Case III:

When $Q(x)$ contains non-repeated irreducible quadratic factors

Definition: A quadratic factor is irreducible if it cannot be written as the product of two linear factors with real coefficients. For example, x^2+x+1 and x^2+3 are irreducible quadratic factors.

If the polynomial $Q(x)$ contains non-repeated irreducible quadratic factors then $\frac{P(x)}{Q(x)}$ may be written as the identity having partial fractions of the form:

$\frac{Ax+B}{ax^2+bx+c}$ where A and B are the numbers to be found.

Case IV:

When $Q(x)$ has repeated irreducible quadratic factors:

If the polynomials $Q(x)$ contains a repeated irreducible quadratic factor $(ax^2+bx+c)^n, n\geq 2$ and n is a positive integer, then $\frac{P(x)}{Q(x)}$ may be written as the following identity:

$$\frac{P(x)}{Q(x)}=\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Where $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ are numbers to be found.

EXERCISE 5.2

Resolve into partial fraction.

1. $\frac{2x^2+3x+3}{(x+1)(x^2+1)}$

Solution:

Suppose $\frac{2x^2+3x+3}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$ (i)

Multiplying by $(x+1)(x^2+1)$ on both sides of equation (i)

$2x^2+3x+3=A(x^2+1)+(Bx+C)(x+1)$ (ii)

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$2(-1)^2+3(-1)+3=A((-1)^2+1)+(B(-1)+C)(-1+1)$

$2-3+3=A(1+1)+0$

$2=2A \Rightarrow A=1$

Expanding equation (ii)

$2x^2+3x+3=Ax^2+A+Bx^2+Bx+Cx+C$

$2x^2+3x+3=(A+B)x^2+(B+C)x+(A+C)x^\circ$

Comparing coefficients of x^2 and x

$A+B=2 \quad \because A=1$

$1+B=2 \Rightarrow B=2-1=1 \Rightarrow B=1$

$B+C=3 \quad \because B=1$

$1+C=3 \Rightarrow C=3-1=2 \Rightarrow C=2$

Equation (i) becomes,

$$\frac{2x^2+3x+3}{(x+1)(x^2+1)}=\frac{1}{x+1}+\frac{x+2}{x^2+1}$$

2. $\dfrac{2x+1}{(x-2)(x^2+3x+5)}$

Solution:

Suppose $\dfrac{2x+1}{(x-2)(x^2+3x+5)}=\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+3x+5}$ (i)

Multiplying by $(x-2)(x^2+3x+5)$ on both side of equation (i)

$2x+1=A(x^2+3x+5)+(Bx+C)(x-2)$ (ii)

Put $x-2=0\Rightarrow x=2$ in equation (ii)

$2(2)+1=A((2)^2+3(2)+5)+(B(2)+C)(2-2)$

$5=A(4+6+5)+0\Rightarrow \dfrac{5}{15}=A\Rightarrow A=\dfrac{1}{3}$

Expanding equation (ii)

$2x+1=Ax^2+3Ax+5A+Bx^2-2Bx+Cx-2C$

$2x+1=(A+B)x^2+(3A-2B+C)x+(5A-2C)x^\circ$

Comparing coefficients of x^2 and x

$A+B=0 \qquad \because A=\dfrac{1}{3}$

$\dfrac{1}{3}+B=0\Rightarrow B=-\dfrac{1}{3}$

Now, $3A-2B+C=2$

$3\left(\dfrac{1}{3}\right)-2\left(-\dfrac{1}{3}\right)+C=2$

$1+\dfrac{2}{3}+C=2$

$\dfrac{5}{3}+C=2\Rightarrow C=2-\dfrac{5}{3}\Rightarrow C=\dfrac{6-5}{3}\Rightarrow C=\dfrac{1}{3}$

Equation (i) becomes: $\dfrac{2x+1}{(x-2)(x^2+3x+5)}=\dfrac{\frac{1}{3}}{x-2}+\dfrac{-\frac{1}{3}x+\frac{1}{3}}{x^2+3x+5}$

Hence partial fractions are: $\dfrac{2x+1}{(x-2)(x^2+3x+5)}=\dfrac{1}{3(x-2)}-\dfrac{x-1}{3(x^2+3x+5)}$

3. $\dfrac{2x+32}{(x-2)(x^2+2)^2}$

Solution:

Suppose $\dfrac{2x+32}{(x-2)(x^2+2)^2}=\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+2}+\dfrac{Ex+D}{(x^2+2)^2}$

Multiplying by $(x-2)(x^2+2)^2$ on both sides of equation (i)

$2x+32=A(x^2+2)^2+(Bx+C)(x-2)(x^2+2)+(Ex+D)(x-2)$ (ii)

Put $x-2=0\Rightarrow x=2$ in equation (ii),

$2(2)+32=A(2^2+2)^2+0+0$

$36=36A\Rightarrow A=\dfrac{36}{36}=1\Rightarrow A=1$

Expanding equation (ii)

$2x+32 = A\left(x^4+4x^2+4\right)+\left(Bx+C\right)\left(x^3+2x-2x^2-4\right)+Ex^2-2Ex+Dx-2D$

$2x+32 = Ax^4+4Ax^2+4A+Bx^4+2Bx^2-2Bx^3-4Bx+Cx^3+2Cx-2Cx^2-4C+Ex^2-2Ex+Dx-2D$

$2x+32 = \left(A+B\right)x^4+\left(-2B+C\right)x^3+\left(4A+2B-2C+E\right)x^2+\left(-4B+2C-2E+D\right)x+\left(4A-4C-2D\right)x^\circ$

Comparing coefficients of x^4, x^3, x^2 and x°

$A+B=0 \quad \because A=1$

$1+B=0 \Rightarrow B=-1$

$-2B+C=0$

$-2\left(-1\right)+C=0 \Rightarrow C=-2$

$4A+2B-2C+E=0$

$4\left(1\right)+2\left(-1\right)-2\left(-2\right)+E=0$

$4-2+4+E=0 \Rightarrow E=-6$

$4A-4C-2D=32$

$4\left(1\right)-4\left(-2\right)-2D=3$

$4+8-32=2D$

$-20=2D$

$D=-10$

Equation (i) becomes: $\dfrac{2x+32}{\left(x-2\right)\left(x^2+2\right)^2} = \dfrac{1}{x-2}+\dfrac{-x+\left(-2\right)}{x^2+2}+\dfrac{-6x-10}{\left(x^2+2\right)^2}$

Hence partial fractions are: $\dfrac{2x+32}{\left(x-2\right)\left(x^2+2\right)^2} = \dfrac{1}{x-2}-\dfrac{x+2}{x^2+2}-\dfrac{2\left(3x+5\right)}{\left(x^2+2\right)^2}$

4. $\dfrac{3x^2+3}{x^3+1}$

Solution:

Consider $\dfrac{3x^2+3}{x^3+1} = \dfrac{3x^2+3}{\left(x+1\right)\left(x^2-x+1\right)}$

Let $\dfrac{3x^2+3}{\left(x+1\right)\left(x^2-x+1\right)} = \dfrac{A}{x+1}+\dfrac{Bx+C}{x^2-x+1}$ (i)

Multiplying by $\left(x+1\right)\left(x^2-x+1\right)$ on both sides of equation (i)

$3x^2+3 = A\left(x^2-x+1\right)+\left(Bx+C\right)\left(x+1\right)$ (ii)

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$3\left(-1\right)^2+3 = A\left(\left(-1\right)^2-\left(-1\right)+1\right)+0$

$3+3 = A\left(1+1+1\right)$

$6=3A \Rightarrow A=2$

Expanding equation (ii)

$3x^2+3 = Ax^2-Ax+A+Bx^2+Bx+Cx+C$

$3x^2+3 = \left(A+B\right)x^2+\left(-A+B+C\right)x+\left(A+C\right)x^\circ$

Comparing coefficients of x^2 and x

$A+B=3 \quad \because A=2$

$2+B=3 \Rightarrow B=1$

$-A+B+C=0$

$-2+1+C=0$

$-1+C=0 \Rightarrow C=1$

Equation (i) becomes: $\dfrac{3x^2+3}{\left(x+1\right)\left(x^2-x+1\right)} = \dfrac{2}{x+1}+\dfrac{x+1}{x^2-x+1}$

5. $\dfrac{x^4}{x^4+2x^2+1}$

Solution:

It is an improper fraction. First we convert it into mixed form.

Now x^4+2x^2+1

$$x^4+2x^2+1\overline{)\,x^4\qquad}\quad 1$$
$$\underline{\pm x^4 \pm 2x^2 \pm 1}$$
$$-2x^2-1$$

$$\frac{x^4}{x^4+2x^2+1}=1+\frac{-2x^2-1}{x^4+2x^2+1}$$

$$=1-\frac{2x^2+1}{x^4+2x^2+1} \qquad \text{(i)}$$

Consider $\dfrac{2x^2+1}{x^4+2x^2+1}=\dfrac{2x^2+1}{\left(x^2+1\right)^2}$

Let $\dfrac{2x^2+1}{\left(x^2+1\right)^2}=\dfrac{Ax+B}{x^2+1}+\dfrac{(Cx+D)}{\left(x^2+1\right)^2}$ (ii)

Multiplying by $\left(x^2+1\right)^2$ on both sides of equation (ii)

$2x^2+1=(Ax+B)\left(x^2+1\right)+(Cx+D)$

$2x^2+1=Ax^3+Ax+Bx^2+B+Cx+D$

$2x^2+1=Ax^3+Bx^2+(A+C)x+(B+D)x^\circ$

Comparing coefficient of x^3, x^2, x and x°

$A=0;\ B=2$

$A+C=0$

$0+C=0 \Rightarrow C=0$

$B+D=1$

$2+D=1 \Rightarrow D=1-2$

$D=-1$

Equation (ii) becomes: $\dfrac{2x^2+1}{\left(x^2+1\right)^2}=\dfrac{0x+2}{x^2+1}+\dfrac{0x-1}{\left(x^2+1\right)^2}=\dfrac{2}{x^2+1}-\dfrac{1}{\left(x^2+1\right)^2}$

So, equation (i) becomes: $\dfrac{x^4}{x^4+2x^2+1}=1-\dfrac{2}{x^2+1}+\dfrac{1}{\left(x^2+1\right)^2}$

6. $\dfrac{6x^4+40x^2}{\left(4-x^2\right)\left(x^2+4\right)^2}$

Solution:

Consider $\dfrac{6x^4+40x^2}{\left(4-x^2\right)\left(x^2+4\right)^2}=\dfrac{6x^4+40x^2}{(2-x)(2+x)\left(x^2+4\right)^2}$

Let $\dfrac{6x^4+40x^2}{(2-x)(2+x)\left(x^2+4\right)^2}=\dfrac{A}{2-x}+\dfrac{B}{2+x}+\dfrac{Cx+D}{x^2+4}+\dfrac{Ex+F}{\left(x^2+4\right)^2}$ (i)

Multiplying by $(2-x)(2+x)\left(x^2+4\right)^2$ on both sides of equation (i)

$6x^4+40x^2=A(2+x)\left(x^2+4\right)^2+B(2-x)\left(x^2+4\right)^2+(Cx+D)(2-x)(2+x)\left(x^2+4\right)+(Ex+F)(2-x)(2+x)$ (ii)

Put $2-x=0 \Rightarrow 2=x$ in equation (ii)

$6(2)^4+40(2)^2=A(2+2)\left((2)^2+4\right)^2+0+0+0$

$6(16)+40(4)=A(4)(4+4)^2$

$96+160=A(4)(8)^2$

$256=256A\Rightarrow A=1$

Put $2+x=0\Rightarrow -2=x$ in equation (ii)

$6(-2)^4+40(-2)^2=0+B\left(2-(-2)\right)\left((-2)^2+4\right)^2+0+0$

$6(16)+40(4)=B(4)(4+4)^2$

$256=256B\Rightarrow B=1$

Expanding equation (ii)

$6x^4+40x^2=A(2+x)(x^4+8x^2+16)+B(2-x)(x^4+8x^2+16)(Cx+D)(4-x^2)(x^2+4)+(Ex+F)(4-x^2)$

$6x^4+40x^2=A\left(2x^4+16x^2+32+x^5+8x^3+16x\right)+B\left(2x^4+16x^2+32-x^5-8x^3-16x\right)$

$+(Cx+D)\left(4x^2+16-x^4-4x^2\right)+4Ex-Ex^3+4F-Fx^2$

$6x^4+40x^2=2Ax^4+16Ax^2+32A+Ax^5+8Ax^3+16Ax+2Bx^4+16Bx^2+32B-Bx^5-8Bx^3-16Bx+16Cx-Cx^5$

$+16D-Dx^4+4Ex-Ex^3+4F-Fx^2$

$6x^4+40x^2=(A-B-C)x^5+(2A+2B-D)x^4+(8A-8B-E)x^3+(16A+16B-F)x^2$

$+(16A-16B+16C+4E)x+(32A+32B+4F)x^{\circ}$

Comparing coefficients of x^5, x^4 and x^3

$A-B-C=0 \quad \because A=1$ and $B=1$

$1-1-C=0\Rightarrow C=0$

$2A+2B-D=6$

$2(1)+2(1)-D=6$

$4-6=D\Rightarrow D=-2$

$8A-8B-E=0$

$8(1)-8(1)=E$

$E=0$

Equation (i) becomes: $\dfrac{6x^4+40x^2}{(4-x^2)(x^2+4)}=\dfrac{1}{2-x}+\dfrac{1}{2+x}+\dfrac{0x-2}{x^2+4}+\dfrac{0x+(-8)}{(x^2+4)^2}$

Hence partial fractions are: $\dfrac{6x^4+40x^2}{(4-x^2)(x^2+4)}=\dfrac{1}{2-x}+\dfrac{1}{2+x}-\dfrac{2}{x^2+4}-\dfrac{8}{(x^2+4)^2}$.