Minor of an Element:

Let us consider a square matrix A of order n, then minor of an element a_{ij}, denoted by M_{ij} is the determinant formed by deleting the ith row and jth column of $A(\text{or } |A|)$.

Cofactor of an Element:

The cofactor of an element a_{ij} of a square matrix A denoted by A_{ij} is defined by $A_{ij}=(-1)^{i+j}M_{ij}$.

Determinant of a square matrix of Order $n=3$

If A is a matrix of order 3, that is, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

$|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+a_{i3}A_{i3}$ for $i=1,2,3$

Or $|A|=a_{1j}A_{1j}+a_{2j}A_{2j}+a_{3j}A_{3j}$ for $j=1,2,3$

For example, for $i=1, j=1$ and $j=2,$ we have

$|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$ (i)

Or $|A|=a_{11}A_{11}+a_{21}A_{21}+a_{31}A_{31}$ (ii)

Or $|A|=a_{12}A_{12}+a_{22}A_{22}+a_{32}A_{32}$ (iii)

Properties of Determinants:

i. For a square matrix $A, |A|=|A^t|$.

ii. If in a square matrix A, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.

iii. If a square matrix A has two identical rows or two identical columns, then $|A|=0$.

iv. If all the entries of a row (or a column) of a square matrix A are zero, then $|A|=0$.

v. If the entries of a row (or a column) in a square matrix A are multiplied by a number $k\in\mathbb{R}$, then the determinant of the resulting matrix is $k|A|$.

vi. If each entry of a row (or a column) of a square matrix consists of two terms, then its determinant can be written as the sum of two determinants, i.e., if

$$B=\begin{bmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & A_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{bmatrix}, \text{then,}$$

$$|B|=\begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

vii. If any row (column) of a determinant is multiplied by a non-zero number k and the result is added to the corresponding entries of another row (column), the value of the determinant does not change.

viii. If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Adjoint of a Square Matrix:

Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the matrix of co-factors of $A=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, and

$$adjA=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Inverse of a square Matrix of Order $n \geq 3$

Let A be a non-singular $(|A| \neq 0)$ square matrix of order n. If there exists a matrix B such that $AB=BA=I_n$, then B is called the multiplicative inverse of A and is denoted by A^{-1}. It is obvious that the order of A^{-1} is $n \times n$.
Thus, $AA^{-1}=I_n$ and $A^{-1}A=I_n$.

If A is non-singular matrix then $A^{-1}=\dfrac{1}{|A|}adjA$

EXERCISE 4.2

1. Evaluate the following determinants:

(i) $\begin{vmatrix} 1 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 3 & 2 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 1 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 3 & 2 \end{vmatrix}$$

Expanding from R_1

$$=1\times\begin{vmatrix} -1 & -3 \\ 3 & 2 \end{vmatrix}-(-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix}-4\begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix}=1\times(-2+9)+2(6-6)-4(9-2)$$

$$=7+2(0)-4(7)=7+0-28$$

$$=-21$$

(ii) $\begin{vmatrix} a+b & a-b & a \\ a & a+b & a-b \\ a-b & a & a+b \end{vmatrix}$

Solution:

Let $A=\begin{vmatrix} a+b & a-b & a \\ a & a+b & a-b \\ a-b & a & a+b \end{vmatrix}$

Expanding from R_1

$$=(a+b)\begin{vmatrix} a+b & a-b \\ a & a+b \end{vmatrix}-(a-b)\begin{vmatrix} a & a-b \\ a-b & a+b \end{vmatrix}+a\begin{vmatrix} a & a+b \\ a-b & a \end{vmatrix}$$

$$=(a+b)\left((a+b)^2-a(a-b)\right)-(a-b)\left(a(a+b)-(a-b)^2\right)+a\left(a^2-(a+b)(a-b)\right)$$

$$=(a+b)\left(a^2+b^2+2ab-a^2+ab\right)-(a-b)\left(a^2+ab-a^2-b^2+2ab\right)+a\left(a^2-a^2+b^2\right)$$

$$=(a+b)\left(b^2+3ab\right)-(a-b)\left(3ab-b^2\right)+ab^2=ab^2+3a^2b+b^3+3ab^2-\left(3a^2b-ab^2-3ab^2+b^3\right)+ab^2$$

$$=4ab^2+3a^2b+b^3-3a^2b+4ab^2-b^3+ab^2$$

$$=9ab^2$$

(iii) $\begin{vmatrix} 2x & x & x \\ y & 2y & y \\ z & z & 2z \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2x & x & x \\ y & 2y & y \\ z & z & 2z \end{vmatrix}$$

Expanding from R_1

$$=2x\begin{vmatrix} 2y & y \\ z & 2z \end{vmatrix}-x\begin{vmatrix} y & y \\ z & 2z \end{vmatrix}+x\begin{vmatrix} y & 2y \\ z & z \end{vmatrix}$$

$$=2x(4yz-yz)-x(2yz-yz)+x(yz-2yz)$$

$$=2x(3yz)-x(yz)+x(-yz)$$

$$=6xyz-xyz-xyz=4xyz$$

2. Without expansion show that:

(i) $\begin{vmatrix} 7 & 8 & 9 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{vmatrix}=0$

Solution:

$$\text{L.H.S}=\begin{vmatrix} 7 & 8 & 9 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{vmatrix}$$

By $\begin{cases} C_2-C_1\to C_2' \\ C_3-C_2\to C_3' \end{cases}$

$$=\begin{vmatrix} 7 & 1 & 1 \\ 5 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$(\because C_2$ and C_3 are identical$)$

$$=\begin{vmatrix} 7 & 1 & 1 \\ 5 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}=0=R.H.S$$

(ii) $\begin{vmatrix} 5 & 6 & -1 \\ 2 & 2 & 0 \\ 2 & -8 & 10 \end{vmatrix}=0$

Solution:

$$\text{L.H.S}=\begin{vmatrix} 5 & 6 & -1 \\ 2 & 2 & 0 \\ 2 & -8 & 10 \end{vmatrix}$$

By $C_1-C_3\to C_1'$

$$=\begin{vmatrix} 6 & 6 & -1 \\ 2 & 2 & 0 \\ -8 & -8 & 10 \end{vmatrix}$$

$(\because C_1 \text{ and } C_2 \text{ are identical})$

$$=\begin{vmatrix} 6 & 6 & -1 \\ 2 & 2 & 0 \\ -8 & -8 & 10 \end{vmatrix}=0=R.H.S$$

(iii) $\begin{vmatrix} -a & 0 & b \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix}=0$

Solution:

$$\text{L.H.S}=\begin{vmatrix} -a & 0 & b \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix}$$

By $cR_1 \to R_1'$

$$=\frac{1}{c}\begin{vmatrix} -ac & 0 & bc \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix}$$

By $R_1+bR_2 \to R_1'$

$$=\frac{1}{c}\begin{vmatrix} -ac & ab & 0 \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix}$$

Taking "$-a$" common from R_1.

$$=-\frac{a}{c}\begin{vmatrix} c & -b & 0 \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix}$$

$(\because R_1 \text{ and } R_3 \text{ are identical})$

$$=-\frac{a}{c}\begin{vmatrix} c & -b & 0 \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix}=0=\text{R.H.S}$$

Alternate solution

$$\text{L.H.S}=\begin{vmatrix} -a & 0 & b \\ 0 & a & -c \\ c & -b & 0 \end{vmatrix}$$

By $cR_1 \to R_1', bR_2 \to R_2'$ and $aR_3 \to R_3'$

$$=\frac{1}{abc}\begin{vmatrix} -ac & 0 & bc \\ 0 & ab & -bc \\ ac & -ab & 0 \end{vmatrix}$$

By $R_1+(R_2+R_2)\to R_1'$

$$=\frac{1}{abc}\begin{vmatrix} -ac+ac & ab-ab & bc-bc \\ 0 & ab & -bc \\ ac & -ab & 0 \end{vmatrix}$$

$$=\frac{1}{abc}\begin{vmatrix} 0 & 0 & 0 \\ 0 & ab & -bc \\ ac & -ab & 0 \end{vmatrix}$$

$=\frac{1}{abc}(0)$ $\because$ each entry of R_1 is zero.

$=0$ = R.H.S

(iv) $\begin{vmatrix} l & m+n & 1 \\ m & n+l & 1 \\ n & l+m & 1 \end{vmatrix}=0$

Solution:

$$\text{L.H.S}=\begin{vmatrix} l & m+n & 1 \\ m & n+l & 1 \\ n & l+m & 1 \end{vmatrix}$$

By $C_1+C_2\to C_1'$

$$=\begin{vmatrix} l+m+n & m+n & 1 \\ l+m+n & n+l & 1 \\ l+m+n & l+m & 1 \end{vmatrix}$$

Taking "$l+m+n$" common from C_1.

$$=(l+m+n)\begin{vmatrix} 1 & m+n & 1 \\ 1 & n+l & 1 \\ 1 & l+m & 1 \end{vmatrix}$$

$(\because C_1$ and C_3 are identical$)$

$=0=R.H.S$

(v) $\begin{vmatrix} 2 & 1 & 3x \\ 2 & 3 & 9x \\ 3 & 5 & 15x \end{vmatrix}=0$

Solution:

$$\text{L.H.S}=\begin{vmatrix} 2 & 1 & 3x \\ 2 & 3 & 9x \\ 3 & 5 & 15x \end{vmatrix}$$

Taking "$3x$" common from C_3.

$$=3x\begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 5 \end{vmatrix}$$

$(\because C_2$ and C_3 are identical$)$

$=0=R.H.S$

(vi) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Solution:

$$\text{L.H.S} = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$$

By $aR_1 \to R_1', bR_2 \to R_2'$ and $cR_3 \to R_3'$

$$= \frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$$

Taking "abc" common from C_1

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \text{R.H.S}$$

3. Using properties of determinants, show that:

(i) $\begin{vmatrix} 3 & 5 & 0 \\ 5 & 25 & 10 \\ 7 & 25 & 1 \end{vmatrix} = 25\begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 7 & 5 & 1 \end{vmatrix}$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 3 & 5 & 0 \\ 5 & 25 & 10 \\ 7 & 25 & 1 \end{vmatrix}$$

Taking “5” common from C_2

$$= 5\begin{vmatrix} 3 & 1 & 0 \\ 5 & 5 & 10 \\ 7 & 5 & 1 \end{vmatrix}$$

Taking “5” common from R_2

$$= 25\begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 7 & 5 & 1 \end{vmatrix} = \text{R.H.S}$$

(ii) $\begin{vmatrix} a+x & a & a \\ a & a+x & a \\ a & a & a+x \end{vmatrix} = x^2(3a+x)$

Solution:

$$\text{L.H.S} = \begin{vmatrix} a+x & a & a \\ a & a+x & a \\ a & a & a+x \end{vmatrix}$$

By $R_1+(R_2+R_3)\to R_1'$

$$=\begin{vmatrix} 3a+x & 3a+x & 3a+x \\ a & a+x & a \\ a & a & a+x \end{vmatrix}$$

Taking $(3a+x)$ common from R_1

$$=(3a+x)\begin{vmatrix} 1 & 1 & 1 \\ a & a+x & a \\ a & a & a+x \end{vmatrix}$$

By $C_2-C_1\to C_2'$ and $C_3-C_1\to C_3'$

$$=(3a+x)\begin{vmatrix} 1 & 0 & 0 \\ a & x & 0 \\ a & 0 & x \end{vmatrix}$$

As the matrix is in triangular form.
So, its determinant is the product of entries of main diagonal.

$$=(3a+x)\times 1\times x\times x = x^2(3a+x) = \text{R.H.S}$$

(iii) $$\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

By $xR_1\to R_1', yR_2\to R_2'$ and $zR_3\to R_3'$

$$=\frac{1}{xyz}\begin{vmatrix} x & x^2 & xyz \\ y & y^2 & xyz \\ z & z^2 & xyz \end{vmatrix}$$

Taking "xyz" common from C_3.

$$=\frac{xyz}{xyz}\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

By $C_2\leftrightarrow C_3$

$$=(-1)\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}$$

By $C_1\leftrightarrow C_2$

$$=(-1)(-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}=\text{R.H.S}$$

(iv) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}=(x-y)(y-z)(z-x)$

Solution:

$$\text{L.H.S}=\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

By $R_1-R_2\to R_1'$ and $R_2-R_3\to R_2'$

$$=\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$=\begin{vmatrix} 0 & x-y & (x-y)(x+y) \\ 0 & y-z & (y-z)(y+z) \\ 1 & z & z^2 \end{vmatrix}$$

Taking $(x-y)$ common from R_1 and $(y-z)$ from R_2.

$$=(x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$$

By $R_2-R_1\to R_2'$

$$=(x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 0 & z-x \\ 1 & z & z^2 \end{vmatrix}$$

Taking $(z-x)$ common from R_2.

$$=(x-y)(y-z)(z-x)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 0 & 1 \\ 1 & z & z^2 \end{vmatrix}$$

Expanding with C_1

$$=(x-y)(y-z)(z-x)\left[0-0+1\begin{vmatrix} 1 & x+y \\ 0 & 1 \end{vmatrix}\right]$$

$$=(x-y)(y-z)(z-x)(1)$$

$$=(x-y)(y-z)(z-x)=\text{R.H.S.}$$

(v) $$\begin{vmatrix} 1 & 1 & 1 \\ a+1 & b+1 & c+1 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & 1 & 1 \\ a+1 & b+1 & c+1 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ (a+1) & (b+1) & (c+1) \\ a^2+1+2a & b^2+1+2b & c^2+1+2c \end{vmatrix}$$

By $\begin{cases} C_1 - C_2 \to C_1' \\ C_2 - C_3 \to C_2' \end{cases}$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c+1 \\ a^2-b^2+2a-2b & b^2-c^2+2b-2c & c^2+1+2c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c+1 \\ (a-b)[a+b+2] & (b-c)[b+c+2] & c^2+1+2c \end{vmatrix}$$

Taking common $(a-b)$ from C_1 and $(b-c)$ from C_2.

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c+1 \\ a+b+2 & b+c+2 & c^2+2c+1 \end{vmatrix}$$

Expanding from R_1.

$$= (a-b)(b-c)\left[0 - 0 + (1)\begin{vmatrix} 1 & 1 \\ a+b+2 & b+c+2 \end{vmatrix}\right]$$

$$= (a-b)(b-c)\left[b+c+2-(a+b+2)\right]$$

$$= (a-b)(b-c)(b+c+2-a-b-2)$$

$$= (a-b)(b-c)(c-a) = \text{R.H.S}$$

(vi) $$\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

By $\begin{cases} R_1 - R_2 \to R_1' \\ R_2 - R_3 \to R_2' \end{cases}$

$$=\begin{vmatrix} b^2 & -b^2 & c^2-a^2\\ a^2-b^2 & c^2 & -c^2\\ b^2 & b^2 & c^2+a^2\end{vmatrix}$$

By $R_1+R_3\to R_1'$

$$=\begin{vmatrix} 2b^2 & 0 & 2c^2\\ a^2-b^2 & c^2 & -c^2\\ b^2 & b^2 & c^2+a^2\end{vmatrix}$$

By $\begin{cases} C_1-C_2\to C_1'\\ C_3+C_2\to C_3'\end{cases}$

$$=\begin{vmatrix} 2b^2 & 0 & 2c^2\\ a^2-b^2-c^2 & c^2 & 0\\ 0 & b^2 & a^2+b^2+c^2\end{vmatrix}$$

Expanding with R_1.

$$=2b^2\begin{vmatrix} c^2 & 0\\ b^2 & a^2+b^2+c^2\end{vmatrix}-0+2c^2\begin{vmatrix} a^2-b^2-c^2 & c^2\\ 0 & b^2\end{vmatrix}$$

$$=2b^2\left[c^2\left(a^2+b^2+c^2\right)-0\right]+2c^2\left[b^2\left(a^2-b^2-c^2\right)-0\right]$$

$$=2b^2\left(c^2a^2+c^2b^2+c^4\right)+2c^2\left(a^2b^2-b^4-b^2c^2\right)$$

$$=2a^2b^2c^2+2b^4c^2+2b^2c^4+2a^2b^2c^2-2b^4c^2-2b^2c^4$$

$$=4^2b^2c^2=\text{R.H.S}$$

(vii) $\begin{vmatrix} a & b & c\\ b+c & c+a & a+b\\ a+b & b+c & c+a\end{vmatrix}=a^3+b^3+c^3-3abc$

Solution:

$$\text{L.H.S}=\begin{vmatrix} a & b & c\\ b+c & c+a & a+b\\ a+b & b+c & c+a\end{vmatrix}$$

By $R_1+R_2\to R_1'$

$$=\begin{vmatrix} a+b+c & a+b+c & a+b+c\\ b+c & c+a & a+b\\ a+b & b+c & c+a\end{vmatrix}$$

Taking "$a+b+c$" common from R_1.

$$=(a+b+c)\begin{vmatrix} 1 & 1 & 1\\ b+c & c+a & a+b\\ a+b & b+c & c+a\end{vmatrix}$$

By $C_2-C_1\to C_2'$ and $C_3-C_1\to C_3'$

$$=(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding with R_1

$$=(a+b+c)\left[1\begin{vmatrix} a-b & a-c \\ c-a & c-b \end{vmatrix}-0+0\right]$$

$$=(a+b+c)\left[(a-b)(c-b)-(c-a)(a-c)\right]$$

$$=(a+b+c)\left[ac-ab-bc+b^2-(ac-c^2-a^2+ac)\right]$$

$$=(a+b+c)\left[ac-ab-bc+b^2-2ac+c^2+a^2\right]$$

$$=(a+b+c)(a^2+b^2+c^2-ac-ab-bc)$$

$$=a^3+b^3+c^3-3abc=R.H.S$$

(viii) $\begin{vmatrix} a+t & a & a \\ b & b+t & b \\ c & c & c+t \end{vmatrix}=t^2(a+b+c+t)$

Solution:

$$\text{L.H.S}=\begin{vmatrix} a+t & a & a \\ b & b+t & b \\ c & c & c+t \end{vmatrix}$$

By $R_1+(R_2+R_3)\rightarrow R_1'$

$$=\begin{vmatrix} a+b+c+t & a+b+c+t & a+b+c+t \\ b & b+t & b \\ c & c & c+t \end{vmatrix}$$

Taking common $(a+b+c+t)$ from R_1.

$$=(a+b+c+t)\begin{vmatrix} 1 & 1 & 1 \\ b & b+t & b \\ c & c & c+t \end{vmatrix}$$

By $\begin{cases} C_2-C_1\rightarrow C_2' \\ C_3-C_1\rightarrow C_3' \end{cases}$

$$=(a+b+c+t)\begin{vmatrix} 1 & 0 & 0 \\ b & t & 0 \\ c & 0 & t \end{vmatrix}$$

As the matrix is in triangular form.
So, its determinant is the product of entries of main diagonal.

$$=(a+b+c+t)\times 1\times t\times t$$

$$t^2(a+b+c+t)=\text{R.H.S}$$

(ix) $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

By $R_1 + (R_2 + R_3) \to R_1'$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking common $(a+b+c)$ from R_1

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

By $\begin{cases} C_2 - C_1 \to C_2' \\ C_3 - C_1 \to C_3' \end{cases}$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

As the matrix is in triangular form.
So, its determinant is the product of entries of main diagonal.

$$= (a+b+c)\times 1 \times [-(a+b+c)] \times [-(a+b+c)]$$

$$= (a+b+c)^3 = \text{R.H.S}$$

(x) $$\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x+y+z)(x-y)(y-z)(z-x)$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

By $R_1 + R_2 \to R_1'$

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Taking $(x+y+z)$ common from R_1

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

By $C_1 - C_2 \to C_1'$ and $C_2 - C_3 \to C_2'$

$$= (x+y+z)\begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$

$$=(x+y+z)\begin{vmatrix} 0 & 0 & 1\\ x-y & y-z & z\\ (x-y)(x+y) & (y-z)(y+z) & z^2\end{vmatrix}$$

Taking $(x-y)$ common from C_1 and $(y-z)$ common from C_2.

$$=(x+y+z)(x-y)(y-z)\begin{vmatrix} 0 & 0 & 1\\ 1 & 1 & z\\ x+y & y+z & z^2\end{vmatrix}$$

Expand from R_1

$$=(x+y+z)(x-y)(y-z)\left[0-0+1\begin{vmatrix} 1 & 1\\ x+y & y+z\end{vmatrix}\right]$$

$$=(x+y+z)(x-y)(y-z)(y+z-x-y)$$

$$=(x+y+z)(x-y)(y-z)(z-x)=R.H.S$$

(xi) $$\begin{vmatrix} 1 & 1 & 1\\ a^2+1 & b^2+1 & c^2+1\\ a^3+a & b^3+b & c^3+c\end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca-1)$$

Solution:

$$\text{L.H.S}=\begin{vmatrix} 1 & 1 & 1\\ a^2+1 & b^2+1 & c^2+1\\ a^3+a & b^3+b & c^3+c\end{vmatrix}$$

By $\begin{cases} C_1-C_2\to C_1' \\ C_2-C_3\to C_2' \end{cases}$

$$=\begin{vmatrix} 0 & 0 & 1\\ a^2-b^2 & b^2-c^2 & c^2+1\\ a^3-b^3+a-b & b^3-c^3+b-c & c^3+c\end{vmatrix}$$

$$=\begin{vmatrix} 0 & 0 & 1\\ (a+b)(a-b) & (b+c)(b-c) & c^2+1\\ (a-b)(a^2+ab+b^2)+(a-b) & (b-c)(b^2+bc+c^2)+(b-c) & c^3+c\end{vmatrix}$$

$$=\begin{vmatrix} 0 & 0 & 1\\ (a-b)(a+b) & (b-c)(b+c) & c^2+1\\ (a-b)\left[(a^2+ab+b^2)+1\right] & (b-c)\left[(b^2+bc+c^2)+1\right] & c^3+c\end{vmatrix}$$

Taking common $(a-b)$ from C_1 and $(b-c)$ from C_2.

$$=(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1\\ a+b & b+c & c^2+1\\ a^2+ab+b^2+1 & b^2+bc+c^2+1 & c^3+c\end{vmatrix}$$

By $C_2-C_1\to C_2'$

$$=(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1\\ a+b & c-a & c^2+1\\ a^2+ab+b^2+1 & c^2-a^2+bc-ab & c^3+c\end{vmatrix}$$

$$=(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1\\ a+b & c-a & c^2+1\\ a^2+ab+b^2+1 & (c-a)(c+a)+(c-a)b & c^3+c\end{vmatrix}$$

$$=(a-b)(b-c)\begin{vmatrix}0&0&1\\a+b&c-a&c^2+1\\a^2+ab+b^2+1&(c-a)\left[(c+a)+b\right]&c^3+c\end{vmatrix}$$

Taking common $(c-a)$ from C_2.

$$=(a-b)(b-c)(c-a)\begin{vmatrix}0&0&1\\a+b&1&c^2+1\\a^2+ab+b^2+1&a+b+c&c^3+c\end{vmatrix}$$

Expanding with R_1.

$$=(a-b)(b-c)(c-a)\left[0-0+(1)\begin{vmatrix}a+b&1\\a^2+ab+b^2+1&a+b+c\end{vmatrix}\right]$$

$$=(a-b)(b-c)(c-a)\left[(a+b)(a+b+c)-\left(a^2+ab+b^2+1\right)\right]$$

$$=(a-b)(b-c)(c-a)\left[a^2+ab+ac+ab+b^2+bc-a^2-ab-b^2-1\right]$$

$$=(a-b)(b-c)(c-a)(ab+bc+ac-1)=\text{R.H.S}$$

(xii) $\begin{vmatrix}1+a&1&1\\1&1+b&1\\1&1&1+c\end{vmatrix}=abc+ab+bc+ca$

Solution:

$$\text{L.H.S}=\begin{vmatrix}1+a&1&1\\1&1+b&1\\1&1&1+c\end{vmatrix}$$

By $\begin{cases}C_1-C_2\to C_1'\\C_2-C_3\to C_2'\end{cases}$

$$=\begin{vmatrix}a&0&1\\-b&b&1\\0&-c&1+c\end{vmatrix}$$

Expanding with R_1

$$=a\begin{vmatrix}b&1\\-c&1+c\end{vmatrix}-0+(1)\begin{vmatrix}-b&b\\0&-c\end{vmatrix}$$

$$=a\left[b(1+c)-(-c)\right]+bc-0$$

$$=a(b+bc+c)+bc$$

$$=ab+abc+ac+bc$$

$$=abc+ab+bc+ca=\text{R.H.S}$$

(xiii) $\begin{vmatrix}1&a&a^2-bc\\1&b&b^2-ca\\1&c&c^2-ab\end{vmatrix}=0$

Solution:

$$\text{L.H.S}=\begin{vmatrix}1&a&a^2-bc\\1&b&b^2-ca\\1&c&c^2-ab\end{vmatrix}$$

By $\begin{cases} C_2 - aC_1 \to C_2' \\ C_3 - a^2C_1 \to C_3' \end{cases}$

$$= \begin{vmatrix} 1 & a-a & a^2-bc-a^2 \\ 1 & b-a & b^2-a^2-ca \\ 1 & c-a & c^2-a^2-ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -bc \\ 1 & b-a & b^2-a^2-ca \\ 1 & c-a & c^2-a^2-ab \end{vmatrix}$$

By $C_3 - bC_2 \to C_3'$

$$= \begin{vmatrix} 1 & 0 & -bc \\ 1 & b-a & b^2-a^2-ca-b(b-a) \\ 1 & c-a & c^2-a^2-ab-b(c-a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -bc \\ 1 & b-a & ab-a^2-ac \\ 1 & c-a & c^2-a^2-bc \end{vmatrix}$$

By $\begin{cases} R_2 - R_1 \to R_2' \\ R_3 - R_1 \to R_3' \end{cases}$

$$= \begin{vmatrix} 1 & 0 & -bc \\ 0 & b-a & ab-a^2-ac+bc \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Taking $(c-a)$ common from R_3.

$$= (c-a)\begin{vmatrix} 1 & 0 & -bc \\ 0 & b-a & ab-a^2-ac+bc \\ 0 & 1 & c+a \end{vmatrix}$$

Expanding with C_1

$$= (c-a)\left[(1)\begin{vmatrix} b-a & ab-a^2-ac+bc \\ 1 & c+a \end{vmatrix} - 0 + 0\right]$$

$$= (c-a)\left[(b-a)(c+a) - (ab-a^2-ac+bc)\right]$$

$$= (c-a)\left[bc+ab-ac-a^2-ab+a^2+ac-bc\right]$$

$$= (c-a)(0) = 0 = \text{R.H.S}$$

(xiv) $\begin{vmatrix} 2x+3 & x+2 & x+a \\ 2x+5 & x+3 & x+b \\ 2x+7 & x+4 & x+c \end{vmatrix} = 0,$ **where** a,b,c **are in A.P.**

Solution:

$$\text{L.H.S} = \begin{vmatrix} 2x+3 & x+2 & x+a \\ 2x+5 & x+3 & x+b \\ 2x+7 & x+4 & x+c \end{vmatrix}$$

By $\begin{cases} R_2 - R_1 \to R_2' \\ R_3 - R_2 \to R_3' \end{cases}$

$$= \begin{vmatrix} 2x+3 & x+2 & x+a \\ 2 & 1 & b-a \\ 2 & 1 & c-b \end{vmatrix}$$

As, a, b and c are in A.P.

So, $b-a = c-b \Rightarrow c = 2b-a$

$$= \begin{vmatrix} 2x+3 & x+2 & x+a \\ 2 & 1 & b-a \\ 2 & 1 & 2b-a-b \end{vmatrix}$$

$$= \begin{vmatrix} 2x+3 & x+2 & x+a \\ 2 & 1 & b-a \\ 2 & 1 & b-a \end{vmatrix} = 0 = \text{R.H.S} \qquad (\because\ R_2 \text{ and } R_3 \text{ are identical})$$

(xv) $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$, **where ω is an imaginary cube root of unity.**

Solution:

$$\text{L.H.S} = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

By $\begin{cases} R_2 + R_3 \to R_2' \\ R_3 + R_1 \to R_3' \end{cases}$

$$= \begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a+b & b+c & c+a \end{vmatrix}$$

By $R_1 + R_2 \to R_1'$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b+c & c+a & a+b \\ a+b & b+c & c+a \end{vmatrix}$$

Taking $(a+b+c)$ common from R_1

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ a+b & b+c & c+a \end{vmatrix}$$

By $\begin{cases} C_2 - C_1 \to C_2' \\ C_3 - C_1 \to C_3' \end{cases}$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding with R_1.

$$= (a+b+c)\left[(1)\begin{vmatrix} a-b & a-c \\ c-a & c-b \end{vmatrix} - 0 + 0\right]$$

$$=(a+b+c)\left[(a-b)(c-b)-(c-a)(a-c)\right]$$
$$=(a+b+c)\left[ac-ab-bc+b^2-\left(ac-c^2-a^2+ac\right)\right]$$
$$=(a+b+c)\left[ac-ab-bc+b^2-2ac+c^2+a^2\right]$$
$$=(a+b+c)\left(a^2+b^2+c^2-ac-ab-bc\right)$$
$$=a^3+b^3+c^3-3abc \qquad \text{(i)}$$

$$\text{R.H.S}=(a+b+c)\left(a+b\omega+c\omega^2\right)\left(a+b\omega^2+c\omega\right)$$
$$=(a+b+c)\left(a^2+ab\omega^2+ac\omega+ab\omega+b^2\omega^3+bc\omega^2+ac\omega^2+bc\omega^4+c^2\omega^3\right)$$
$$=(a+b+c)\left(a^2+b^2\omega^3+c^2\omega^3+\omega(ab+ac)+\omega^2(ab+bc+ac)+bc\omega^4\right)$$
$$=(a+b+c)\left(a^2+b^2+c^2+\left(-1-\omega^2\right)(ab+ac)+ab\omega^2+bc\omega^2+ac\omega^2+bc\omega\right)$$
$$\left(\because \omega^3=1,\ 1+\omega+\omega^2=0,\ \omega=-1-\omega^2\right)$$
$$=(a+b+c)\left(a^2+b^2+c^2-ab-ac-ab\omega^2-ac\omega^2+ab\omega^2+bc\omega^2+ac\omega^2+bc\omega\right)$$
$$=(a+b+c)\left(a^2+b^2+c^2-ab-ac+bc\left(\omega+\omega^2\right)\right)$$
$$=(a+b+c)\left(a^2+b^2+c^2-ab-ac-bc\right) \qquad \left(\because 1+\omega+\omega^2=0,\ \omega+\omega^2=-1\right)$$
$$=a^3+b^3+c^3-3abc \qquad \text{(ii)}$$

From (i) and (ii) L.H.S = R.H.S

4. **If** $A=\begin{bmatrix}1 & 2 & -3\\ 0 & -5 & 0\\ -2 & -2 & 7\end{bmatrix}$ **and** $B=\begin{bmatrix}-5 & -2 & 5\\ -3 & -1 & 4\\ -2 & -1 & 2\end{bmatrix}$**, the find:**

(i) A_{13}, A_{23}, A_{33} **and** $|A|$

Solution:

Given that :

$$A=\begin{bmatrix}1 & 2 & -3\\ 0 & -5 & 0\\ -2 & -2 & 7\end{bmatrix}$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}0 & -5\\ -2 & -2\end{vmatrix}$$
$$=(-1)^4(0-10)=1\times(-10)$$
$$A_{13}=-10$$

Now, $A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ -2 & -2\end{vmatrix}$

$$=(-1)^5(-2+4)=-1\times 2=-2$$

Similarly,

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 2\\ 0 & -5\end{vmatrix}$$
$$=(-1)^6(-5-0)=1\times(-5)$$
$$A_{33}=-5$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -5 & 0 \\ -2 & -2 & 7 \end{vmatrix}$$

Expand from R_1

$$= 1 \times \begin{vmatrix} -5 & 0 \\ -2 & 7 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ -2 & 7 \end{vmatrix} - 3\begin{vmatrix} 0 & -5 \\ -2 & -2 \end{vmatrix}$$

$$= 1(-35-0) - 2(0-0) - 3(0-10)$$

$$|A| = -35 - 0 + 30 = -5$$

(ii) B_{31}, B_{32}, B_{33} **and** $|B|$

Solution:

Given that :

$$B = \begin{bmatrix} -5 & -2 & 5 \\ -3 & -1 & 4 \\ -2 & -1 & 2 \end{bmatrix}$$

$$B_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 5 \\ -1 & 4 \end{vmatrix}$$

$$= (-1)^4(-8+5)$$

$$B_{31} = 1 \times (-3) = -3$$

$$\text{Now, } B_{32} = (-1)^{3+2}\begin{vmatrix} -5 & 5 \\ -3 & 4 \end{vmatrix}$$

$$= (-1)^5(-20+15)$$

$$B_{32} = (-1)(-5) = 5$$

$$\text{Similarly, } B_{33} = (-1)^{3+3}\begin{vmatrix} -5 & -2 \\ -3 & -1 \end{vmatrix}$$

$$= (-1)^6(5-6) = 1 \times (-1)$$

$$B_{33} = -1$$

$$\text{Now, } |B| = \begin{vmatrix} -5 & -2 & 5 \\ -3 & -1 & 4 \\ -2 & -1 & 2 \end{vmatrix}$$

Expand from R_1

$$= -5\begin{vmatrix} -1 & 4 \\ -1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} -3 & 4 \\ -2 & 2 \end{vmatrix} + 5\begin{vmatrix} -3 & -1 \\ -2 & -1 \end{vmatrix}$$

$$= -5(-2+4) + 2(-6+8) + 5(3-2)$$

$$= -5(2) + 2(2) + 5(1) = -10 + 4 + 5$$

$$|B| = -1$$

5. Find the values of x if:

(i) $\begin{vmatrix} 2 & 1 & x \\ -1 & -4 & -3 \\ x & 1 & 0 \end{vmatrix} = 5$

Solution

Given that: $\begin{vmatrix} 2 & 1 & x \\ -1 & -4 & -3 \\ x & 1 & 0 \end{vmatrix} = 5$

Expand from R_1.

$$2\begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ x & 0 \end{vmatrix} + x\begin{vmatrix} -1 & -4 \\ x & 1 \end{vmatrix} = 5$$

$$2(0+3)-1(0+3x)+x(-1+4x)=5$$

$$6-3x-x+4x^2-5=0$$

$$4x^2-4x+1=0$$

$$4x^2-2x-2x+1=0$$

$$2x(2x-1)-1(2x-1)=0$$

$$(2x-1)(2x-1)=0$$

$$(2x-1)^2=0$$

$$2x-1=0$$

$$x=\frac{1}{2}$$

(ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -3 & x \end{vmatrix} = 9$

Solution:

Given that: $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -3 & x \end{vmatrix} = 9$

Expand from R_1

$$1\begin{vmatrix} x+1 & 2 \\ -3 & x \end{vmatrix} - (x-1)\begin{vmatrix} -1 & 2 \\ 2 & x \end{vmatrix} + 3\begin{vmatrix} -1 & x+1 \\ 2 & -3 \end{vmatrix} = 9$$

$$(x^2+x+6)-(x-1)(-x-4)+3(3-2x-2)=9$$

$$x^2+x+6-(-x^2-4x+x+4)+3(1-2x)=9$$

$$x^2+x+6+x^2+3x-4+3-6x-9=0$$

$$2x^2-2x-4=0$$

$$x^2-x-2=0$$

$$x^2-2x+x-2=0$$

$$x(x-2)+1(x-2)=0$$

$$(x-2)(x+1)=0$$

$$\begin{array}{l|l} \text{Either} & \text{Or} \\ x-2=0 & x+1=0 \\ x=2 & x=-1 \end{array}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix}=0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix}=0$$

Expand from R_1

$$1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix}-1\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix}+1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix}=0$$

$$(x^2-12)-(2x-6)+(12-3x)=0$$

$$x^2-12-2x+6+12-3x=0$$

$$x^2-5x+6=0$$

$$x^2-3x-2x+6=0$$

$$x(x-3)-2(x-3)=0$$

$$(x-2)(x-3)=0$$

$$\begin{array}{l|l} \text{Either} & \text{Or} \\ x-2=0 & x-3=0 \\ x=2 & x=3 \end{array}$$

6. **Find $|AA^t|$ and $|A^tA|$ if:**

(i) $A=\begin{bmatrix} -3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Solution:

Given that: $A=\begin{bmatrix} -3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Taking transpose on both sides:

$$A^t=\begin{bmatrix} -3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}^t$$

$$A^t=\begin{bmatrix} -3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Now, $AA^t=\begin{bmatrix} -3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 9+4+1 & -6+2-3 \\ -6+2-3 & 4+1+9 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 14 & -7 \\ -7 & 14 \end{bmatrix}$$

Taking determinant on both sides.

$$\left|AA^t\right| = \begin{vmatrix} 14 & -7 \\ -7 & 14 \end{vmatrix} = 196 - 49$$

$$\left|AA^t\right| = 147$$

Similarly, $A^t A = \begin{bmatrix} -3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9+4 & -6+2 & -3+6 \\ -6+2 & 4+1 & 2+3 \\ 3+6 & -2+3 & -1+9 \end{bmatrix}$

$$A^t A = \begin{bmatrix} 13 & -4 & 3 \\ -4 & 5 & 5 \\ 9 & 1 & 8 \end{bmatrix}$$

Taking determinant on both sides.

$$\left|A^t A\right| = \begin{vmatrix} 13 & -4 & 3 \\ -4 & 5 & 5 \\ 9 & 1 & 8 \end{vmatrix}$$

$$= 13(40-5) + 4(-32-45) + 3(-4-45)$$

$$= 13(35) + 4(-77) + 3(-49)$$

$$= 455 - 308 - 147 = 455 - 455$$

$$\left|A^t A\right| = 0$$

(ii) $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

Solution:

Given that: $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

Taking transpose on both sides

$$A^t = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}^t \Rightarrow A^t = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Now, $AA^t = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 9+1 & 6+2 & 3+3 \\ 6+2 & 4+4 & 2+6 \\ 3+3 & 2+6 & 1+9 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 10 & 8 & 6 \\ 8 & 8 & 8 \\ 6 & 8 & 10 \end{bmatrix}$$

Taking determinant on both sides.

$$\left|AA^t\right| = \begin{vmatrix} 10 & 8 & 6 \\ 8 & 8 & 8 \\ 6 & 8 & 10 \end{vmatrix}$$

$$= 10(80-64) - 8(80-48) + 6(64-48)$$

$$= 10(16) - 8(32) + 6(16) = 160 - 256 + 96$$

$$\left|AA^t\right| = 0$$

Similarly, $A^t A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 9+4+1 & 3+4+3 \\ 3+4+3 & 1+4+9 \end{bmatrix}$$

$$A^t A = \begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix}$$

Taking determinant on both sides.

$$\left|A^t A\right| = \begin{vmatrix} 14 & 10 \\ 10 & 14 \end{vmatrix} = 196 - 100$$

$$\left|A^t A\right| = 96$$

7. If A is a square matrix of order 3, then show that $|kA| = k^3|A|$.

Solution:

Since A is a square matrix of order 3.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

To show that $|kA| = k^3|A|$

Now we multiply by k (any non-zero scalar) by matrix A.

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

$$|kA| = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

Taking common k from R_1, R_2 and R_3

$$|kA| = k.k.k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|kA| = k^3\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Hence $|kA| = k^3|A|$

8. Find the values of λ if A and B are singular.

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 7 & \lambda & 6 \\ 2 & 3 & 1 \end{bmatrix},\ B = \begin{bmatrix} -2 & 4 & 5 \\ 1 & -2 & 1 \\ 2 & \lambda & 0 \end{bmatrix}$$

Solution:

Given that: $A = \begin{bmatrix} 4 & 2 & 3 \\ 7 & \lambda & 6 \\ 2 & 3 & 1 \end{bmatrix}$

Since A is a singular matrix.

i.e., $|A| = 0$

$$\begin{vmatrix} 4 & 2 & 3 \\ 7 & \lambda & 6 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

Expand from R_1

$$4(\lambda-18) - 2(7-12) + 3(21-2\lambda) = 0$$

$$4\lambda - 72 + 10 + 63 - 6\lambda = 0$$

$$-2\lambda + 1 = 0 \Rightarrow -2\lambda = -1 \Rightarrow \lambda = \frac{1}{2}$$

Also given that: $B = \begin{bmatrix} -2 & 4 & 5 \\ 1 & -2 & 1 \\ 2 & \lambda & 0 \end{bmatrix}$

Since B is a singular matrix.

i.e., $|B| = 0$

$$\begin{vmatrix} -2 & 4 & 5 \\ 1 & -2 & 1 \\ 2 & \lambda & 0 \end{vmatrix} = 0$$

Expand from R_1

$$-2(0-\lambda) - 4(0-2) + 5(\lambda+4) = 0$$

$$2\lambda + 8 + 5\lambda + 20 = 0$$

$$7\lambda + 28 = 0 \Rightarrow 7\lambda = -28 \Rightarrow \lambda = -4$$

9. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ -5 & 0 & 4 \\ 5 & 4 & 0 \end{bmatrix}$ and show that $A^{-1}A = I_3$.

Solution:

Given that: $A = \begin{bmatrix} 1 & 2 & 1 \\ -5 & 0 & 4 \\ 5 & 4 & 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ -5 & 0 & 4 \\ 5 & 4 & 0 \end{vmatrix}$$

$$|A| = 1(0-16) - 2(0-20) + 1(-20-0)$$

$$|A| = -16+40-20$$

$$|A| = 4 \neq 0$$

Since A is non-singular matrix so, its inverse exists.

$$adj\,A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \begin{bmatrix} \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} & -\begin{vmatrix} -5 & 4 \\ 5 & 0 \end{vmatrix} & \begin{vmatrix} -5 & 0 \\ 5 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 5 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -5 & 0 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} (0-16) & -(0-20) & (-20-0) \\ -(0-4) & (0-5) & -(4-10) \\ (8-0) & -(4+5) & (0+10) \end{bmatrix}^t$$

$$= \begin{bmatrix} -16 & 20 & -20 \\ 4 & -5 & 6 \\ 8 & -9 & 10 \end{bmatrix}^t$$

$$adj\;A = \begin{bmatrix} -16 & 4 & 8 \\ 20 & -5 & -9 \\ -20 & 6 & 10 \end{bmatrix}$$

So,

$$A^{-1} = \frac{adj\,A}{|A|} = \frac{1}{4}\begin{bmatrix} -16 & 4 & 8 \\ 20 & -5 & -9 \\ -20 & 6 & 10 \end{bmatrix}$$

To show that $A^{-1}A = I_3$

Let L.H.S $= A^{-1}A$

$$= \frac{1}{4}\begin{bmatrix} -16 & 4 & 8 \\ 20 & -5 & -9 \\ -20 & 6 & 10 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ -5 & 0 & 4 \\ 5 & 4 & 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -16-20+40 & -32+0+32 & -16+16+0 \\ 20+25-45 & 40-0-36 & 20-20-0 \\ -20-30+50 & -40+0+40 & -20+24+0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$=\begin{bmatrix}\frac{4}{4} & 0 & 0\\ 0 & \frac{4}{4} & 0\\ 0 & 0 & \frac{4}{4}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I_3=\text{R.H.S}$$

Hence $A^{-1}A=I_3$

10. **Verify that $(AB)^t=B^tA^t$ if:**

(i) $A=\begin{bmatrix}1 & -1 & 2\\ 0 & -3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ -3 & -2\\ 0 & 1\end{bmatrix}$

Solution:

Given that: $A=\begin{bmatrix}1 & -1 & 2\\ 0 & -3 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 1\\ -3 & -2\\ 0 & 1\end{bmatrix}$

First we will find AB.

$$AB=\begin{bmatrix}1 & -1 & 2\\ 0 & -3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -3 & -2\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1+3+0 & 1+2+2\\ 0+9+0 & 0+6+1\end{bmatrix}=\begin{bmatrix}4 & 5\\ 9 & 7\end{bmatrix}$$

Taking transpose on both sides,

$$(AB)^t=\begin{bmatrix}4 & 9\\ 5 & 7\end{bmatrix} \qquad \text{(i)}$$

Now, taking transpose of matrix A and B.

$$A^t=\begin{bmatrix}1 & 0\\ -1 & -3\\ 2 & 1\end{bmatrix}, B^t=\begin{bmatrix}1 & -3 & 0\\ 1 & -2 & 1\end{bmatrix}$$

$$B^tA^t=\begin{bmatrix}1 & -3 & 0\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & -3\\ 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1+3+0 & 0+9+0\\ 1+2+2 & 0+6+1\end{bmatrix}$$

$$=\begin{bmatrix}4 & 9\\ 5 & 7\end{bmatrix} \qquad \text{(ii)}$$

From equation (i) and (ii)

$(AB)^t=B^tA^t$

(ii) $A=\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix},\ B=\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}$

Solution:

Given that: $A=\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix},\ B=\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}$

First we will find AB.

$$AB=\begin{bmatrix}1 & 2\\ 1 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & -3\\ -2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1-4 & -3+2\\ 1-8 & -3+4\\ 2-2 & -6+1\end{bmatrix}$$

$$AB=\begin{bmatrix}-3 & -1\\ -7 & 1\\ 0 & -5\end{bmatrix}$$

Taking transpose on both sides $(AB)^t=\begin{bmatrix}-3 & -7 & 0\\ -1 & 1 & -5\end{bmatrix}$ (i)

Now, taking transpose of matrix A and B,

$$A^t=\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & 1\end{bmatrix}, B^t=\begin{bmatrix}1 & -2\\ -3 & 1\end{bmatrix}$$

$$B^tA^t=\begin{bmatrix}1 & -2\\ -3 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1-4 & 1-8 & 2-2\\ -3+2 & -3+4 & -6+1\end{bmatrix}$$

$$=\begin{bmatrix}-3 & -7 & 0\\ -1 & 1 & -5\end{bmatrix} \qquad \text{(ii)}$$

From equation (i) and (ii)

$$(AB)^t=B^tA^t$$