{0.22} + i\sqrt{0.92}\right)$$

$$= \pm\left(0.46 + i\,0.95\right)$$

4. If $z_1 = 2+3i$ and $z_2 = 1-\alpha$, find the value of α such that $\operatorname{Im}(z_1 z_2) = 7$.

Sol

$$z_1 . z_2 = (2+3i).(1-\alpha)$$

$$z_1 z_2 = 2(1-\alpha) + 3(1-\alpha)i$$

$$\operatorname{Im}(z_1 z_2) = 3(1-\alpha)$$

Since $\operatorname{Im}(z_1 z_2) = 7$

So,

$$3(1-\alpha) = 7$$

$$1-\alpha = \frac{7}{3}$$

$$-\alpha = \frac{7}{3} - 1$$

$$-\alpha = \frac{7-3}{3}$$

$$-\alpha = \frac{4}{3}$$

$$\boxed{\alpha = -\frac{4}{3}}$$

5. If $z_1 = x + yi$ and $z_2 = a + bi$, find x, y, a and b such that $z_1 + z_2 = 10 + 4i$ and $z_1 - z_2 = 6 + 2i$.

Sol

Given that $z_1 = x+yi$

$z_2 = a+bi$

Since

$z_1 + z_2 = 10+4i$

$x+yi+a+bi = 10+4i$

$x+a+(y+b)i = 10+4i$

$z_1 - z_2 = 6+2i$

$x+yi-(a+bi) = 6+2i$

$x+yi-a-bi = 6+2i$

$(x-a)+(y-b)i = 6+2i$

Equate the real and imaginary parts:

$x+a=10$ —— ①

$y+b=4$ —— ②

$x-a=6$ —— ③

$y-b=2$ —— ④

Adding ① & ③

$$\begin{array}{r} x+\cancel{a} = 10 \\ x-\cancel{a} = 6 \\ \hline 2x = 16 \end{array}$$

$\boxed{x=8}$

Put in ①,

$8+a=10$

Adding ② & ④

$$\begin{array}{r} y+\cancel{b} = 4 \\ y-\cancel{b} = 2 \\ \hline 2y = 6 \end{array}$$

$\boxed{y=3}$

Put in ②,

$3+b=4$

$a = 10 - 8$

$\boxed{a = 2}$

$b = 4 - 3$

$\boxed{b = 1}$

Hence

✓ $z_1 = 8 + 3i$

✓ $z_2 = 2 + i$

6. Show that $\forall\, z_1, z_2 \in C$, $\overline{z_1\, z_2} = \overline{z_1}\;\overline{z_2}$

Sol

Let $z_1 = a + bi$ & $z_2 = c + di$

then

$\bar{z}_1 = a - bi$ $\bar{z}_2 = c - di$

L.H.S.

$z_1 z_2 = (a+bi).(c+di)$

$= ac + adi + bci + bdi^2$

$z_1 z_2 = (ac - bd) + (ad + bc)i$

$\overline{z_1 z_2} = (ac-bd) - (ad+bc)i$

R.H.S

$\bar{z}_1 \bar{z}_2 = (a-bi).(c-di)$

$= ac - adi - bci + bdi^2$

$= ac - (ad+bc)i - bd$

$\bar{z}_1 \bar{z}_2 = (ac-bd) - (ad+bc)i$

Hence

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

7. Find the square root of the following complex numbers:

(i) $-7-24i$

Sol

$$z = -7-24i$$

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$$

$$|z| = \sqrt{(-7)^2+(-24)^2}$$

$$= \sqrt{49+576}$$

$$= \sqrt{625}$$

$$|z| = 25$$

$x=-7, \quad y=-24$

So,

$$\sqrt{-7-24i} = \pm\left(\sqrt{\frac{25+(-7)}{2}} + \frac{i(-24)}{|-24|}\sqrt{\frac{25-(-7)}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{18}{2}} - \frac{i\cancel{24}}{\cancel{24}}\sqrt{\frac{32}{2}}\right)$$

$$= \pm\left(\sqrt{9} - i\sqrt{16}\right)$$

$$\sqrt{-7-24i} = \pm(3-4i)$$

$$\begin{cases} 3-4i \\ -3+4i \end{cases}$$

(ii) $8 - 6i$

Sol

$z = 8 - 6i$,

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$$

$$|z| = \sqrt{8^2 + (-6)^2}$$

$$= \sqrt{64+36}$$

$$= \sqrt{100}$$

$$|z| = 10$$

So,

$$\sqrt{8-6i} = \pm\left(\sqrt{\frac{10+8}{2}} + i\,\frac{(-6)}{|-6|}\sqrt{\frac{10-8}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{\cancel{18}^{\,9}}{\cancel{2}}} - i\,\frac{\cancel{6}}{\cancel{6}}\sqrt{\frac{\cancel{2}}{\cancel{2}}}\right)$$

$$= \pm\left(\sqrt{9} - i\sqrt{1}\right)$$

$$\sqrt{8-6i} = \pm(3 - i)$$

$3 - i$, $\quad -3 + i$

(iii) $-15 - 36i$

Sol

$$z = -15 - 36i$$

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$$

$$|z| = \sqrt{(-15)^2 + (-36)^2}$$

$$= \sqrt{225 + 1296}$$

$$= \sqrt{1521}$$

$$|z| = 39$$

$x = -15$

Now,

$$\sqrt{-15-36i} = \pm\left(\sqrt{\frac{39+(-15)}{2}} + i\frac{(-36)}{|-36|}\sqrt{\frac{39-(-15)}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{24}{2}} - i\frac{\cancel{36}}{\cancel{36}}\sqrt{\frac{54}{2}}\right)$$

$$= \pm\left(\sqrt{12} - i\sqrt{27}\right)$$

$$\sqrt{-15-36i} = \pm\left(2\sqrt{3} - 3\sqrt{3}\, i\right)$$

(iv) $119 + 120i$

Sol

$$z = 119 + 120i$$

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$$

$$|z| = \sqrt{119^2 + 120^2}$$

$$= \sqrt{14161 + 14400}$$

$$= \sqrt{28561}$$

$$|z| = 169$$

Now,

$$\sqrt{119+120i} = \pm\left(\sqrt{\frac{169+119}{2}} + i\,\frac{\cancel{(120)}}{\cancel{|120|}}\sqrt{\frac{169-119}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{288}{2}} + i\sqrt{\frac{50}{2}}\right)$$

$$= \pm\left(\sqrt{144} + i\sqrt{25}\right)$$

$$\sqrt{119+120i} = \pm(12+5i)$$

8. Find the square root of $13-20\sqrt{3}i$ and represent them on an Argand diagram.

Sol

Let $z = 13 - 20\sqrt{3}\,i$ $\qquad x = 13, \quad y = -20\sqrt{3}$

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$$

$$|z| = \sqrt{13^2 + (-20\sqrt{3})^2}$$

$$= \sqrt{169 + (400\times 3)}$$

$$= \sqrt{169+1200}$$

$$= \sqrt{1369}$$

$$|z| = 37$$

Now

$$\sqrt{13-20\sqrt{3}} = \pm\left(\sqrt{\frac{37+13}{2}} + i\,\frac{(-20\sqrt{3})}{|-20\sqrt{3}|}\sqrt{\frac{37-13}{2}}\right)$$

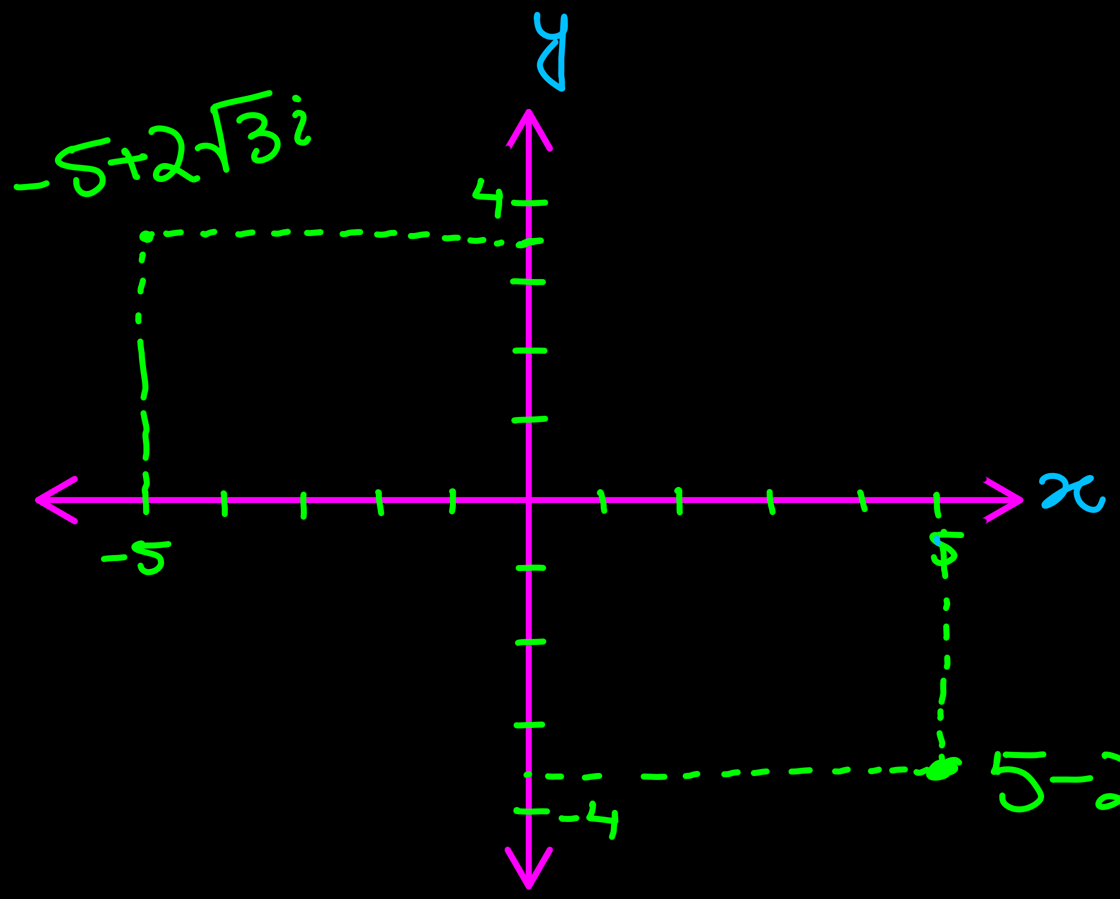
$$= \pm\left(\sqrt{\frac{50}{2}} - i\,\frac{\cancel{20\sqrt{3}}}{\cancel{20\sqrt{3}}}\sqrt{\frac{24}{2}}\right)$$

$$= \pm\left(\sqrt{25} - i\sqrt{12}\right)$$

$$\sqrt{13-20\sqrt{3}} = \pm\left(5 - 2\sqrt{3}\,i\right)$$

$\underline{5-2\sqrt{3}\,i}$ $\qquad -(5-2\sqrt{3}\,i)=\underline{-5+2\sqrt{3}\,i}$

$\because 2\sqrt{3}\approx 3.5$



9. Find the value of x and y if $(-7+i)(x+iy)+(-1-5i)=i(11-i)$

Sol

$$(-7+i)(x+iy)+(-1-5i)=i(11-i)$$

$$-7x-7yi+xi+i^2y-1-5i=11i-i^2$$

$$-7x+(-7y+x)i-y-1-5i=11i-(-1)$$

$$-7x-y-1+(x-7y-5)i=1+11i$$

Equate the real and imaginary parts:

$-7x-y-1=1 \qquad\qquad x-7y-5=11$

$-7x-y=1+1$

$-7x - y = 2$ —— ①　　　　$x - 7y = 16$ —— ②

From ①

$$-7x - 2 = y \quad \text{—— ③}$$

Substitute $y = -7x - 2$ into equ.②

$$x - 7(-7x - 2) = 16$$

$$x + 49x + 14 = 16$$

$$50x = 16 - 14$$

$$50x = 2$$

$$x = \frac{2}{50}$$

$$x = \frac{1}{25}$$

Substitute $x = \frac{1}{25}$ into equ.③

$$y = -7\left(\frac{1}{25}\right) - 2$$

$$= \frac{-7}{25} - 2$$

$$y = \frac{-7 - 50}{25} = \frac{-57}{25} \ *$$

10. Find the value of x and y if $(5-2i)(x+iy)+3=i(11-i)-4i$

Sol.

$$(5-2i)(x+iy)+3=i(11-i)-4i$$

$$5x+5yi-2xi-2yi^2+3=11i-i^2-4i$$

$$5x+(5y-2x)i-2y(-1)+3=7i-(-1)$$

$$5x+2y+3+(5y-2x)i=1+7i$$

Equate the real and imaginary parts:

$$5x+2y+3=1 \qquad 5y-2x=7$$

$$5x+2y=-2 \quad \text{—— ①} \qquad -2x+5y=7 \quad \text{—— ②}$$

Multiply equ.① by '2', equ.② by '5' and then add,

$$\begin{array}{r} \cancel{10x}+4y=-4 \\ -\cancel{10x}+25y=35 \\ \hline 29y=31 \end{array}$$

$$y=\frac{31}{29} \qquad \text{Put in ①}$$

$$5x+2\left(\frac{31}{29}\right)=-2$$

$$5x+\frac{62}{29}=-2$$

$$5x=-2-\frac{62}{29}$$

$$5x = \frac{-58-62}{29}$$

$$5x = \frac{-120}{29}$$

$$x = \frac{1}{\cancel{5}} \times \frac{-\overset{24}{\cancel{120}}}{29}$$

$$x = \frac{-24}{29}$$

11) Find the real values of u and v if

$$\frac{u-2}{2+i} + \frac{v-3}{2-i} = 4i$$

Sol

$$\frac{u-2}{2+i} + \frac{v-3}{2-i} = 4i$$

$$\frac{(u-2)(2-i)+(v-3)(2+i)}{(2+i)(2-i)} = 4i$$

$$\frac{2u-ui-4+2i+2v+vi-6-3i}{2^2-i^2} = 4i$$

$$\frac{(2u+2v-10-ui+vi-i}{4-(-1)}=4i$$

$$\frac{2u+2v-10+i(-u+v-1)}{5}=4i$$

$$2u+2v-10+i(-u+v-1)=20i$$

Comparing real and imaginary parts:

$2u+2v-10=0$ —— ①

$-u+v-1=20$

$v=20+u+1$

$v=21+u$ —— ②

Substitute into equ①,

$$2u+2(21+u)-10=0$$

$$2u+42+2u-10=0$$

$$4u+32=0$$

$$4u=-32$$

$$u=\frac{-32}{4}=-8$$

Put in ②

$$v=21+(-8)$$

$$v=21-8=13$$

12. If $z_1 = 4+5i$ and $z_2 = \alpha - 2i$, find the value of α such that $\text{Re}(z_1 z_2) = 20$.

Sol

$$z_1 z_2 = (4+5i)(\alpha-2i)$$

$$= 4\alpha - 8i + 5\alpha i - 10i^2$$

$$= 4\alpha + (-8+5\alpha)i - 10(-1)$$

$$z_1 z_2 = 4\alpha + 10 + (5\alpha - 8)i$$

Given that

$$\text{Re}(z_1 z_2) = 20$$

$$\rightarrow 4\alpha + 10 = 20$$

$$4\alpha = 10$$

$$\alpha = \frac{\cancel{10}^{\,5}}{\cancel{4}_{\,2}}$$

$$\alpha = \frac{5}{2}$$

بِسْمِ اللّٰهِ الرَّحْمٰنِ الرَّحِيْمِ

(In the Name of Allah, the Most Compassionate, the Most Merciful)

Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Complex Polynomials as a Product of Linear Factors
- Exercise 1.3: Concepts and Examples

1.2 Complex Polynomials as a Product of Linear Factors

A **complex polynomial** $P(z)$ is a polynomial function of the complex variable z with complex coefficients. It is expressed in the general form as

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

Where $a_n, a_{n-1}, \dots, a_1, a_0$ are complex numbers $(a_n \neq 0)$, and $n \geq 0$ is an integer representing the degree of the polynomial.

For examples $P_1(z) = (1 - i)z + 3i$, $P_2(z) = (5 - 4i)z^2 + (2 + i)z + (3 - 4i)$ and $P_3(z) = (2 - i)z^3 + 2z^2 i + (5 + 3i)$ are the examples of linear, quadratic and cubic complex polynomials respectively. If $n = 0$, then $P(z)$ becomes a constant polynomial.

A fundamental property of complex polynomials is that they can always be factored into a product of linear factors.

According to the **Fundamental theorem of algebra**, a polynomial of degree $n \geq 1$ has exactly n roots in complex numbers system C.

A **corollary** to this theorem states that any polynomial $P(z)$ of degree n can be factored completely into a constant a and n linear factor over C in the form

$$P(z) = a(z - z_1)(z - z_2)\dots(z - z_n) \qquad (1)$$

where $z_1, z_2, \dots, z_n$ are complex roots of the polynomial. Once we know the roots of a polynomial equation, we can apply equation (1) to factored the polynomial $P(z)$ into n linear factors. Specifically, if z_1 and z_2 are roots of the polynomial equation $P(z)$, then the equation must be $P(z) = (z - z_1)(z - z_2)$. For examples, the polynomial $P(x) = x^2 + 4$ consists of real coefficient has no real roots, so it cannot be factored into linear polynomials with real coefficients. However, if we considered as a complex polynomial $P(z) = z^2 + 4$, we can easily be factored into two linear factors as

$$z^2 + 4 = (z + 2i)(z - 2i)$$

$a^2 - b^2 = (a-b)(a+b)$

where $2i$ and $-2i$ are the complex roots of $z^2 + 4 = 0$

$z^2 + 4 = z^2 - (-1)2^2$

$= z^2 - i^2 2^2 = z^2 - (2i)^2 = (z + 2i)(z - 2i)$

Example 6: Factorize the polynomial $P(z) = z^2 + (1 - i)\, z - i$.

Solution:
$$\begin{aligned} P(z) &= z^2 + (1-i)\,z - i \\ &= z^2 + z - iz - i \\ &= z(z+1) - i(z+1) \\ &= (z+1)(z-i) \end{aligned}$$

Example 7: Factorize the polynomial $P(z) = z^2 - 4iz + 12$

Solution:
$$\begin{aligned} P(z) &= z^2 - 4iz + 12 \\ &= z^2 - 4iz - (-12) \\ &= z^2 - 4iz - i^2 12 && \because \quad i^2 = -1 \\ &= z^2 - i6z + i2z - i^2 12 && -4 = 2-6 \\ &= z(z - 6i) + 2i(z - 6i) \\ &= (z - 6i)(z + 2i) \end{aligned}$$

Example 8: Factorize the polynomial $P(z) = z^3 + (1 + i)z^2 + iz$.

Solution:
$$\begin{aligned} P(z) &= z^3 + (1 + i)\, z^2 + iz \\ &= z[z^2 + (1 + i)z + i] \\ &= z[z^2 + z + iz + i] \\ &= z[z(z + 1) + i(z + 1)] \\ &= z[(z + 1)(z + i)] \\ &= z(z + 1)(z + i) \text{ are linear factors.} \end{aligned}$$

Key Concept

The Rational Root Theorem is a mathematical tool used to find all possible rational roots of a polynomial equation with integer coefficients. According to rational root theorem:

If a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients, then every rational root $\frac{p}{q}$ (in simplest terms) satisfies:

(i) p is a factor of the constant term a_0. (ii) q is a factor of the leading coefficient a_n.

$z^3 - 3z^2 + z + 5$ $\quad a_0 = 5, \quad a_3 = 1$

$p = 5, -5, 1, -1 \quad q = 1, -1$

$\frac{p}{q} = \frac{5}{1}, \frac{5}{-1}, \frac{-5}{1}, \frac{-5}{-1}, \frac{1}{+1}, \frac{1}{-1}, \frac{-1}{1}, \frac{-1}{1}$

$= 5, -5, 1, -1$ $\quad P(5), P(-5), P(1), P(-1)$

Example 9: Factorize the + polynomial $P(z) = z^3 - 3z^2 + z + 5$.

Solution: According to rational root theorem the possible root of the equation are ± 1 and ± 5. On checking, we see that $z = -1$ is the root of the polynomial $P(z)$ because

$$P(-1) = (-1)^3 - 3(-1)^2 + (-1) + 5 = 0.$$

$z - (-1) = z + 1$

So $z + 1$ is a factor of the $P(z)$. Using synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 1 & 5 \\ & & -1 & 4 & -5 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

= Remainder

Therefore, $z^3 - 3z^2 + z + 5 = (z+1)(z^2 - 4z + 5)$ ...(i)

Next find the factors of $z^2 - 4z + 5$ using quadratic formula

$z^2 - 4z + 5 = 0$, here $a = 1, b = -4, c = 5$

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2}$$

$\Rightarrow \quad z = 2 \pm 2i$ ✓

The quadratic factors are $z^2 - 4z + 5 = \underbrace{(z-(2+i))}_{z_1}\underbrace{(z-(2-i))}_{z_2} = (z-2-i)(z-2+i)$

Substitutes in equation (i), we have the

$$z^3 - 3z^2 + z + 5 = (z+1)(z-2-i)(z-2+i)$$ ✓

1.3.1 Solution of Quadratic Equations by Completing the Square

Example 10: Solve the equation $2z^2 - 12z + 50 = 0$ by completing square method and hence express it as a product of its linear factors.

Sol

$$2z^2 - 12z + 50 = 0$$

Divide by 2,

$$z^2 - 6z + 25 = 0$$

$$\underbrace{z^2 - 2(z)(3) + 3^2} - 3^2 + 25 = 0$$

$$(z-3)^2 + 16 = 0$$

$$(z-3)^2 = -16$$

Square root

$$z - 3 = \pm\sqrt{-16}$$

$$= \pm\sqrt{16}\sqrt{-1}$$

$$z - 3 = \pm 4i$$

$$z = 3 \pm 4i$$

$$3 + 4i, \quad 3 - 4i$$

$$2z^2 - 12z + 50 = 2(z - (3+4i))(z - (3-4i))$$

$$= 2(z - 3 - 4i)(z - 3 + 4i)$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Complex Polynomials as a Product of Linear Factors
- Exercise 1.3: Q1 - Q4

EXERCISE 1.3

1. Factorize the following:

(i) $a^2 + 4b^2$

Sol

$$a^2 + 4b^2 = a^2 - (-1)4b^2$$
$$= a^2 - i^2 4b^2$$
$$= a^2 - (2bi)^2$$
$$= (a - 2bi)(a + 2bi)$$

(ii) $9a^2 + 16b^2$

Sol

$$9a^2 + 16b^2 = 9a^2 - (-1)16b^2$$
$$= 9a^2 - i^2 16b^2$$
$$= (3a)^2 - (4bi)^2$$
$$= (3a - 4bi)(3a + 4bi)$$

(iii) $3x^2 + 3y^2$

Sol

$$3x^2+3y^2 = 3\left(x^2+y^2\right)$$

$$= 3\left(x^2-(-1)\,y^2\right)$$

$$= 3\left(x^2-i^2y^2\right)$$

$$= 3\left[x^2-(iy)^2\right]$$

$$= 3\,(x-iy)(x+iy)$$

(iv) $144x^2 + 225y^2$

Sol

$$144x^2+225y^2 = 9\left(16x^2+25y^2\right)$$

$$= 16x^2-(-1)\,25y^2$$

$$= (4x)^2-i^2(5y)^2$$

$$= (4x)^2-(5yi)^2$$

$$= (4x-5yi)(4x+5yi)$$

(v) $z^2 - 2iz - 1$

Sol

$$z^2 - 2iz - 1 = z^2 - 2iz + i^2$$
$$= z^2 - iz - iz + i^2$$
$$= z(z-i) - i(z-i)$$
$$= (z-i)(z-i)$$

(vi) $z^2 + 6z + 13$

Sol

$$z^2 + 6z + 13 = z^2 + 2(z)(3) + 3^2 + 4$$
$$= (z+3)^2 + 4$$
$$= (z+3)^2 - (-1)4$$
$$= (z+3)^2 - (2i)^2$$
$$= (z+3-2i)(z+3+2i)$$

(vii) $z^2 + 4z + 5$

Sol

$$z^2+4z+5 = z^2+2(z)(2)+2^2+1$$
$$= (z+2)^2+1$$
$$= (z+2)^2-(-1)$$
$$= (z+2)^2-i^2$$
$$= (z+2-i)(z+2+i)$$

(viii) $2z^2 - 22z + 65$

Sol

$$2z^2-22z+65 = 2\left(z^2-11z+\frac{65}{2}\right)$$
$$= 2\left[z^2-2(z)\left(\frac{11}{2}\right)+\left(\frac{11}{2}\right)^2-\left(\frac{11}{2}\right)^2+\frac{65}{2}\right]$$
$$= 2\left[\left(z-\frac{11}{2}\right)^2-\frac{121}{4}+\frac{65}{2}\right]$$
$$= 2\left[\left(z-\frac{11}{2}\right)^2-\frac{121-130}{4}\right]$$
$$= 2\left[\left(z-\frac{11}{2}\right)^2-(-1)\frac{9}{4}\right]$$
$$= 2\left[\left(z-\frac{11}{2}\right)^2-i^2\frac{3^2}{2^2}\right]$$

$$= 2\left[\left(z-\frac{11}{2}\right)^2-\left(\frac{3i}{2}\right)^2\right]$$

$$= 2\left[\left(z-\frac{11}{2}-\frac{3i}{2}\right)\left(z-\frac{11}{2}+\frac{3i}{2}\right)\right]$$

$$= 2\left(z-\frac{11+3i}{2}\right)\left(z-\frac{11-3i}{2}\right)$$

2. Factorize the following polynomial into its linear factors:

(i) z^3+8

Sol

$$z^3+8 = z^3+2^3$$

$$= (z+2)(z^2-(z)(2)+2^2) \qquad \because a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$= (z+2)(z^2-2z+4)$$

$$= (z+2)(z^2-2(z)(1)+1^2+3)$$

$$= (z+2)\left[(z-1)^2+3\right]$$

$$= (z+2)\left[(z-1)^2-(-1)3\right]$$

$$= (z+2)\left[(z-1)^2-i^2(\sqrt{3})^2\right]$$

$$= (z+2)\left[(z-1)^2 - (\sqrt{3}\,i)^2\right]$$

$$= (z+2)(z-1-\sqrt{3}\,i)(z-1+\sqrt{3}\,i)$$

(ii) $z^3 + 27$

Sol

$$z^3+27 = z^3+3^3$$

$$= (z+3)(z^2-(z)(3)+3^2) \qquad \because a^3+b^3 = (a+b)(a^2-ab+b^2)$$

$$= (z+3)(z^2-3z+9)$$

$$= (z+3)\left(z^2-2z\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+9\right)$$

$$= (z+3)\left[\left(z-\frac{3}{2}\right)^2-\frac{9}{4}+9\right]$$

$$= (z+3)\left[\left(z-\frac{3}{2}\right)^2-\frac{9-36}{4}\right]$$

$$= (z+3)\left[\left(z-\frac{3}{2}\right)^2-\frac{(-27)}{4}\right]$$

$$= (z+3)\left[\left(z-\frac{3}{2}\right)^2-\frac{i^2 27}{4}\right]$$

$$= (z+3)\left[\left(z-\frac{3}{2}\right)^2-\left(\frac{\sqrt{27}\,i}{2}\right)^2\right]$$

$$= (z+3)\left(z-\frac{3}{2}-\frac{\sqrt{27}}{2}i\right)\left(z-\frac{3}{2}+\frac{\sqrt{27}}{2}i\right)$$

$$= (z+3)\left(z - \frac{3+\sqrt{27}\,i}{2}\right)\left(z - \frac{3-\sqrt{27}\,i}{2}\right)$$

$$= (z+3)\left(z - \frac{3+3\sqrt{3}\,i}{2}\right)\left(z - \frac{3-3\sqrt{3}\,i}{2}\right)$$

(iii) $z^3 - 2z^2 + 16z - 32$

Sol

$$z^3 - 2z^2 + 16z - 32 = z^2(z-2) + 16(z-2)$$

$$= (z-2)(z^2+16)$$

$$= (z-2)(z^2 - (-1)16)$$

$$= (z-2)(z^2 - i^2 4^2)$$

$$= (z-2)\left[z^2 - (4i)^2\right]$$

$$= (z-2)(z-4i)(z+4i)$$

(iv) $z^4 + 21z^2 - 100$

Sol

$$z^4+21z^2-100 = z^4+25z^2-4z^2-100$$

$$= z^2(z^2+25)-4(z^2+25)$$

$$= (z^2+25)(z^2-4)$$

$$= [z^2-(-1)5^2](z^2-2^2)$$

$$= [z^2-(5i)^2](z-2)(z+2)$$

$$= (z-5i)(z+5i)(z-2)(z+2)$$

(v) $z^4 - 16$

Sol

$$z^4-16 = (z^2)^2-4^2$$

$$= (z^2-4)(z^2+4)$$

$$= (z^2-2^2)[z^2-(-1)4]$$

$$= (z-2)(z+2)[z^2-i^2 \cdot 2^2]$$

$$= (z-2)(z+2)\left[z^2-(2i)^2\right]$$

$$= (z-2)(z+2)(z-2i)(z+2i)$$

(vi) $z^4 + 3z^2 - 4$

Sol

$$z^4+3z^2-4 = z^4+4z^2-z^2-4$$

$$= z^2(z^2+4)-(z^2+4)$$

$$= (z^2+4)(z^2-1)$$

$$= \left[z^2-(-1)4\right](z^2-1^2)$$

$$= \left[z^2-i^2 2^2\right](z-1)(z+1)$$

$$= \left[z^2-(2i)^2\right](z-1)(z+1)$$

$$= (z-2i)(z+2i)(z-1)(z+1)$$

(vii) $z^4 + 5z^2 + 6$

Sol

$$z^4+5z^2+6 = z^4+3z^2+2z^2+6$$
$$= z^2(z^2+3)+2(z^2+3)$$
$$= (z^2+3)(z^2+2)$$
$$= [z^2-(-1)3][z^2-(-1)2]$$
$$= [z^2-i^2(\sqrt{3})^2][z^2-i^2(\sqrt{2})^2]$$
$$= [z^2-(\sqrt{3}\,i)^2][z^2-(\sqrt{2}\,i)^2]$$
$$= (z-\sqrt{3}\,i)(z+\sqrt{3}\,i)(z-\sqrt{2}\,i)(z+\sqrt{2}\,i)$$

(viii) $z^4 - 32z^2 - 3969$

Sol

$$z^4-32z^2-3969 = z^4+49z^2-81z^2-3969$$
$$= z^2(z^2+49)-81(z^2+49)$$
$$= (z^2+49)(z^2-81)$$
$$= [z^2-(-1)49](z^2-9^2)$$

$$= \left[z^2 - i^2 7^2\right](z-9)(z+9)$$

$$= \left[z^2 - (7i)^2\right](z-9)(z+9)$$

$$= (z-7i)(z+7i)(z-9)(z+9)$$

3. Find the roots of $z^4 + 7z^2 - 144 = 0$ and hence express it as a product of linear factors.

Sol

$$z^4 + 16z^2 - 9z^2 - 144 = 0$$

$$z^2(z^2+16) - 9(z^2+16) = 0$$

$$(z^2+16)(z^2-9) = 0$$

$$[z^2 - (-1)4^2](z^2 - 3^2) = 0$$

$$[z^2 - (4i)^2](z-3)(z+3) = 0$$

$$(z-4i)(z+4i)(z-3)(z+3) = 0$$

$z-4i=0$, $z+4i=0$, $z-3=0$, $z+3=0$

$z=4i$, $z=-4i$, $z=3$, $z=-3$

$$\{-4i,\ 4i,\ -3,\ 3\}$$

So, given polynomial can be written as:

$$= (z-(-4i))(z-4i)(z-(-3))(z-3)$$

4. Solve the following complex quadratic equation by completing square method:

(i) $2z^2 - 3z + 4 = 0$

Sol

$$2z^2 - 3z + 4 = 0$$

$$2\left[z^2 - \frac{3}{2}z + 2\right] = 0$$

$$z^2 - \frac{3}{2}z + 2 = \frac{0}{2}$$

$$z^2 - 2(z)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 + 2 = 0$$

$$\left(z - \frac{3}{4}\right)^2 - \frac{9}{16} + 2 = 0$$

$$\left(z - \frac{3}{4}\right)^2 = \frac{9}{16} - 2$$

$$= \frac{9-32}{16}$$

$$\left(z - \frac{3}{4}\right)^2 = \frac{-23}{16}$$

Taking square root, $z - \frac{3}{4} = \pm\sqrt{\frac{-23}{16}}$

$$z - \frac{3}{4} = \pm\frac{\sqrt{23}}{4}i$$

$$z = \frac{3}{4} \pm \frac{\sqrt{23}}{4}i$$

$$z = \frac{3 \pm \sqrt{23}i}{4}$$

(ii) $z^2 - 6z + 30 = 0$

Sol

$$z^2 - 2(z)(3) + 3^2 + 21 = 0$$

$$(z-3)^2 + 21 = 0$$

$$(z-3)^2 = -21$$

Taking square root

$$z - 3 = \pm\sqrt{-21}$$

$$= \pm\sqrt{21}\, i$$

$$z = 3 \pm \sqrt{21}\, i$$

(iii) $3z^2 - 18z + 50 = 0$

Sol

$$3\left(z^2 - 6z + \frac{50}{3}\right) = 0$$

$$z^2 - 6z + \frac{50}{3} = \frac{0}{3}$$

$$z^2 - 2(z)(3) + 3^2 - 3^2 + \frac{50}{3} = 0$$

$$(z-3)^2 - 9 + \frac{50}{3} = 0$$

$$(z-3)^2 = 9 - \frac{50}{3}$$

$$(z-3)^2 = \frac{27-50}{3}$$

$$(z-3)^2 = \frac{-23}{3}$$

$$z-3 = \pm\sqrt{\frac{-23}{3}}$$

$$z-3 = \pm\sqrt{\frac{23}{3}}\,i$$

$$z = 3 \pm \sqrt{\frac{23}{3}}\,i$$

$$z = 3 \pm \frac{\sqrt{69}}{3}\,i$$

(iv) $z^2 + 4z + 13 = 0$

Sol

$$z^2 + 2(z)(2) + 2^2 + 9 = 0$$

$$(z+2)^2 + 9 = 0$$

$$(z+2)^2 = -9$$

Taking square root,

$$z+2 = \pm\sqrt{-9}$$

$$z+2 = \pm 3i$$

$$z = -2 \pm 3i$$

(v) $2z^2+6z+9=0$

Sol

$$\frac{2z^2+6z+9}{2}=\frac{0}{2}$$

$$z^2+3z+\frac{9}{2}=0$$

$$z^2+2z\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+\frac{9}{2}=0$$

$$\left(z+\frac{3}{2}\right)^2-\frac{9}{4}+\frac{9}{2}=0$$

$$\left(z+\frac{3}{2}\right)^2=\frac{9}{4}-\frac{9}{2}$$

$$=\frac{9-18}{4}$$

$$\left(z+\frac{3}{2}\right)^2=\frac{-9}{4}$$

Taking square root,

$$z+\frac{3}{2}=\pm\sqrt{\frac{-9}{4}}$$

$$=\pm\frac{3}{2}i$$

$$z=-\frac{3}{2}\pm\frac{3}{2}i$$

(vi) $3z^2 - 5z + 7 = 0$

Sol

$$3\left[z^2 - \frac{5}{3}z + \frac{7}{3}\right] = 0$$

$$z^2 - \frac{5}{3}z + \frac{7}{3} = \frac{0}{3}$$

$$z^2 - 2z\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2 + \frac{7}{3} = 0$$

$$\left(z - \frac{5}{6}\right)^2 - \frac{25}{36} + \frac{7}{3} = 0$$

$$\left(z - \frac{5}{6}\right)^2 = \frac{25}{36} - \frac{7}{3}$$

$$= \frac{25 - 84}{36}$$

$$\left(z - \frac{5}{6}\right)^2 = -\frac{59}{36}$$

Taking square root,

$$z - \frac{5}{6} = \pm\sqrt{\frac{-59}{36}}$$

$$= \pm\frac{\sqrt{-59}}{6}$$

$$z = \frac{5}{6} \pm \frac{\sqrt{-59}}{6}$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Complex Polynomials as a Product of Linear Factors
- Exercise 1.3: Q5 - Q8

5. Solve the following equations:

(i) $2z^4 - 32 = 0$

Sol

$$2.(z^4 - 16) = 0$$

$$z^4 - 16 = \frac{0}{2}$$

$$(z^2)^2 - 4^2 = 0$$

$$(z^2 + 4).(z^2 - 4) = 0$$

$$z^2 + 4 = 0$$

$$z^2 + 2^2 = 0$$

$$z^2 - (-1)\,2^2 = 0$$

$$z^2 - (2i)^2 = 0$$

$$(z + 2i).(z - 2i) = 0$$

$$z + 2i = 0, \qquad z - 2i = 0$$

$$z = -2i, \qquad z = 2i$$

$$z^2 - 4 = 0$$

$$z^2 - 2^2 = 0$$

$$(z+2).(z-2) = 0$$

$$z + 2 = 0, \qquad z - 2 = 0$$

$$z = -2, \qquad z = 2$$

$$S.S. = \{\pm 2, \pm 2i\}$$

(ii) $3z^5 - 243z = 0$

Sol

$$3z(z^4 - 81) = 0$$

$$z.(z^4 - 81) = 0/3$$

$\boxed{z = 0}$, $\quad z^4 - 81 = 0$

$$(z^2)^2 - 9^2 = 0$$

$$(z^2 + 9).(z^2 - 9) = 0$$

$$z^2 + 9 = 0$$

$$z^2 + 3^2 = 0$$

$$z^2 - (-1)3^2 = 0$$

$$z^2 - i^2 3^2 = 0$$

$$z^2 - (3i)^2 = 0$$

$$(z + 3i).(z - 3i) = 0$$

$z + 3i = 0$, $\quad z - 3i = 0$

$z = -3i$, $\quad z = 3i$

$$z^2 - 9 = 0$$

$$z^2 - 3^2 = 0$$

$$(z+3)(z-3) = 0$$

$z + 3 = 0$, $\quad z - 3 = 0$

$z = -3$, $\quad z = 3$

$$S.S. = \{0, -3, 3, -3i, 3i\}$$

(iii) $5z^5 - 5z = 0$

Sol

$$5z(z^4-1)=0$$

$$z(z^4-1)=\frac{0}{5}$$

$$z.(z^4-1)=0$$

$z=0,$ $\quad$ $z^4-1=0$

$$(z^2)^2-1^2=0$$

$$(z^2+1).(z^2-1)=0$$

$z^2+1=0,$ | $z^2-1=0$

$z^2=-1$ | $z^2-1^2=0$

Taking square root, | $(z+1).(z-1)=0$

$z=\pm\sqrt{-1}$ | $z+1=0,$ | $z-1=0$

| $z=-1,$ | $z=1$

$z=\pm i$

$$S.S.=\{0,\ \pm 1,\ \pm i\}$$

(iv) $z^3 - 5z^2 + z - 5 = 0$

Sol

$$z^2(z-5) + (z-5) = 0$$

$$(z-5).(z^2+1) = 0$$

$z-5=0,$

$z = 5$

$z^2+1=0$

$z^2 = -1$

Taking square root,

$z = \pm\sqrt{-1}$

$z = \pm i$

$$S.S. = \{5, \pm i\}$$

(v) $4z^4 - 25z^2 - 21 = 0$

$-25 = -28 + 3$

Sol

$$4z^4 - 25z^2 - 21 = 0$$

$$4z^4 - 28z^2 + 3z^2 - 21 = 0$$

$$4z^2(z^2 - 7) + 3(z^2 - 7) = 0$$

$$(z^2 - 7).(4z^2 + 3) = 0$$

$z^2 - 7 = 0,$ $\qquad$ $4z^2 + 3 = 0$

$z^2 = 7$ $\qquad$ $4z^2 = -3$

$z^2 = \frac{-3}{4}$

$z = \pm\sqrt{7}$

$z = \pm\sqrt{\frac{-3}{4}}$

$z = \pm\frac{\sqrt{3}}{2}i$

$$S.S = \left\{\pm\sqrt{7},\ \pm\frac{\sqrt{3}}{2}i\right\}$$

(vi) $z^3 + z^2 + z + 1 = 0$

Sol

$$z^2.(z+1) + (z+1) = 0$$

$$(z+1).(z^2+1) = 0$$

$$z+1=0, \qquad z^2+1=0$$

$$z = -1 \qquad z^2 = -1$$

$$z = \pm\sqrt{-1}$$

$$z = \pm i$$

$$S.S. = \{-1, \pm i\}$$

6. Find a polynomial of degree 3 with zeros 3, $-2i$, $2i$ and satisfying $P(1) = 20$.

Sol

Let $z_0 = 3$, $z_1 = -2i$, $z_2 = 2i$

Then,

$$P(z) = c(z-z_0)(z-z_1)(z-z_2)$$

$$= c(z-3)(z-(-2i))(z-2i)$$

$$= c(z-3)(z+2i)(z-2i)$$

$$= c(z-3)(z^2-(2i)^2)$$

$$= c(z-3)(z^2-4i^2)$$

$$= c(z-3)(z^2-4(-1))$$

$$= c(z-3)(z^2+4)$$

$$P(z) = c(z^3-3z^2+4z-12) \quad \text{—— } 1$$

Using $P(1) = 20$,

$$1 \Longrightarrow \quad P(1) = c[1^3-3(1^2)+4(1)-12]$$

$$20 = c(1-3+4-12)$$

$$20 = c(-10)$$

$$\frac{20}{-10} = c$$

$c = -2$ Substitute into ①

$$P(z) = -2(z^3 - 3z^2 + 4z - 12)$$

$$P(z) = -2z^3 + 6z^2 - 8z + 24$$

7. Find a polynomial of degree 4 with zeros $2i$, $-2i$, 1, -1, and satisfying $P(2) = 240$.

Sol

Let $z_0 = 2i$, $z_1 = -2i$, $z_2 = 1$, $z_3 = -1$

Then

$$P(z) = c(z - z_0)(z - z_1)(z - z_2)(z - z_3)$$

$$= c(z - 2i)(z - (-2i))(z - 1)(z - (-1))$$

$$= c(z - 2i)(z + 2i)(z - 1)(z + 1)$$

$$= c(z^2 - (2i)^2)(z^2 - 1^2)$$

$$= c(z^2 - 4i^2)(z^2 - 1)$$

$$= c(z^2 - 4(-1))(z^2 - 1)$$

$$P(z) = c(z^2+4)(z^2-1)$$

$$= c(z^4 - z^2 + 4z^2 - 4)$$

$$P(z) = c(z^4 + 3z^2 - 4) \quad \text{———} \quad (1)$$

Using $P(2) = 240$,

$$P(2) = c(2^4 + 3(2^2) - 4)$$

$$240 = c(16 + 12 - 4)$$

$$240 = c(24)$$

$$\frac{240}{24} = c$$

$c = 10$ Put in (1)

$$P(z) = 10(z^4 + 3z^2 - 4)$$

$$P(z) = 10z^4 + 30z^2 - 40$$

8. Find a polynomial of degree 4 with zeros 4, -4, $1+i$, $1-i$ and satisfying $P(2) = 72$.

Sol

Let $z_0 = 4$, $z_1 = -4$, $z_2 = 1+i$, $z_3 = 1-i$

Then

$$P(z) = c(z-z_0)(z-z_1)(z-z_2)(z-z_3)$$

$$= c(z-4)(z-(-4))(z-(1+i))(z-(1-i))$$

$$= c(z-4)(z+4)(z-1-i)(z-1+i)$$

$$= c(z^2-4^2)((z-1)^2-i^2)$$

$$= c(z^2-4^2)(z^2-2z+1-(-1))$$

$$= c(z^2-16)(z^2-2z+2)$$

$$= c(z^4-2z^3+2z^2-16z^2+32z-32)$$

$$P(z) = c(z^4-2z^3-14z^2+32z-32) \quad \text{—— } ①$$

Using $P(2) = 72$,

$$P(2) = c(2^4-2(2^3)-14(2^2)+32(2)-32)$$

$$72 = c(\cancel{16}-\cancel{16}-56+64-32)$$

$$72 = c(-24)$$

$$\frac{72}{-24} = c$$

$c = -3$ Put in ①

$$P(z) = -3(z^4 - 2z^3 - 14z^2 + 32z - 32)$$

$$P(z) = -3z^4 + 6z^3 + 42z^2 - 96z + 96$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Cube and Fourth Roots of Unity
- Exercise 1.4: Concepts and Examples

1.4 Three Cube Roots of Unity

Let x be a cube root of unity

$$\therefore \quad x = (1)^{\frac{1}{3}}$$

$$\Rightarrow \quad x^3 = 1$$

$$\Rightarrow \quad x^3 - 1 = 0$$

$$\Rightarrow \quad (x-1)(x^2+x+1) = 0$$

Either $\quad x - 1 = 0 \Rightarrow \quad x = 1$

or $\quad x^2 + x + 1 = 0$

$$\therefore \quad a=1=b=c \quad x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$\Rightarrow \quad \boxed{x = \frac{-1 \pm \sqrt{3}i}{2}} \qquad (\because \ \sqrt{-1} = i)$$

> **Note:**
>
> We know that the numbers containing i are called **Complex numbers**. So $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$ are called complex or imaginary cube roots of unity.

$$\sqrt[3]{x} = x^{1/3}$$

$$x^{1/4}$$

Thus, the three cube roots of unity are:

$$1, \quad \frac{-1+\sqrt{3}i}{2} \quad \text{and} \quad \frac{-1-\sqrt{3}i}{2}$$

$$\because a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$(\omega^2)^2 = \omega$$

1.4.1 Properties of Cube Roots of Unity

(i) Each complex cube root of unity is square of the other

If $\frac{-1+\sqrt{3}i}{2} = \omega$, then $\frac{-1-\sqrt{3}i}{2} = \omega^2$,

and if $\frac{-1+\sqrt{3}i}{2} = \omega$, then $\frac{-1+\sqrt{3}i}{2} = \omega^2$ [ω is read as omega]

(ii) The sum of all the three cube roots of unity is zero i.e., $1 + \omega + \omega^2 = 0$

(iii) The product of all the three cube roots of unity is unity i.e., $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$

Four Fourth Roots of Unity

Let x be the fourth root of unity

$$\therefore \qquad x = (1)^{\frac{1}{4}}$$

$$\Rightarrow \qquad x^4 = 1$$

$$\Rightarrow \qquad x^4 - 1 = 0 \longrightarrow (x^2)^2 - 1^2 = 0$$

$$\Rightarrow \qquad (x^2 - 1)(x^2 + 1) = 0$$

$$\Rightarrow \qquad x^2 - 1 = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$

and $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm i$.

Hence four fourth roots of unity are: $1, -1, i, -i$.

Properties of four Fourth Roots of Unity

We have found that the four fourth roots of unity are: $1, -1, +i, -i$

$1 + (-1) = 0$

$i, -i$

(i) Sum of all the four fourth roots of unity is zero

$\because \quad 1 + (-1) + i + (-i) = 0$

(ii) The real fourth roots of unity are additive inverses of each other

1 and -1 are the real fourth roots of unity and $1 + (-1) = 0 = (-1) + 1$

(iii) Both the imaginary fourth roots of unity are conjugate of each other

i and $-i$ are imaginary fourth roots of unity, which are obviously conjugates of each other.

(iv) Product of all the fourth roots of unity is -1 i.e., $1 \times (-1) \times i \times (-i) = -1$

$1(-1) \times i \times (-i) = i^2 = -1$

Example 11: Prove that: $(x^3 + y^3) = (x + y)(x + \omega y)(x + \omega^2 y)$

$1 + \omega + \omega^2 = 0$

Solution: R.H.S $= (x + y)(x + \omega y)(x + \omega^2 y)$

$$= (x + y)[x^2 + (\omega + \omega^2)yx + \omega^3 y^2]$$

$$= (x + y)(x^2 - xy + y^2) = x^3 + y^3 \quad \{\because \omega^3 = 1, \omega + \omega^2 = -1\} = \text{L.H.S.}$$

Hence proved.

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Cube and Forth Roots of Unity
- Exercise 1.4: Q1 - Q4

EXERCISE 1.4

1. Find the three cube roots of:

$8^{1/3}$

(i) 8

Sol:

Let x be a cube root of 8,

$$x = 8^{1/3}$$

$$\rightarrow x^3 = 8$$

$$x^3 - 8 = 0$$

$$x^3 - 2^3 = 0$$

$$(x-2).(x^2+(x)(2)+2^2) = 0 \qquad \because a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$x - 2 = 0$$

$$x = 2$$

$$x^2 + 2x + 4 = 0$$

$$x^2 + 2(x)(1) + 1^2 + 3 = 0$$

$$(x+1)^2 + 3 = 0$$

$$(x+1)^2 = -3$$

Taking square root,

$$x + 1 = \pm\sqrt{-3}$$

$$x + 1 = \pm\sqrt{3}\, i$$

$$x = 2\left(\frac{-1 \pm \sqrt{3}\, i}{2}\right)$$

$$x = 2\omega,\quad 2\omega^2$$

$$\{2, 2\omega, 2\omega^2\}$$

(ii) -8

Sol

$$x = (-8)^{1/3}$$

$$x^3 = -8$$

$$x^3 + 8 = 0$$

$$x^3 + 2^3 = 0 \qquad \because a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$(x+2).(x^2 - 2x + 2^2) = 0$$

$$x + 2 = 0$$

$$x = -2$$

$$x^2 - 2x + 4 = 0$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$\because \sqrt{12} = \sqrt{2^2 \times 3}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$x = \frac{2 \pm 2\sqrt{3}\,i}{2}$$

$$x = 2\left(\frac{1 \pm \sqrt{3}\,i}{2}\right)$$

$$\{-2, -2\omega, -2\omega^2\}$$

$$2\left(\frac{1 + \sqrt{3}\,i}{2}\right), \quad 2\left(\frac{1 - \sqrt{3}\,i}{2}\right)$$

$$-2\left(\frac{-1 - \sqrt{3}\,i}{2}\right), \quad -2\left(\frac{-1 + \sqrt{3}\,i}{2}\right)$$

(iii) -27

Sol

$$x = (-27)^{1/3}$$

$$x^3 = -27$$

$$x^3 + 27 = 0$$

$$x^3 + 3^3 = 0$$

$$(x+3)\left(x^2 - (x)(3) + 3^2\right) = 0$$

$$x + 3 = 0$$

$$x = -3$$

$$x^2 - 3x + 9 = 0$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{9-36}}{2}$$

$$= \frac{3 \pm \sqrt{-27}}{2}$$

$$= \frac{3 \pm 3\sqrt{3}\, i}{2}$$

$$= -3\left(\frac{-1 \pm \sqrt{3}\, i}{2}\right)$$

$$-3\left(\frac{-1+\sqrt{3}\, i}{2}\right), \quad -3\left(\frac{-1-\sqrt{3}\, i}{2}\right)$$

$$-3\omega \qquad -3\omega^2$$

$$\{-3, -3\omega, -3\omega^2\}$$

(iv) 64

Sol

$$x = 64^{1/3}$$

$$x^3 = 64$$

$$x^3 - 64 = 0$$

$$x^3 - 4^3 = 0$$

$$(x-4)\left(x^2 + (x)(4) + 4^2\right) = 0$$

$$x - 4 = 0$$

$$x = 4$$

$$x^2 + 4x + 16 = 0$$

$$x^2 + 2(x)(2) + 2^2 + 12 = 0$$

$$(x+2)^2 = -12$$

$$x + 2 = \pm\sqrt{-12}$$

$$x + 2 = \pm 2\sqrt{3}\,i$$

$$x = -2 \pm 2\sqrt{3}\,i$$

$$x = 2\left(-1 \pm \sqrt{3}\,i\right)$$

$$x = 4\left(\frac{-1 \pm \sqrt{3}\,i}{2}\right)$$

$$\{4,\ 4\omega,\ 4\omega^2\}$$

(v) -125

Sol

$$x = (-125)^{1/3}$$

$$x^3 = -125$$

$$x^3 + 125 = 0$$

$$x^3 + 5^3 = 0$$

$$(x+5).\left(x^2 - (x)(5) + 5^2\right) = 0$$

$$x + 5 = 0$$

$$x = -5$$

$$x^2 - 5x + 25 = 0$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(25)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25 - 100}}{2}$$

$$= \frac{5 \pm \sqrt{-75}}{2}$$

$$= \frac{5 \pm 5\sqrt{3}i}{2}$$

$$\sqrt{75} = \sqrt{3 \times 5^2}$$

$$= 5\sqrt{3}$$

$$x = -5\left(\frac{-1 \pm \sqrt{3}i}{2}\right)$$

$$\{-5, -5\omega, -5\omega^2\}$$

2. Find the four fourth roots of 16, 81, 625. Also show that their sum is zero in each case.

Sol (a)

Let x be a fourth root of 16,

$$x = 16^{1/4}$$

$$x^4 = 16$$

$$x^4 - 16 = 0$$

$$x^4 - 2^4 = 0$$

$$(x^2)^2 - (2^2)^2 = 0$$

$$(x^2 - 2^2).(x^2 + 2^2) = 0$$

$$x^2 - 2^2 = 0$$

$$x^2 = 2^2$$

$$x = \pm\sqrt{2^{\cancel{2}}}$$

$$x = \pm 2$$

$$x^2 + 2^2 = 0$$

$$x^2 = -2^2$$

$$x = \pm\sqrt{-2^{\cancel{2}}}$$

$$x = \pm\sqrt{2^{\cancel{2}}}\, i$$

$$x = \pm 2i$$

$$\{-2, 2, -2i, 2i\}$$

Now,

$$-2 + 2 + (-2i) + 2i = 0$$

Sol (b)

$$x = 81^{1/4}$$

$$x^4 = 81$$

$$x^4 - 81 = 0$$

$$x^4 - 3^4 = 0$$

$$(x^2)^2 - (3^2)^2 = 0$$

$$(x^2 - 3^2)(x^2 + 3^2) = 0$$

$x^2 - 3^2 = 0$ or $x^2 + 3^2 = 0$

$(x-3)(x+3) = 0$ $\qquad$ $x^2 = -3^2$

$x - 3 = 0$, $\quad x + 3 = 0$ $\qquad$ $x = \pm\sqrt{-3^2}$

$x = 3$, $\quad x = -3$ $\qquad$ $x = \pm 3i$

$$\{-3,\ 3,\ -3i,\ 3i\}$$

Now,

$$-3 + 3 + (-3i) + 3i = 0$$

Sol (c)

Let x be a fourth root of 625,

$$x = 625^{1/4}$$

$$x^4 = 625$$

$$x^4 - 625 = 0$$

$$x^4 - 5^4 = 0$$

$$(x^2)^2 - (5^2)^2 = 0$$

$$(x^2 - 5^2).(x^2 + 5^2) = 0$$

$$x^2 - 5^2 = 0$$

$$(x-5)(x+5) = 0$$

$$x - 5 = 0, \quad x + 5 = 0$$

$$x = 5, \quad x = -5$$

$$x^2 + 5^2 = 0$$

$$x^2 = -5^2$$

$$x = \pm\sqrt{-5^2}$$

$$x = \pm 5i$$

$$\{-5, 5, -5i, 5i\}$$

Now,

$$-5 + 5 + (-5i) + 5i = 0$$

3. If $1, \omega, \omega^2$ are the cube roots of unity, show that $1+\omega^n+\omega^{2n}=3$ where 'n' is a multiple of 3 respectively.

Sol

Since 'n' is a multiple of 3, so

$n = 3k$ where $k \in \mathbb{Z}$

$$\text{L.H.S.} = 1+\omega^n+\omega^{2n}$$

$$= 1+\omega^{3k}+\omega^{2(3k)}$$

$$= 1+(\omega^3)^k+(\omega^3)^{2k} \qquad \because \omega^3 = 1$$

$$= 1+1^k+1^{2k}$$

$$= 1+1+1$$

$$= 3 = \text{R.H.S.}$$

4. Evaluate:

(i) $\left(\frac{-1+\sqrt{-3}}{2}\right)^7 + \left(\frac{-1-\sqrt{-3}}{2}\right)^7$

Sol

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^7 + \left(\frac{-1-\sqrt{-3}}{2}\right)^7 = \left(\frac{-1+\sqrt{3}\,i}{2}\right)^7 + \left(\frac{-1-\sqrt{3}\,i}{2}\right)^7$$

$$= \omega^7 + (\omega^2)^7$$

$$= \omega^6 \cdot \omega + \omega^{14}$$

$$= (\omega^3)^2 \omega + \omega^{12} \cdot \omega^2$$

$$= (1)^2 \cdot \omega + (\omega^3)^4 \cdot \omega^2$$

$$= \omega + (1)^4 \omega^2$$

$$= \omega + \omega^2$$

$\because 1 + \omega + \omega^2 = 0$

$$= -1$$

(ii) $\left(-1+\sqrt{-3}\right)^5+\left(-1-\sqrt{-3}\right)^5$

Sol

$$\omega = \frac{-1+\sqrt{3}i}{2}$$

$$\therefore 2\omega = -1+\sqrt{3}i$$

$$\omega^2 = \frac{-1-\sqrt{3}i}{2}$$

$$\therefore 2\omega^2 = -1-\sqrt{3}i$$

$$\left(-1+\sqrt{-3}\right)^5+\left(-1-\sqrt{-3}\right)^5 = (2\omega)^5+\left(2\omega^2\right)^5$$

$$= 32\omega^5+32\omega^{10}$$

$$= 32\left(\omega^5+\omega^{10}\right)$$

$$= 32\left(\omega^3.\omega^2+\omega^9.\omega\right)$$

$$= 32\left(1.\omega^2+\left(\omega^3\right)^3\omega\right)$$

$$= 32\left(\omega^2+(1)^3\omega\right)$$

$$= 32\left(\omega^2+\omega\right)$$

$$= 32(-1)$$

$$= -32$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Cube and Forth Roots of Unity
- Exercise 1.4: Q5 - Q9

YouTube Channel: The Mathematics Outlet

5. Show that:

$(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})\ldots$ to $2n$ factors $= 2^{2n}$.

$\because \omega^3 = 1$

$\because 1+\omega+\omega^2 = 0$

Sol

$$\text{L.H.S.} = (1-\omega+\omega^2)(1-\omega^2+\omega^3\cdot\omega)(1-\omega^3\cdot\omega+(\omega^3)^2\omega^2)$$
$$(1-(\omega^3)^2\omega^2+(\omega^3)^5\omega)\ldots 2n \text{ factors}$$

$$= (1-\omega+\omega^2)(1-\omega^2+1\cdot\omega)(1-1\cdot\omega+(1^2)\omega^2)$$
$$(1-(1^2)\omega^2+(1^5)\omega)\ldots 2n \text{ factors}$$

$$= (1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)(1-\omega^2+\omega)\ldots 2n$$

$\because 1+\omega = -\omega^2$ $\quad\because 1+\omega^2 = -\omega$

$$= (-\omega-\omega)(-\omega^2-\omega^2)(-\omega-\omega)(-\omega^2-\omega^2)\ldots 2n$$

$$= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)\ldots 2n$$

$$= (-2)^{2n}\,\omega\,\omega^2\,\omega\,\omega^2 \ldots 2n$$

$$= 2^{2n}\,\omega^3\cdot\omega^3\cdot\omega^3 \ldots n$$

$$= 2^{2n}(1\cdot 1\cdot 1 \ldots n)$$

$$= 2^{2n} = \text{R.H.S.}$$

6. Prove that $\left(\frac{i+\sqrt{3}}{2}\right)^8 + \left(\frac{i-\sqrt{3}}{2}\right)^8 = -1.$

Sol

$$\text{L.H.S.} = \left(\frac{i+\sqrt{3}}{2}\right)^8 + \left(\frac{i-\sqrt{3}}{2}\right)^8 \qquad \because \omega = \frac{-1+\sqrt{3}i}{2}$$

$$= \left(\frac{i}{i} \times \frac{i+\sqrt{3}}{2}\right)^8 + \left(\frac{i}{i} \times \frac{i-\sqrt{3}}{2}\right)^8$$

$$= \left(\frac{1}{i} \times \frac{i^2+\sqrt{3}i}{2}\right)^8 + \left(\frac{1}{i} \times \frac{i^2-\sqrt{3}i}{2}\right)^8$$

$$= \left(\frac{1}{i} \times \frac{-1+\sqrt{3}i}{2}\right)^8 + \left(\frac{1}{i} \times \frac{-1-\sqrt{3}i}{2}\right)^8$$

$$= \frac{1}{i^8}\,\omega^8 + \frac{1}{i^8}\left(\omega^2\right)^8$$

$$= \frac{1}{(i^2)^4}\left(\omega^3\right)^2\omega^2 + \frac{1}{(i^2)^4}\,\omega^{16}$$

$$= \frac{1}{(-1)^4}(1)^2\omega^2 + \frac{1}{(-1)^4}\left(\omega^3\right)^5\omega$$

$$= \omega^2 + (1)^5\omega$$

$$= \omega^2 + \omega \qquad \because 1+\omega+\omega^2=0$$

$$= -1 = \text{R.H.S.}$$

7. Evaluate $\sum_{K=0}^{5} \omega^{2K}$, where ω is an imaginary cube root of unity.

Sol

$$\sum_{K=0}^{5} \omega^{2K} = \omega^{2(0)} + \omega^{2(1)} + \omega^{2(2)} + \omega^{2(3)} + \omega^{2(4)} + \omega^{2(5)}$$

$$= \omega^{0} + \omega^{2} + \omega^{4} + \omega^{6} + \omega^{8} + \omega^{10} \qquad \because \omega^3 = 1$$

$$= 1 + \omega^2 + \omega^3 . \omega + (\omega^3)^2 + (\omega^3)^2 \omega^2 + (\omega^3)^3 \omega$$

$$= 1 + \omega^2 + 1 . \omega + (1^2) + (1^2)\omega^2 + (1^3)\omega$$

$$= 1 + \omega^2 + \omega + 1 + \omega^2 + \omega$$

$$= 0 + 0$$

$$= 0$$

8. If ω is an imaginary cube root of unity, prove that

$$\frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}=\omega$$

Sol

$$\text{L.H.S.} = \frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}$$

$$= \frac{\omega(a+b\omega^2+c\omega)}{\omega(a\omega^2+b\omega+c)}$$

$$= \frac{\omega(a+b\omega^2+c\omega)}{a\omega^3+b\omega^2+c\omega} \qquad \because \omega^3=1$$

$$= \frac{\omega\cancel{(a+b\omega^2+c\omega)}}{\cancel{a+b\omega^2+c\omega}}$$

$$= \omega = \text{R.H.S.}$$

9. If ω is an cube root of unity, prove that:

$$\frac{a\omega^{12}+b\omega^{17}+c\omega^{19}}{a\omega^{14}+b\omega^{22}+c\omega^{30}}=\omega$$

Sol

$$L.H.S. = \frac{a(\omega^3)^4+b(\omega^3)^5\omega^2+c(\omega^3)^6\omega}{a(\omega^3)^4\omega^2+b(\omega^3)^7\omega+c(\omega^3)^{10}}$$

$$=\frac{a(1^4)+b(1^5)\omega^2+c(1^6)\omega}{a(1^4)\omega^2+b(1^7)\omega+c(1^{10})}$$

$$=\frac{a+b\omega^2+c\omega}{a\omega^2+b\omega+c}$$

$$=\frac{\omega(a+b\omega^2+c\omega)}{\omega(a\omega^2+b\omega+c)}$$

$$=\frac{\omega(a+b\omega^2+c\omega)}{a\omega^3+b\omega^2+c\omega} \qquad \because \omega^3=1$$

$$=\frac{\omega\cancel{(a+b\omega^2+c\omega)}}{\cancel{a+b\omega^2+c\omega}}$$

$$=\omega = R.H.S.$$

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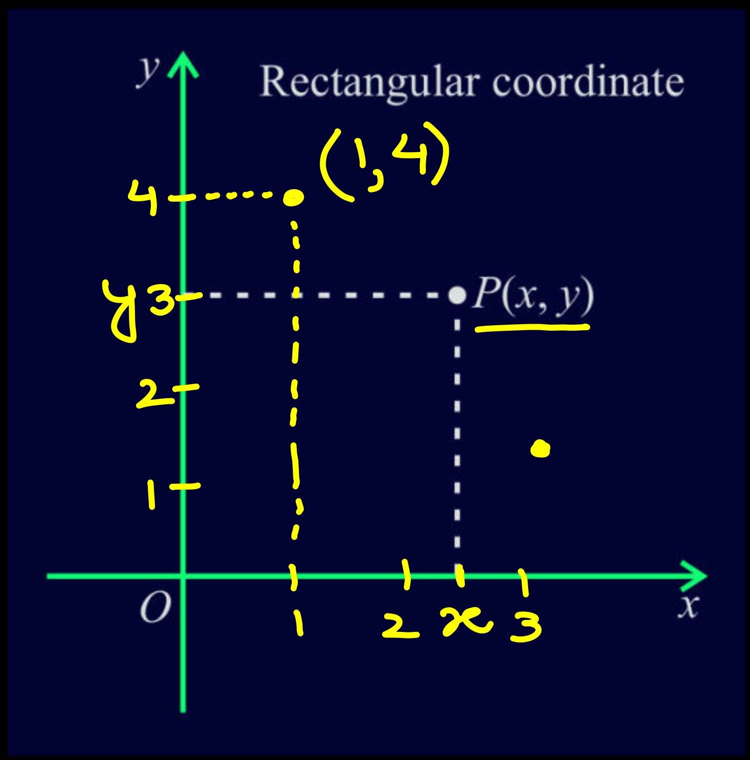
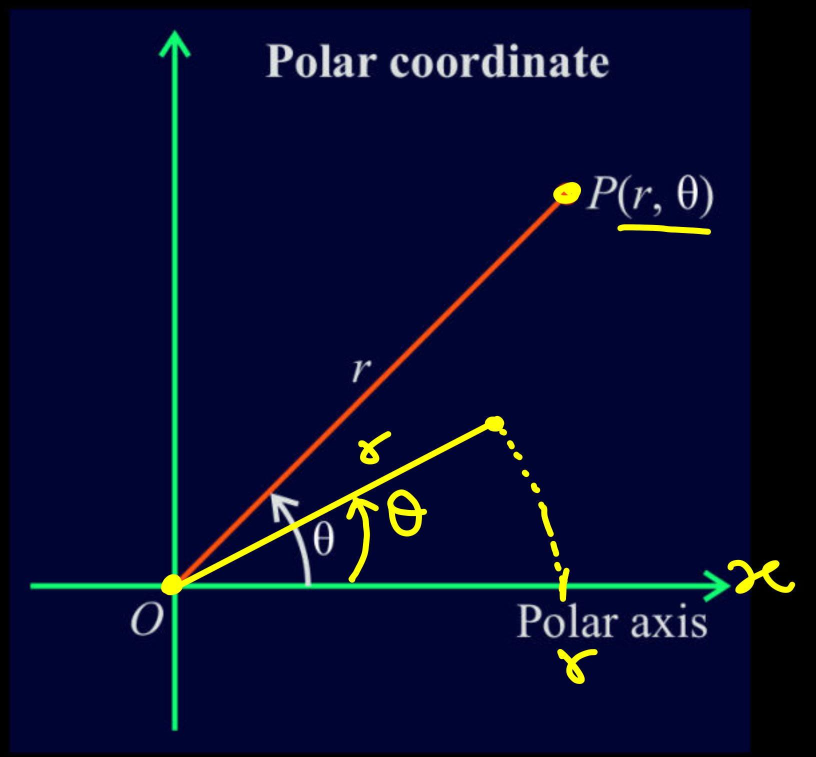
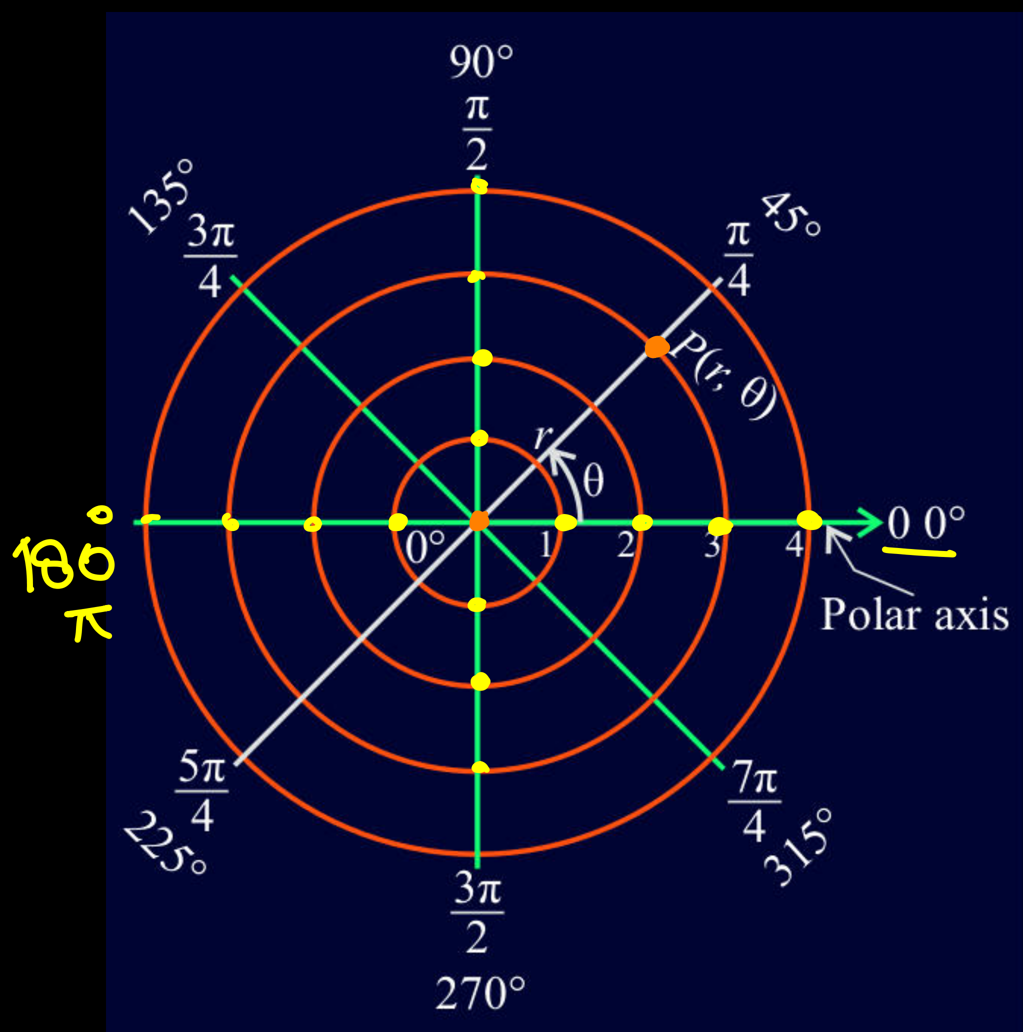
Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Polar Coordinate System
- Exercise 1.5: Concepts and Examples

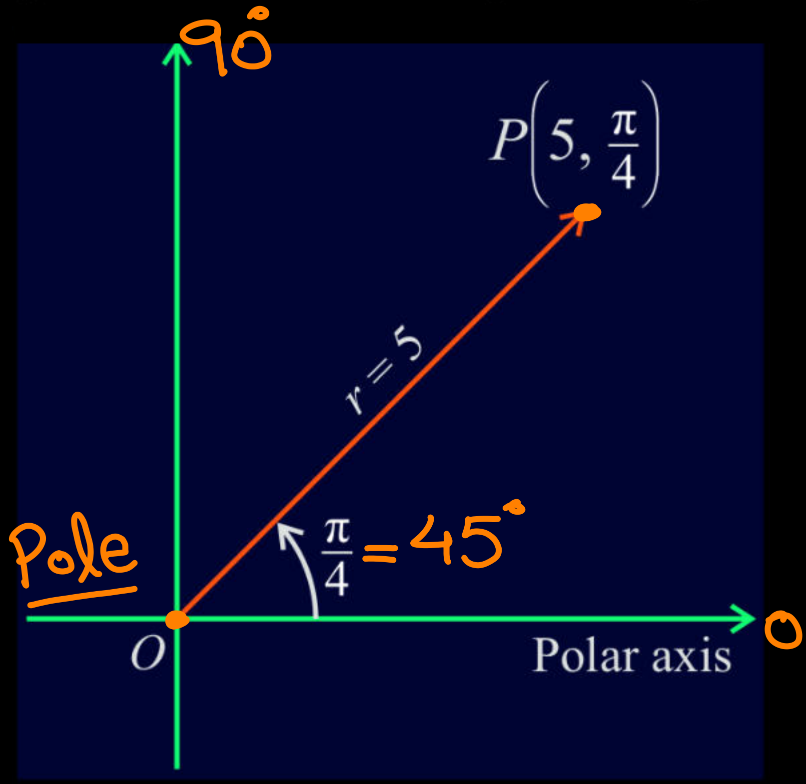
Polar Coordinates System



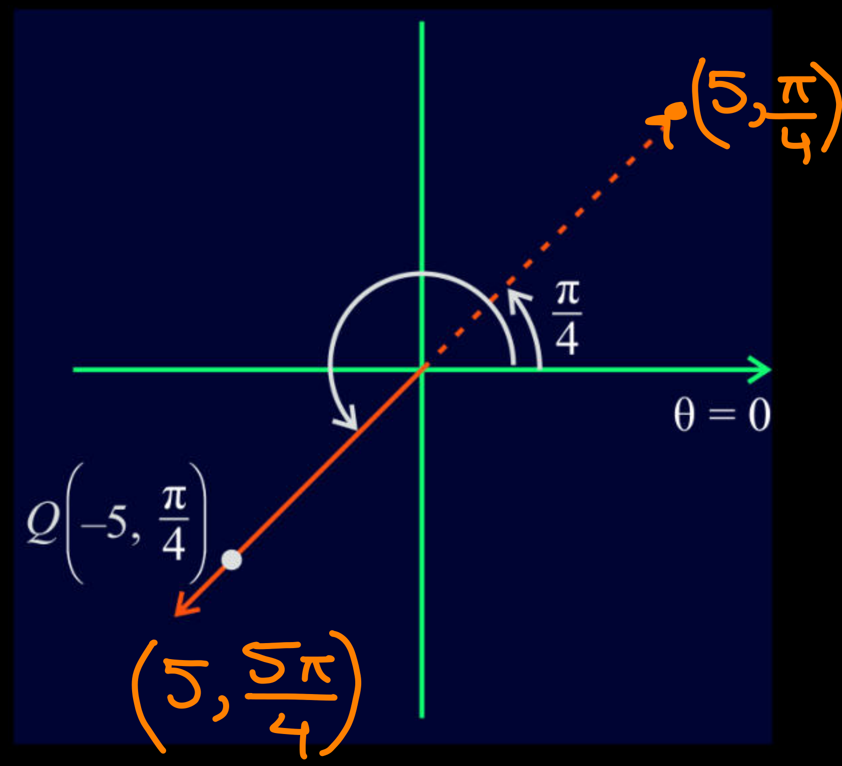




For example, the polar coordinates $\left(5, \dfrac{\pi}{4}\right)$ represent a point 5 units away from pole at an angle of $\dfrac{\pi}{4}$ radians.



$\left(-5, \dfrac{\pi}{4}\right)$



For example, the polar coordinates $\left(-5, \dfrac{\pi}{4}\right)$ represent a point 5 units away from the pole, but in the direction of $\dfrac{\pi}{4} + \pi = \dfrac{5\pi}{4}$ radians.

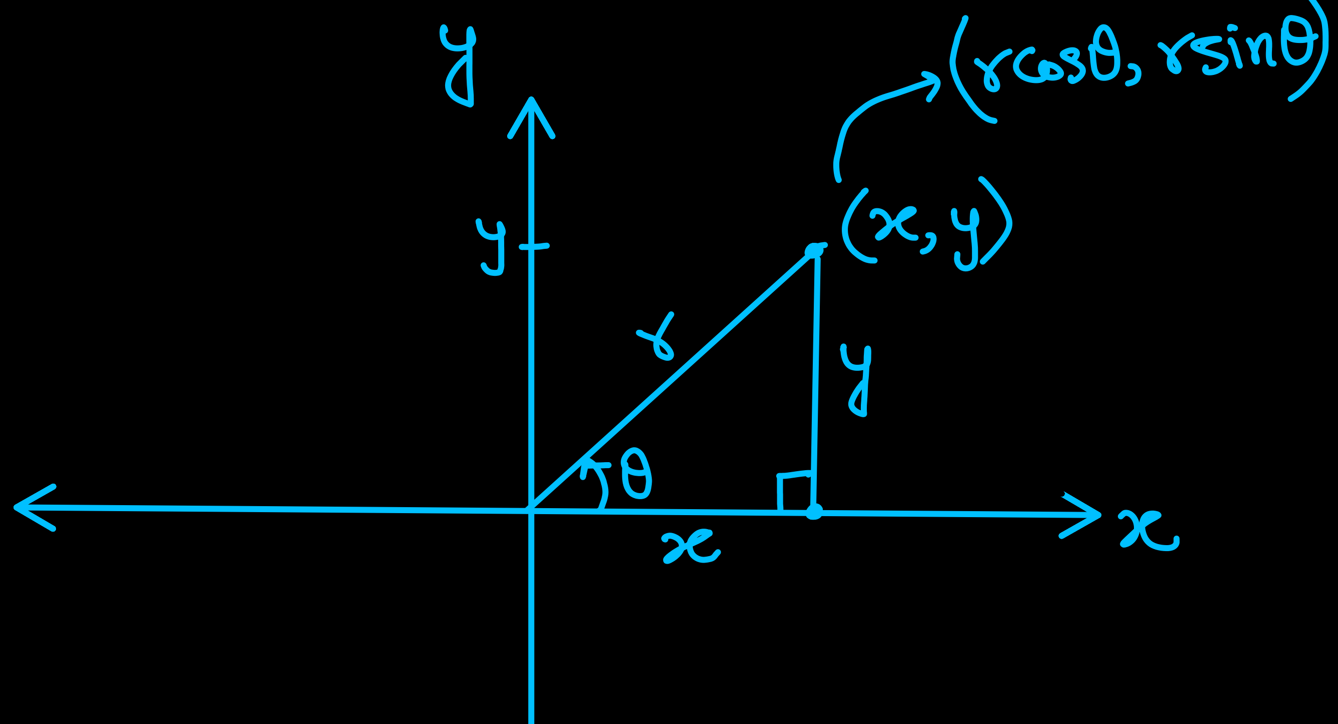
$z = x + iy$

$r^2 = x^2 + y^2$

$r = \sqrt{x^2 + y^2}$

$\sin\theta = \dfrac{y}{r} \implies y = r.\sin\theta$

$\cos\theta = \dfrac{x}{r} \implies x = r\cos\theta$



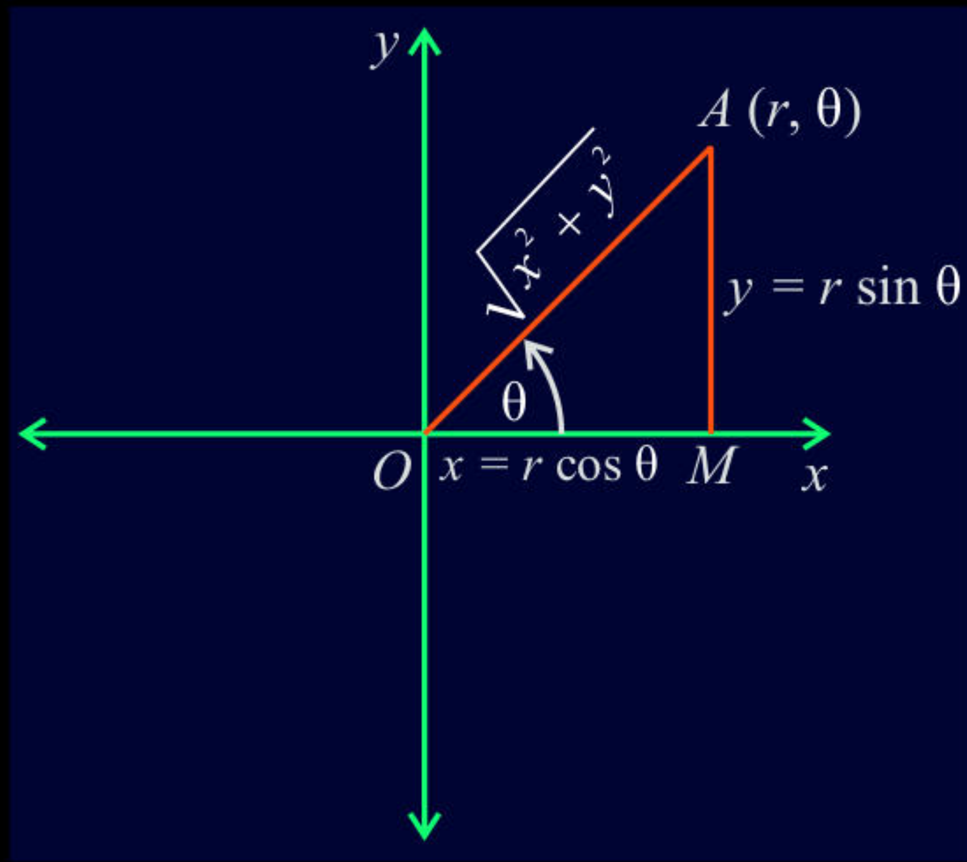
$\rightarrow z = x + iy$

$= r\cos\theta + ir\sin\theta$

$\rightarrow z = r(\cos\theta + i\sin\theta)$

The Polar Form of a Complex Number

Consider the adjoining diagram representing the complex number $z = x + iy$. From the diagram, we see that $x = r\cos\theta$ and $y = r\sin\theta$, where $r = |z|$ is modulus and θ is called an argument of z.



Hence $\quad x + iy = r\cos\theta + i\,r\sin\theta \qquad$ (i)

where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$ $(x \neq 0)$

Equation (i) is called the polar form of the complex number z.

→ **Note:** We can write $\cos\theta + i\sin\theta = \text{cis}\,\theta$

$\tan\theta = \frac{y}{x}$

argument: $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Example 12: Express the complex number $1 + i\sqrt{3}$ in polar form.

$x = r\cos\theta$

$y = r\sin\theta$

Solution: **Step – I** : Put $r\cos\theta = 1$ and $r\,\cancel{\cos\theta}\,\sin\theta = \sqrt{3}$

Step – II : $r^2 = (1)^2 + \left(\sqrt{3}\right)^2$

$\Rightarrow \quad r^2 = 1 + 3 = 4$

$\Rightarrow \quad r = 2$

Step – III : $\theta = \tan^{-1}\frac{\sqrt{3}}{1} = \tan^{-1}\sqrt{3} = 60°$

Thus → $1 + i\sqrt{3} = 2\cos 60° + i\,2\sin 60°$

$r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$

Note:

- If $x = 0, y > 0$ then $\theta = \frac{\pi}{2}$
- If $x = 0, y < 0$ then $\theta = -\frac{\pi}{2}$
- If $x = 0, y = 0$ then θ is undefined.

Principal Argument: The principal argument θ of a complex number $z = a + bi$ is the angle between the positive real axis and the line joining (a, b) to the origin in the Argand plane.

$$\text{Arg}\, z = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$\arg z \in (0, 2\pi]$

It is denoted by *arg*. It is a single, specific value of the argument, typically chosen within a standard range: $\text{Arg}\, z \in (-\pi, \pi]$.

Operations on Complex Numbers in Polar Form

Addition and Subtraction of Complex number in Polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form. The addition and subtraction of two numbers can be computed simply as

$$z_1 + z_2 = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$$

and $z_1 - z_2 = r_1(\cos\theta_1 + i\sin\theta_1) - r_2(\cos\theta_2 + i\sin\theta_2)$

Multiplication of Complex number in Polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form. The product of two complex numbers can be derived by multiplying them directly and simplifying

$$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \qquad \because\ i^2 = -1$$

$$z_1 \cdot z_2 = r_1 \cdot r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{(Using trigonometric identities)}$$

Thus, multiplying two complex numbers in polar form involves multiplying their moduli and summing their arguments i.e., $arg(z_1 \cdot z_2) = arg(z_1) + arg(z_2)$

Example 13: Find the product of $5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$.

Solution: Let $z_1 = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

Here, $r_1 = 5$ and $\theta_1 = \frac{\pi}{6}$, while $r_2 = 4$ and $\theta_2 = \frac{3\pi}{2}$

Substitute this value in the product formula

$$z_1 \cdot z_2 = r_1 \cdot r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$= 5 \times 4\left[\cos\left(\frac{\pi}{6} + \frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{6} + \frac{3\pi}{2}\right)\right] = 20\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Thus, the required product is $20\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.

Division of Complex Number in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form. The formula for division of two complex numbers in polar form can be derived by rationalizing the denominator.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \qquad \left(\begin{array}{l}\text{Multiply and divide the equation}\\ \text{by conjugate of } \cos\theta_2 + i\sin\theta_2\end{array}\right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right] \qquad \text{(Using trigonometric identities)}$$

Thus, the modulus of the division of two complex numbers equals the quotient of their moduli, while the arguments of the quotient is the difference between their arguments.

Thus, when dividing two complex numbers, the modulus of the result is the ratio of their moduli, and the argument of the result is the difference between their arguments

i.e., $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

Example 14: Divide $\frac{2}{7}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$ by $\frac{3}{5}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

Solution: Let $z_1 = \frac{2}{7}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$ and $z_2 = \frac{3}{5}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

Here, $r_1 = \frac{2}{7}, \theta_1 = \frac{7\pi}{6}, r_2 = \frac{3}{5}$ $and\, \theta_2 = -\frac{\pi}{2}$.

Substitute value in the quotient formula

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$= \frac{2}{7} \times \frac{5}{3}\left[\cos\left(\frac{7\pi}{6} - \left(-\frac{\pi}{2}\right)\right) + i\sin\left(\frac{7\pi}{6} - \left(-\frac{\pi}{2}\right)\right)\right]$$

$$\frac{z_1}{z_2} = \frac{10}{21}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$ is the required polar form of division of two complex number.

Example 15: If $z = x + iy$, then write the equation $|3z - i| = |3\bar{z} + 7|$ in term of x and y.

Solution: Given $|3z - i| = |3\bar{z} + 7|$ ...(i)

$$|3z - i| = |3(x+iy) - i| = |3x + i(3y-1)| = \sqrt{(3x)^2 + (3y-1)^2}$$

$$|3\bar{z} + 7| = |\overline{3x+3iy} + 7| = |3x - 3iy + 7| = |3x + 7 + i(-3y)| = \sqrt{(3x+7)^2 + (-3y)^2}$$

Substitutes these values in (i)

$$\left(\sqrt{(3x)^2 + (3y-1)^2}\right)^2 = \left(\sqrt{(3x+7)^2 + (-3y)^2}\right)^2$$

Taking square on both sides

$$(3x)^2 + (3y-1)^2 = (3x+7)^2 + (-3y)^2$$

$$\cancel{9x^2} + \cancel{9y^2} - 6y + 1 = \cancel{9x^2} + 42x + 49 + \cancel{9y^2}$$

$$\Rightarrow \quad -6y + 1 = 42x + 49$$

$$\Rightarrow \quad -6y = 42x + 48$$

$$\text{or} \quad \boxed{y = -7x - 8}$$

The equation $y = -7x - 8$ represents a straight line in the complex plane.

Example 16: Show that $(x+2)^2+y^2=8$ if $\arg\left(\frac{z+2i}{z-2i}\right)=\frac{3\pi}{4}$ for $z=x+iy$.

Solution: $\frac{z+2i}{z-2i}=\frac{x+iy+2i}{x+iy-2i}=\frac{x+i(y+2)}{x+i(y-2)}=\frac{x+i(y+2)}{x+i(y-2)}\times\frac{x-i(y-2)}{x-i(y-2)}$

$$\Rightarrow \quad \frac{z+2i}{z-2i}=\frac{(x^2+y^2-4)+4ix}{x^2+(y-2)^2}=\left(\frac{x^2+y^2-4}{x^2+(y-2)^2}\right)+i\left(\frac{4x}{x^2+(y-2)^2}\right)$$

As $\quad \rightarrow \arg\left(\frac{z+2i}{z-2i}\right)=\frac{3\pi}{4}$ $\qquad \tan^{-1}\left(\frac{y}{x}\right)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{4x}{\cancel{x^2+(y-2)^2}}}{\frac{x^2+y^2-4}{\cancel{x^2+(y-2)^2}}}\right)=\frac{3\pi}{4} \quad \Rightarrow \quad \frac{4x}{x^2+y^2-4}=\tan\frac{3\pi}{4}=-1$$

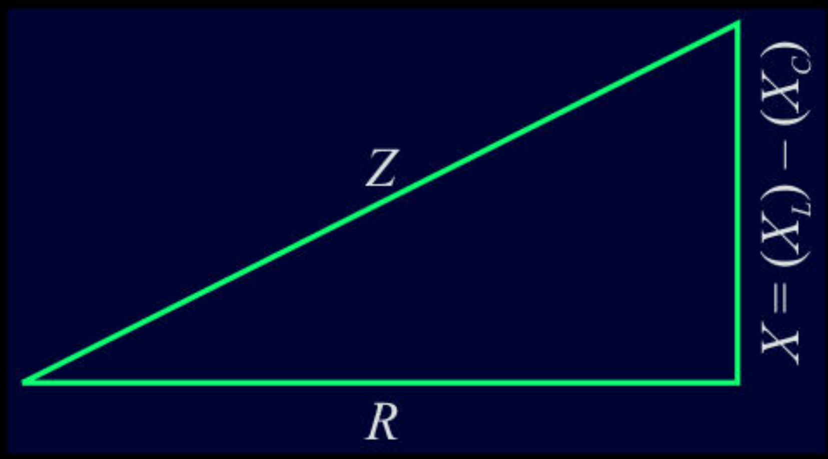
$$\Rightarrow \quad 4x=-1(x^2+y^2-4) \quad \Rightarrow \quad x^2+4x+y^2=4$$

Completing the square for x^2, we have

$$(x+2)^2+y^2=8$$

Complex Numbers in the Real World (Voltage, Current and Resistance)

Ohm's Law is a fundamental principle in physics that describes the relationship between voltage 'v', current 'I' and resistance 'R' in an electrical circuit. Mathematically Ohm's Law can be expressed by the formula $V = IR$.
when dealing with alternating current (AC) circuits, resistance generalizes to impedance (Z). Resistance in a circuit is due to inductor (X_L) and capacitor (X_C). Their difference is reactance $X = (X_L) - (X_C)$. Geometrically it is shown in the adjacent figure. Here $Z = R + iX$



Then for AC circuits, Ohm's Law in Terms of Impedance is expressed by the formula $V = I \cdot Z$.

Example 17: If the impedance of circuit is $11(\cos 55.35° + i\sin 55.35°)$ ohms at a voltage of $25\left(\cos 30° + i\sin 30°\right)$ V, find the value of current in the circuit.

Solution: Substitute the voltage 25(cos 30° + I sin 30°) and impedance 11(cos 55.35° + I sin 55.35°) into the equation $V = IZ$, where V is voltage, I denote the current and Z is impedance.

$$25\left(\cos 30° + i\sin 30°\right) = I\,.11(\cos 55.35° + i\sin 55.35°)$$

$$\text{or} \qquad I = \frac{25\left(\cos 30° + i\sin 30°\right)}{11(\cos 55.35° + i\sin 55.35°)}$$

$$I = \frac{25}{11}\left[\cos\left(30° - 55.35°\right) + i\sin\left(30° - 55.35°\right)\right]$$

$$I = 2.27\left[\cos\left(-25.35°\right) + i\sin\left(-25.35°\right)\right]$$

Express into rectangular form

$$I = 2.27\left[0.90 + i(-0.42)\right] = 2.04 - 0.95i$$

Thus, current is $6 - 4.21i$.

Cryptography: It is the science of securing information by transforming readable messages called plaintext into secrete code called ciphertext using mathematical algorithms and encryption keys. It consists of two main processes i.e., encryption to lock message with complex math, and decryption to unlock it with the right key.

Example 18: The word "MATH" is to be encrypted by multiplying a complex number $k = 2 + 3i$ and then decrypted back to its original form using the concept of multiplicative inverse in complex numbers.

Each letter of the alphabet is assigned a numerical value as follows:

$$A = 1, B = 2, C = 3, \ldots, Z = 26$$

Solution: First, we assign each letter in the word "MATH" a complex number with zero imaginary part. The encryption and decryption shown in the table below

Letter	Complex Number (z)	z encrypted = $z \times k$	z decrypted = z encrypted / k	Letter
M	$13 + 0i$	$(13 + 0i)(2 + 3i) = 26 + 39i$	$(26 + 39i) / (2 + 3i) = 13 + 0i$	M
A	$1 + 0i$	$(1 + 0i)(2 + 3i) = 2 + 3i$	$(2 + 3i) / (2 + 3i) = 1 + 0i$	A
T	$20 + 0i$	$(20 + 0i)(2 + 3i) = 40 + 60i$	$(40 + 60i) / 2 + 3i = 20 + 0i$	T
H	$8 + 0i$	$(8 + 0i)(2 + 3i) = 16 + 24i$	$16+24i / 2 + 3i = 8 + 0i$	H

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Polar Coordinate System
- Exercise 1.5: Q1 & Q2

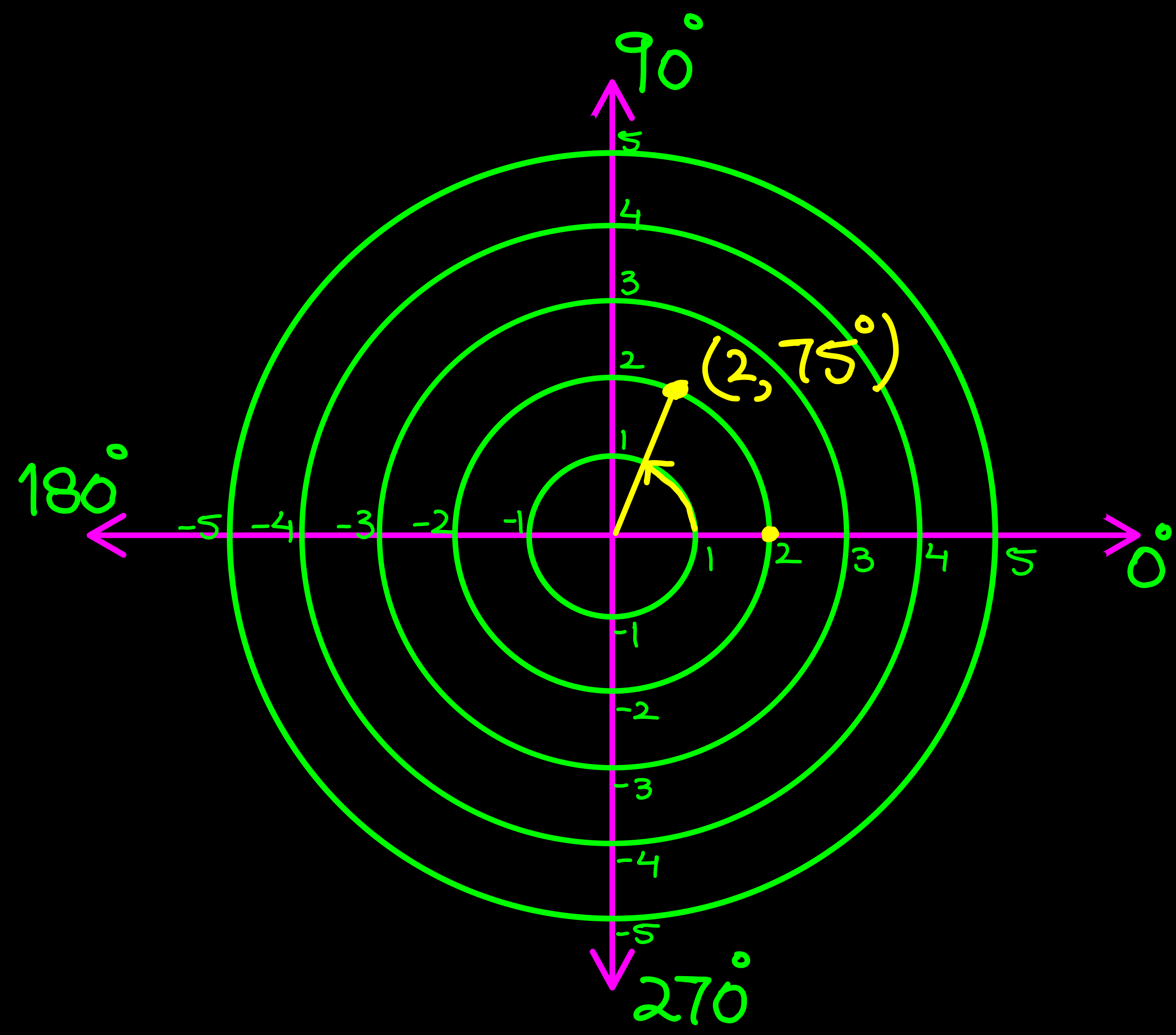
EXERCISE 1.5

1. Plot the following points:

(i) $(2, 75^\circ)$

Sol

$r = 2, \quad \theta = 75^\circ$

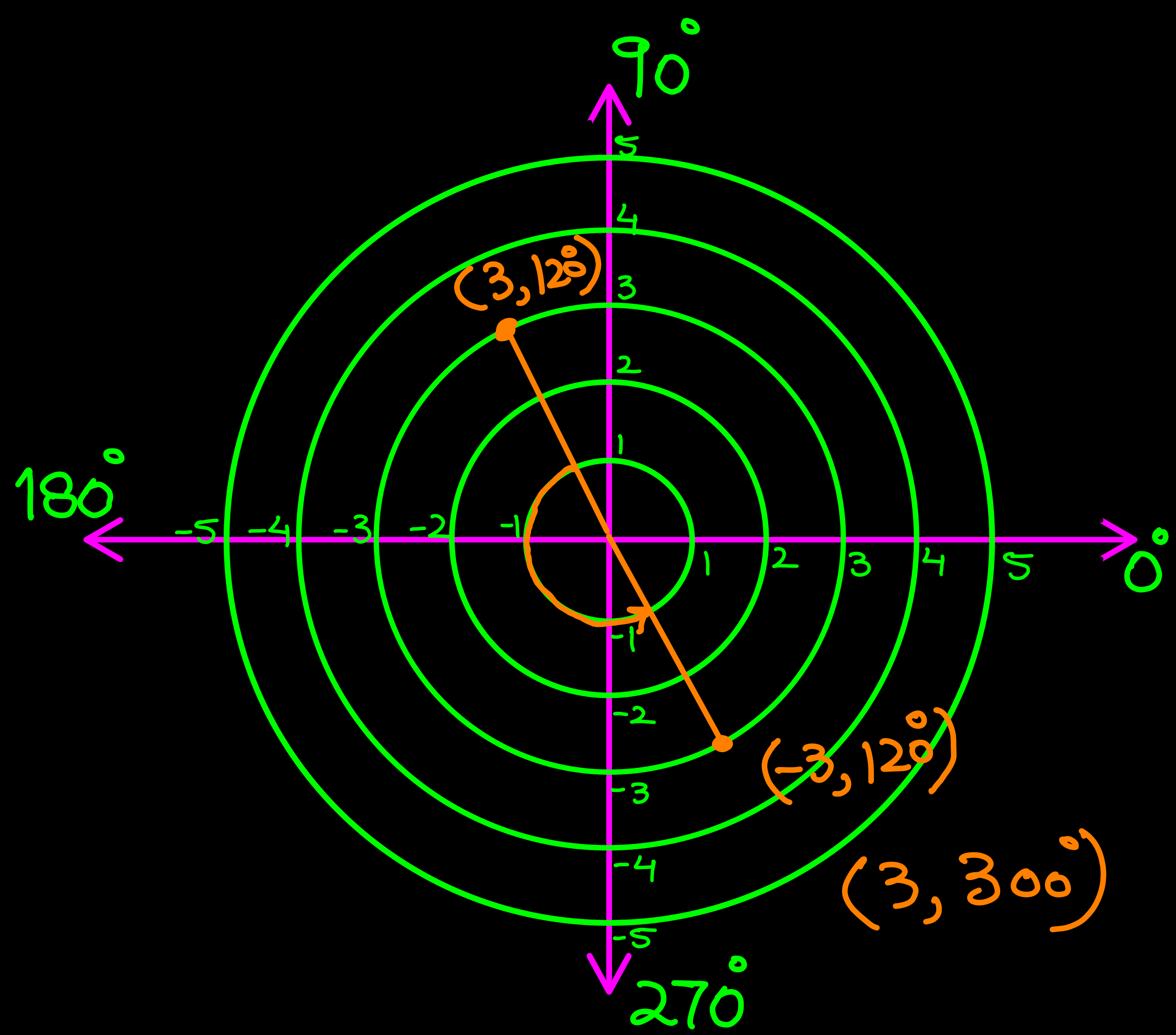


(ii) $(-3, 120^\circ)$

Sol

$r = -3 \qquad \theta = 120^\circ$

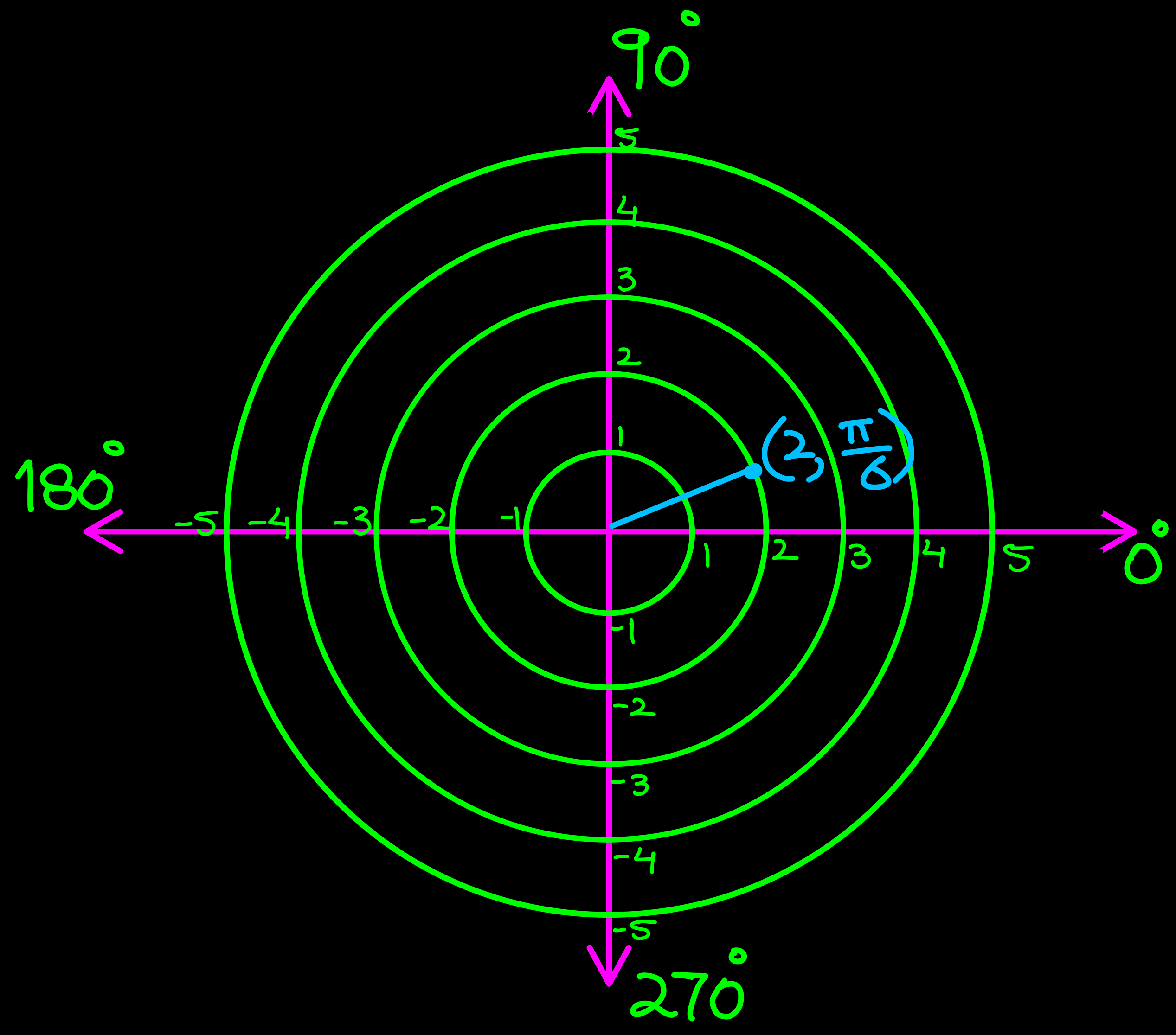
$120^\circ + 180^\circ = 300^\circ$



(iii) $\left(2, \frac{\pi}{6}\right)$

Sol

$r = 2, \quad \theta = \frac{\pi}{6} = 30°$

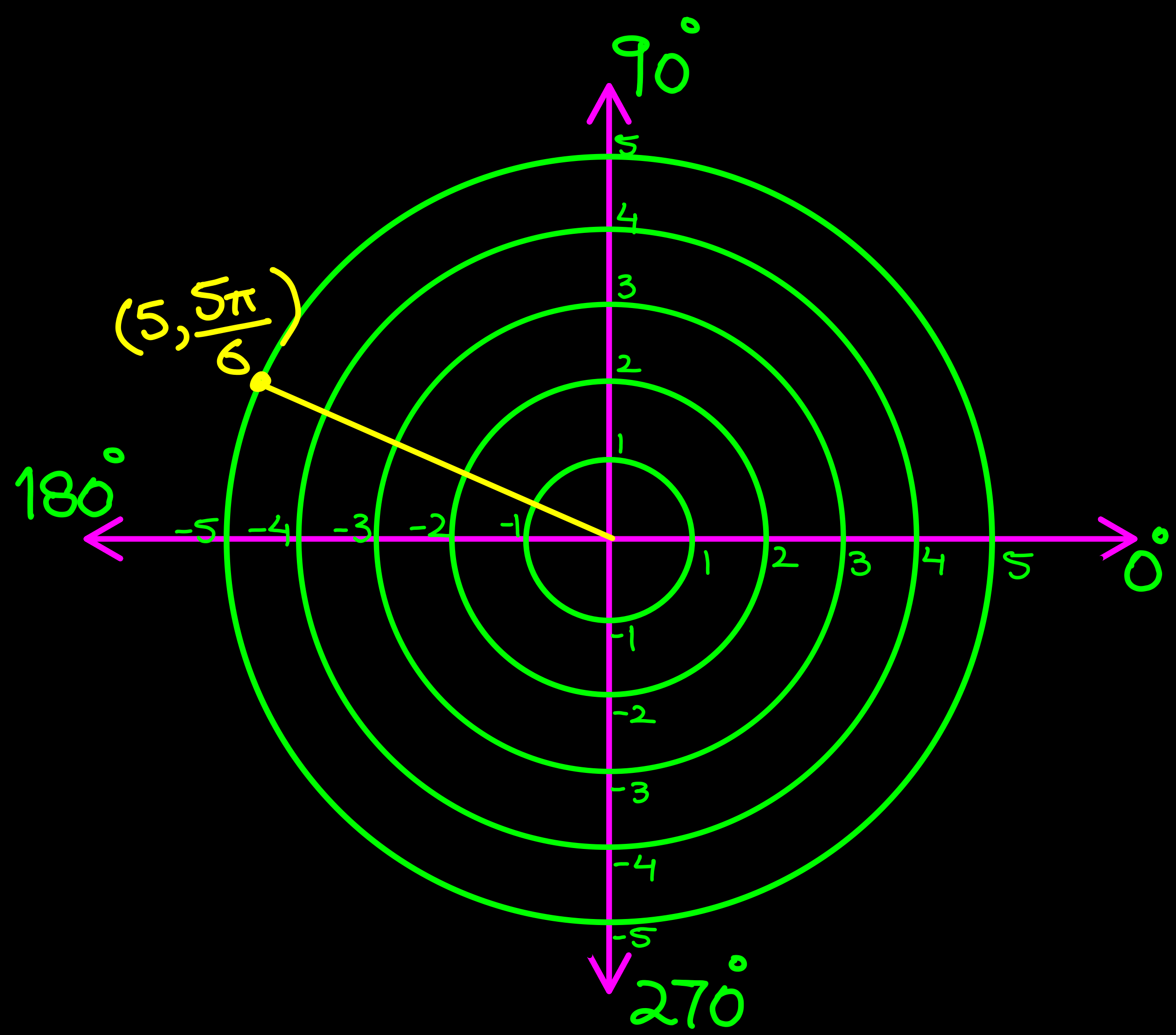


(iv) $\left(5, \frac{5\pi}{6}\right)$

Sol

$r = 5$

$\theta = \frac{5\pi}{6} = 150°$

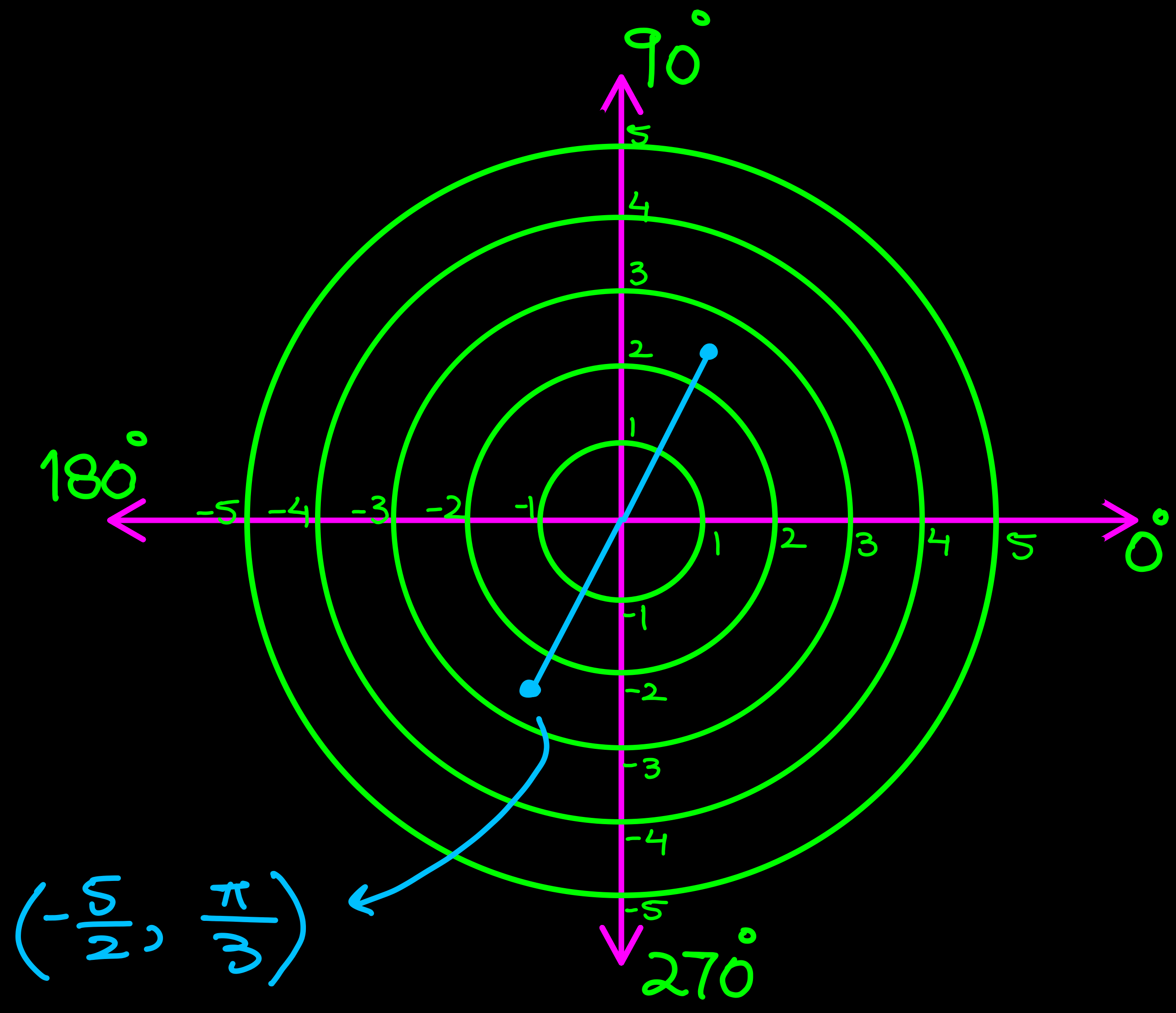


(v) $\left(-\frac{5}{2}, \frac{\pi}{3}\right)$

Sol

$r = -\frac{5}{2}$, $\theta = \frac{\pi}{3} = 60^\circ$

$60^\circ + 180^\circ = 240^\circ$



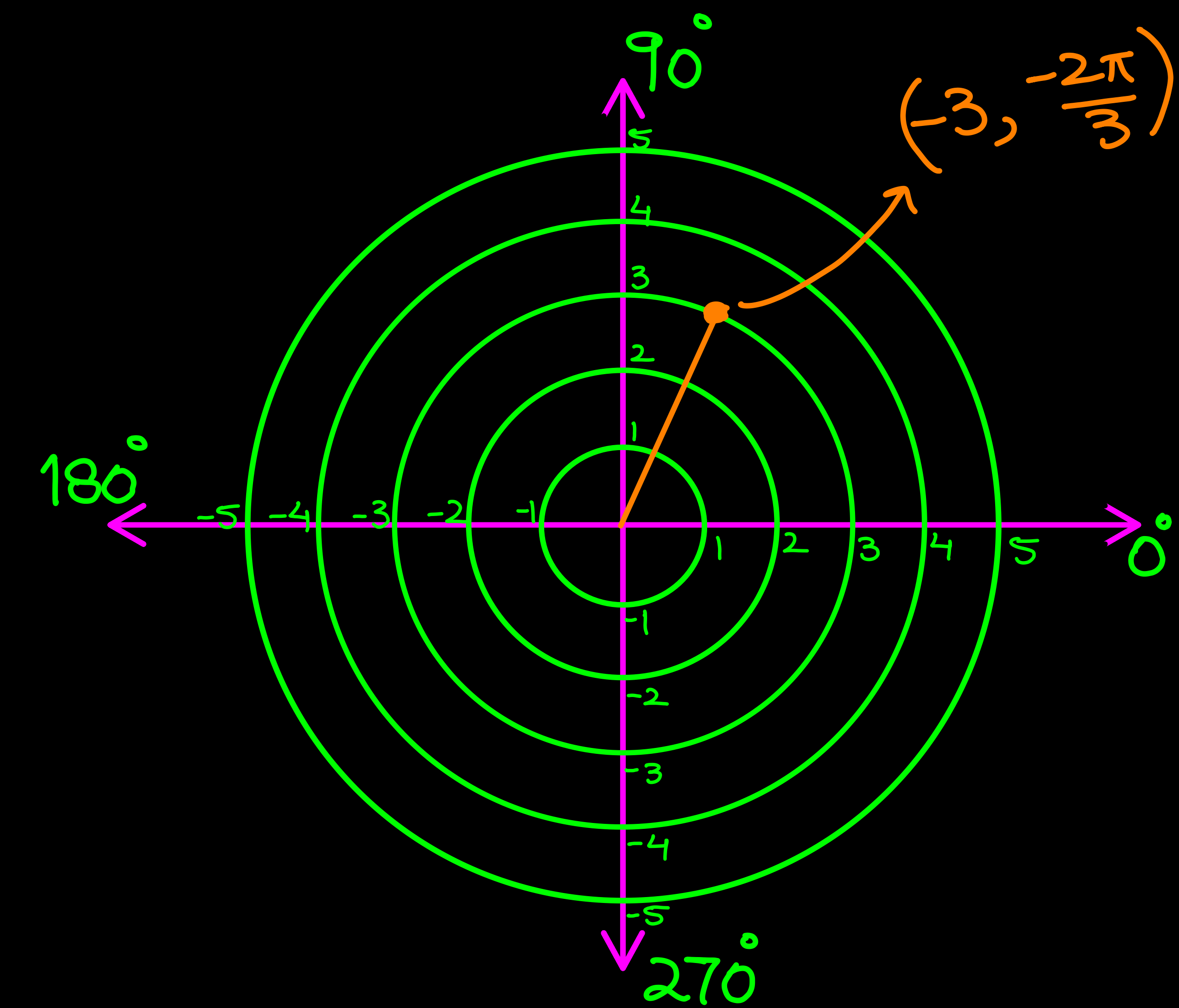
(vi) $\left(-3, -\frac{2\pi}{3}\right)$

Sol

$r = -3$, $\theta = -\frac{2\pi}{3}$

$\theta = -120^\circ$

$-120^\circ + 180^\circ = 60^\circ$



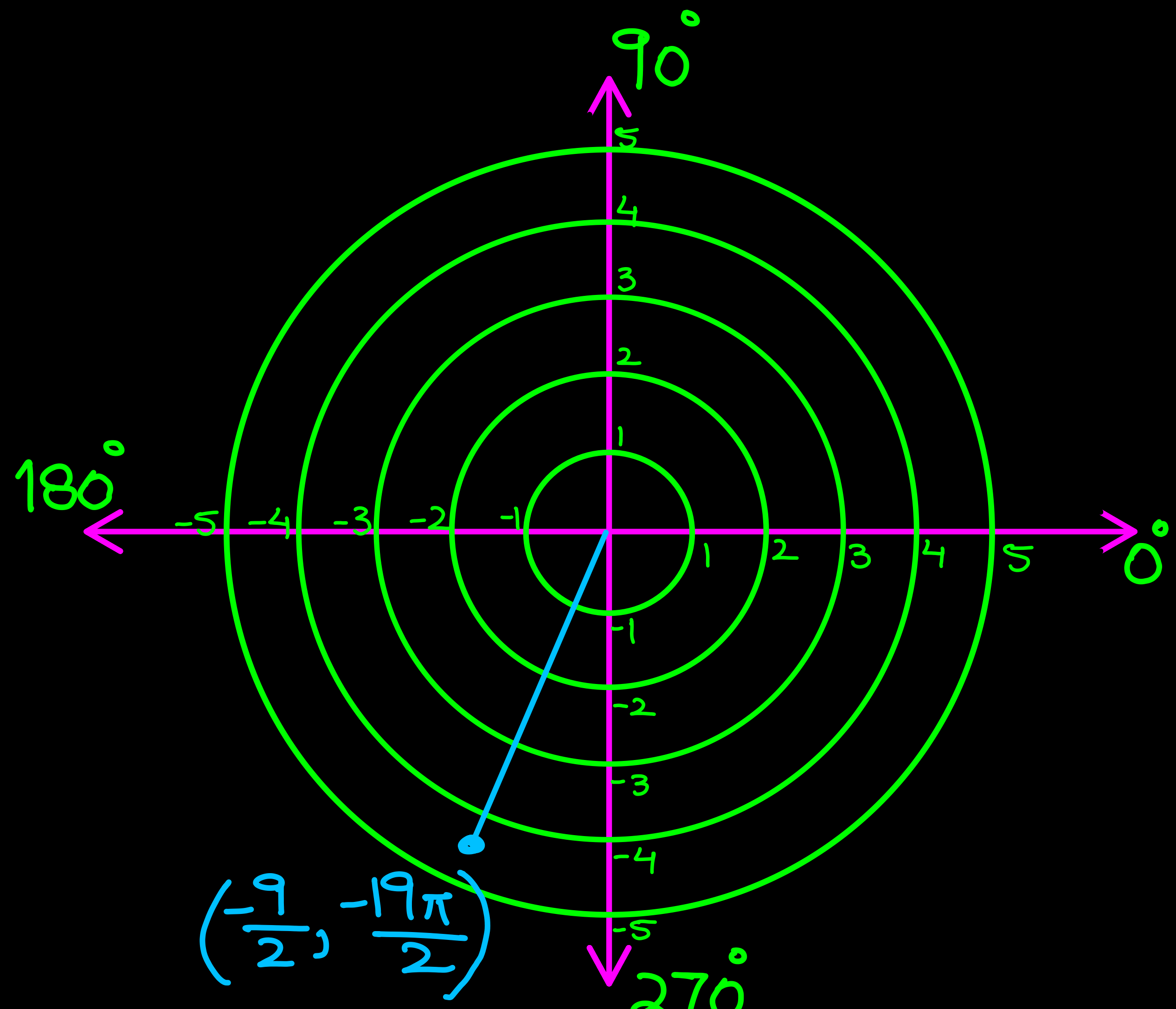
(vii) $\left(-\frac{9}{2}, -\frac{19\pi}{12}\right)$

Sol

$r = -\frac{9}{2}$

$\theta = -\frac{19\pi}{12} = -285°$

$-285° + 180° = -105°$



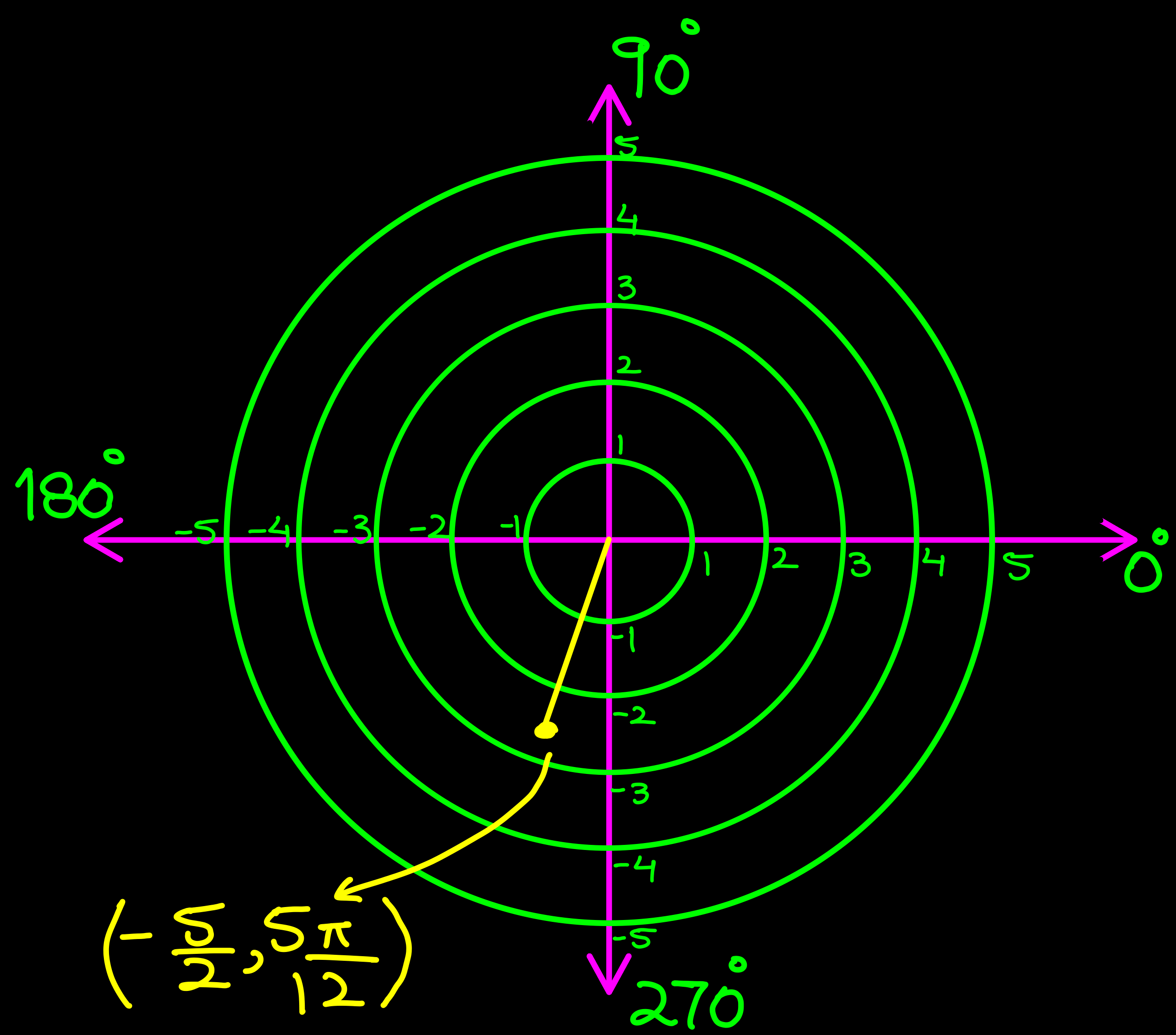
(viii) $\left(-\frac{5}{2}, \frac{5\pi}{12}\right)$

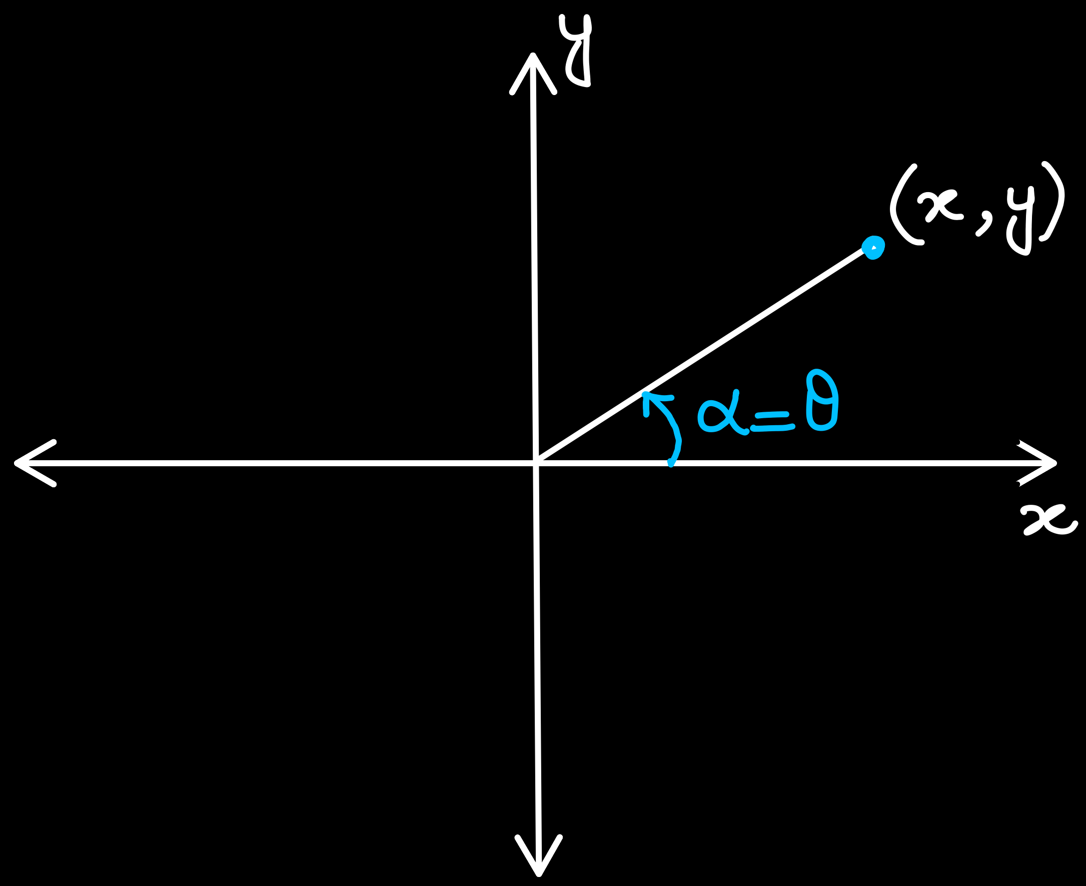
Sol

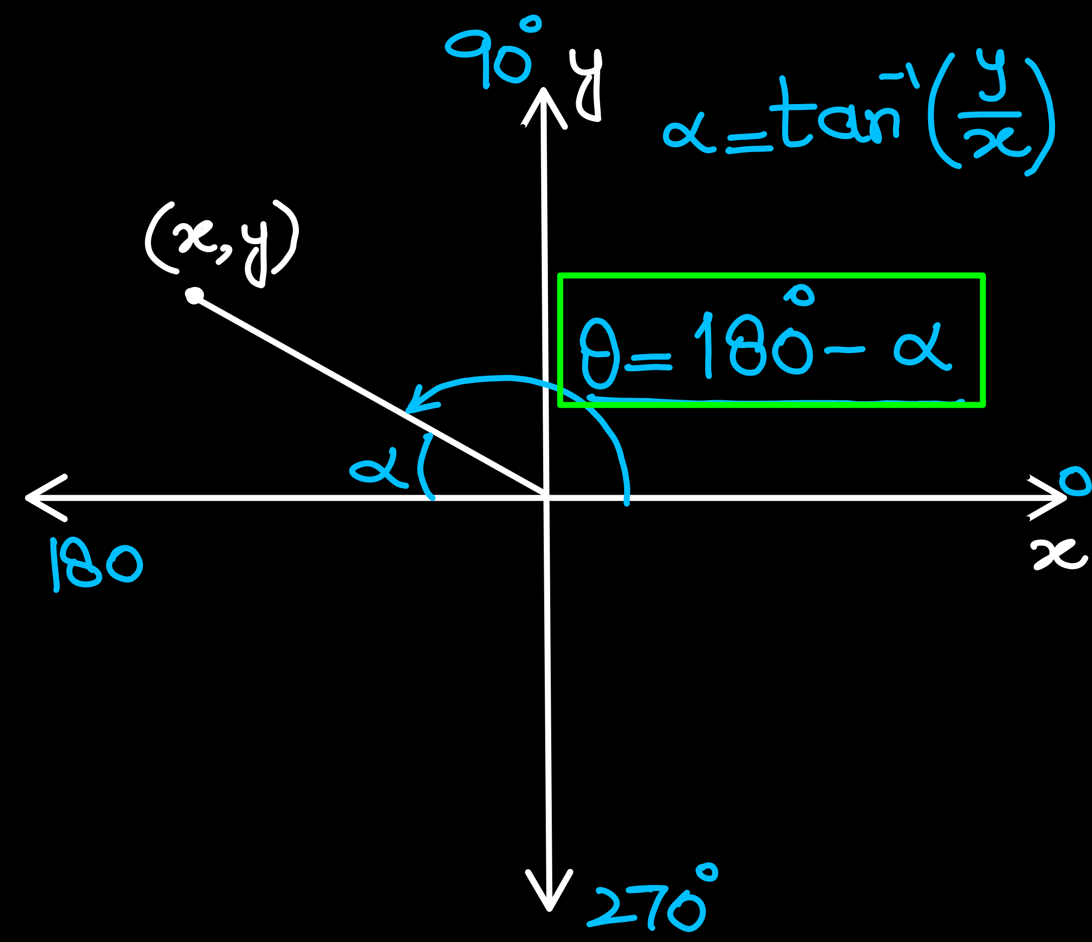
$r = -\frac{5}{2} = -2.5$

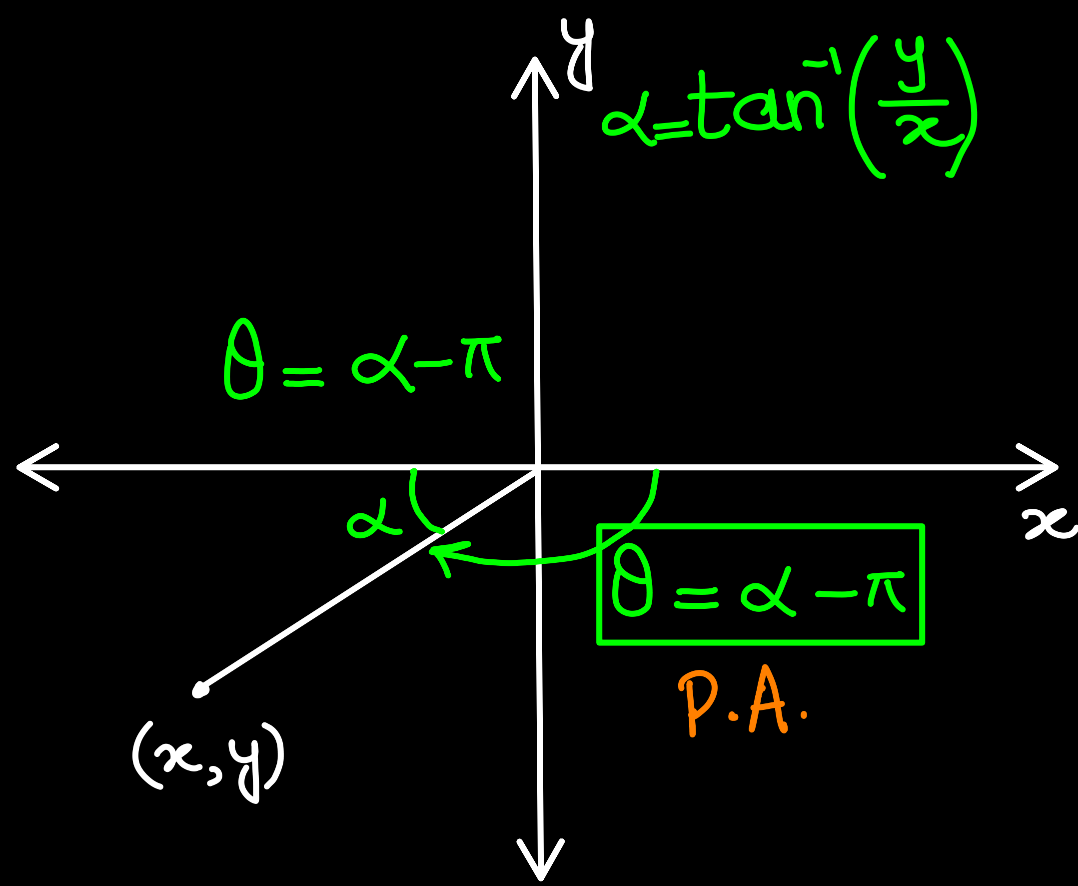
$\theta = \frac{5\pi}{12} = 75°$

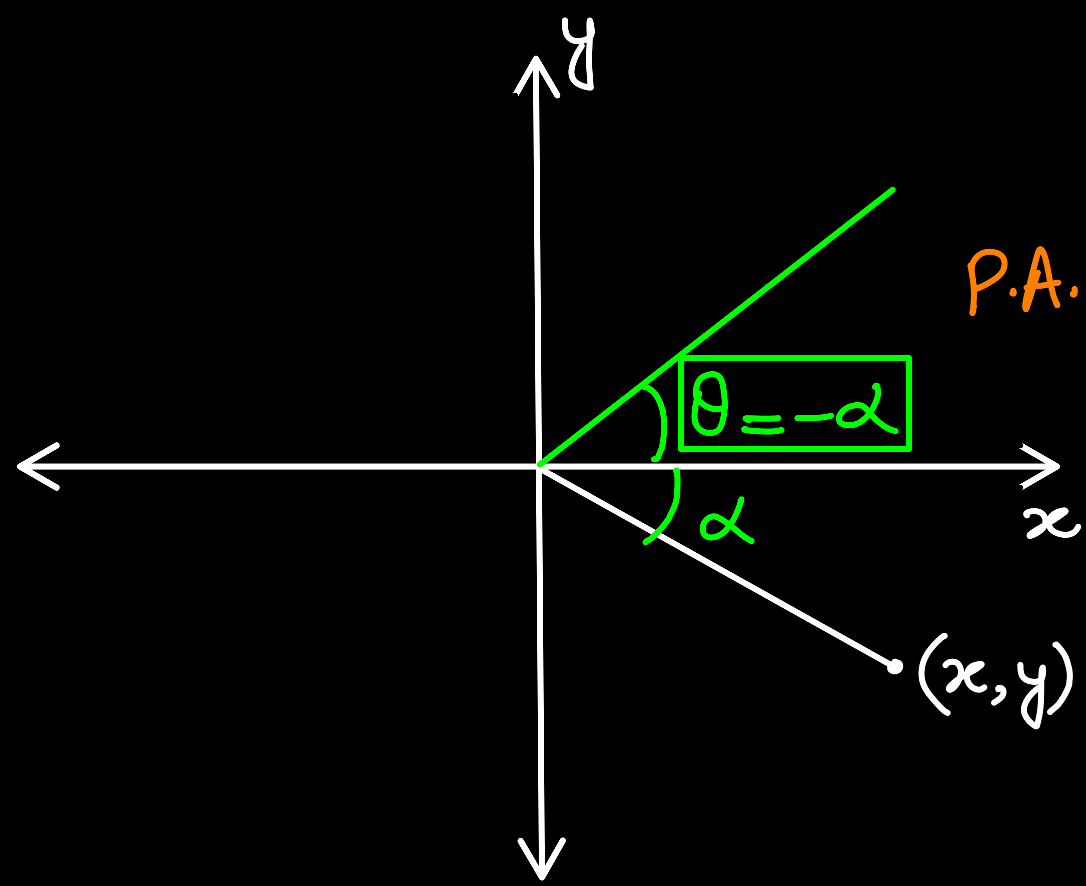
$75° + 180° = 255°$











$\operatorname{Arg} z \in (-\pi, \pi]$

S.A. $\boxed{\theta = \pi + \alpha}$

$\boxed{\theta = 2\pi - \alpha}$

S.A.

2. Express the following complex numbers in polar form :

(i) $4+3i$

Sol

$$z = r(\cos\theta + i\sin\theta) \text{ —— } ①$$

Here $x = 4$, $y = 3$

$$r = \sqrt{x^2 + y^2}$$

$$\alpha = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r = \sqrt{4^2 + 3^2}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25}$$

$$r = 5$$

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^\circ$$

$$\operatorname{Arg} z = \theta = \alpha$$

From ①

$$4 + 3i = 5(\cos 36.87^\circ + i\sin 36.87^\circ)$$

(ii) $1+i$

Sol

$$z = r(\cos\theta + i\sin\theta) \quad \text{——} \quad (1)$$

$x = 1, \quad y = 1$

$$r = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = 45^\circ = \frac{\pi}{4}$$

So,

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(iii) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Sol

$$z = r(\cos\theta + i\sin\theta)$$

Here

$$x = \frac{1}{2}, \qquad y = \frac{\sqrt{3}}{2}$$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$r = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{1+3}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/\cancel{2}}{1/\cancel{2}}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

From ①,

$$\frac{1}{2}+\frac{\sqrt{3}}{2}i = 1.\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

(iv) $-\frac{5}{2}-\frac{5\sqrt{3}}{2}i$

Sol

$$z = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{\left(-\frac{5}{2}\right)^2 + \left(-\frac{5\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{25}{4} + \frac{75}{4}}$$

$$r = \sqrt{\frac{100}{4}} = \frac{10}{2} = 5$$

$$\alpha = \tan^{-1}\left(\frac{|y|}{|x|}\right)$$

$$= \tan^{-1}\left(\frac{\left|-\frac{5\sqrt{3}}{2}\right|}{\left|-\frac{5}{2}\right|}\right)$$

$$\alpha = \tan^{-1}\left(\frac{\cancel{5}\sqrt{3}/\cancel{2}}{\cancel{5}/\cancel{2}}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$\theta = \alpha - \pi = \frac{\pi}{3} - \pi = \frac{\pi - 3\pi}{3} = \frac{-2\pi}{3}$$

① $\Longrightarrow$ $-\frac{5}{2} - \frac{5\sqrt{3}}{2}i = 5\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$

P.A.

S.A. $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

(v) $\dfrac{1-i}{1+i}$

Sol

$$\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i}$$

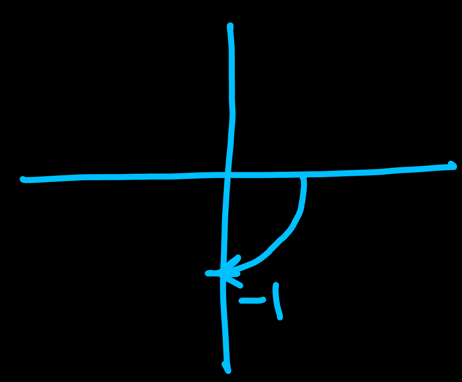
$$= \frac{(1-i)^2}{1^2 - i^2}$$

$$= \frac{1^2 - 2(1)(i) + i^2}{1-(-1)}$$

$$= \frac{\cancel{1} - 2i - \cancel{1}}{1+1}$$

$$= \frac{-2i}{2}$$

$$\frac{1-i}{1+i} = 0 - i$$



$$r = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$$

$$\alpha = \tan^{-1}\left(\frac{|-1|}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{2}$$

So,

$$\frac{1-i}{1+i} = 1.\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$$

(vi) $\dfrac{\sqrt{3}+i}{1+\sqrt{3}\,i}$

Sol

$$\frac{\sqrt{3}+i}{1+\sqrt{3}i} = \frac{\sqrt{3}+i}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}\,i}{1-\sqrt{3}\,i}$$

$$= \frac{(\sqrt{3}+i)(1-\sqrt{3}i)}{1^2 - (\sqrt{3}\,i)^2}$$

$$= \frac{\sqrt{3}-3i+i-\sqrt{3}\,i^2}{1-3i^2}$$

$$= \frac{\sqrt{3}-2i+\sqrt{3}}{1+3}$$

$$= \frac{2\sqrt{3}-2i}{4}$$

$$= \frac{\cancel{2}(\sqrt{3}-i)}{\cancel{4}\,2}$$

$$\frac{\sqrt{3}+i}{1+\sqrt{3}i} = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = \sqrt{\frac{4}{4}} = 1$$

$$\alpha = \tan^{-1}\left(\frac{|-1/2|}{|\sqrt{3}/2|}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}, \qquad \theta = -\frac{\pi}{6}$$

So,
$$\frac{\sqrt{3}+i}{1+\sqrt{3}i} = 1.\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$$

(vii) $\dfrac{3+4i}{4+3i}$

Sol

$$\frac{3+4i}{4+3i} = \frac{3+4i}{4+3i} \times \frac{4-3i}{4-3i}$$

$$= \frac{12-9i+16i-12i^2}{4^2-(3i)^2}$$

$$= \frac{12+7i+12}{16-9i^2}$$

$$= \frac{24+7i}{25}$$

$$\frac{3+4i}{4+3i} = \frac{24}{25} + \frac{7}{25}i$$

$$r = \sqrt{\left(\frac{24}{25}\right)^2 + \left(\frac{7}{25}\right)^2}$$

$$= \sqrt{\frac{576}{625} + \frac{49}{625}}$$

$$= \sqrt{\frac{625}{625}}$$

$$r = 1$$

$$\theta = \tan^{-1}\left(\frac{7/\cancel{25}}{24/\cancel{25}}\right)$$

$$= \tan^{-1}\left(\frac{7}{24}\right)$$

$$\theta = 16.3^\circ$$

So,

$$\frac{3+4i}{4+3i} = 1.\left(\cos 16.3^\circ + i.\sin 16.3^\circ\right)$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Polar Coordinate System
- Exercise 1.5: Q3 - Q5

EXERCISE 1.5

3. Convert each of the complex number z in the rectangular form $x+iy$:

(i) $4\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$

$z=r(\cos\theta+i\sin\theta)$

Sol

$$4\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)=4\left(\frac{1}{2}+i\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$=2-i2\sqrt{3}$$

(ii) $\frac{3}{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

Sol

$$\frac{3}{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=\frac{3}{2}\left(-\frac{\sqrt{3}}{2}+i\left(-\frac{1}{2}\right)\right)$$

$$=-\frac{3\sqrt{3}}{4}-i\frac{3}{4}$$

(iii) $|z| = 7,\ arg(z) = \frac{23\pi}{12}$

Sol

$$z = |z|\left(\cos\theta + i\sin\theta\right)$$

$$= 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$$

$$= 7\left(\frac{\sqrt{6}+\sqrt{2}}{4} + i\left(\frac{-\sqrt{6}+\sqrt{2}}{4}\right)\right)$$

$$= 7\left(0.96 - i\,0.26\right)$$

$$= 6.76 - i\,1.81$$

(iv) $|z| = 11,\ arg(z) = -\dfrac{11\pi}{12}$

Sol

$$z = |z|\left(\cos\theta + i\sin\theta\right)$$

$$= 11\left(\cos\left(\frac{-11\pi}{12}\right) + i\sin\left(\frac{-11\pi}{12}\right)\right)$$

$$= 11\left(\cos\frac{11\pi}{12} - i\cdot\sin\frac{11\pi}{12}\right)$$

$$= 11\left(\frac{-\sqrt{6}-\sqrt{2}}{4} - i\,\frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

$$= -10.62 - i2.85$$

(v) $\quad |z| = \dfrac{10}{3},\ arg(z) = -\dfrac{17\pi}{12}$

Sol

$$z = |z|\left(\cos\theta + i\sin\theta\right)$$

$$= \frac{10}{3}\left[\cos\left(\frac{-17\pi}{12}\right) + i\sin\left(\frac{-17\pi}{12}\right)\right]$$

$$= \frac{10}{3}\left[\cos\frac{17\pi}{12} - i\sin\frac{17\pi}{12}\right]$$

$$= \frac{10}{3}\left[-\frac{\sqrt{6}+\sqrt{2}}{4} - i\left(-\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right]$$

$$= -0.86 + 3.21\, i$$

(vi) $2\cos(-33) + i\,2\sin(-33)$

Sol

$$2\cos(-33^\circ) + i\,2\sin(-33^\circ) = 2\cos 33^\circ - i\,2\sin 33^\circ$$

$$= 2(0.84) - i\,2(0.54)$$

$$= 1.68 - i\,1.09$$

4. If $z_1 = 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ and $z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ then find

(i) $z_1 + z_2$

Sol

$$z_1 + z_2 = 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) + 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 9\left(-\frac{\sqrt{2}}{2}\right) + 9i\left(-\frac{\sqrt{2}}{2}\right) + 5\left(\frac{1}{2}\right) + 5i\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{-9\sqrt{2}+5}{2} + i\,\frac{-9\sqrt{2}+5\sqrt{3}}{2}$$

$$= -3.86 - i\,2.03$$

(ii) $z_1 - z_2$

Sol

$$z_1 - z_2 = 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) - 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 9\left(-\frac{\sqrt{2}}{2}\right) + 9i\left(-\frac{\sqrt{2}}{2}\right) - 5\left(\frac{1}{2}\right) - 5i\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{-9\sqrt{2}-5}{2} - i\,\frac{9\sqrt{2}+5\sqrt{3}}{2}$$

$$= -8.86 - i\,10.69$$

(iii) $z_1 \cdot z_2$

$$z_1 \cdot z_2 = 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \cdot 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\because z_1 \cdot z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

$$= 9 \times 5\left(\cos\left(\frac{5\pi}{4}+\frac{\pi}{3}\right) + i\sin\left(\frac{5\pi}{4}+\frac{\pi}{3}\right)\right)$$

$$= 45\left[\cos\left(\frac{15\pi+4\pi}{12}\right) + i\sin\left(\frac{15\pi+4\pi}{12}\right)\right]$$

$$= 45\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$

(iv) $\frac{z_1}{z_2}$

Sol

$$\frac{z_1}{z_2} = \frac{9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)}{5\left(\cos\frac{\pi}{3} + i\sin \pi/3\right)}$$

$$= \frac{9}{5}\left[\cos\left(\frac{5\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{5\pi}{4} - \frac{\pi}{3}\right)\right]$$

$$= \frac{9}{5}\left[\cos\left(\frac{15\pi - 4\pi}{12}\right) + i\sin\left(\frac{15\pi-4\pi}{12}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{9}{5}\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right]$$

5. If $z_1 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$ and $z_2 = 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$ then find the following and express the result into $x + iy$ form

(i) $z_1 + z_2$

Sol

$$z_1 + z_2 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) + 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$= 7\cos\frac{23\pi}{12} + i\,7\sin\frac{23\pi}{12} + 11\cos\frac{11\pi}{12} + i\,11\sin\frac{11\pi}{12}$$

$$= \left(7\cos\frac{23\pi}{12} + 11\cos\frac{11\pi}{12}\right) + i\left(7\sin\frac{23\pi}{12} + 11\sin\frac{11\pi}{12}\right)$$

$$= -3.86 + i\,1.03$$

(ii) $z_1 - z_2$

Sol

$$z_1 - z_2 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) - 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$= 7\cos\frac{23\pi}{12} + i\,7\sin\frac{23\pi}{12} - 11\cos\frac{11\pi}{12} - i\,11\sin\frac{11\pi}{12}$$

$$= \left(7\cos\frac{23\pi}{12} - 11\cos\frac{11\pi}{12}\right) + i\left(7\sin\frac{23}{12} - 11\sin\frac{11\pi}{12}\right)$$

$$z_1 - z_2 = 17.38 - i4.65$$

(iii) $z_1 \cdot z_2$

Sol

$$z_1 \cdot z_2 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) \cdot 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$= 7 \times 11 \left(\cos\left(\frac{23\pi}{12} + \frac{11\pi}{12}\right) + i\sin\left(\frac{23\pi}{12} + \frac{11\pi}{12}\right)\right)$$

$$= 77\left[\cos\left(\frac{34\pi}{12}\right) + i\sin\frac{34\pi}{12}\right]$$

$$= 77\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$z_1 \cdot z_2 = -66.68 + i\,38.5$$

(iv) $\frac{z_1}{z_2}$

Sol

$$\frac{z_1}{z_2} = \frac{7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)}{11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)}$$

$$= \frac{7}{11}\left[\cos\left(\frac{23\pi}{12} - \frac{11\pi}{12}\right) + i\sin\left(\frac{23\pi}{12} - \frac{11\pi}{12}\right)\right]$$

$$= \frac{7}{11}\left[\cos\left(\frac{\cancel{12}\pi}{\cancel{12}}\right) + i\sin\left(\frac{\cancel{12}\pi}{\cancel{12}}\right)\right]$$

$$= \frac{7}{11}(-1 + i0)$$

$$= -\frac{7}{11} + 0i$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Polar Coordinate System
- Exercise 1.5: Q6 - Q10

EXERCISE 1.5

6. If z_1 and z_2 are two complex numbers, show that:

$\arg z, \ 0<\theta\leq 2\pi$

$\operatorname{Arg} z, \ -\pi<\theta\leq\pi$

(i) $\operatorname{Arg}(z_1 \cdot z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$

Sol:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form. The product of two complex numbers can be derived by multiplying them directly and simplifying

$$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\left(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\right)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right] \qquad \because \ i^2 = -1$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right] \qquad \text{(Using trigonometric identities)}$$

Thus, multiplying two complex numbers in polar form involves multiplying their moduli and summing their arguments i.e., $arg(z_1 \cdot z_2) = arg(z_1) + arg(z_2)$

$$\operatorname{Arg}(z_1 \cdot z_2) = \theta_1 + \theta_2$$

$$= \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

(ii) $\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$

Sol

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex number in polar form. The formula for division of two complex numbers in polar form can be derived by rationalizing the denominator.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \qquad \left(\begin{array}{l}\text{Multiply and divide the equation}\\ \text{by conjugate of } \cos\theta_2 + i\sin\theta_2\end{array}\right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$\because (a+b)(a-b) = a^2 - b^2$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right] \qquad \text{(Using trigonometric identities)}$$

Thus, the modulus of the division of two complex numbers equals the quotient of their moduli, while the arguments of the quotient is the difference between their arguments.

Thus, when dividing two complex numbers, the modulus of the result is the ratio of their moduli, and the argument of the result is the difference between their arguments

i.e., $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$$

7. Divide $z_1 = 6(\cos 150° + i \sin 150°)$ by $z_2 = 3(\cos 30° + i \sin 30°)$ and express in $x + iy$ form.

Sol

$$\frac{z_1}{z_2} = \frac{6(\cos 150° + i \sin 150°)}{3(\cos 30° + i \sin 30°)}$$

$$= 2[\cos(150° - 30°) + i \sin(150° - 30°)]$$

$$= 2(\cos 120° + i \sin 120°)$$

$$= 2\left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)$$

$$\frac{z_1}{z_2} = -1 + i\sqrt{3}$$

8. Multiply $z_1 = 2(\cos 60° + i \sin 60°)$ and $z_2 = 5(\cos 90° + i \sin 90°)$ and express in $x + iy$ form.

Sol

$$z_1 z_2 = 2(\cos 60° + i \sin 60°)\, 5(\cos 90° + i \sin 90°)$$

$$= 2 \times 5 [\cos(60° + 90°) + i \sin(60° + 90°)]$$

$$= 10(\cos 150° + i \sin 150°)$$

$$= 10\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$$

$$z_1 z_2 = -5\sqrt{3} + 5i$$

9. Find the modulus and argument of $z = -2 - 2i$.

Sol

$$|z| = \sqrt{(-2)^2 + (-2)^2}$$

$$= \sqrt{4+4}$$

$$= \sqrt{2(4)}$$

$$|z| = 2\sqrt{2}$$

$$\alpha = \tan^{-1}\left(\frac{|-2|}{|-2|}\right)$$

$$= \tan^{-1}\left(\frac{2}{2}\right)$$

$$= \tan^{-1}(1)$$

$$\alpha = \frac{\pi}{4}$$

Since z is in 3^{rd}-Quad.,

$z = r(\cos\theta + i\sin\theta)$

2π

$$\arg z = \alpha + \pi$$

$$= \frac{\pi}{4} + \pi$$

$$\arg z = \frac{\pi + 4\pi}{4} = \frac{5\pi}{4}$$

In general,

$$\arg z = \frac{5\pi}{4} + 2\pi n$$

10. Write the equation $\arg(\bar{z}-2+i)=\frac{2\pi}{3}$ in cartesian form, if $z=x+iy$.

Sol

Given $\arg(\bar{z}-2+i)=\frac{2\pi}{3}$

Since $z=x+iy$,

$$\arg(\overline{x+iy}-2+i)=\frac{2\pi}{3}$$

$$\arg(x-iy-2+i)=\frac{2\pi}{3}$$

$$\arg(x-2+i(1-y))=\frac{2\pi}{3}$$

$$\tan^{-1}\left(\frac{1-y}{x-2}\right)=\frac{2\pi}{3}$$

$$\frac{1-y}{x-2}=\tan\left(\frac{2\pi}{3}\right)$$

$$\frac{1-y}{x-2}=-\sqrt{3}$$

$$1-y=-\sqrt{3}(x-2)$$

$$1-y=-\sqrt{3}x+2\sqrt{3}$$

$$\sqrt{3}x-y+1-2\sqrt{3}=0$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Polar Coordinate System
- Exercise 1.5: Q11 - Q14

EXERCISE 1.5

11. If $z = x + iy$ and $\arg\left(\frac{\bar{z}-1+2i}{\bar{z}+1-2i}\right) = \frac{9\pi}{4}$, show that $x^2 + y^2 - 4x + 2y - 5 = 0$.

Sol

$$\arg\left(\frac{\bar{z}-1+2i}{\bar{z}+1-2i}\right) = \frac{9\pi}{4}$$

Since $z = x+iy$ $(\bar{z} = x-iy)$

$$\arg\left(\frac{x-iy-1+2i}{x-iy+1-2i}\right) = 9\frac{\pi}{4}$$

$$\arg\left(\frac{x-1+i(2-y)}{x+1+i(-2-y)}\right) = \frac{9\pi}{4}$$

$$\because \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

$$\arg\left(x-1+i(2-y)\right) - \arg\left(x+1+i(-2-y)\right) = \frac{9\pi}{4}$$

$$\tan^{-1}\left(\frac{2-y}{x-1}\right) - \tan^{-1}\left(\frac{-2-y}{x+1}\right) = \frac{9\pi}{4}$$

$$\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$$\tan^{-1}\left[\frac{\frac{2-y}{x-1} - \frac{(-2-y)}{x+1}}{1+\frac{2-y}{x-1}\cdot\frac{(-2-y)}{x+1}}\right] = \frac{9\pi}{4}$$

$$\frac{\frac{2-y}{x-1} - \frac{(-2-y)}{x+1}}{1+\frac{2-y}{x-1}\cdot\frac{(-2-y)}{x+1}} = \tan\frac{9\pi}{4}$$

$$\frac{\dfrac{(2-y)(x+1)+(2+y)(x-1)}{\cancel{(x-1)(x+1)}}}{\dfrac{(x-1)(x+1)-(2-y)(2+y)}{\cancel{(x-1)(x+1)}}}=1$$

$$\frac{2x+\cancel{2}-\cancel{xy}-y+2x-\cancel{2}+\cancel{xy}-y}{(x^2-1^2)-(2^2-y^2)}=1$$

$$\frac{4x-2y}{x^2-1-4+y^2}=1$$

$$\frac{4x-2y}{x^2+y^2-5}=1$$

$$4x-2y=x^2+y^2-5$$

$$0=x^2+y^2-4x+2y-5$$

$$x^2+y^2-4x+2y-5=0$$

12. If $z = x + iy$ and $\arg(z-2-3i) - \arg(z+2+3i) = 2\pi$, show that $2y = 3x$.

Sol

$$\arg(x+iy-2-3i) - \arg(x+iy+2+3i) = 2\pi$$

$$\arg[x-2+i(y-3)] - \arg[x+2+i(y+3)] = 2\pi$$

$$\tan^{-1}\frac{y-3}{x-2} - \tan^{-1}\frac{y+3}{x+2} = 2\pi$$

$$\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$$\tan^{-1}\left[\frac{\frac{y-3}{x-2} - \frac{y+3}{x+2}}{1+\frac{y-3}{x-2}\cdot\frac{y+3}{x+2}}\right] = 2\pi$$

$$\frac{\frac{y-3}{x-2} - \frac{y+3}{x+2}}{1+\frac{y-3}{x-2}\cdot\frac{y+3}{x+2}} = \tan 2\pi$$

$$\frac{(x-2)(x+2)\left(\frac{y-3}{x-2} - \frac{y+3}{x+2}\right)}{(x-2)(x+2)\left(1+\frac{y-3}{x-2}\cdot\frac{y+3}{x+2}\right)} = 0$$

$$\frac{(x+2)(y-3) - (x-2)(y+3)}{(x-2)(x+2) + (y-3)(y+3)} = 0$$

$$(x+2)(y-3) - (x-2)(y+3) = 0$$

$$xy - 3x + 2y - 6 - (xy + 3x - 2y - 6) = 0$$

$$\cancel{xy} - 3x + 2y - \cancel{6} - \cancel{xy} - 3x + 2y + \cancel{6} = 0$$

$$-6x + 4y = 0$$

$$4y = 6x$$

$$\Rightarrow \quad 2y = 3x$$

13. Solve the equation $|z-2i| = |\bar{z}+2|$ for $z = x+iy$.

Sol

$$|x+iy-2i| = |x-iy+2|$$

$$|x+i(y-2)| = |x+2-iy|$$

$$\sqrt{x^2+(y-2)^2} = \sqrt{(x+2)^2+(-y)^2}$$

Squaring,

$$\left(\cancel{\sqrt{x^2+(y-2)^2}}\right)^{\cancel{2}} = \left(\cancel{\sqrt{(x+2)^2+(-y)^2}}\right)^{\cancel{2}}$$

$$x^2+(y-2)^2 = (x+2)^2+y^2$$

$$\cancel{x^2}+\cancel{y^2}-4y+\cancel{4} = \cancel{x^2}+4x+\cancel{4}+\cancel{y^2}$$

$$-4y = 4x$$

$$\frac{\cancel{-4}y}{\cancel{-4}} = \frac{\cancel{4}x}{\cancel{-4}}$$

$$\Rightarrow \quad y = -x$$

14. For $z = x + iy$, solve the equation $|5z+4+i| = |5\bar{z}-3+2i|$.

Sol

$$|5(x+iy)+4+i| = |5(x-iy)-3+2i|$$

$$|5x+5yi+4+i| = |5x-5yi-3+2i|$$

$$|5x+4+i(5y+1)| = |5x-3+i(2-5y)|$$

$$\sqrt{(5x+4)^2+(5y+1)^2} = \sqrt{(5x-3)^2+(2-5y)^2}$$

Squaring,

$$\left(\sqrt{(5x+4)^2+(5y+1)^2}\right)^{\cancel{2}} = \left(\sqrt{(5x-3)^2+(2-5y)^2}\right)^{\cancel{2}}$$

$$(5x+4)^2+(5y+1)^2 = (5x-3)^2+(2-5y)^2$$

$$\cancel{25x^2}+2(5x)(4)+16+\cancel{25y^2}+2(5y)(1)+1^2 = \cancel{25x^2}-2(5x)(3)$$
$$+9+4-2(2)(5y)+\cancel{25y^2}$$

$$40x+10y+17 = -30x-20y+13$$

$$40x+10y+17+30x+20y-13=0$$

$$70x+30y+4=0$$

$$30y = -70x-4$$

$$y = \frac{-70x}{30}-\frac{4}{30} \implies y = \frac{-7}{3}x-\frac{2}{15}$$

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Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Polar Coordinate System
- Exercise 1.5: Q15 - Q18

EXERCISE 1.5

15. Determine the set of points $z = x + iy$ that satisfy the equation $|3\bar{z} - 2 + i| = |3z + i|$.

Sol

Since $z = x + iy \quad (\bar{z} = x - iy)$

$$|3(x-iy) - 2 + i| = |3(x+iy) + i|$$

$$|3x - 3yi - 2 + i| = |3x + 3yi + i|$$

$$|3x - 2 + i(1-3y)| = |3x + i(1+3y)|$$

$$\sqrt{(3x-2)^2 + (1-3y)^2} = \sqrt{(3x)^2 + (1+3y)^2}$$

Squaring,

$$\left(\sqrt{(3x-2)^2 + (1-3y)^2}\right)^{\cancel{2}} = \left(\sqrt{(3x)^2 + (1+3y)^2}\right)^{\cancel{2}}$$

$$(3x-2)^2 + (1-3y)^2 = (3x)^2 + (1+3y)^2$$

$$\cancel{9x^2} - 2(3x)(2) + 4 + \cancel{1} - 2(1)(3y) + \cancel{9y^2} = \cancel{9x^2} + \cancel{1} + 2(1)(3y) + \cancel{9y^2}$$

$$-12x - 6y + 4 = 6y$$

$$-12x + 4 = 12y$$

$$-x + \frac{4}{12} = y$$

$$y = -x + \frac{1}{3}$$

16. If $z = x + iy$ and $\omega = \frac{1-iz}{z-i}$, show that $|\omega| = 1$

$\Rightarrow z$ is real.

Sol

$$\omega = \frac{1-iz}{z-i} = \frac{1-i(x+iy)}{x+iy-i}$$

$$= \frac{1-xi-i^2y}{x+i(y-1)}$$

$$\omega = \frac{1+y-xi}{x+i(y-1)}$$

$$|\omega| = \left|\frac{1+y-xi}{x+i(y-1)}\right|$$

$$|\omega| = \frac{|1+y-xi|}{|x+i(y-1)|} \qquad \because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Since $|\omega| = 1$

$$1 = \frac{|1+y-xi|}{|x+i(y-1)|}$$

$$|x+i(y-1)| = |1+y-xi|$$

$$\sqrt{x^2+(y-1)^2} = \sqrt{(1+y)^2+(-x)^2}$$

Squaring, $\left(\cancel{\sqrt{x^2+(y-1)^2}}\right)^{\cancel{2}} = \left(\cancel{\sqrt{(1+y)^2+(-x)^2}}\right)^{\cancel{2}}$

$$\cancel{x^2} + (y-1)^2 = (1+y)^2 + \cancel{x^2}$$

$$\cancel{y^2} - 2y + \cancel{1} = \cancel{1} + 2y + \cancel{y^2}$$

$$-2y = 2y$$

$$-2y - 2y = 0$$

$$-4y = 0$$

$$y = 0$$

Hence

$$z = x + 0i = x \quad \text{(real)}$$

17. If z_1 and z_2 are different complex numbers with $|z_2|=1$, find $\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|$.

Sol

$$\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right| = \frac{|z_2-z_1|}{|1-\bar{z}_1 z_2|}$$

$\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Multiply & divide by $|\bar{z}_2|$.

$$= \frac{|\bar{z}_2|\,|z_2-z_1|}{|\bar{z}_2|\,|1-\bar{z}_1 z_2|}$$

$\because |z_1||z_2| = |z_1 z_2|$

$$= \frac{|\bar{z}_2|\,|z_2-z_1|}{|\bar{z}_2-\bar{z}_1 z_2 \bar{z}_2|}$$

$$= \frac{|\bar{z}_2|\,|z_2-z_1|}{|\bar{z}_2-\bar{z}_1\,|z_2|^2|}$$

$\because z\bar{z} = |z|^2$

Since $|z_2|=1$,

$$= \frac{|\bar{z}_2|\,|z_2-z_1|}{|\bar{z}_2-\bar{z}_1(1)|}$$

$$= \frac{|\bar{z}_2|\,|z_2-z_1|}{|\bar{z}_2-\bar{z}_1|}$$

$\because \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$$= \frac{|\bar{z}_2|\,|z_2 - z_1|}{|\overline{z_2 - z_1}|}$$

$\because\ |z| = |\bar{z}|$

$$= \frac{|z_2|\,\cancel{|z_2 - z_1|}}{\cancel{|z_2 - z_1|}}$$

$$\left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right| = 1$$

18. An AC source supplies a voltage of $V = 120\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ volts to a circuit with impedance $Z = \frac{1+i\sqrt{3}}{2}$ ohms. Calculate the current in polar form.

Sol

As $\quad V = IZ$

$$\Rightarrow I = \frac{V}{Z} \quad \text{—— (1)}$$

$$Z = \frac{1+i\sqrt{3}}{2} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad \because z = r(\cos\theta + i\sin\theta)$$

$$r = |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

$$\arg z = \tan^{-1}\left(\frac{\sqrt{3}/\cancel{2}}{1/\cancel{2}}\right) = \frac{\pi}{3}$$

$$Z = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

From (1)

$$I = \frac{120\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$$

$$= 120\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right]$$

$$= 120\left[\cos\left(\frac{3\pi - 4\pi}{12}\right) + i\sin\left(\frac{3\pi - 4\pi}{12}\right)\right]$$

$$I = 120\left[\cos\left(\frac{-\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)\right]$$

$\because \quad \cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta$

$$I = 120\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$$

بِسْمِ اللهِ الرَّحْمٰنِ الرَّحِيْمِ

(In the Name of Allah, the Most Compassionate, the Most Merciful)

Learning Outcomes

- Class 11: Mathematics (PECTAA)
- Unit 1: Complex Numbers
- Polar Coordinate System
- Exercise 1.5: Q19 - Q22

EXERCISE 1.5

19. An AC circuit has an impedance of $Z = 3 - 6i$ ohms and is connected to a voltage source of $V = 90 + 30i$ volts. Find the current in both rectangular and polar form.

Sol

$$V = IZ$$

$$\Longrightarrow \quad I = \frac{V}{Z}$$

$$I = \frac{90+30i}{3-6i}$$

$$= \frac{\cancel{3}(30+10i)}{\cancel{3}(1-2i)}$$

$$I = \frac{30+10i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{30+60i+10i+20i^2}{(1-2i)(1+2i)}$$

$$= \frac{30+70i-20}{1^2-(2i)^2}$$

$$= \frac{10+70i}{1+4} = \frac{10+70i}{5}$$

$$I = 2+14i \longrightarrow \text{Rectangular form}$$

To convert into polar form,

$$I = r(\cos\theta + i\sin\theta)$$

$r = |I| = \sqrt{2^2 + 14^2}$

$= \sqrt{4 + 196}$

$= \sqrt{200}$

$\because 200 = 2 \times 100$

$= 2 \times 10^2$

$r = 10\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{14}{2}\right) = \tan^{-1}(7)$

$\theta = 81.87^\circ$

Polar form: $I = 10\sqrt{2}\left(\cos 81.87^\circ + i \sin 81.87^\circ\right)$

20. Encrypt the word "CODE" by multiplying the complex encryption key $k = 2 - i$. Then decrypt it back to the original word.

Sol

We start by representing the letters in "MATH" a complex number with zero imaginary part.

Letter	Complex Number (z)	z encrypted $= z \times k$	z decrypted $= z$ encrypted $/ k$	Letter
M	$13+0i$	$(13+0i)(2-i) = 26-13i$	$(26-13i)/(2-i) = 13+0i$	M
A	$1+0i$	$(1+0i)(2-i) = 2-i$	$(2-i)/(2-i) = 1+0i$	A
T	$20+0i$	$(20+0i)(2-i) = 40-20i$	$(40-20i)/(2-i) = 20+0i$	T
H	$8+0i$	$(8+0i)(2-i) = 16-8i$	$(16-8i)/(2-i) = 8+0i$	H

$$\frac{26-13i}{2-i} = \frac{13\cancel{(2-i)}}{\cancel{2-i}} = 13+0i$$

$$\frac{40-20i}{2-i} = \frac{20\cancel{(2-i)}}{\cancel{2-i}} = 20+0i$$

21. Consider the complex encryption key $k = 3 - 3i$. Encrypt the word "QUIZ", and then recover the original word using the inverse of the key.

Sol

We start by representing the letters in "QUIZ" a complex number with zero imaginary part.

Letter	Complex Number (z)	z encrypted = z × k	z decrypted = z encrypted / k	Letter
Q	$17+0i$	$(17+0i)(3-3i)=51-51i$	$(51-51i)/(3-3i)= 17+0i$	Q
U	$21+0i$	$(21+0i)(3-3i)=63-63i$	$(63-63i)/(3-3i)= 21+0i$	U
I	$9+0i$	$(9+0i)(3-3i)=27-27i$	$(27-27i)/(3-3i)= 9+0i$	I
Z	$26+0i$	$(26+0i)(3-3i)=78-78i$	$(78-78i)/(3-3i)= 26+0i$	Z

$$\frac{51-51i}{3-3i} = \frac{17\cancel{(3-3i)}}{\cancel{3-3i}} = 17+0i$$

$$\frac{63-63i}{3-3i} = \frac{21\cancel{(3-3i)}}{\cancel{(3-3i)}} = 21+0i$$

22. Encrypt the word "CLASS" by adding the complex encryption key $k = -3 + 4i$. Then decrypt it back to the original word.

Sol

We start by representing the letters in "CLASS" a complex number with zero imaginary part.

Letter	Complex Number (z)	z encrypted $= z + k$	z decrypted $= z$ encrypted $- k$	Letter
C	$3+0i$	$(3+0i)+(-3+4i) = 0+4i$	$(0+4i)-(-3+4i) = 3+0i$	C
L	$12+0i$	$(12+0i)+(-3+4i) = 9+4i$	$(9+4i)-(-3+4i) = 12+0i$	L
A	$1+0i$	$(1+0i)+(-3+4i) = -2+4i$	$(-2+4i)-(-3+4i) = 1+0i$	A
S	$19+0i$	$(19+0i)+(-3+4i) = 16+4i$	$(16+4i)-(-3+4i) = 19+0i$	S
S	$19+0i$	$(19+0i)+(-3+4i) = 16+4i$	$(16+4i)-(-3+4i) = 19+0i$	S

$$0+4i-(-3+4i) = 0+\cancel{4i}+3-\cancel{4i} = 3+0i$$