14.4

1. Find the volume of parallelepiped for which the given vectors are three edges

(i) $\underline{u} = 3\underline{i} + 2\underline{k}$; $\underline{v} = \underline{i} + 2\underline{j} + \underline{k}$; $\underline{w} = -\underline{j} + 4\underline{k}$

Volume $= u.(\underline{v} \times w)$

$$= \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$$

$$= 3(8+1) - 0(4-1) + 2(-1-0)$$

$$= 3(9) - 0(3) + 2(-1)$$

$$= 27 - 0 - 2$$

$= 25$ cubic units

ii) $\underline{u} = \underline{i} - 4\underline{j} - \underline{k}$; $\underline{v} = \underline{i} - \underline{j} - 2\underline{k}$; $\underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Volume $= \underline{u}.(\underline{v} \times \underline{w})$

$$= \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & -2 \\ -3 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= 1(-1-6) + 4(1+4) - 1(-3+2)$$

$$= -7 + 4(5) - 1(-1)$$

$$= -7 + 20 + 1$$

$= 14$ cubic units

(iii) $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$; $\underline{v} = 2\underline{i} - \underline{j} - \underline{k}$; $\underline{w} = \underline{j} + \underline{k}$

volume $= \underline{u} \cdot (\underline{v} \times \underline{w})$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 1(-1+1) + 2(2-0) + 3(2-0)$$

$$= 1(0) + 2(2) + 3(2)$$

$= 0 + 4 + 6 = 10$ cubic units

2. Verify that $\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$

If $\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}$; $\underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$

and $\underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$.

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$$

$$= 3(3+10) - (-1)(4+4) + 5(20-6)$$

$$= 3(13) + 1(8) + 5(14)$$

$$= 39 + 8 + 70$$

$$= 117$$

$$\underline{b} \cdot \underline{c} \times \underline{a} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$$

$$= 4(25+1) - 3(10-3) + (-2)(-2-15)$$

$$= 4(26) - 3(7) - 2(-17)$$

$$= 104 - 21 + 34$$

$$= 117$$

$$\underline{c} \cdot \underline{a} \times \underline{b} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

$$= 2(2-15) - 5(-6-20) + 1(9+4)$$

$$= 2(-13) - 5(-26) + 1(13)$$

$= -26 + 130 + 13$

$= 117$

3. Prove that the vectors $\underline{i} - 2\underline{j} + 3\underline{k}$, $-2\underline{i} + 3\underline{j} - 4\underline{k}$ and $\underline{i} - 3\underline{j} + 5\underline{k}$ are coplanar.

For coplanar.

$$= \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 0$$

$$= 1\begin{vmatrix} 3 & -4 \\ -3 & 5 \end{vmatrix} - (-2)\begin{vmatrix} -2 & -4 \\ 1 & 5 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix} = 0$$

$= 1(15 - 12) + 2(-10 + 4) + 3(6 - 3) = 0$

$= 1(3) + 2(-6) + 3(3) = 0$

$= 3 - 12 + 9 = 0$

$= \cancel{12} - \cancel{12} = 0$

$= 0 = 0$

4 Find the constant α such that the vectors are coplanar.

(i) $\underline{i} - \underline{j} + \underline{k}$, $\underline{i} - 2\underline{j} - 3\underline{k}$ and $3\underline{i} - \alpha\underline{j} + 5\underline{k}$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$$

$$1\begin{vmatrix}-2 & -3\\ -\alpha & 5\end{vmatrix} - (-1)\begin{vmatrix}1 & -3\\ 3 & 5\end{vmatrix} + 1\begin{vmatrix}1 & -2\\ 3 & -\alpha\end{vmatrix} = 0$$

$1(-10-3\alpha)+1(5+9)+1(-\alpha+6)=0$

$-10-3\alpha+14-\alpha+6=0$

$-4\alpha+10=0$

$+4\alpha=+10$

$\alpha=\frac{10}{4}$

$\boxed{\alpha=5/2}$

(ii) $\underline{i}-2\alpha\underline{j}-k$; $\underline{i}-2\underline{j}+2\underline{k}$ and $\alpha\underline{i}-2\underline{j}+\underline{k}$

$$\begin{vmatrix}1 & -2\alpha & -1\\ 1 & -2 & 2\\ \alpha & -2 & 1\end{vmatrix} = 0$$

$$1\begin{vmatrix}-2 & 2\\ -2 & 1\end{vmatrix} - (-2\alpha)\begin{vmatrix}1 & 2\\ \alpha & 1\end{vmatrix} + (-1)\begin{vmatrix}1 & -2\\ \alpha & -2\end{vmatrix} =$$

$1(-2+4)+2\alpha(1-2\alpha)-1(-2+2\alpha)=0$

$2+\cancel{2\alpha}-4\alpha^2+2-\cancel{2\alpha}=0$

$4-4\alpha^2=0$

$4(1-\alpha^2)=0$

$1-\alpha^2=0$

$\sqrt{1}=\sqrt{\alpha^2}$

$\alpha=\pm 1$

5. Prove that points whose position vectors are $A(-6\underline{i}+3\underline{j}+2\underline{k})$, $B(3\underline{i}-2\underline{j}+4\underline{k})$, $C(5\underline{i}+7\underline{j}+3\underline{k})$, $D(-13\underline{i}+17\underline{j}-\underline{k})$ are coplanar.

$$\vec{AB} = \vec{OB} - \vec{OA}$$
$$= (3\underline{i}-2\underline{j}+4\underline{k})-(-6\underline{i}+3\underline{j}+2\underline{k})$$
$$= (3+6)\underline{i}+(-2-3)\underline{j}+(4-2)\underline{k}$$
$$= 9\underline{i}-5\underline{j}+2\underline{k}$$

$$\vec{AC} = \vec{OC} - \vec{OA}$$
$$= (5\underline{i}+7\underline{j}+3\underline{k})-(-6\underline{i}+3\underline{j}+2\underline{k})$$
$$= (5+6)\underline{i}+(7-3)\underline{j}+(3-2)\underline{k}$$
$$= 11\underline{i}+4\underline{j}+\underline{k}$$

$$\vec{AD} = \vec{OD} - \vec{OA}$$
$$= (-13\underline{i}+17\underline{j}-\underline{k})-(-6\underline{i}+3\underline{j}+2\underline{k})$$
$$= (-13+6)\underline{i}+(17-3)\underline{j}+(-1-2)\underline{k}$$
$$= -7\underline{i}+14\underline{j}-3\underline{k}$$

Given in question

$$\begin{vmatrix} 9 & -5 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix}$$

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$$= 9\begin{vmatrix} 4 & 1 \\ 14 & -3 \end{vmatrix} - (-5)\begin{vmatrix} 11 & 1 \\ -7 & -3 \end{vmatrix} + 2\begin{vmatrix} 11 & 4 \\ -7 & 14 \end{vmatrix}$$

$= 9(-12-14)+5(-33+7)+2(154+28)$

$= 9(-26)+5(-26)+2(182)$

$= -234-130+364$

$= -364+364$

$= 0$

So the points are Coplanar.

6. Find the value of:

(a)(i) $2\underline{i} \times 2\underline{j} \cdot \underline{k}$ (ii) $3\underline{j} \cdot \underline{k} \times \underline{i}$

$= 4\underline{k} \cdot \underline{k}$ $= 3\underline{j} \cdot \underline{j}$

$= 4(1)$ $= 3(1)$

$= 4$ $= 3$

(iii) $[k \quad i \quad j]$ (iv) $[\underline{i} \quad \underline{i} \quad \underline{k}]$

$= k \cdot \underline{i} \times \underline{j}$

$= k \cdot \underline{k}$

$= 1$

(b) Prove that $\underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{v} \cdot (\underline{w} \times \underline{u}) + \underline{w} \cdot (\underline{u} \times \underline{v})$
$= 3\underline{u} \cdot (\underline{v} \times \underline{w})$

$\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$, $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$

$\underline{w} = w_1\underline{i} + w_2\underline{j} + w_3\underline{k}$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix} + \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + (-1) \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + (-1) \begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - (-1) \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - (-1) \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$= \underline{u} \cdot (\underline{v} \times \underline{w}) + (u \cdot (\underline{v} \times \underline{w}) + \underline{u} \cdot (\underline{v} \times \underline{w})$

$= 3\ \underline{u} \cdot (\underline{v} \times \underline{w})$

It is proved

7. Find volume of tetrahedron with the vertices.

(i) $(0, 1, 2)$, $(3, 2, 1)$, $(1, 2, 1)$ and $(5, 5, 6)$

Volume $= \frac{1}{8} (\overline{AB} \cdot (\overline{AC} \times \overline{AD}))$

$A(0, 1, 2)$, $B(3, 2, 1)$, $C(1, 2, 1)$, $D(5, 5, 6)$

$\overline{AB} = \overline{OB} - \overline{OA}$

$= (3-0) + (2-1) + (1-2)$

$= 3, 1, -1$

$\overline{AB} = (3, 1, -1)$

$\overline{AC} = \overline{OC} - \overline{OA}$

$= (1, 1, -1)$

$\overline{AD} = \overline{OD} - \overline{OA}$

$= (5, 4, 4)$

Volume of tetrahedron $= \frac{1}{6} \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$

$= \frac{1}{6} [3(4+4) - 1(4+5) + (-1)(4-5)]$

$= \frac{1}{6} [3(8) - 1(9) - 1(-1)]$

$= \frac{1}{6} (24 - 9 + 1)$

$= \frac{1}{\cancel{6}_3} (\cancel{16}) 8$

$= \frac{8}{3}$ Cubic units

(ii) $(2, 1, 8)$, $(3, 2, 9)$, $(2, 1, 4)$ and $(3, 3, 10)$

$A(2, 1, 8)$, $B(3, 2, 9)$, $C(2, 1, 4)$, $D(3, 3, 10)$

$\overline{AB} = \overline{OB} - \overline{OA}$

$= (1, 1, 1)$

$\overline{AC} = \overline{OC} - \overline{OA}$

$= (0, 0, -4)$

$\overline{AD} = \overline{OD} - \overline{OA}$

$= (1, 2, 2)$

Volume of tetrahedron $= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$

$= \frac{1}{6}\left[1(0+8) - 1(0+4) + 1(0-0)\right]$

$= \frac{1}{6}\left(1(8) - 1(4) + 1(0)\right]$

$= \frac{1}{6}(8 - 4 + 0)$

$= \frac{\cancel{8}\,4}{6} = 2/3$ cubic units

8. Prove that the point whose position vectors are $A(3\underline{i} + 2\underline{j} - \underline{k})$, $B(\underline{i} - 2\underline{j} + \underline{k})$, $C(6\underline{i} + 4\underline{j} - 2\underline{k})$, $D(9\underline{i} + 6\underline{j} - 3\underline{k})$ are coplanar.

$\overline{AB} = \overline{OB} - \overline{OA}$

$= (\underline{i} - 2\underline{j} + \underline{k}) - (3\underline{i} + 2\underline{j} - \underline{k})$

$= (1-3)\underline{i} + (-2+2)\underline{j} + (1+1)\underline{k}$

$= -2\underline{i} + 0\underline{j} + 2\underline{k}$

$$\overline{AC} = \overline{OC} - \overline{OA}$$
$$= (6\underline{i} + 4\underline{j} - 2\underline{k}) - (3\underline{i} + 2\underline{j} - \underline{k})$$
$$= (6-3)\underline{i} + (4-2)\underline{j} + (-2+1)\underline{k}$$
$$= 3\underline{i} + 2\underline{j} - \underline{k}$$

$$\overline{AD} = \overline{OD} - \overline{OA}$$
$$= (9\underline{i} + 6\underline{j} - 3\underline{k}) - (3\underline{i} + 2\underline{j} - \underline{k})$$
$$= (9-3)\underline{i} + (6-2)\underline{j} + (-3+1)\underline{k}$$
$$= 6\underline{i} + 4\underline{j} - 2\underline{k}$$

$$\begin{vmatrix} -2 & 0 & 2 \\ 3 & 2 & -1 \\ 6 & 4 & -2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} - 0\begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix}$$

$$= -2(-4+4) - 0(-6+6) + 2(12-12)$$
$$= -2(0) - 0(0) + 2(0)$$
$$= 0 - 0 + 0 = 0$$

So, points are coplanar.

9. Prove that for any three non-zero vector $\underline{u}, \underline{v}$ and $\underline{w}$,

$$(\underline{u}+\underline{v}).[(\underline{v}+\underline{w}) \times (\underline{w}+\underline{u})] = 2[\underline{u}\ \underline{v}\ \underline{w}]$$

$(\underline{u}+\underline{v}).[\underline{v}\times\underline{w}+\underline{v}\times\underline{u}+\underline{w}\times\underline{w}+\underline{w}\times\underline{u}]$

$=(\underline{u}+\underline{v}).[\underline{v}\times\underline{w}+\underline{v}\times\underline{u}+\underline{w}\times\underline{u}]$

$=\underline{u}.(\underline{v}\times\underline{w})+\underline{u}.(\underline{v}\times\underline{u})+\underline{u}.(\underline{w}\times\underline{u})+\underline{v}.(\underline{v}\times\underline{w})$
$+\underline{v}.(\underline{v}\times\underline{u})+\underline{v}.(\underline{w}\times\underline{u})$

$=\underline{u}.(\underline{v}\times\underline{w})+0+0+0+0+\underline{v}.(\underline{w}\times\underline{u})$

$=\underline{u}.(\underline{v}\times\underline{w})+\underline{u}.(\underline{v}\times\underline{w})\quad\because\ \underline{v}.(\underline{w}\times\underline{u})=\underline{u}.(\underline{v}\times\underline{w})$

$=[u\ v\ w]+[u\ v\ w]$

$=2[\underline{u}\ \underline{v}\ \underline{w}]$

It is proved

10. Consider a parallelepiped determined by the vector $\underline{u}=2\underline{i}+4\underline{j}-3\underline{k}$, $\underline{v}=5\underline{i}-3\underline{j}+6\underline{k}$ and $w=4\underline{i}-7\underline{j}-2\underline{k}$. If the base of parallelepiped is define by the vectors $\underline{u}$ and $\underline{v}$ then find the height of the parallelepiped.

Volume of parallelepiped $=[u\ v\ w]$

$$\begin{vmatrix} 2 & 4 & -3 \\ 5 & -3 & 6 \\ 4 & -7 & -2 \end{vmatrix}$$

$= 2(6+42) - 4(-10-24) - 3(-35+12)$

$= 2(48) - 4(-34) - 3(-23)$

$= 96 + 136 + 69$

$= 301$ cubic units

Area of base $= |\underline{u} \times \underline{v}|$

$$\underline{u} \times \underline{v} = \begin{vmatrix} i & j & k \\ 2 & 4 & -3 \\ 5 & -3 & 6 \end{vmatrix}$$

$= i(24-9) - j(12+15) + k(-6-20)$

$= 15\underline{i} - 27\underline{j} - 26\underline{k}$

$|\underline{u} \times \underline{v}| = \sqrt{(15)^2 + (-27)^2 + (-26)^2}$

$= \sqrt{1630}$

$V =$ length $\times$ werth $\times$ height

$=$ (Area of base) $\times$ height

$301 = \sqrt{1630} \times$ height

$\dfrac{301}{\sqrt{1630}} =$ height

11. A mechanic applies a force of 50 pounds along the positive x-axis on a wrench connected to a bolt. The pivot point of the wrench is at the origin

(0, 0, 0) and the force is applied at the point (0ft, 2ft, 3 ft). Determine the torque produced by this force about the pivot point.

$F = 50$

$O = (0, 0, 0)$

$M = (0, 2, 3)$

$\tau = r \times F$

$= (OM) \times F$

$$= \begin{vmatrix} i & j & k \\ 0 & 2 & 3 \\ 50 & 0 & 0 \end{vmatrix}$$

$= i(0) - j(-150) + k(-100)$

$= 150\underline{j} - 100\underline{k}$

12. A drone flies from point (1, 2, 5) to point (4, 6, 9), which each unit representing a meter. What is the magnitude of the displacement the drone experienced during this flight?

$A(1, 2, 5)$, $B(4, 6, 9)$

$\vec{d} = \overline{AB}$

$= (4-1)\underline{i} + (6-2)\underline{j} + (9-5)\underline{k}$

$= 3\underline{i} + 4\underline{j} + 4\underline{k}$

$|\vec{d}| = \sqrt{(3)^2 + (4)^2 + (4)^2}$

$= \sqrt{9+16+16}$

$|\vec{d}| = \sqrt{41}$

13. The vector $\underline{u} = 50\underline{i} + 75\underline{j} + 65\underline{k}$ Show how many belts, pants, and shirts were sold at a store. The vector $\underline{w} = 1500\underline{i} + 3500\underline{j} + 3000\underline{k}$ Show the price (in rupees) of each item. Find $\underline{u}.\underline{w}$ and explain what the result tells us in real life.

$\underline{u} = 50\underline{i} + 75\underline{j} + 65\underline{k}$, $\underline{w} = 1500\underline{i} + 3500\underline{j} + 3000\underline{k}$

$\underline{u}.\underline{w} = (50\underline{i} + 75\underline{j} + 65\underline{k}) \cdot (1500\underline{i} + 3500\underline{j} + 3000\underline{k})$

$= 50 \times 100 + 75 \times 3500 + 65 \times 3000$

$\underline{u}.\underline{w} = Rs\ 532500$

$\underline{u}.\underline{w}$ gives us total revenue store gets.

14. A force $F = (20, -10, 30)N$ is applied at a point $P(2, -1, 4)$ in 3D space. The pivot point is at $M(1, 2, -3)$. Calculate the torque produced by this force about the pivot point M.

$$\underline{F} = 20\underline{i} - 10\underline{j} + 30\underline{k}$$

$$\underline{r} = \overline{PM}$$

$$= \overline{OM} - \overline{OP}$$

$$= (2-1, -1-2, 4+3)$$

$$\underline{r} = \underline{i} - 3\underline{j} + 7\underline{k}$$

$$\tau = \underline{r} \times \underline{F}$$

$$= \begin{vmatrix} i & j & k \\ 1 & -3 & 7 \\ 20 & -10 & 30 \end{vmatrix}$$

$$= i \begin{vmatrix} -3 & 7 \\ -10 & 30 \end{vmatrix} - j \begin{vmatrix} 1 & 7 \\ 20 & 30 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & -3 \\ 20 & -10 \end{vmatrix}$$

$$= i(-90+70) - \underline{j}(30-140) + \underline{k}(-10+60)$$

$$\tau = -20\underline{i} + 110\underline{j} + 50\underline{k}$$

15. An electric shop sells three types of appliances: Fans, Heaters, and Ovens. The monthly sales quantities are 500 units of Fans, 300 units of Heaters and 200 units of ovens. The profit per unit for each appliance is Rs 500 for Fans, Rs 400 for Heaters and Rs 2,000 for Ovens.

(a) Represent the monthly sales quantities and the profit per unit as vectors.

Take Fans along x-axis

Heaters along y-axis

Ovens along z-axis

Sales vector $= \vec{S} = 500i + 300j + 200k$

Profit vector $= \vec{P} = 500i + 400j + 2000k$

(b) Calcute the total monthly profit using vector operations.

Monthly Profit $= \vec{S} \cdot \vec{P}$

$= 250000 + 120000 + 400000$

Monthly Profit $=$ Rs 770,000

HASSAN MEHBOOB

S.S.S(Math)