Partial Fractions:

To express a single rational fraction as a sum of two or more single rational fractions is called **Partial Fractions.**

For Example:

(i) $$\frac{3x}{(x-1)(x+2)}=\frac{1}{x-1}+\frac{2}{x+2}$$

(ii) $$\frac{5x^2+5x-3}{(x+1)^2(x-2)}=\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x-2}$$

Partial Fraction Resolution:

Expressing a rational fractions as a sum of partial fractions is called **Partial Fraction Resolution.**

For Example:

$$\frac{7x+25}{(x+3)(x+4)}=\frac{A}{x+3}+\frac{B}{x+4}$$

Equation:

An open sentence formed by using the sign of equality '=' is called an equation.

Types of Equation:

There are two types of equations.

Conditional Equation	Identity
It is an equation in which **two** algebraic expressions are **equal** for particular values of the variable. **For Example:** a) $2x=3$ is a conditional equation and it is true only if $x=\frac{3}{2}$. b) $x^2+x-6=0$ is a conditional equation and it is true for $x=2,-3$ only.	It is an equation which holds good for all values of the variable. **For Example:** a) $(a+b)x\equiv ax+bx$ is an identity and its two sides are equal for all values of x. b) $(x+3)(x+4)\equiv x^2+7x+12$ is also an identity which is true for all values of x.

Rational Fractions:

An expression of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x with real coefficients and $Q(x)\neq 0$, is called **Rational Fraction**.

Rational fractions can be divided into following two types.

Proper Rational Fraction	Improper Rational Fraction
A rational fraction $\frac{P(x)}{Q(x)}$ is called a **Proper Rational Fraction** if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator. **For Example:** $\frac{3}{x+1},\frac{2x-5}{x^2+4}$ and $\frac{9x^2}{x^3+1}$	A rational fraction $\frac{P(x)}{Q(x)}$ is called **Improper Rational Fraction** if degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator. **For Example:** $\frac{x}{2x-3},\frac{(x-2)(x+1)}{(x-1)(x+4)}$ and $\frac{x^2-3}{3x+1}$

Theorem:

"If two polynomials are equal for all values of the variable, then the polynomials have same degree and the coefficients of like powers of the variable in both the polynomials must be equal."

For Example:

If $px^3+qx^2-ax+b=2x^3-3x^2-4x+5, \forall\, x$ then $p=2,\ q=-3,\ a=4$ and $b=5$.

Resolution of a Rational Fraction $\frac{P(x)}{Q(x)}$ into Partial Fractions

(i) The degree of $P(x)$ must be less than that of $Q(x)$. If not, divide and work with remainder theorem.

(ii) Factorize the denominator $Q(x)$ into its irreducible factor, write the rational fraction into partial fractions.

(iii) Multiply the identity with the denominator of left hand side.

(iv) Equate the coefficients of like terms $(\text{powers of } x)$.

(v) Solve the resulting equations for the coefficients.

Case-I:

Resolution of $\frac{P(x)}{Q(x)}$ into partial fraction when $Q(x)$ has only non-repeated linear factors:

The polynomial $Q(x)$ may be written as:

$Q(x)=(x-a_1)(x-a_2)...(x-a_n)$, where $a_1 \neq a_2 \neq ... \neq a_n$

$\therefore \frac{P(x)}{Q(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+...+\frac{A_n}{x-a_n}$ is an identity.

Where $A_1, A_2, ..., A_n$ are numbers to be found.

Case-II:

When $Q(x)$ has repeated linear factors:

If the polynomial $Q(x)$ has repeated linear factors $(x-a)^n, n \geq 2$ and n is a positive integer, then $\frac{P(x)}{Q(x)}$ may be written as the following identity:

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+...+\frac{A_n}{(x-a)^n}$$

Where A_1, A_2, …, A_n are numbers to be found.

EXERCISE 5.1

Resolve the following into partial fractions.

1. $\dfrac{2}{x^2-1}$

Solution:

$$\frac{2}{x^2-1}=\frac{2}{(x-1)(x+1)}$$

Suppose $\dfrac{2}{(x-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}$ (i)

Multiplying by $(x-1)(x+1)$ on both sides of equation (i), we get

$2=A(x+1)+B(x-1)$ (ii)

Put $x-1=0\Rightarrow x=1$ in equation (ii)

$2=A(1+1)+B(1-1)$

$2=2A+0\Rightarrow A=1$

Put $x+1=0\Rightarrow x=-1$ in equation (ii), we get

$2=A(-1+1)+B(-1-1)$

$2=0+B(-2)\Rightarrow B=-1$

So, equation (i) becomes $\dfrac{2}{(x-1)(x+1)}=\dfrac{1}{x-1}+\dfrac{-1}{x+1}$

Hence, partial fractions are: $\dfrac{2}{x^2-1}=\dfrac{1}{x-1}-\dfrac{1}{x+1}$.

2. $\dfrac{a-b}{(x-a)(x-b)}$

Solution:

Suppose $\dfrac{a-b}{(x-a)(x-b)}=\dfrac{A}{x-a}+\dfrac{B}{x-b}$ (i)

Multiplying by $(x-a)(x-b)$ on both sides of equation (i)

$a-b=A(x-b)+B(x-a)$ (ii)

Put $x-a=0\Rightarrow x=a$ in equation (i)

$a-b=A(a-b)+B(a-a)$

$a-b=A(a-b)+0\Rightarrow A=\dfrac{a-b}{a-b}=1\Rightarrow A=1$

Put $x-b=0\Rightarrow x=b$ in equation (ii), we get

$a-b=A(b-b)+B(b-a)$

$a-b=0-B(a-b)\Rightarrow B=\dfrac{a-b}{-(a-b)}=-1\Rightarrow B=-1$

So, equation (i) becomes: $\dfrac{a-b}{(x-a)(x-b)}=\dfrac{1}{x-a}+\dfrac{-1}{x-b}$

Hence partial fractions are: $\dfrac{a-b}{(x-a)(x-b)}=\dfrac{1}{x-a}-\dfrac{1}{x-b}$.

3. $\dfrac{x^2+1}{(x+1)(x-1)}$

Solution:

Since the **degree** of polynomial of numerator and denominator is **equal** so, it is improper Rational Fraction. First transform it into **mixed form** by **division**.

$$\frac{x^2+1}{x^2-1}$$

$$\begin{array}{r|l} & \quad 1 \\ x^2-1 & \ \ x^2+1 \\ & -x^2 \mp 1 \\ \hline & \quad 2 \end{array}$$

Dividing x^2+1 by x^2-1, we have

quotient = 1 and remainder = 2, therefore

$$\frac{x^2+1}{x^2-1}=1+\frac{2}{x^2-1}=1+\frac{2}{(x+1)(x-1)} \qquad \text{(i)}$$

Let, $\dfrac{2}{(x+1)(x-1)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}$ (ii)

Multiplying by $(x+1)(x-1)$ on both sides of equation (ii), we get

$$2=A(x-1)+B(x+1) \qquad \text{(iii)}$$

Putting $x-1=0 \Rightarrow x=-1$ in equation (iii)

$$2=A(0)+B(1+1)$$

$$2=2B \Rightarrow B=1$$

Putting $x+1=0 \Rightarrow x=-1$ in equation (iii)

$$2=A(-1-1)+B(0)$$

$$2=-2A \Rightarrow A=-1$$

Putting the values of A and B in equation (ii), we get,

$$\frac{2}{(x+1)(x-1)}=\frac{-1}{x+1}+\frac{1}{x-1}$$

By using equation (i)

The partial fractions are: $\dfrac{x^2+1}{x^2-1}=1+\dfrac{1}{x-1}-\dfrac{1}{x+1}$

4. $\dfrac{2x+3}{(x+1)(x+2)(x+3)}$

Solution:

Suppose $\dfrac{2x+3}{(x+1)(x+2)(x+3)}=\dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{x+3}$ (i)

Multiplying by $(x+1)(x+2)(x+3)$ on both sides of equation (i) we get

$$2x+3=A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2) \qquad \text{(ii)}$$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$$2(-1)+3=A(-1+2)(-1+3)+B(-1+1)(-1+3)+C(-1+1)(-1+2)$$

$$-2+3=A(1)(2)+0+0 \Rightarrow 1=2A \Rightarrow A=\frac{1}{2}$$

Put $x+2=0 \Rightarrow x=-2$ in equation (ii)

$2(-2)+3 = A(-2+2)(-2+3)+B(-2+1)(-2+3)+C(-2+1)(-2+2)$

$-4+3 = 0+B(-1)(1)+0 \Rightarrow -1=-B \Rightarrow B=1$

Put $x+3=0 \Rightarrow x=-3$ in equation (ii)

$2(-3)+3 = A(-3+2)(-3+3)+B(-3+1)(-3+3)+C(-3+1)(-3+2)$

$-6+3 = 0+0+C(-2)(-1) \Rightarrow -3=2C \Rightarrow C=-\dfrac{3}{2}$

So, equation (i) becomes: $\dfrac{2x+3}{(x+1)(x+2)(x+3)} = \dfrac{\frac{1}{2}}{x+1}+\dfrac{1}{x+2}+\dfrac{-\frac{3}{2}}{x+3}$

Hence, partial fractions are: $\dfrac{2x+3}{(x+1)(x+2)(x+3)} = \dfrac{1}{2(x+1)}+\dfrac{1}{x+2}-\dfrac{3}{2(x+3)}$.

5. $\dfrac{x^2+4x+5}{(x+1)(x^2+5x+6)}$

Solution:

Consider $x^2+5x+6 = x^2+3x+2x+6$

$= x(x+3)+2(x+3) = (x+2)(x+3)$

Suppose $\dfrac{x^2+4x+5}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{x+3}$ (i)

Multiplying by $(x+1)(x+2)(x+3)$ on both sides of equation (i), we get

$x^2+4x+5 = A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii), we get,

$(-1)^2+4(-1)+5 = A(-1+2)(-1+3)+B(-1+1)(-1+3)+(-1+1)(-1+2)$

$1-4+5 = A(1)(2)+0+0$

$2 = 2A \Rightarrow A=1$

Put $x+2=0 \Rightarrow x=-2$ in equation (ii), we get

$(-2)^2+4(-2)+5 = A(-2+2)(-2+3)+B(-2+1)(-2+3)+C(-2+1)(-2+2)$

$4-8+5 = 0+B(-1)(1)+0$

$1=-B \Rightarrow B=-1$

Put $x+3=0 \Rightarrow x=-3$ in equation (ii), we get

$(-3)^2+4(-3)+5 = A(-3+2)(-3+3)+B(-3+1)(-3+3)+C(-3+1)(-3+2)$

$9-12+5 = 0+0+C(-2)(-1)$

$2=2C \Rightarrow C=1$

So, equation (i) becomes $\dfrac{x^2+4x+5}{(x+1)(x+2)(x+3)} = \dfrac{1}{x+1}+\dfrac{-1}{x+2}+\dfrac{1}{x+3}$

Hence partial fractions are: $\dfrac{x^2+4x+5}{(x+1)(x+2)(x+3)} = \dfrac{1}{x+1}-\dfrac{1}{x+2}+\dfrac{1}{x+3}$.

6. $\dfrac{4x^3+5x^2-3x-2}{x^2-1}$

Solution:

It is improper rational fraction, so first we transform it into mixed form

So, $\dfrac{4x^3+5x^2-3x-2}{x^2-1}$

$$\begin{array}{r|l} & 4x+5 \\ x^2-1 & 4x^3+5x^2-3x-2 \\ & \pm 4x^3 \quad \mp 4x \\ \hline & 5x^2+x-2 \\ & \pm 5x^2 \quad \mp 5 \\ \hline & x+3 \end{array}$$

Hence $\dfrac{4x^3+5x^2-3x-2}{x^2-1}=4x+5+\dfrac{x+3}{x^2-1}=4x+5+\dfrac{x+3}{(x-1)(x+1)}$ (i)

Suppose $\dfrac{x+3}{(x-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}$ (ii)

Multiplying by $(x-1)(x+1)$ on both sides of equation (ii)

$x+3=A(x+1)+B(x-1)$ (iii)

Put $x-1=0\Rightarrow x=1$ in equation (iii)

$1+3=A(1+1)+B(1-1)$

$4=2A+0\Rightarrow A=\dfrac{4}{2}=2$

Put $x+1=0\Rightarrow x=-1$ in equation (iii)

$-1+3=A(-1+1)+B(-1-1)$

$2=0+B(-2)$

$2=-2B\Rightarrow B=-1$

Put the values of A and B in equation (ii)

$\dfrac{x+3}{(x-1)(x+1)}=\dfrac{2}{x-1}+\dfrac{-1}{x+1}$

So, equation (i) becomes $\dfrac{4x^3+5x^2-3x-2}{x^2-1}=4x+5+\dfrac{2}{x-1}-\dfrac{1}{x+1}$.

7. $\dfrac{3x^2-12x+11}{(x-1)(x-2)(x-3)}$

Solution:

Suppose $\dfrac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$ (i)

Multiplying by $(x-1)(x-2)(x-3)$ on both sides of equation (i)

$3x^2-12x+11=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$ (ii)

Put $x-1=0\Rightarrow x=1$ in equation (ii)

$3(1)^2-12(1)+11=A(1-2)(1-3)+B(1-1)(1-3)+C(1-1)(1-2)$

$3-12+11=A(-1)(-2)+0+0$

$2=2A\Rightarrow A=1$

Put $x-2=0 \Rightarrow x=2$ in equation (ii)

$3(2)^2-12(2)+11=0+B(2-1)(2-3)+0$

$12-24+11=B(1)(-1)$

$-1=-B \Rightarrow B=1$

Put $x-3=0 \Rightarrow x=3$ in equation (ii)

$3(3)^2-12(3)+11=0+0+C(3-1)(3-2)$

$27-36+11=C(2)(1)$

$2=2C \Rightarrow C=1$

Put the values of A, B and C in equation (i)

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$$

8. $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$

Solution:

As $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}=\dfrac{(x^2-3x+2)(x-3)}{(x^2-9x+20)(x-6)}$

$$=\frac{x^3-3x^2-3x^2+9x+2x-6}{x^3-6x^2-9x^2+54x+20x-120}=\frac{x^3-6x^2+11x-6}{x^3-15x^2+74x-120}$$

It is improper rational fraction, so first we transform it into mixed form.

$$\begin{array}{r|l} & \quad 1 \\ x^3-15x^2+74x-120 & x^3-6x^2+11x-6 \\ & \underline{\pm x^3 \mp 15x^2 \pm 74x \mp 120} \\ & \quad 9x^2-63x+114 \end{array}$$

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}=1+\frac{9x^2-63x+114}{(x-4)(x-5)(x-6)} \qquad \text{(i)}$$

Let $\dfrac{9x^2-63x+114}{(x-4)(x-5)(x-6)}=\dfrac{A}{x-4}+\dfrac{B}{x-5}+\dfrac{C}{x-6}$ (ii)

Multiplying by $(x-4)(x-5)(x-6)$ on both sides of equation (ii)

$$9x^2-63x+114=A(x-5)(x-6)+B(x-4)(x-6)+C(x-4)(x-5) \qquad \text{(iii)}$$

Put $x-4=0 \Rightarrow x=4$ in equation (iii)

$9(4)^2-63(4)+114=A(4-5)(4-6)+B(4-4)(4-6)+C(4-4)(4-5)$

$144-252+114=A(-1)(-2)+0+0$

$6=2A \Rightarrow A=3$

Put $x-5=0 \Rightarrow x=5$ in equation (iii)

$9(5)^2-63(5)+114=0+B(1)(-1)+0$

$24=-B \Rightarrow B=-24$

Put $x-6=0 \Rightarrow x=6$ in equation (iii)

$9(6)^2-63(6)+114=A(6-5)(6-6)+B(6-4)(6-6)+C(6-4)(6-5)$

$324-378+114=0+0+C(2)(1)$

$60=2C \Rightarrow C=30$

So, equation (ii) becomes: $\dfrac{9x^2-63x+114}{(x-4)(x-5)(x-6)}=\dfrac{3}{x-4}+\dfrac{-24}{x-5}+\dfrac{30}{x-6}$

Hence partial fractions are: $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}=1+\dfrac{3}{x-4}-\dfrac{24}{x-5}+\dfrac{30}{x-6}$

9. $\dfrac{x^2}{(x^2+a)(x^2+b)(x^2+c)}$

Solution:

Let $x^2=y$ then $\dfrac{y}{(y+a)(y+b)(y+c)}$

Suppose $\dfrac{y}{(y+a)(y+b)(y+c)}=\dfrac{A}{y+a}+\dfrac{B}{y+b}+\dfrac{C}{y+c}$ (i)

Multiplying by $(y+a)(y+b)(y+c)$ on both sides of equation (i)

$y=A(y+b)(y+c)+B(y+a)(y+c)+C(y+a)(y+b)$ (ii)

Put $y+a=0 \Rightarrow y=-a$ in equation (ii)

$-a=A(-a+b)(-a+c)+0+0$

$-a=-A(a-b)(c-a)$

$\Rightarrow A=\dfrac{a}{(a-b)(c-a)}$

Put $y+b=0 \Rightarrow y=-b$ in equation (ii)

$-b=0+B(-b+a)(-b+c)+0$

$-b=-B(a-b)(b-c)$

$B=\dfrac{b}{(a-b)(b-c)}$

Put $y+c=0 \Rightarrow y=-c$ in equation (ii)

$-c=0+0+C(-c+a)(-c+b)$

$-c=-C(c-a)(b-c)$

$C=\dfrac{c}{(c-a)(b-c)}$

Put the values of A, B and C in equation (i)

$$\frac{y}{(y+a)(y+b)(y+c)}=\frac{\frac{a}{(a-b)(c-a)}}{y+a}+\frac{\frac{b}{(a-b)(b-c)}}{y+b}+\frac{\frac{c}{(c-a)(b-c)}}{y+c}$$

Using $x^2=y,$ partial fractions are

$$\frac{x^2}{(x^2+9)(x^2+b)(x^2+c)}=\frac{a}{(a-b)(c-a)(x^2+a)}+\frac{b}{(a-b)(b-c)(x^2+b)}+\frac{c}{(c-a)(b-c)(x^2+c)}$$

10. $\dfrac{x+1}{(x-1)^2}$

Solution:

Suppose $\dfrac{x+1}{(x-1)^2}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}$ (i)

Multiplying by $(x-1)^2$ on both sides of equation (i)

$x+1=A(x-1)+B$ (ii)

Expanding equation (ii)

$x+1=Ax-A+B$

$1x+1=Ax+(-A+B)$

Comparing coefficients of x and x°

$\Rightarrow 1=A$ and $1=-A+B$

$1=-1+B \Rightarrow B=2$

So, equation (i) becomes

$$\frac{x+1}{(x-1)^2}=\frac{1}{x-1}+\frac{2}{(x-1)^2}$$

11. $\dfrac{x^2+x}{(x^2-1)^2}$

Solution:

Consider $\dfrac{x^2+x}{(x^2-1)^2}=\dfrac{x(x+1)}{((x-1)(x+1))^2}=\dfrac{x(x+1)}{(x-1)^2(x+1)^2}=\dfrac{x}{(x-1)^2(x+1)}$

Let $\dfrac{x}{(x-1)^2(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}$ (i)

Multiplying by $(x-1)^2(x+1)$ on both sides of equation (i)

$x=A(x-1)(x+1)+B(x+1)+C(x-1)^2$ (ii)

Put $x-1=0 \Rightarrow x=1$ in equation (ii)

$1=0+B(1+1)+0 \Rightarrow 1=2B \Rightarrow B=\dfrac{1}{2}$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$-1=0+0+C(-1-1)^2 \Rightarrow -1=C(4) \Rightarrow C=\dfrac{-1}{4}$

Expanding equation (ii)

$x=A(x^2-1)+B(x+1)+C(x^2-2x+1)$

$x=Ax^2-A+Bx+B+Cx^2-2Cx+C$

$x=(A+C)x^2+(B-2C)x+(-A+B+C)$

Comparing co-efficients of x^2

$0=A+C \Rightarrow 0=A-\dfrac{1}{4} \Rightarrow A=\dfrac{1}{4}$

Equation (i) becomes $\dfrac{x}{(x-1)^2(x+1)}=\dfrac{\frac{1}{4}}{(x-1)}+\dfrac{\frac{1}{2}}{(x-1)^2}+\dfrac{-\frac{1}{4}}{(x+1)}$

Hence, partial fractions are: $\dfrac{x^2+x}{(x^2-1)^2}=\dfrac{1}{4(x-1)}+\dfrac{1}{2(x-1)^2}-\dfrac{1}{4(x+1)}$

12. $\dfrac{3x^2+4x-5}{(x-1)^3}$

Solution:

Suppose $\dfrac{3x^2+4x-5}{(x-1)^3}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}$ (i)

Multiplying by $(x-1)^3$ on both sides of equation (i), we get

$3x^2+4x-5=A(x-1)^2+B(x-1)+C$ (ii)

Put $x-1=0 \Rightarrow x=1$ in equation (ii)

$3(1)^2+4(1)-5=0+0+C$

$3+4-5=C \Rightarrow 2=C$

Expanding equation (ii)

$3x^2+4x-5=A(x^2-2x+1)+Bx-B+C$

$3x^2+4x-5=Ax^2-2Ax+A+Bx-B+C$

$3x^2+4x-5=Ax^2+(-2A+B)x+(A-B+C)x^\circ$

Comparing coefficients of x^2, x and x°

$3=A$

$-2A+B=4 \quad \because A=3$

$-2(3)+B=4$

$-6+B=4 \Rightarrow B=4+6=10$

$A-B+C=-5$

$3-10+C=-5$

$-7+C=-5 \Rightarrow C=-5+7=2$

So, equation (i) becomes

$$\frac{3x^2+4x-5}{(x-1)^3}=\frac{3}{x-1}+\frac{10}{(x-1)^2}+\frac{2}{(x-1)^3}$$

13. $\dfrac{1}{x(x+1)^3}$

Solution:

Suppose $\dfrac{1}{x(x+1)^3}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}+\dfrac{D}{(x+1)^3}$ (i)

Multiplying by $x(x+1)^3$ on both sides of equation (i)

$1=A(x+1)^3+Bx(x+1)^2+Cx(x+1)+Dx$ (ii)

Put $x=0$ in equation (ii)

$1=A(0+1)^3+0+0+0 \Rightarrow A=1$

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$1=0+0+0+D(-1)$

$1=-D \Rightarrow D=-1$

Expanding equation (ii)

$1=A(x^3+3x^2+3x+1)+Bx(x^2+2x+1)+Cx^2+Cx+Dx$

$1=Ax^3+3Ax^2+3Ax+A+Bx^3+2Bx^2+Bx+Cx^2+Cx+Dx$

$1=(A+B)x^3+(3A+2B+C)x^2+(3A+B+C+D)x+A$

Comparing coefficients of x^3, x^2, x and x°

$A+B=0$ (1)

$3A+2B+C=0$ (2)

$3A+B+C+D=0$

$A=1$

Put $A=1$ in equation (1), $1+B=0 \Rightarrow B=-1$

Equation (2) becomes: $3(1)+2(-1)+C=0$

$1+C=0 \Rightarrow C=-1$

Hence partial fractions are: $\dfrac{1}{x(x+1)^3}=\dfrac{1}{x}-\dfrac{1}{x+1}-\dfrac{1}{(x+1)^2}-\dfrac{1}{(x+1)^3}$

14. $\dfrac{4x^2-3x+1}{(x+1)(x-1)^2}$

Solution:

Suppose $\dfrac{4x^2-3x+1}{(x+1)(x-1)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$

Multiplying by $(x+1)(x-1)^2$ on both sides of equation (i)

$4x^2-3x+1=A(x-1)^2+B(x+1)(x-1)+C(x+1)$ (ii)

Put $x+1=0 \Rightarrow x=-1$ in equation (ii)

$4(-1)^2-3(-1)+1=A(-1-1)^2+0+0$

$4+3+1=4A$

$8=4A \Rightarrow A=2$

Put $x-1=0 \Rightarrow x=1$ in equation (ii)

$4(1)^2-3(1)+1=0+0+C(1+1)$

$4-3+1=2C$

$2=2C \Rightarrow C=1$

Expanding equation (ii)

$4x^2-3x+1=A(x^2+1-2x)+B(x^2-1)+Cx+C$

$4x^2-3x+1=Ax^2+A-2Ax+Bx^2-B+Cx+C$

$4x^2-3x+1=(A+B)x^2+(-2A+C)x+(A-B+C)x^\circ$

Comparing coefficients of x^2 and x

$A+B=4$ (1)

$-2A+C=-3$ (2)

Equation (1) becomes

$2+B=4 \Rightarrow B=2$ $(\because A=2)$

Put the value of A in equation (2)

$-2(2)+C=-3$

$C=-3+4 \Rightarrow C=1$

Put the values of A, B and C in equation (i)

$\dfrac{4x^2-3x+1}{(x+1)(x-1)^2}=\dfrac{2}{x+1}+\dfrac{2}{x-1}+\dfrac{1}{(x-1)^2}$

15. $\dfrac{12x^2-48}{(x-2)^2(x+2)^2}$

Solution:

$\dfrac{12x^2-48}{(x-2)^2(x+2)^2}=\dfrac{12(x^2-4)}{(x-2)^2(x+2)^2}=\dfrac{12(x-2)(x+2)}{(x-2)^2(x+2)^2}=\dfrac{12}{(x-2)(x+2)}$

Suppose $\dfrac{12}{(x-2)(x+2)}=\dfrac{A}{x-2}+\dfrac{B}{x+2}$ (i)

Multiplying by $(x-2)(x+2)$ on both sides of equation (i)

$12 = A(x+2) + B(x-2)$ (ii)

Put $x-2=0 \Rightarrow x=2$ in equation (ii)

$12 = A(2+2)+0$

$12 = A(4) \Rightarrow A=3$

Put $x+2=0 \Rightarrow x=-2$ in equation (ii)

$12 = 0 + B(-2-2)$

$12 = B(-4) \Rightarrow B=-3$

Equation (i) becomes $\dfrac{12}{(x-2)(x+2)} = \dfrac{3}{x-3} + \dfrac{-3}{x+2}$

Hence partial fractions are $\dfrac{12x^2-48x}{(x-2)^2(x+2)^2} = \dfrac{3}{x-2} - \dfrac{3}{x+2}$.