Elementary Row Operations on a Matrix:

Usually, a given system of linear equations is reduced to a simple equivalent system by applying elementary operations which are stated as below:

(i) Interchanging two equations.
(ii) Multiplying an equation by a non-zero number
(iii) Adding a multiple of one equation to another equation.

Corresponding to these three elementary operations, the following elementary row operations are applied to matrices to obtain equivalent matrices:

(i) Interchanging two rows
(ii) Multiplying a row by a non-zero number
(iii) Adding a multiple of one row to another row

Notations that are used to represent row operations for (i) to (iii) are given below:

- Interchanging R_i and R_j is expressed as $R_i \leftrightarrow R_j$.
- k times R_i is denoted by $kR_i \to R_i'$.
- Adding k times R_j to R_i is expressed as $R_i + kR_j \to R_i'$

Upper Triangular Matrix:

A square matrix $A = \left[a_{ij}\right]$ is called **upper triangular** if all elements below the principal diagonal are zero, that is, $a_{ij} = 0$ for all $i > j$.

Lower Triangular Matrix:

A square matrix $A = \left[a_{ij}\right]$ is said to be **lower triangular** if all elements above the principal diagonal are zero, that is, $a_{ij} = 0$ for all $i < j$

Triangular Matrix:

A square matrix A is named as **triangular** whether it is upper triangular or lower triangular matrix. For example, the matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -1 & 2 & 3 & 1 \end{bmatrix} \text{are triangular matrices.}$$

Echelon and Reduced Echelon Forms of Matrix:

In any non-zero row of a matrix, the first non-zero entry is called the leading entry of that row.

Echelon Form of a Matrix

An $m \times n$ matrix A is called in (row) echelon form if:

(i) The number of zeros before the leading entry is greater than the number zeros in the preceding row.
(ii) Every non-zero row in A precedes every zero row (if any).
(iii) The leading entry element in each row is 1.

The matrices $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in echelon form.

Reduced Echelon Form of a Matrix:

An $m \times n$ matrix A is said to be in reduced (row) echelon form if the first non-zero entry (or leading entry) in R_i lies in C_j, then all other entries of C_j are zero.

The matrices $\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in (row) reduced echelon form.

Inverse of a Matrix:

Let A be a non-singular matrix. If the application of elementary row operations on $A \vdots I$ in succession reduces A to I, then the resulting matrix is $I \vdots A^{-1}$.

Rank of Matrix:

Let *A* be a non-zero matrix, if *r* is the number of non-zero rows when it is reduced to the reduced echelon form, then *r* is called the (row) rank of the matrix *A*.

System of Non-Homogeneous Linear Equation:

Three linear equations in three variables such as:

$$\left.\begin{aligned} a_1x+b_1y+c_1z&=k_1 \\ a_2x+b_2y+c_2z&=k_2 \\ a_3x+b_3y+c_2z&=k_3 \end{aligned}\right\} \qquad \text{(i)}$$

is called a system of non-homogeneous linear equations in the three variables x, y and z. If constant terms k_1, k_2 and k_3 are not all zero.

Consistent: A system of linear equations is said to be consistent if the system has a unique solution or it has infinitely many solutions.

Inconsistent:

A system of linear equations is said to be inconsistent if the system has no solution.

Now we will solve the system of non-homogeneous linear equations with the help of the following methods.

(i) Using reduced echelon form

(ii) Using matrix inversion method

(iii) Using Cramer's rule

System of Homogeneous Linear Equation:

The system of following homogeneous linear equations:

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3&=0 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3&=0 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3&=0 \end{aligned}\right\} \qquad \text{(i)}$$

is always satisfied by $x_1=0, x_2=0$ and $x_3=0,$ so such a system is always consistent.

Trivial solution: The solution $(0,0,0)$ of the above homogeneous system is called the trivial solution.

Non-Trivial solution: Any other solution of system (i) other than the trivial solution is called a non-trivial solution.

EXERCISE 4.3

1. Find the inverse of the following matrices by using row operations.

(i) $\begin{bmatrix} 2 & 6 & -3 \\ 0 & -2 & 0 \\ -2 & 5 & 6 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 2 & 6 & -3 \\ 0 & -2 & 0 \\ -2 & 5 & 6 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 6 & -3 \\ 0 & -2 & 0 \\ -2 & 5 & 6 \end{vmatrix}$$

$$= 2(-12-0) - 6(0-0) - 3(0-4)$$

$$= -24 - 0 + 12 = -12 \neq 0$$

As $|A| \neq 0$, so A is non-singular matrix.

Appending I_3 on the right of the matrix A, we have

$$\underset{\sim}{R} \begin{bmatrix} 2 & 6 & -3 & : & 1 & 0 & 0 \\ 0 & -2 & 0 & : & 0 & 1 & 0 \\ -2 & 5 & 6 & : & 0 & 0 & 1 \end{bmatrix}$$

By $\frac{1}{2}R_1 \to R_1'$, we get

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 & -\frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & -2 & 0 & : & 0 & 1 & 0 \\ -2 & 5 & 6 & : & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + 2R_1 \to R_3'$, we get

$$\underset{\sim}{R} \begin{bmatrix} 1 & 3 & -\frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & -2 & 0 & : & 0 & 1 & 0 \\ 0 & 11 & 3 & : & 1 & 0 & 1 \end{bmatrix}$$

By $-\frac{1}{2}R_2 \to R_2'$, we get,

$$\underset{\sim}{R}\begin{bmatrix} 1 & 3 & -\frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 11 & 3 & : & 0 & 1 & 0 \end{bmatrix}$$

By $R_1 - 3R_2 \to R_1'$ and $R_3 - 11R_2 \to R_3'$, we get

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -\frac{3}{2} & : & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 & : & 1 & \frac{11}{2} & 1 \end{bmatrix}$$

By $\frac{1}{3}R_3 \to R_3'$, we get

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -\frac{3}{2} & : & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{11}{6} & \frac{1}{3} \end{bmatrix}$$

By $R_1 + \frac{3}{2}R_3 \to R_1'$, we get

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & 1 & \frac{17}{4} & \frac{1}{2} \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{11}{6} & \frac{1}{3} \end{bmatrix}$$

Thus, the inverse of A is $\begin{bmatrix} 1 & \frac{17}{4} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{11}{6} & \frac{1}{3} \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 8 \\ 1 & 0 & 2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 8 \\ 1 & 0 & 2 \end{bmatrix}$

Inverse of A by Row operations Appending I_3 on the right of matrix A

$$\begin{bmatrix} 1 & 2 & -1 & : & 1 & 0 & 0 \\ 0 & -2 & 8 & : & 0 & 1 & 0 \\ 1 & 0 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 - R_1 \to R_3'$, $\left(-\frac{1}{2}\right)R_2 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & : & 1 & 0 & 0 \\ 0 & 1 & -4 & : & 0 & -\frac{1}{2} & 0 \\ 0 & -2 & 3 & : & -1 & 0 & 1 \end{bmatrix}$$

By $R_3 + 2R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & : & 1 & 0 & 0 \\ 0 & 1 & -4 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -5 & : & -1 & -1 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{5}\right)R_3 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & : & 1 & 0 & 0 \\ 0 & 1 & -4 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

By $R_2 + 4R_3 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & \frac{4}{5} & \frac{3}{10} & -\frac{4}{5} \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

By $R_1 + R_3 \to R_1'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 0 & : & \frac{6}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & : & \frac{4}{5} & \frac{3}{10} & -\frac{4}{5} \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

By $R_1 - 2R_2 \to R_1'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & -\frac{2}{5} & -\frac{2}{5} & \frac{7}{5} \\ 0 & 1 & 0 & : & \frac{4}{5} & \frac{3}{10} & -\frac{4}{5} \\ 0 & 0 & 1 & : & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} -\frac{2}{5} & -\frac{2}{5} & \frac{7}{5} \\ \frac{4}{5} & \frac{3}{10} & -\frac{4}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

(iii) $\begin{bmatrix} 1 & 6 & 2 \\ 2 & 13 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 6 & 2 \\ 2 & 13 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Inverse of A by Row operations Appending I_3 on the right of matrix A

$$\begin{bmatrix} 1 & 6 & 2 & : & 1 & 0 & 0 \\ 2 & 13 & 0 & : & 0 & 1 & 0 \\ 0 & -1 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 6 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -4 & : & -2 & 1 & 0 \\ 0 & -1 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 6 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -4 & : & -2 & 1 & 0 \\ 0 & 0 & -3 & : & -2 & 1 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{3}\right)R_3 \to R_3'$ $\underset{\sim}{R} \begin{bmatrix} 1 & 6 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -4 & : & -2 & 1 & 0 \\ 0 & 0 & 1 & : & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

By $R_2 + 4R_3 \to R_2'$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 6 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{4} \\ 0 & 0 & 1 & : & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

By $R_1 - 2R_3 \to R_1'$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 6 & 0 & : & -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 0 & : & \frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & : & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

By $R_1 - 6R_2 \to R_1'$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & : & -\frac{13}{3} & \frac{8}{3} & \frac{26}{3} \\ 0 & 1 & 0 & : & \frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & : & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} -\frac{13}{3} & \frac{8}{3} & \frac{26}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

2. Find the rank of the following matrices:

(i) $\begin{bmatrix} 1 & -1 & 3 & 1 \\ -2 & -6 & 1 & -1 \\ 3 & 1 & 4 & -2 \end{bmatrix}$

Solution:

We reduce it into echelon form by using row operation

$$\begin{bmatrix} 1 & -1 & 3 & 1 \\ -2 & -6 & 1 & -1 \\ 3 & 1 & 4 & -2 \end{bmatrix}$$

By $R_2 + 2R_1 \to R_2'$, $R_3 - 3R_1 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & -8 & 7 & 1 \\ 0 & 4 & -5 & -5 \end{bmatrix}$$

By $\left(-\frac{1}{8}\right)R_2 \to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -\frac{7}{8} & -\frac{1}{8} \\ 0 & 4 & -5 & -5 \end{bmatrix}$$

By $R_3 - 4R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -\frac{7}{8} & -\frac{1}{8} \\ 0 & 0 & -\frac{3}{2} & -\frac{9}{2} \end{bmatrix}$$

By $\left(-\frac{2}{3}\right)R_3 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -\frac{7}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Which is in echelon form.

Numbers of non-zero rows $= 3$

i.e Rank $= 3$

(ii) $$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution:

We reduce it into echelon form by using row operations.

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

By $R_2 - 2R_1 \to R_2'$, $R_3 + R_1 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & 5 \\ 0 & 1 & -1 \end{bmatrix} \text{By } R_2 \leftrightarrow R_4$$

By $R_3 + 2R_2 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

By $\frac{1}{2}R_3 \to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form.
Numbers of non-zero rows $= 3$
Rank of given matrix is 3

(iii) $\begin{bmatrix} 3 & -1 & 3 & 0 & 1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 1 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 3 & -1 & 3 & 0 & 1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 1 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3 & -1 & 3 & 0 & 1 \\ 2 & 3 & 4 & 2 & 1 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix} \text{By } R_1 \leftrightarrow R_2$$

By $R_2 - 3R_1 \to R_2'$, $R_3 - 2R_1 \to R_3'$, $R_4 - 2R_1 \to R_4'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & -7 & 6 & 9 & 7 \\ 0 & -1 & 6 & 8 & 5 \\ 0 & 1 & 0 & 3 & 7 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1 & 6 & 8 & 5 \\ 0 & -7 & 6 & 9 & 7 \end{bmatrix} \text{By } R_2 \leftrightarrow R_4$$

By $\begin{matrix} R_3 + R_2 \to R_3' \\ R_4 + 7R_2 \to R_4' \end{matrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 12 \\ 0 & 0 & 6 & 30 & 56 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & \frac{11}{6} & 2 \\ 0 & 0 & 6 & 30 & 56 \end{bmatrix} \text{ By } \frac{1}{6}R_3$$

By $R_4 - 6R_3 \to R_4'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & \frac{11}{6} & 2 \\ 0 & 0 & 0 & 19 & 44 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & \frac{11}{6} & 2 \\ 0 & 0 & 0 & 1 & \frac{44}{19} \end{bmatrix} \text{ By } \frac{1}{19}R_4$$

Which is in echelon form.

Numbers of non-zero rows $= 4$

Rank of given matrix is 4

3. Solve the following system of linear equations by Cramer's Rule:

(i) $\left.\begin{matrix} 2x + y - z = 1 \\ x - y + 2z = 3 \\ 3x + 2y + z = 4 \end{matrix}\right\}$

Solution:

The given system of linear equations can be written in matrix form

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Here $|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$

$= 2(-1-4) - 1(1-6) + (-1)(2+3)$

$= 2(-5) - 1(-5) - (5)$

$|A| = -10 + 5 - 5 = -10 \neq 0$

Since $|A| \neq 0$ so solution is possible

Now by Cramer's rule

$$x = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 3 & -1 & 2 \\ 4 & 2 & 1 \end{vmatrix}}{|A|} = \frac{1(-1-4)-1(3-8)-1(6+4)}{-10} \quad x = \frac{-5+5-10}{-10} = \frac{-10}{-10} = 1$$

$$y = \frac{\begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 3 & 4 & 1 \end{vmatrix}}{|A|} = \frac{2(3-8)-1(1-6)-1(4-9)}{-10} \quad y = \frac{2(-5)+5+5}{-10} = \frac{-10+10}{-10} = 0$$

$$Z = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 3 \\ 3 & 2 & 4 \end{vmatrix}}{|A|} = \frac{2(-4-6)-1(4-9)+1(2+3)}{-10} = \frac{-20+5+5}{-10} = \frac{-10}{-10} = 1$$

$x = 1, y = 0, z = 1$

Hence solution set is $\{(1,0,1)\}$.

(ii) $$\left.\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 5 \\ -2x_1 + 3x_2 + 2x_3 &= 3 \end{aligned}\right\}$$

Solution:

The given system of linear equations can be written in matrix form

$$\begin{bmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ -2 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ -2 & 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ y_2 \\ z_3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Here $|A| = \begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ -2 & 3 & 2 \end{vmatrix}$

$= 1(-2-3)-2(8+2)-3(12-2)$

$|A| = -5-20-30 = -55 \neq 0$

Since $|A| \neq 0$ so solution is possible.

Now by Cramer rule

$$x_1 = \frac{\begin{vmatrix} 0 & 2 & -3 \\ 5 & -1 & 1 \\ 3 & 3 & 2 \end{vmatrix}}{|A|} = \frac{0-2(10-3)-3(15+3)}{-55}$$

$$x_1 = \frac{-14-54}{-55} = \frac{-68}{-55} = \frac{68}{55}$$

$$x_2 = \frac{\begin{bmatrix} 1 & 0 & -3 \\ 4 & 5 & 1 \\ -2 & 3 & 2 \end{bmatrix}}{|A|} = \frac{1(10-3)-0-3(12+10)}{-55} \quad x_2 = \frac{7-66}{-55} = \frac{-59}{-55} = \frac{59}{55}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 4 & -1 & 5 \\ -2 & 3 & 3 \end{vmatrix}}{|A|} = \frac{1(-3-15)-2(12+10)+0}{-55} \quad x_3 = \frac{-18-44}{-55} = \frac{-62}{-55} = \frac{62}{55}$$

$$x_1 = \frac{68}{55}, x_2 = \frac{59}{55}, x_3 = \frac{62}{55}$$

Hence solution set is $\left\{\left(\frac{68}{55}, \frac{59}{55}, \frac{62}{55}\right)\right\}$

(iii)
$$\left.\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ x_1 + 2x_2 + 2x_3 &= 2 \\ x_1 - 2x_2 - x_3 &= 1 \end{aligned}\right\}$$

Solution:

The given system of linear equations can be written in matrix form

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$

$$= 2(-2+4)+1(-1-2)+1(-2-2)$$

$$|A| = 4-3-4 = -3 \neq 0$$

Since $|A| \neq 0$ So Solution is possible. Now by Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}}{|A|}$$

$$= \frac{1(-2+4)+1(-2-2)+1(-4-2)}{-3}$$

$$x_1 = \frac{2-4-6}{-3} = \frac{-8}{-3} = \frac{8}{3}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & -1 \end{vmatrix}}{|A|} \qquad x_2 = \frac{-8+3-1}{-3} = \frac{-6}{-3} = 2$$

$$= \frac{2(-2-2)-1(-1-2)+1(1-2)}{-3}$$

$$x_3=\frac{\begin{vmatrix}2&-1&1\\1&2&2\\1&-2&1\end{vmatrix}}{|A|}=\frac{2(2+4)+1(1-2)+1(-2-2)}{-3}\quad x_3=\frac{12-1-4}{-3}=\frac{7}{-3}=\frac{-7}{3}$$

$$x_1=\frac{8}{3},x_2=2,x_3=\frac{-7}{3}$$

Hence solution set is $\left\{\left(\frac{8}{3},2,\frac{-7}{3}\right)\right\}$

4. Solve the following system of linear equations by matrix inversion method:

(i) $\left.\begin{aligned}x-2y+z&=-1\\3x+y-2z&=4\\y-z&=1\end{aligned}\right\}$

Solution:

The matrix form of the given system of linear equations is:

$$\begin{bmatrix}1&-2&1\\3&1&-2\\0&1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\4\\1\end{bmatrix}$$

Let $A=\begin{bmatrix}1&-2&1\\3&1&-2\\0&1&-1\end{bmatrix},X=\begin{bmatrix}x\\y\\z\end{bmatrix},B=\begin{bmatrix}-1\\4\\1\end{bmatrix}$

$AX=B$

$X=A^{-1}B\ldots$(i)

$$|A|=\begin{vmatrix}1&-2&1\\3&1&-2\\0&1&-1\end{vmatrix}$$

$$=1(-1+2)+2(-3-0)+1(3-0)$$

$$|A|=1(1)-6+3=-2\neq 0$$

Since $|A|\neq 0$ so inverse of A is possible and given by $A^{-1}=\frac{adjA}{|A|}$

Now $adjA=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}^t$

$$=\begin{bmatrix}\begin{vmatrix}1&-2\\1&-1\end{vmatrix}&-\begin{vmatrix}3&-2\\0&-1\end{vmatrix}&\begin{vmatrix}3&1\\0&1\end{vmatrix}\\-\begin{vmatrix}-2&1\\1&-1\end{vmatrix}&\begin{vmatrix}1&1\\0&-1\end{vmatrix}&-\begin{vmatrix}1&-2\\0&1\end{vmatrix}\\\begin{vmatrix}-2&1\\1&-2\end{vmatrix}&-\begin{vmatrix}1&1\\3&-2\end{vmatrix}&\begin{vmatrix}1&-2\\3&1\end{vmatrix}\end{bmatrix}^t$$

$$=\begin{bmatrix}(-1+2)&-(-3)&3\\-(2-1)&(-1-0)&-(1)\\(4-1)&-(-2-3)&(1+6)\end{bmatrix}^t$$

$$=\begin{bmatrix}1 & 3 & 3\\ -1 & -1 & -1\\ 3 & 5 & 7\end{bmatrix}^t=\begin{bmatrix}1 & -1 & 3\\ 3 & -1 & 5\\ 3 & -1 & 7\end{bmatrix}$$

$$A^{-1}=\frac{adjA}{|A|}=\frac{-1}{2}\begin{bmatrix}1 & -1 & 3\\ 3 & -1 & 5\\ 3 & -1 & 7\end{bmatrix}$$

Put the values of A^{-1} and B in (i) we get

$$X=A^{-1}B=-\frac{1}{2}\begin{bmatrix}1 & -1 & 3\\ 3 & -1 & 5\\ 3 & -1 & 7\end{bmatrix}\begin{bmatrix}-1\\ 4\\ 1\end{bmatrix}$$

$$=\frac{-1}{2}\begin{bmatrix}-1-4+3\\ -3-4+5\\ -3-4+7\end{bmatrix}=\frac{-1}{2}\begin{bmatrix}-2\\ -2\\ 0\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$x=1, y=1, z=0$

Hence solution set is $\{(1,1,0)\}$.

(ii)
$$\left.\begin{aligned}2x_1+x_2+3x_3&=3\\ x_1+3x_2-2x_3&=0\\ -3x_1-x_2+2x_3&=4\end{aligned}\right\}$$

Solution:

The matrix form of the given system of linear equations is:

$$\begin{bmatrix}2 & 1 & 3\\ 1 & 3 & -2\\ -3 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}3\\ 0\\ 4\end{bmatrix}$$

Let $A=\begin{bmatrix}2 & 1 & 3\\ 1 & 3 & -2\\ -3 & -1 & 2\end{bmatrix}, X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, B=\begin{bmatrix}3\\ 0\\ 4\end{bmatrix}$

$AX=B$

$X=A^{-1}B \ldots$(i)

$$|A|=\begin{vmatrix}2 & 1 & 3\\ 1 & 3 & -2\\ -3 & -1 & 2\end{vmatrix}$$

$$=2(6-2)-1(2-6)+3(-1+9)$$

$$=8+4+24=36\neq 0$$

Since $|A|\neq 0$ So inverse of A is possible and given by $A^{-1}=\dfrac{adj\,A}{|A|}$

$$adjA=\begin{bmatrix}A_1 & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^t$$

$$adjA=\begin{bmatrix}\begin{vmatrix}3 & -2\\ -1 & 2\end{vmatrix} & -\begin{vmatrix}1 & -2\\ -3 & 2\end{vmatrix} & \begin{vmatrix}1 & 3\\ -3 & -1\end{vmatrix}\\ -\begin{vmatrix}1 & 3\\ -1 & 2\end{vmatrix} & \begin{vmatrix}2 & 3\\ -3 & 2\end{vmatrix} & -\begin{vmatrix}2 & 1\\ -3 & -1\end{vmatrix}\\ \begin{vmatrix}1 & 3\\ 3 & -2\end{vmatrix} & -\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix} & \begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix}\end{bmatrix}^t \quad adjA=\begin{bmatrix}(6-2) & -(2-6) & (-1+9)\\ -(2+3) & (4+9) & -(-2+3)\\ (-2-9) & -(-4-3) & (6-1)\end{bmatrix}^t$$

$$adjA=\begin{bmatrix}4 & 4 & 8\\ -5 & 13 & -1\\ -11 & 7 & 5\end{bmatrix}^t$$

$$adjA=\begin{bmatrix}4 & -5 & -11\\ 4 & 13 & 7\\ 8 & -1 & 5\end{bmatrix}$$

$$A^{-1}=\frac{adjA}{|A|}=\frac{1}{36}\begin{bmatrix}4 & -5 & -11\\ 4 & 13 & 7\\ 8 & -1 & 5\end{bmatrix}$$

Put the values of A^{-1} and B in equation (i)

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{36}\begin{bmatrix}4 & -5 & -11\\ 4 & 13 & 7\\ 8 & -1 & 5\end{bmatrix}\begin{bmatrix}3\\ 0\\ 4\end{bmatrix}$$

$$=\frac{1}{36}\begin{bmatrix}12-0-44\\ 12+0+28\\ 24-0+20\end{bmatrix}=\frac{1}{36}\begin{bmatrix}-32\\ 40\\ 44\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{32}{36}\\ \frac{40}{36}\\ \frac{44}{36}\end{bmatrix}=\begin{bmatrix}-\frac{8}{9}\\ \frac{10}{9}\\ \frac{11}{9}\end{bmatrix}$$

$$x=-\frac{8}{9},y=\frac{10}{9},z=\frac{11}{9}$$

Hence solution set is $\left\{\left(-\frac{8}{9},\frac{10}{9},\frac{11}{9}\right)\right\}$.

(iii) $\left.\begin{aligned}x+y&=2\\ 2x-z&=1\\ 2y-3z&=-1\end{aligned}\right\}$

Solution:

The matrix form of the given system of linear equations is:

$$\begin{bmatrix}1 & 1 & 0\\ 2 & 0 & -1\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$$

Let $A=\begin{bmatrix}1 & 1 & 0\\ 2 & 0 & -1\\ 0 & 2 & -3\end{bmatrix},X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},B=\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$

$$AX=B$$

$X = A^{-1}B \ldots$(i)

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix} = 1(0+2) - 1(-6-0) + 0$$

$$|A| = 2 + 6 = 8 \neq 0$$

Since $|A| \neq 0$ so inverse of A is possible and given by $A^{-1} = \dfrac{adjA}{|A|}$

$$adjA = \begin{bmatrix} A_1 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$adjA = \begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \end{bmatrix}^t$$

$$adjA = \begin{bmatrix} (0+2) & -(-6+0) & (4-0) \\ -(-3-0) & (-3-0) & -(2-0) \\ (-1-0) & -(-1-0) & (0-2) \end{bmatrix}^t$$

$$adjA = \begin{bmatrix} 2 & 6 & 4 \\ 3 & -3 & -2 \\ -1 & 1 & -2 \end{bmatrix}^t$$

$$adjA = \begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

Put the values of A^{-1} and B in equation (i)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 4+3+1 \\ 12-3-1 \\ 8-2+2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$x = 1, y = 1, z = 1$

Hence solution set is $\{(1,1,1)\}$.

5. Solve the following system by reducing their augmented matrices to the echelon form and the reduced echelon form:

(i) $\left.\begin{aligned} x_1+2x_2-2x_3&=-1\\ 2x_1+3x_2+x_3&=1\\ 5x_1+4x_2-3x_3&=1 \end{aligned}\right\}$

Solution:

The augmented matrix of the given system is:

$$\begin{bmatrix} 1 & 2 & -2 & : & -1\\ 2 & 3 & 1 & : & 1\\ 5 & 4 & -3 & : & 1 \end{bmatrix}$$

We reduced the above matrix to echelon and reduced echelon form by applying elementry row operations, that is:

$$\begin{bmatrix} 1 & 2 & -2 & : & -1\\ 2 & 3 & 1 & : & 1\\ 5 & 4 & -3 & : & 1 \end{bmatrix}$$

By $R_2-2R_1\to R_2',R_3-5R_1\to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & : & -1\\ 0 & -1 & 5 & : & 3\\ 0 & -6 & 7 & : & 6 \end{bmatrix}$$

By $\mathrm{R}_3-6R_2\to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & : & -1\\ 0 & -1 & 5 & : & 3\\ 0 & 0 & -23 & : & -12 \end{bmatrix}$$

By $\left(-\dfrac{1}{23}\right)R_3\to R_3'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & : & -1\\ 0 & -1 & 5 & : & 3\\ 0 & 0 & 1 & : & \dfrac{12}{23} \end{bmatrix}$$

By $(-1)R_2\to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & : & -1\\ 0 & 1 & -5 & : & -3\\ 0 & 0 & 1 & : & \dfrac{12}{23} \end{bmatrix}\ldots\ldots.(\mathrm{I})$$

By $R_2+5R_3\to R_2'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -2 & : & -1\\ 0 & 1 & 0 & : & -\dfrac{9}{23}\\ 0 & 0 & 1 & : & \dfrac{12}{23} \end{bmatrix}$$

By $R_1+2R_3\to R_1'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 0 & : & \frac{1}{23} \\ 0 & 1 & 0 & : & -\frac{9}{23} \\ 0 & 0 & 1 & : & \frac{12}{23} \end{bmatrix}$$

By $R_1 - 2R_2 \to R_1'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & \frac{19}{23} \\ 0 & 1 & 0 & : & -\frac{9}{23} \\ 0 & 0 & 1 & : & \frac{12}{23} \end{bmatrix} \text{........(II)}$$

(I) is in echelon form the equivalent system is

$x_1 + 2x_2 - 2x_3 = -1 \ldots\ldots(i)$

$x_2 - 5x_3 = -3 \ldots.(ii)$

$x_3 = \frac{12}{23} \ldots\ldots(iii)$

From equation (ii)

$$x_2 - 5\left(\frac{12}{23}\right) = -3$$

$$\Rightarrow x_2 = -3 + \frac{60}{23} = \frac{-69+60}{23} = -\frac{9}{23}$$

Form equation (i)

$$x_1 + 2\left(-\frac{9}{23}\right) - 2\left(\frac{12}{23}\right) = 1$$

$$\Rightarrow x_1 = -1 + \frac{18}{23} + \frac{24}{23} = \frac{-23+18+24}{23} = \frac{19}{23}$$

$$x_1 = \frac{19}{23}, x_2 = -\frac{9}{23} \text{ and } x_3 = \frac{12}{23}$$

(II) is in reduced echelon form the equivalent system in row reduced echelon form:

$$x_1 = \frac{19}{23}, x_2 = -\frac{9}{23} \text{ and } x_3 = \frac{12}{23}$$

Hence solution set is $\left\{\left(\frac{19}{23}, -\frac{9}{23}, \frac{12}{23}\right)\right\}$.

(ii)
$$\left.\begin{aligned} x + 2y + z &= 2 \\ 2x + y + 2z &= 3 \\ 2x + 3y - z &= 7 \end{aligned}\right\}$$

Solution:

The augmented matrix of the given system is:

$$\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2 & 1 & 2 & : & 3 \\ 2 & 3 & -1 & : & 7 \end{bmatrix}$$

We reduced the above matrix to echelon and reduced echelon form by applying elementary row operations that is:

By $R_3 - 2R_1 \to R_3', R_2 - 2R_1 \to R_1'$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -3 & 0 & : & -1 \\ 0 & -1 & -3 & : & 3 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{1}{3} \\ 0 & -1 & -3 & : & 3 \end{bmatrix} \left(-\frac{1}{3}\right)R_2 \to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{1}{3} \\ 0 & 0 & -3 & : & \frac{10}{3} \end{bmatrix} \text{By } R_3 + R_2 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{1}{3} \\ 0 & 0 & 1 & : & \frac{-10}{9} \end{bmatrix} \text{By } \left(-\frac{1}{3}\right)R_3 \to R_3' \text{ .... (I)}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 0 & : & \frac{28}{9} \\ 0 & 1 & 0 & : & \frac{1}{3} \\ 0 & 0 & 1 & : & \frac{-10}{9} \end{bmatrix} \text{By } R_1 - R_3 \to R_1'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & \frac{22}{9} \\ 0 & 1 & 0 & : & \frac{1}{3} \\ 0 & 0 & 1 & : & \frac{-10}{9} \end{bmatrix} \text{By } R_1 - 2R_2 \to R_1' \text{ ...(II)}$$

(I) is in echelon form the equivalent system is:

$x + 2y + z = 2$ .....(i)

$y = \frac{1}{3}$ ....(ii)

$z = -\frac{10}{9}$ ....(iii)

From equation (i)

$$x + 2\left(\frac{1}{3}\right) - \frac{10}{9} = 2$$

$$x = 2 - \frac{2}{3} + \frac{10}{9} = \frac{18 - 6 + 10}{9} = \frac{22}{9}$$

$$x = \frac{22}{9}, y = \frac{1}{3} \text{ and } z = -\frac{10}{9}$$

(II) is in reduced echelon form the equivalent system in row reduced echelon form:

$$x = \frac{22}{9}, y = \frac{1}{3} \text{ and } z = -\frac{10}{9}$$

Hence solution set is $\left\{\left(\frac{22}{9}, \frac{1}{3}, \frac{-10}{9}\right)\right\}$.

(iii) $\left.\begin{aligned} x_1+4x_2+x_3&=2\\ 2x_1+x_2-2x_3&=9\\ 3x_1+x_2-x_3&=12\end{aligned}\right\}$

Solution:

The augmented matrix of the given system is:

$$\begin{bmatrix} 1 & 4 & 1 & : & 2\\ 2 & 1 & -2 & : & 9\\ 3 & 1 & -1 & : & 12\end{bmatrix}$$

We reduced the above matrix to echelon and reduced echelon form by applying element row operations that is:

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & : & 2\\ 0 & -7 & -4 & : & 5\\ 0 & -11 & -4 & : & 6\end{bmatrix} \text{By } \begin{matrix} R_2-2R_1\to R_2' \\ R_3-3R_1\to R_3'\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & : & 2\\ 0 & -7 & -4 & : & 5\\ 0 & -4 & 0 & : & 1\end{bmatrix} \text{By } \mathrm{R}_3-R_2\to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & : & 2\\ 0 & 7 & 4 & : & -5\\ 0 & 1 & 0 & : & -\frac{1}{4}\end{bmatrix} \text{By } \begin{matrix} (-1)R_2\to R_2' \\ \left(-\frac{1}{4}\right)R_3\to R_3'\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & : & 2\\ 0 & 1 & 0 & : & -\frac{1}{4}\\ 0 & 7 & 4 & : & -5\end{bmatrix} \text{By } R_2\leftrightarrow R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & : & 2\\ 0 & 1 & 0 & : & -\frac{1}{4}\\ 0 & 0 & 4 & : & -\frac{13}{4}\end{bmatrix} \text{By } R_3-7R_2\to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 1 & : & 2\\ 0 & 1 & 0 & : & -\frac{1}{4}\\ 0 & 0 & 1 & : & \frac{-13}{16}\end{bmatrix} \text{By } \left(\frac{1}{4}\right)R_3\to R_3' \text{ ....(I)}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 0 & : & \frac{45}{16}\\ 0 & 1 & 0 & : & -\frac{1}{4}\\ 0 & 0 & 1 & : & -\frac{13}{16}\end{bmatrix} \text{By } R_1-R_3\to R_1'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & \frac{61}{16}\\ 0 & 1 & 0 & : & -\frac{1}{4}\\ 0 & 0 & 1 & : & -\frac{13}{16}\end{bmatrix} \text{By } \mathrm{R}_1-4R_2\to R_1' \text{ ....(II)}$$

(I) is in echelon form the equivalent system is:

$x_1 + 4x_2 + x_3 = 2$ .....(i)

$x_2 = -\frac{1}{4}$ .....(ii)

$x_3 = -\frac{13}{16}$ ......(iii)

From equation (i)

$$x_1 + 4\left(-\frac{1}{4}\right) + \left(\frac{-13}{16}\right) = 2$$

$$\Rightarrow x_1 = 2 + 1 + \frac{13}{16} = \frac{32+16+13}{16} = \frac{61}{16}$$

$x_1 = \frac{61}{16}, x_2 = -\frac{1}{4}$ and $x_3 = -\frac{13}{16}$

(II) is in reduced echelon form the equivalent system in row reduced echelon form:

$x_1 = \frac{61}{16}, x_2 = -\frac{1}{4}$, and $x_3 = -\frac{13}{16}$

Hence solution set is $\left\{\left(\frac{61}{16}, -\frac{1}{4}, -\frac{13}{16}\right)\right\}$.

6. Solve the following systems of homogenous linear equations by using Gaussian elimination method:

(i)
$$\left.\begin{aligned} x + 4y - 2z &= 0 \\ 2x + y + 5z &= 0 \\ 5x + 2y + 8z &= 0 \end{aligned}\right\}$$

Solution:

Let $A_b = \begin{bmatrix} 1 & 4 & -2 & : & 0 \\ 2 & 1 & 5 & : & 0 \\ 5 & 2 & 8 & : & 0 \end{bmatrix}$

By $\begin{matrix} R_2 - 2R_1 \to R_2' \\ R_3 - 5R_1 \to R_3' \end{matrix}$

$$A_b = \begin{bmatrix} 1 & 4 & -2 & : & 0 \\ 0 & -7 & 9 & : & 0 \\ 0 & -18 & 18 & : & 0 \end{bmatrix}$$

By $\left(-\frac{1}{18}\right)R_3 \to R_3'$

$$= \begin{bmatrix} 1 & 4 & -2 & : & 0 \\ 0 & -7 & 9 & : & 0 \\ 0 & 1 & -1 & : & 0 \end{bmatrix}$$

By $R_2 \leftrightarrow R_3$

$$= \begin{bmatrix} 1 & 4 & -2 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & -7 & 9 & : & 0 \end{bmatrix}$$

By $R_3 + 7R_2 \to R_3'$

$$=\begin{bmatrix}1 & 4 & -2 & : & 0\\0 & 1 & -1 & : & 0\\0 & 0 & 2 & : & 0\end{bmatrix}$$

By $\left(\frac{1}{2}\right)R_3 \to R_3'$

$$=\begin{bmatrix}1 & 4 & -2 & : & 0\\0 & 1 & -1 & : & 0\\0 & 0 & 1 & : & 0\end{bmatrix}$$

Which is the required echelon form. So rank of $A=$ rank of A_b then solution must be exist therefore:

$x+4y-2z=0$ .....(i)

$y-z=0$ ....(ii)

$z=0$ ....(iii)

Form equation (i)

$x+4(0)-2(0)=0 \Rightarrow x=0$

Form equation (ii)

$y-0=0 \Rightarrow y=0$

$x=0, y=0, z=0$

Hence solution set is $\{(0,0,0)\}$.

(ii) $\left.\begin{aligned}x_1+4x_2+2x_3=0\\2x_1+x_2-3x_3=0\\3x_1+2x_2-4x_3=0\end{aligned}\right\}$

Solution:

Let $A_b=\begin{bmatrix}1 & 4 & 2 & : & 0\\2 & 1 & -3 & : & 0\\3 & 2 & -4 & : & 0\end{bmatrix}$

By $\begin{matrix}R_2-2R_1 \to R_2'\\R_3-3R_1 \to R_3'\end{matrix}$

$$=\begin{bmatrix}1 & 4 & 2 & : & 0\\0 & -7 & -7 & : & 0\\0 & -10 & -10 & : & 0\end{bmatrix}$$

By $\begin{matrix}\left(-\frac{1}{7}\right)R_2 \to R_2'\\\left(-\frac{1}{10}\right)R_3 \to R_3'\end{matrix}$

$$=\begin{bmatrix}1 & 4 & 2 & : & 0\\0 & 1 & 1 & : & 0\\0 & 1 & 1 & : & 0\end{bmatrix}$$

By $R_3-R_2 \to R_3'$

$$=\begin{bmatrix} 1 & 4 & 2 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Which is required echelon form. So rank of $A =$ rank of A_b is less than number of variables, so we have infinite many solution of given system.

$x_1 + 4x_2 + 2x_3 = 0$ …..(i)

$x_2 + x_3 = 0$ …..(ii)

Choose $x_3 = t$ is any arbitrary constant so that:

From equation (ii) $x_2 + t = 0 \Rightarrow x_2 = -t$

From equation (i)

$x_1 + 4(-t) + 2t = 0$

$x_1 - 4t + 2t = 0 \Rightarrow x_1 = 2t$

$x_1 = 2t, x_2 = -t, x_3 = t$

For any value of t.

Hence solution set is $\{(2t, -t, t)\}$.

(iii) $$\left.\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + 5x_3 &= 0 \\ 2x_1 + x_2 + 4x_3 &= 0 \end{aligned}\right\}$$

Solution:

Let $$A_b = \begin{bmatrix} 1 & 2 & -1 & : & 0 \\ 1 & -1 & 5 & : & 0 \\ 2 & 1 & 4 & : & 0 \end{bmatrix}$$

By $R_2 - R_1 \to R_2'$

$R_3 - 2R_1 \to R_3'$

$$A_b = \begin{bmatrix} 1 & 2 & -1 & : & 0 \\ 0 & -3 & 6 & : & 0 \\ 0 & -3 & 6 & : & 0 \end{bmatrix}$$

By $R_3 - R_2 \to R_3'$

$$A_b = \begin{bmatrix} 1 & 2 & -1 & : & 0 \\ 0 & -3 & 6 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Which is required echelon form. So rank of $A =$ rank of A_b is less than number of variables. So we have infinite many solution of given system.

$x_1 + 2x_2 - x_3 = 0$ …(i)

$-3x_2 + 6x_3 = 0$ ….(ii)

Choose $x_3 = t$ is any arbitrary constant so that:

From equation (ii)

$-3x_2 + 6t = 0 - 3x_2 = -6t \Rightarrow x_2 = 2t$

From equation(i)

$x_1 + 2(2t) - t = 0 \Rightarrow x_1 + 4t - t = 0 \Rightarrow x_1 = -3t$ $x_1 = -3t, x_2 = 2t, x_3 = t$

For any value of t.

Hence solution set is $\{(-3t, 2t, t)\}$.

7. A triangle has vertices at $A(4,1), B(-2,5)$ and $C(0,-3)$. Find the vertices of the reflected triangle over the $y-axis$ using a transformation matrix.

Solution:

Given vertices of the triangle are:

$A(4,1), B(-2,5)$ and $(0,-3)$

Here to reflect the given point over the $y-$axis is we use the transformation matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

We write the point as column matrices $A = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, B = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, C = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

The vertex A' of the reflected image

$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} = (-4,1)$

The vertex B' of the reflected image

$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0+5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = (2,5)$

The vertex C' of the reflected image

$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} = (0,-3)$

Thus the vertices of the reflected triangle are

$A'(-4,1), B'(2,5)$ and $C'(0,-3)$

8. The point A is mapped to $(30, 20, -5)$ by the scaling matrix $P = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$. Find the coordinates of A.

Solution:

Let the coordinates of point A be (x, y, z). We can write this as a column matrix**:** $A = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The transformed point A' with coordinate $(30, 20, -5)$ can be written as a column matrix: $A' = \begin{bmatrix} 30 \\ 20 \\ -5 \end{bmatrix}$

Since A is mapped to A' by scaling matrix P, then $PA = A'$

$$\begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} -5x+0+0 \\ 0-5y+0 \\ 0+0-5z \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} -5x \\ -5y \\ -5z \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ -5 \end{bmatrix}$$

$-5x = 30; -5y = 20; -5z = -5$

$x = -6; y = -4; z = 1$

Hence the coordinates of point

A is $(-6, -4, 1)$.

9. Find the equation of the image of the curve with equation $y = x^2$ under the transformation with associated matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution:

Equation $y = x^2$

Associated matrix = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Let original coordinates are (x, y) and new coordinates are (X, Y).

By transformation

$X = x + 2y$ ……(i)

$Y = 3x + 4y$ ……(ii)

Multiply equation (i) by 3 on both sides

$3X = 3x + 6y$ …..(iii)

By subtracting equation (ii) and (iii)

$3X - Y = 3x + 6y - (3x + 4y)$

$3X - Y = 3x + 6y - 3x - 4y$

$3X - Y = 2y$

$\Rightarrow y = \dfrac{3X - Y}{2}$

As $y = x^2$, then

$\dfrac{3X - Y}{2} = x^2$

$3X - Y = 2x^2$

Image of the curve $y = x^2$ is

$y = \dfrac{3X - Y}{2}$ **(No Answer in Book)**

10. Use the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ to encode the message: KEEP IT UP, where letter *A* to *Z* are corresponding to the numbers 1 to 26.

Solution:

As we know that

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Message: KEEP IT UP

$K = 11, E = 5, P = 16, I = 9, T = 20, U = 21$ Making group each of length 3

$(KEE), (PIT), (UP)$

Uncoded matrix are

$$\begin{bmatrix}11\\5\\5\end{bmatrix}, \begin{bmatrix}16\\9\\20\end{bmatrix} \text{ and } \begin{bmatrix}21\\16\\0\end{bmatrix}$$

Encoding matrix	Uncoded matrix	Coded matrix
$\begin{bmatrix}1&0&1\\2&-1&1\\0&1&2\end{bmatrix}$	$\begin{bmatrix}11\\5\\5\end{bmatrix}$	$\begin{bmatrix}16\\22\\15\end{bmatrix}$
$\begin{bmatrix}1&0&1\\2&-1&1\\0&1&2\end{bmatrix}$	$\begin{bmatrix}16\\9\\20\end{bmatrix}$	$\begin{bmatrix}36\\43\\19\end{bmatrix}$
$\begin{bmatrix}1&0&1\\2&-1&1\\0&1&2\end{bmatrix}$	$\begin{bmatrix}21\\16\\0\end{bmatrix}$	$\begin{bmatrix}21\\26\\16\end{bmatrix}$

Coded matrix are

$$\begin{bmatrix}16\\22\\15\end{bmatrix}, \begin{bmatrix}36\\43\\19\end{bmatrix}, \begin{bmatrix}21\\26\\16\end{bmatrix}$$

11. Decode the message $\begin{bmatrix}11\\20\\43\end{bmatrix}\begin{bmatrix}25\\10\\41\end{bmatrix}\begin{bmatrix}22\\14\\41\end{bmatrix}$ that was encode using matrix $A=\begin{bmatrix}1&1&-1\\1&0&1\\2&1&1\end{bmatrix}$, where the numbers 1 to 26 are corresponding to the letter *A* to *Z*, and 27 is representing space or "-"

Solution:

The encoded message is given three column matrices:

$$\begin{bmatrix}11\\20\\43\end{bmatrix}, \begin{bmatrix}25\\10\\41\end{bmatrix} \text{ and } \begin{bmatrix}22\\14\\41\end{bmatrix}$$

To decode, we need to find the inverse of matrix *A*.

Here, $|A|=\begin{vmatrix}1&1&-1\\1&0&1\\2&1&1\end{vmatrix}$

$=1(0-1)-1(1-2)-1(1-0)=-1+1-1$

$=-1\neq 0$

Since $|A|\neq 0$, so inverse of matrix *A* is possible and given by $A^{-1}=\dfrac{adjA}{|A|}$

$$adjA=\begin{bmatrix}A_{1}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}^t$$

$$adjA=\begin{bmatrix}\begin{vmatrix}0&1\\1&1\end{vmatrix}&-\begin{vmatrix}1&1\\2&1\end{vmatrix}&\begin{vmatrix}1&0\\2&1\end{vmatrix}\\-\begin{vmatrix}1&-1\\1&1\end{vmatrix}&\begin{vmatrix}1&-1\\2&1\end{vmatrix}&-\begin{vmatrix}1&1\\2&1\end{vmatrix}\\\begin{vmatrix}1&-1\\0&1\end{vmatrix}&-\begin{vmatrix}1&-1\\1&1\end{vmatrix}&\begin{vmatrix}1&1\\1&0\end{vmatrix}\end{bmatrix}^t$$

$$=\begin{bmatrix}(0-1) & -(1-2) & (1-0)\\ -(1+1) & (1+2) & -(1-2)\\ (1-0) & -(1+1) & (0-1)\end{bmatrix}^t$$

$$adjA=\begin{bmatrix}-1 & -2 & 1\\ 1 & 3 & -2\\ 1 & 1 & -1\end{bmatrix}$$

So, $A^{-1}=\dfrac{adjA}{|A|}=\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}$

Now we multiply each encoded matrix by A^{-1}

$$\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}11\\ 20\\ 43\end{bmatrix}=\begin{bmatrix}11+40-43\\ -11-60+86\\ -11-20+43\end{bmatrix}=\begin{bmatrix}8\\ 15\\ 12\end{bmatrix}$$

8 corresponds to *H*, 15 corresponds to *O*, 12 corresponds to *L*.

$$\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}25\\ 10\\ 41\end{bmatrix}=\begin{bmatrix}25+20-41\\ -25-30+82\\ -25-10+41\end{bmatrix}=\begin{bmatrix}4\\ 27\\ 6\end{bmatrix}$$

4 corresponds to *D*, 27 corresponds to *A*, 6 corresponds to *F*.

$$\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}22\\ 14\\ 41\end{bmatrix}=\begin{bmatrix}22+28-41\\ -22-42+82\\ -22-14+41\end{bmatrix}=\begin{bmatrix}9\\ 18\\ 5\end{bmatrix}$$

9 corresponds to *I*, 18 corresponds to *R* and 5 corresponds to *E*.

Hence required decoded message is

HOLD A FIRE.

(Wrong Answer in Book)