

Unit # 04, Matrices and Determinants

Matrix:

A rectangular array of numbers enclosed by a pair of bracket is called a matrix such as:

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 2 & 6 \\ 4 & 1 & -1 \end{bmatrix}$$

The numbers used in rows or columns are said to be the entries or elements of the matrix.

Generally bracketed rectangular array of $m.n$ elements a_{ij} ($1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n$) arranged in m rows and n columns

such as: $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$ is called an m by n or (written as $m \times n$ matrix). The matrices are usually represented by

the capital letters such as A, B, C, X, Y etc., and small letters such as a, b, c, l, m, n or $a_{11}, a_{12}, a_{13}, \dots$, etc., are used to indicate the entries of the matrices.

Row Matrix or Row Vector:

A matrix, which has only one row, i.e, a $1 \times n$ matrix of the form $[a_{i1} \ a_{i2} \ a_{i3} \ \dots \ a_{in}]$ is said to be a **row matrix** or **row vector**. For example $[1 \ -1 \ 3 \ 4]$ is a row matrix having 4 columns.

Column Matrix or Column Vector:

A matrix which has only one column i.e., an $m \times 1$ matrix of the form $\begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is said to be a **column matrix** or a **column vector**.

For example $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is a column matrix having 3 rows.

Rectangular Matrix:

If $m \neq n$, then the matrix is called a rectangular matrix of order $m \times n$, that is, the matrix in which number of rows is not equal to the number of columns, is said to be a rectangular matrix.

For example; $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ is rectangular matrix of order 2×3 .

Square Matrix:

If $m = n$, then the matrix of the order $m \times n$ is said to be a square matrix of order n or m . i.e., the matrix which has **same number of rows and columns** is called a square matrix. For example: $\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$ is a square matrix of order 2.



Diagonal Matrix:

Let $A = [a_{ij}]$ be a square matrix of order n . If $a_{ij} = 0 \forall i \neq j$ and at least one $a_{ij} \neq 0$ for $i = j$, that is, some elements of the principal diagonal of A may be zero but not all, then the matrix A is called a **diagonal matrix**.

The matrices $[7]$, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ are diagonal matrices.

Scalar Matrix:

Let $A = [a_{ij}]$ be a square matrix of order n . If $a_{ij} = 0 \forall i \neq j$ and $a_{ij} = k$ (some non-zero scalar) for all $i = j$, then the matrix is called a **scalar matrix** of order n . For examples

$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ are scalar matrices of order 2 and 4.

Unit Matrix or Identity Matrix:

Let $A = [a_{ij}]$ be a square matrix of order n . If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$, then the matrix A is called a **unit matrix** or **identity matrix** of order n . We denote such matrix by I_n or simply I . For example The identity matrix of order is 3 is

denoted by I_3 , that is, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Null Matrix or Zero Matrix:

A square or rectangular matrix whose each element is zero, is called a **null** or **zero** matrix. An $m \times n$ matrix with all its elements equal to zero is denoted by $O_{m \times n}$. Null matrices may be of any order. Here are some examples:

$[0]$, $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Equal Matrices:

Two matrices of the same order are said to be **equal** if their corresponding entries are equal e.g.

$A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal i.e., $A = B$ iff $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \dots, m, j = 1, 2, 3, \dots, n$.

Transpose of a matrix:

If A is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of A , is called the transpose of A . it is denoted by A^t . Let $A = [a_{ij}]_{m \times n}$, then the transpose of A is defined as:

$A^t = [a'_{ij}]_{n \times m}$ where $a'_{ij} = a_{ji}$ for $i = 1, 2, 3, \dots, n$ and $j = 1, 2, 3, \dots, m$



For example, if $B = [b_{ij}]_{3 \times 4} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$, then

$B' = [b'_{ij}]_{4 \times 3}$ where $b'_{ij} = b_{ji}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ i.e.,

$$B' = \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} \\ b'_{21} & b'_{22} & b'_{23} \\ b'_{31} & b'_{32} & b'_{33} \\ b'_{41} & b'_{42} & b'_{43} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \\ b_{14} & b_{24} & b_{34} \end{bmatrix}$$

Properties of Matrix Addition, Scalar Multiplication and Matrix Multiplication:

If A, B and C are conformable for the indicated sum or product of matrices and c and d are scalars, then following properties are true:

1. Commutative property w.r.t. addition: $A + B = B + A$
2. Associative property w.r.t addition: $(A + B) + C = A + (B + C)$
3. Associative property of scalar multiplication: $(cd)A = c(dA)$.
4. Existence of additive identity: $A + O = O + A = A$ (O is null matrix and A is a square matrix)
5. Existence of multiplicative identity: $IA = AI = A$ (I is unit/identity matrix)
6. Distributive property w.r.t scalar multiplication:
 - (a) $c(A + B) = cA + cB$
 - (b) $(c + d)A = cA + dA$
7. Associative property w.r.t multiplication: $A(BC) = (AB)C$
8. Left distributive property: $A(B + C) = AB + AC$
9. Right distributive property: $(A + B)C = AC + BC$
10. $c(AB) = (cA)B = A(cB)$.

EXERCISE 4.1

1. If $A = [a_{ij}]_{3 \times 4}$, then show that
 - i) $I_3A = A$
 - ii) $AI_4 = A$

Solution:

We have $A = [a_{ij}]_{3 \times 4}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Also we know that $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- (i) $I_3A = A$



Consider L.H.S = I_3A

$$\begin{aligned}
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\
 &= \begin{bmatrix} 1 \times a_{11} + 0 \times a_{21} + 0 \times a_{31} & 1 \times a_{12} + 0 \times a_{22} + 0 \times a_{32} & 1 \times a_{13} + 0 \times a_{23} + 0 \times a_{33} & 1 \times a_{14} + 0 \times a_{24} + 0 \times a_{34} \\ 0 \times a_{11} + 1 \times a_{21} + 0 \times a_{31} & 0 \times a_{12} + 1 \times a_{22} + 0 \times a_{32} & 0 \times a_{13} + 1 \times a_{23} + 0 \times a_{33} & 0 \times a_{14} + 1 \times a_{24} + 0 \times a_{34} \\ 0 \times a_{11} + 0 \times a_{21} + 1 \times a_{31} & 0 \times a_{12} + 0 \times a_{22} + 1 \times a_{32} & 0 \times a_{13} + 0 \times a_{23} + 1 \times a_{33} & 0 \times a_{14} + 0 \times a_{24} + 1 \times a_{34} \end{bmatrix} \\
 &= \begin{bmatrix} a_{11} + 0 + 0 & a_{12} + 0 + 0 & a_{13} + 0 + 0 & a_{14} + 0 + 0 \\ 0 + a_{21} + 0 & 0 + a_{22} + 0 & 0 + a_{23} + 0 & 0 + a_{24} + 0 \\ 0 + 0 + a_{31} & 0 + 0 + a_{32} & 0 + 0 + a_{33} & 0 + 0 + a_{34} \end{bmatrix} \\
 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A = \text{R.H.S}
 \end{aligned}$$

L.H.S = R.H.S, Hence $I_3A = A$

(ii) $AI_4 = A$

L.H.S = AI_4

$$\begin{aligned}
 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} a_{11} \times 1 + a_{12} \times 0 + a_{13} \times 0 + a_{14} \times 0 & a_{11} \times 0 + a_{12} \times 1 + a_{13} \times 0 + a_{14} \times 0 & a_{11} \times 0 + a_{12} \times 0 + a_{13} \times 1 + a_{14} \times 0 & a_{11} \times 0 + a_{12} \times 0 + a_{13} \times 0 + a_{14} \times 1 \\ a_{21} \times 1 + a_{22} \times 0 + a_{23} \times 0 + a_{24} \times 0 & a_{21} \times 0 + a_{22} \times 1 + a_{23} \times 0 + a_{24} \times 0 & a_{21} \times 0 + a_{22} \times 0 + a_{23} \times 1 + a_{24} \times 0 & a_{21} \times 0 + a_{22} \times 0 + a_{23} \times 0 + a_{24} \times 1 \\ a_{31} \times 1 + a_{32} \times 0 + a_{33} \times 0 + a_{34} \times 0 & a_{31} \times 0 + a_{32} \times 1 + a_{33} \times 0 + a_{34} \times 0 & a_{31} \times 0 + a_{32} \times 0 + a_{33} \times 1 + a_{34} \times 0 & a_{31} \times 0 + a_{32} \times 0 + a_{33} \times 0 + a_{34} \times 1 \end{bmatrix} \\
 &= \begin{bmatrix} a_{11} + 0 + 0 + 0 & 0 + a_{12} + 0 + 0 & 0 + 0 + a_{13} + 0 & 0 + 0 + 0 + a_{14} \\ a_{21} + 0 + 0 + 0 & 0 + a_{22} + 0 + 0 & 0 + 0 + a_{23} + 0 & 0 + 0 + 0 + a_{24} \\ a_{31} + 0 + 0 + 0 & 0 + a_{32} + 0 + 0 & 0 + 0 + a_{33} + 0 & 0 + 0 + 0 + a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A = \text{R.H.S}
 \end{aligned}$$

L.H.S = R.H.S, Hence Proved.

2. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 5 & 0 \\ 3 & 4 & -1 \end{bmatrix}$, then find

(i) $A - B$ (ii) $B - C$

(iii) $(A - B) - C$ (iv) $A - (B - C)$

(i) $A - B$

Solution:



$$A - B = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-2 & -1-1 & 2-(-1) \\ 3-1 & 2-2 & 1-4 \\ -1-(-1) & 0-2 & 4-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & 2+1 \\ 2 & 0 & -3 \\ -1+1 & -2 & 3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix}$$

(ii) $B - C$

Solution:

$$B - C = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ -1 & 5 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1-0 & -1-(-2) \\ 1-(-1) & 2-5 & 4-0 \\ -1-3 & 2-4 & 1-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1+2 \\ 1+1 & -3 & 4 \\ -4 & -2 & 1+1 \end{bmatrix}$$

$$B - C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ -4 & -2 & 2 \end{bmatrix}$$

(iii) $(A - B) - C$

Solution:

First we find $A - B$.

$$A - B = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$



$$= \begin{bmatrix} 0-2 & -1-1 & 2-(-1) \\ 3-1 & 2-2 & 1-4 \\ -1-(-1) & 0-2 & 4-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & 2+1 \\ 2 & 0 & -3 \\ -1+1 & -2 & 3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix}$$

Now,

$$(A - B) - C = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ -1 & 5 & 0 \\ 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} -2-1 & -2-0 & 3-(-2) \\ 2-(-1) & 0-5 & -3-0 \\ 0-3 & -2-4 & 3-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 & 3+2 \\ 2+1 & -5 & -3 \\ -3 & -6 & 3+1 \end{bmatrix}$$

$$(A - B) - C = \begin{bmatrix} -3 & -2 & 5 \\ 3 & -5 & -3 \\ -3 & -6 & 4 \end{bmatrix}$$

(iv) $A - (B - C)$

Solution:

First we find $(B - C)$

$$B - C = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ -1 & 5 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1-0 & -1-(-2) \\ 1-(-1) & 2-5 & 4-0 \\ -1-3 & 2-4 & 1-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1+2 \\ 1+1 & -3 & 4 \\ -4 & -2 & 1+1 \end{bmatrix}$$

$$B - C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ -4 & -2 & 2 \end{bmatrix}$$

Now,



$$A - (B - C) = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ -4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 0-1 & -1-1 & 2-1 \\ 3-2 & 2-(-3) & 1-4 \\ -1-(-4) & 0-(-2) & 4-2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 1 \\ 1 & 2+3 & -3 \\ -1+4 & 0+2 & 2 \end{bmatrix}$$

$$A - (B - C) = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 5 & -3 \\ 3 & 2 & 2 \end{bmatrix}$$

3. If $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2i & 1 \\ -i & i \end{bmatrix}$, then show that

(i) $(AB)C = A(BC)$

Solution:

Given that: $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2i & 1 \\ -i & i \end{bmatrix}$

L.H.S = $(AB)C$

First, we will find AB .

$$AB = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} -i & 1 \\ 2i & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (i)(-i) + (2i)(2i) & (i)(1) + (2i)(1) \\ (1)(-i) + (-i)(2i) & (1)(1) + (-i)(1) \end{bmatrix} \quad AB = \begin{bmatrix} -i^2 + 4i^2 & i + 2i \\ -i - 2i^2 & 1 - i \end{bmatrix}$$

$$AB = \begin{bmatrix} 3i^2 & 3i \\ -i + 2 & 1 - i \end{bmatrix} = \begin{bmatrix} -3 & 3i \\ 2 - i & 1 - i \end{bmatrix}$$

Now, $(AB)C = \begin{bmatrix} -3 & 3i \\ 2 - i & 1 - i \end{bmatrix} \begin{bmatrix} 2i & 1 \\ -i & i \end{bmatrix}$

$$= \begin{bmatrix} (-3)(2i) + (3i)(-i) & (-3)(1) + (3i)(i) \\ (2-i)(2i) + (1-i)(-i) & (2-i)(1) + (1-i)(i) \end{bmatrix} = \begin{bmatrix} -6i - 3i^2 & -3 + 3i^2 \\ 4i - 2i^2 - i + i^2 & 2 - i + i - i^2 \end{bmatrix}$$

$$= \begin{bmatrix} -6i + 3 & -3 - 3 \end{bmatrix} \quad (\because i^2 = -1)$$

$$(AB)C = \begin{bmatrix} 3 - 6i & -6 \\ 1 + 3i & 3 \end{bmatrix}$$

R.H.S = $A(BC)$

First we will find BC .



$$BC = \begin{bmatrix} -i & 1 \\ 2i & 1 \end{bmatrix} \begin{bmatrix} 2i & 1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} (-i)(2i) + (1)(-i) & (-i)(1) + (1)(i) \\ (2i)(2i) + (1)(-i) & (2i)(1) + (1)(i) \end{bmatrix}$$

$$= \begin{bmatrix} -2i^2 - i & -i + i \\ 4i^2 - i & 2i + i \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 - i & 0 \\ -4 - i & 3i \end{bmatrix} \quad (\because i^2 = -1)$$

Now, $A(BC) = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 2 - i & 0 \\ -4 - i & 3i \end{bmatrix}$

$$= \begin{bmatrix} (i)(2 - i) + (2i)(-4 - i) & (i)(0) + (2i)(3i) \\ (1)(2 - i) + (-i)(-4 - i) & (1)(0) + (-i)(3i) \end{bmatrix}$$

$$= \begin{bmatrix} 2i - i^2 - 8i - 2i^2 & 0 + 6i^2 \\ 2 - i + 4i + i^2 & 0 - 3i^2 \end{bmatrix}$$

$$= \begin{bmatrix} -6i + 3 & -6 \\ 1 + 3i & 3 \end{bmatrix} \quad (\because i^2 = -1)$$

$$A(BC) = \begin{bmatrix} 3 - 6i & -6 \\ 1 + 3i & 3 \end{bmatrix}$$

Hence, proved that L.H.S = R.H.S

(ii) $A(B + C) = AB + AC$

Solution:

Given that: $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2i & 1 \\ -i & i \end{bmatrix}$

L.H.S = $A(B + C)$

First we will find $B + C$.

$$B + C = \begin{bmatrix} -i & 1 \\ 2i & 1 \end{bmatrix} + \begin{bmatrix} 2i & 1 \\ -i & i \end{bmatrix} = \begin{bmatrix} -i + 2i & 1 + 1 \\ 2i - i & 1 + i \end{bmatrix} \quad B + C = \begin{bmatrix} i & 2 \\ i & 1 + i \end{bmatrix}$$

Now, $A(B + C) = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} i & 2 \\ i & 1 + i \end{bmatrix}$

$$= \begin{bmatrix} (i)(i) + (2i)(i) & (i)(2) + (2i)(1 + i) \\ (1)(i) + (-i)(i) & (1)(2) + (-i)(1 + i) \end{bmatrix}$$



$$= \begin{bmatrix} i^2 + 2i^2 & 2i + 2i + 2i^2 \\ i - i^2 & 2 - i - i^2 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} -3 & -2+4i \\ 1+i & 3-i \end{bmatrix} (\because i^2 = -1)$$

R.H.S = AB + AC

First, we will find AB and AC.

$$AB = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} -i & 1 \\ 2i & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (i)(-i) + (2i)(2i) & (i)(1) + (2i)(1) \\ (1)(-i) + (-i)(2i) & (1)(1) + (-i)(1) \end{bmatrix}$$

$$AB = \begin{bmatrix} -i^2 + 4i^2 & i + 2i \\ -i - 2i^2 & 1 - i \end{bmatrix}$$

$$AB = \begin{bmatrix} 3i^2 & 3i \\ -i + 2 & 1 - i \end{bmatrix}$$

$$AB = \begin{bmatrix} -3 & 3i \\ 2 - i & 1 - i \end{bmatrix}$$

Similarly, $AC = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 2i & 1 \\ -i & i \end{bmatrix}$

$$= \begin{bmatrix} (i)(2i) + (2i)(-i) & (i)(1) + (2i)(i) \\ (1)(2i) + (-i)(-i) & (1)(1) + (-i)(i) \end{bmatrix}$$

$$= \begin{bmatrix} 2i^2 - 2i^2 & i + 2i^2 \\ 2i + i^2 & 1 - i^2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 0 & -2+i \\ -1+2i & 2 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} -3 & 3i \\ 2 - i & 1 - i \end{bmatrix} + \begin{bmatrix} 0 & -2+i \\ -1+2i & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3+0 & 3i-2+i \\ 2-i-1+2i & 1-i+2 \end{bmatrix}$$

$$AB + BC = \begin{bmatrix} -3 & -2+4i \\ 1+i & 3-i \end{bmatrix} = \text{R.H.S}$$



4. If A and B are square matrices of the same order, then explain why in general

i) $(A+B)^2 \neq A^2 + 2AB + B^2$

ii) $(A-B)^2 \neq A^2 - 2AB + B^2$

iii) $(A+B)(A-B) \neq A^2 - B^2$

(i) $(A+B)^2 \neq A^2 + 2AB + B^2$

Solution:

$$\text{L.H.S} = (A+B)^2 = (A+B)(A+B)$$

$$= A.A + A.B + B.A + B.B$$

$$= A^2 + AB + BA + B^2$$

$$= A^2 + (AB + BA) + B^2$$

In matrices $AB \neq BA$ in general

Therefore $AB + BA \neq 2AB$

$$(A+B)^2 = A^2 + (AB + BA) + B^2$$

$$\neq A^2 + 2AB + B^2$$

Hence $(A+B)^2 \neq A^2 + 2AB + B^2$

(ii) $(A-B)^2 \neq A^2 - 2AB + B^2$

Solution:

$$\text{Consider } (A-B)^2 = (A-B)(A-B)$$

$$= A.A - AB - BA + B.B$$

$$= A^2 - (AB + BA) + B^2$$

In matrices $AB \neq BA$ in general

Therefore $AB + BA \neq 2AB$

So $(A-B)^2 \neq A^2 - 2AB + B^2$



Hence $(A - B)^2 \neq A^2 - 2AB + B^2$

(iii) $(A + B)(A - B) \neq A^2 - B^2$

Solution:

Consider $(A + B)(A - B)$

$$= A.A - AB + BA - B.B$$

$$= A^2 - A.B + BA - B^2$$

In matrices $AB \neq BA$ in general

Therefore $-AB + BA \neq 0$

So, $(A + B)(A - B) = A^2 - A.B + BA - B^2$

$$\neq A^2 - B^2$$

Hence $(A + B)(A - B) \neq A^2 - B^2$

5. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \\ -3 & 5 & 3 \end{bmatrix}$, then find $A + A', A - A', AA', A'A$ and $(A')'$.

Solution:

Given that: $A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \\ -3 & 5 & 3 \end{bmatrix}$

First we will find A'

$$A' = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \\ -3 & 5 & 3 \end{bmatrix}'$$

$$A' = \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 5 \\ 3 & 2 & 3 \end{bmatrix}$$

$$\text{Now, } A + A' = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \\ -3 & 5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 5 \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1-1 & 2+1 & 3-3 \\ 1+2 & 0+0 & 2+5 \\ -3+3 & 5+2 & 3+3 \end{bmatrix}$$



$$A + A^t = \begin{bmatrix} -2 & 3 & 0 \\ 3 & 0 & 7 \\ 0 & 7 & 6 \end{bmatrix}$$

$$\text{Similarly, } A - A^t = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \\ -3 & 5 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 5 \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1+1 & 2-1 & 3+3 \\ 1-2 & 0-0 & 2-5 \\ -3-3 & 5-2 & 3-3 \end{bmatrix}$$

$$A - A^t = \begin{bmatrix} 0 & 1 & 6 \\ -1 & 0 & -3 \\ -6 & 3 & 0 \end{bmatrix}$$

$$\text{Now, } AA^t = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \\ -3 & 5 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 5 \\ 3 & 2 & 3 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} (-1)(-1) + (2)(2) + (3)(3) & (-1)(1) + (2)(0) + (3)(2) & (-1)(-3) + (2)(5) + (3)(3) \\ (1)(-1) + (0)(2) + (2)(3) & (1)(1) + (0)(0) + (2)(2) & (1)(-3) + (0)(5) + (2)(3) \\ (-3)(-1) + (5)(2) + (3)(3) & (-3)(1) + (5)(0) + (3)(2) & (-3)(-3) + (5)(5) + (3)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & -1+0+6 & 3+10+9 \\ -1+0+6 & 1+0+4 & -3+0+6 \\ 3+10+9 & -3+0+6 & 9+25+9 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 14 & 5 & 22 \\ 5 & 5 & 3 \\ 22 & 3 & 43 \end{bmatrix}$$

$$\text{Similarly, } A^t A = \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 5 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \\ -3 & 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-1) + (1)(1) + (-3)(-3) & (-1)(2) + (1)(0) + (-3)(5) & (-1)(3) + (1)(2) + (-3)(3) \\ (2)(-1) + (0)(1) + (5)(-3) & (2)(2) + (0)(0) + (5)(5) & (2)(3) + (0)(2) + (5)(3) \\ (3)(-1) + (2)(1) + (3)(-3) & (3)(2) + (2)(0) + (3)(5) & (3)(3) + (2)(2) + (3)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+9 & -2+0-15 & -3+2-9 \\ -2+0-15 & 4+0+25 & 6+0+15 \\ -3+2-9 & 6+0+15 & 9+4+9 \end{bmatrix}$$

$$A^t A = \begin{bmatrix} 11 & -17 & -10 \\ -17 & 29 & 21 \\ -10 & 21 & 22 \end{bmatrix}$$



$$\text{Now, } (A^t)^t = \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 5 \\ 3 & 2 & 3 \end{bmatrix}^t$$

$$(A^t)^t = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \\ -3 & 5 & 3 \end{bmatrix}$$

6. Solve the matrix equation $A^2 - 5A + 4I - X = 0$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Solution:

Given that: $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

First we will find A^2

$$A^2 = A.A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

Now, $A^2 - 5A + 4I - X = 0$

$$X = A^2 - 5A + 4I$$

$$X = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} 5-10 & -1-0 & 2-5 \\ 9-10 & -2-5 & 5-15 \\ 0-5 & -1-(-5) & -2-0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$



$$X = \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} -5+4 & -1+0 & -3+0 \\ -1+0 & -7+4 & -10+0 \\ -5+0 & 4+0 & -2+4 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

7. If A and B are two matrices such that $AB = B$ and $BA = A$, show that $A^2 + B^2 = A + B$.

Solution:

Given that: $AB = B$ and $BA = A$

$$B.AB = B.B \quad \text{and} \quad A.BA = A.A$$

$$(B.A)B = B.B \quad \text{and} \quad (A.B)A = A.A$$

$$AB = B^2 \quad \text{and} \quad BA = A^2 \quad (\because AB = B \text{ and } BA = A)$$

$$B = B^2 \quad \text{and} \quad A = A^2 \quad (\because AB = B \text{ and } BA = A)$$

$$A^2 + B^2 = A + B$$

Hence proved that, L.H.S. = R.H.S.

