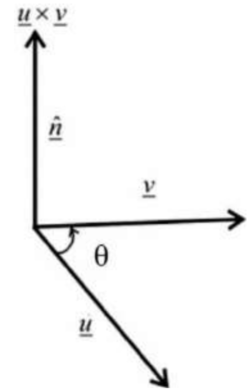
The Cross Product or Vectors Product of Two Vector:

(i) Definition 1:



The vector or cross product of two vectors $\underline{u}$ and $\underline{v}$ in space is

$\underline{u}\times\underline{v}=|\underline{u}|\,|\underline{v}|\sin\theta\,\underline{n}$

where θ is the angle between $\underline{u}$ *and* $\underline{v}$ and $0\le\theta\le\pi$. $\underline{n}$ is a unit vector perpendicular to the plane of $\underline{u}$ *and* $\underline{v}$. It is important to note that $\underline{u}\times\underline{v}\ne\underline{v}\times\underline{u}$, rather $\underline{u}\times\underline{v}=-(\underline{v}\times\underline{u})$.

(i) The Unit Vectors:

$\underline{i},\underline{j},\underline{k}$

(a) $\underline{i}\times\underline{i}=0$, $\underline{j}\times\underline{j}=0$, $\underline{k}\times\underline{k}=0$

(b) $\underline{i}\times\underline{j}=\underline{k}$, $\underline{j}\times\underline{k}=\underline{i}$, $\underline{k}\times\underline{i}=\underline{j}$

Definition 2:

If $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$ and $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ are two vectors in space, then cross product of $\underline{u}$ and $\underline{v}$ is : $\underline{u}\times\underline{v}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$ ("Determinant formula" for $\underline{u}\times\underline{v}$.)

Parallel Vectors:

If two vectors $\underline{u}$ and $\underline{v}$ are parallel, then $\underline{u}\times\underline{v}=\underline{0}$

Angle between two Vectors:

The angle θ between two vectors $\underline{u}$ and $\underline{v}$ is $\sin\theta=\dfrac{|\underline{u}\times\underline{v}|}{|\underline{u}|\,|\underline{v}|}$

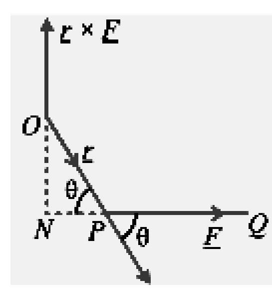
Area of Parallelogram:

Area of Parallelogram $=$ $Base\times Height=|\underline{u}\times\underline{v}|$

Area of a Triangle:

Area of Triangle = $\frac{1}{2}$ (Area of Parallelogram) $=\frac{1}{2}|\underline{u}\times\underline{v}|$ where $\underline{u}$ and $\underline{v}$ are vectors along to adjacent sides of triangle.

Moment of Force:



Let a force $\underline{F}\left(\overrightarrow{PQ}\right)$ act at a point P, then moment of $\underline{F}$ about O

$=$ Product of force $\underline{F}$ and perpendicular $\overrightarrow{ON}$ the direction of $\underline{n}$

$=\left(\overrightarrow{PQ}\right)\left(\overrightarrow{ON}\right)\underline{n}=(PQ)(OP)\sin\theta\underline{n}$

$=\overrightarrow{OP}\times\overrightarrow{PQ}=\underline{r}\times\underline{F}$

Prof: MUHAMMAD IRFAN
MATHEMATICS INSTRUCTOR /
(F.Sc. / I.Cs. / O/A-Level / Entry Test Expert)

Visit us at: MathCity.org
https://www.youtube.com/@mirfandogar
https://www.facebook.com/mirfandogar786
0300-1920009

EXERCISE 14.3

Q.1 **Compute the cross product $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$. Check your answer by showing that each $\underline{a}$ and $\underline{b}$ are perpendicular to $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$.**

(i) $\underline{a}=2\underline{i}+\underline{j}-\underline{k}$, $\underline{b}=\underline{i}-\underline{j}+\underline{k}$

Solution:

$\underline{a}=2\underline{i}+\underline{j}-\underline{k}$, $\underline{b}=\underline{i}-\underline{j}+\underline{k}$

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & 1 & -1\\ 1 & -1 & 1\end{vmatrix}$$

$=\underline{i}(1-1)-\underline{j}(2+1)+\underline{k}(-2-1)$

$=-3\underline{j}-3\underline{k}$

$\underline{a}.(\underline{a}\times\underline{b})=(2\underline{i}+\underline{j}-\underline{k}).(-3\underline{j}-3\underline{k})$

$=0-3+3=0 \quad\Rightarrow\quad \underline{a}\perp\underline{a}\times\underline{b}$

$\underline{b}.(\underline{a}\times\underline{b})=(\underline{i}-\underline{j}+\underline{k}).(-3\underline{j}-3\underline{k})$

$=(1)(0)+(-1)(-3)+(1)(-3)$

$=0+3-3=0$

$\Rightarrow \underline{b}\perp\underline{a}\times\underline{b}$

Thus $\underline{a}\times\underline{b}$ is perpendicular to both vectors $\underline{a}$ and $\underline{b}$.

$$\underline{b}\times\underline{a}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & -1 & 1\\ 2 & 1 & -1\end{vmatrix}$$

$=\underline{i}(1-1)-\underline{j}(-1-2)+\underline{k}(1+2)$

$=3\underline{j}+3\underline{k}$

$\underline{a}.(\underline{b}\times\underline{a})=(2\underline{i}+\underline{j}-\underline{k}).(3\underline{j}+3\underline{k})$

$=2(0)+(1)(3)+(-1)(3)$

$=3-3=0\Rightarrow \underline{a}\perp\underline{b}\times\underline{a}$

$\underline{b}.(\underline{b}\times\underline{a})=(\underline{i}-\underline{j}+\underline{k}).(3\underline{j}+3\underline{k})$

$=(1)(0)+(-1)(3)+(1)(3)=0-3+3=0$

Thus $\underline{b}\times\underline{a}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

(ii) $\underline{a}=\underline{i}+3\underline{j}+2\underline{k}$, $\underline{b}=2\underline{i}-\underline{j}+\underline{k}$

Solution:

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 3 & 2\\ 2 & -1 & 1\end{vmatrix}$$

$=\underline{i}(3+2)-\underline{j}(1-4)+\underline{k}(-1-6)=5\underline{i}+3\underline{j}-7\underline{k}$

$\underline{a}.(\underline{a}\times\underline{b})=(\underline{i}+3\underline{j}+2\underline{k}).(5\underline{i}+3\underline{j}-7\underline{k})=0$ Also

$\underline{b}.(\underline{a}\times\underline{b})=(2\underline{i}-\underline{j}+\underline{k}).(5\underline{i}+3\underline{j}-7\underline{k})$

$\underline{b}.(\underline{a}\times\underline{b})=10-3-7=0$

Thus $\underline{a}\times\underline{b}$ is perpendicular to both vector $\underline{a}$ and $\underline{b}$.

$$\underline{b}\times\underline{a}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & -1 & 1\\ 1 & 3 & 2\end{vmatrix}$$

$= \underline{i}(-2-3) - \underline{j}(4-1) + \underline{k}(6+1) = -5\underline{i} - 3\underline{j} + 7\underline{k}$

$\underline{a}.(\underline{b}\times\underline{a}) = (\underline{i}+3\underline{j}+2\underline{k}).(-5\underline{i}-3\underline{j}+7\underline{k})$ $\underline{a}.(\underline{b}\times\underline{a}) = -5-9+14 = 0$

Also

$\underline{b}.(\underline{b}\times\underline{a}) = (2\underline{i}-\underline{j}+\underline{k}).(-5\underline{i}+3\underline{j}+7\underline{k}) = 0$

Thus $\underline{b}\times\underline{a}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

(iii) $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$, $\underline{b} = -\underline{i} + \underline{j} + 3\underline{k}$

Solution:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 1 \\ -1 & 1 & 3 \end{vmatrix}$$

$= \underline{i}(-6-1) - \underline{j}(6+1) + \underline{k}(2-2)$

$= -7\underline{i} - 7\underline{j}$

$\underline{a}.(\underline{a}\times\underline{b}) = (2\underline{i}-2\underline{j}+\underline{k}).(-7\underline{i}-7\underline{j})$

$\underline{a}.(\underline{a}\times\underline{b}) = -14+14 = 0 \Rightarrow \underline{a}\perp\underline{a}\times\underline{b}$

$\underline{b}.(\underline{a}\times\underline{b}) = (-\underline{i}+\underline{j}+3\underline{k}).(-7\underline{i}-7\underline{j})$

$= 0 \Rightarrow \underline{b}\perp\underline{a}\times\underline{b}$

Thus $\underline{a}\times\underline{b}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 1 & 3 \\ 2 & -2 & 1 \end{vmatrix}$$

$= \underline{i}(1+6) - \underline{j}(-1-6) + \underline{k}(2-2)$

$= 7\underline{i} + 7\underline{j}$ $\underline{a}.(\underline{b}\times\underline{a}) = (2\underline{i}-2\underline{j}+\underline{k}).(7\underline{i}+7\underline{j})$

$\underline{a}.(\underline{b}\times\underline{a}) = 14-14 = 0 \Rightarrow \underline{a}\perp\underline{b}\times\underline{a}$

$\underline{b}.(\underline{b}\times\underline{a}) = (-\underline{i}+\underline{j}+3\underline{k}).(7\underline{i}+7\underline{j})$

$\underline{b}.(\underline{b}\times\underline{a}) = -7+7 = 0 \Rightarrow \underline{b}\perp\underline{b}\times\underline{a}$

Thus $\underline{b}\times\underline{a}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

(iv) $\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$

Solution:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix}$$

$= \underline{i}(1+2) - \underline{j}(-4+4) + \underline{k}(-4-2)$

$= 3\underline{i} - 6\underline{k}$

$\underline{a}.(\underline{a}\times\underline{b}) = (-4\underline{i}+\underline{j}-2\underline{k}).(3\underline{i}-6\underline{k})$

$= (-4)(3) + 0 + (-2)(-6) = 0$

$\Rightarrow \quad \underline{a}\perp\underline{a}\times\underline{b}$

$\underline{b}.(\underline{a}\times\underline{b}) = (2\underline{i}+\underline{j}+\underline{k}).(3\underline{i}-6\underline{k})$

$= (2)(3) + (1)(0) + (1)(-6)$

$= 6+0-6=0 \Rightarrow \underline{b} \perp \underline{a}\times\underline{b}$

Thus $\underline{a}\times\underline{b}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ -4 & 1 & -2 \end{vmatrix}$$

$= \underline{i}(-2-1)-\underline{j}(-4+4)+\underline{k}(2+4)$

$= -3\underline{i}+6\underline{k}$

$\underline{a}.(\underline{b}\times\underline{a}) = (-4\underline{i}+\underline{j}-2\underline{k}).(-3\underline{i}+6\underline{k}) = (-4)(-3)+1(0)+(-2)(6)=0$

$\Rightarrow \quad \underline{a} \perp \underline{b}\times\underline{a}$

$\underline{b}.(\underline{b}\times\underline{a}) = (2\underline{i}+\underline{j}+\underline{k}).(-3\underline{i}+6\underline{k})$

$= (2)(-3)+1(0)+(1)(6)=0$

$\Rightarrow \quad \underline{b} \perp \underline{b}\times\underline{a}$

Thus $\underline{b}\times\underline{a}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$

Q.2 Find a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$. Also find sine of the angle between them.

(i) $\underline{a} = \underline{i}+6\underline{j}-3\underline{k}$, $\underline{b} = 2\underline{i}+\underline{j}+3\underline{k}$

Solution:

$\underline{a} = \underline{i}+6\underline{j}-3\underline{k}$, $\underline{b} = 2\underline{i}+\underline{j}+3\underline{k}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 6 & -3 \\ 2 & 1 & 3 \end{vmatrix}$$

$= \underline{i}(18+3)-\underline{j}(3+6)+\underline{k}(1-12)$

$= 21\underline{i}-9\underline{j}-11\underline{k}$

$|\underline{a}\times\underline{b}| = |21\underline{i}-9\underline{j}-11\underline{k}|$

$= \sqrt{(21)^2+(-9)^2+(-11)^2}$

$= \sqrt{441+81+121} = \sqrt{643}$

Let $\hat{n}$ is unit vector

$$\hat{n} = \frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \frac{21\underline{i}-9\underline{j}-11\underline{k}}{\sqrt{643}}$$

$$\hat{n} = \frac{21}{\sqrt{643}}\underline{i} - \frac{9}{\sqrt{643}}\underline{j} - \frac{11}{\sqrt{643}}\underline{k}$$

$$\Rightarrow \sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$= \frac{\sqrt{643}}{\sqrt{(1)^2+(6)^2+(-3)^2}\sqrt{(2)^2+(1)^2+(3)^2}}$$

$$= \frac{\sqrt{643}}{\sqrt{1+36+9}\sqrt{4+1+9}} = \frac{\sqrt{643}}{\sqrt{644}}$$

$$\sin\theta = \frac{\sqrt{643}}{\sqrt{644}}$$

(ii) $\underline{a}=-\underline{i}-\underline{j}-\underline{k}$, $\underline{b}=2\underline{i}-3\underline{j}+4\underline{k}$

Solution:

$\underline{a}=-\underline{i}-\underline{j}-\underline{k}$, $\underline{b}=2\underline{i}-3\underline{j}+4\underline{k}$

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ -1 & -1 & -1\\ 2 & -3 & 4\end{vmatrix}$$

$=\underline{i}(-4-3)-\underline{j}(-4+2)+\underline{k}(3+2)$

$=-7\underline{i}+2\underline{j}+5\underline{k}$

$|\underline{a}\times\underline{b}|=\sqrt{(-7)^2+(2)^2+(5)^2}$

$=\sqrt{49+4+25}=\sqrt{78}$

Let $\hat{n}$ is unit vector

$$\hat{n}=\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}=\frac{-7\underline{i}+2\underline{j}+5\underline{k}}{\sqrt{78}}$$

$$\hat{n}=\frac{-7}{\sqrt{78}}\underline{i}+\frac{2}{\sqrt{78}}\underline{j}+\frac{5}{\sqrt{78}}\underline{k}$$

$$\sin\theta=\frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$=\frac{\sqrt{78}}{\sqrt{1+1+1}\sqrt{4+9+16}}$$

$$=\frac{\sqrt{78}}{\sqrt{3}\sqrt{29}}=\frac{\sqrt{78}}{\sqrt{87}}$$

(iii) $\underline{a}=\underline{i}+\underline{j}+\underline{k}$, $\underline{b}=\underline{i}-\underline{j}-\underline{k}$

Solution:

$\underline{a}=\underline{i}+\underline{j}+\underline{k}$, $\underline{b}=\underline{i}-\underline{j}-\underline{k}$

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 1 & 1\\ 1 & -1 & -1\end{vmatrix}$$

$=\underline{i}(-1+1)-\underline{j}(-1-1)+\underline{k}(-1-1)$

$=2\underline{j}-2\underline{k}$

$=\sqrt{(2)^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}$

$|\underline{a}\times\underline{b}|=2\sqrt{2}$

Let $\hat{n}$ is unit vector.

$$\hat{n}=\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}=\frac{2\underline{j}-2\underline{k}}{2\sqrt{2}}=\frac{\underline{j}-\underline{k}}{\sqrt{2}}$$

$$\sin\theta=\frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$=\frac{2\sqrt{2}}{\sqrt{(1)^2+(1)^2+(1)^2}\sqrt{(1)^2+(-1)^2+(-1)^2}}$$

$$=\frac{2\sqrt{2}}{\sqrt{3}\sqrt{3}}$$

$$\sin\theta = \frac{2\sqrt{2}}{3}$$

(iv) $\underline{a} = 5\underline{i} + \underline{j} - 3\underline{k}$, $\underline{b} = -2\underline{i} + 4\underline{j} + \underline{k}$

Solution:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 1 & -3 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= \underline{i}(1+12) - \underline{j}(5-6) + \underline{k}(20+2) = 13\underline{i} + \underline{j} + 22\underline{k}$$

$$|\underline{a}\times\underline{b}| = |13\underline{i} + \underline{j} + 22\underline{k}|$$

$$\sqrt{169+1+484} = \sqrt{654}$$

Let $\hat{n}$ is unit vector

$$\hat{n} = \frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \frac{13\underline{i} + \underline{j} + 22\underline{k}}{\sqrt{654}}$$

$$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|} = \frac{\sqrt{654}}{\sqrt{25+1+9}\sqrt{4+16+1}}$$

$$= \frac{\sqrt{654}}{\sqrt{35}\sqrt{21}} = \sqrt{\frac{654}{735}}$$

$$\sin\theta = \frac{\sqrt{654}}{\sqrt{735}}$$

Q.3 Find the area of the triangle, formed by the points *P*, *Q* and *R*.

(i) $P(2,3,5)$, $Q(1,2,0)$; $R(4,1,2)$

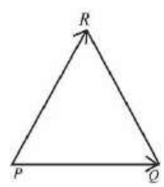
Solution:

$P(2,3,5)$, $Q(1,2,0)$; $R(4,1,2)$

$$\overrightarrow{PQ} = (\underline{i} + 2\underline{j}) - (2\underline{i} + 3\underline{j} + 5\underline{k})$$

$$= -\underline{i} - \underline{j} - 5\underline{k}$$

$$\overrightarrow{PR} = (4\underline{i} + \underline{j} + 2\underline{k}) - (2\underline{i} + 3\underline{j} + 5\underline{k}) = 2\underline{i} - 2\underline{j} - 3\underline{k}$$



Now $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -5 \\ 2 & -2 & -3 \end{vmatrix}$

$$= \underline{i}(3-10) - \underline{j}(3+10) + \underline{k}(2+2)$$

$$= -7\underline{i} - 13\underline{j} + 4\underline{k}$$

$$|\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{(-7)^2 + (-13)^2 + (4)^2}$$

$$= \sqrt{49+169+16} = \sqrt{234}$$

$\therefore$ Area of triangle *PQR* is

$$\Delta = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}(\sqrt{234}) = \frac{3\sqrt{26}}{2}\text{ sq.units}$$

(ii) $P(0,0,1),\ Q(2,-1,2);\ R(-1,3,2)$

Solution:

$$\overrightarrow{PQ} = (2\underline{i} - \underline{j} + 2\underline{k}) - (0\underline{i} + 0\underline{j} + \underline{k})$$
$$= 2\underline{i} - \underline{j} + \underline{k}$$
$$\overrightarrow{PR} = (-\underline{i} + 3\underline{j} + 2\underline{k}) - (0\underline{i} + 0\underline{j} + \underline{k})$$
$$= -\underline{i} + 3\underline{j} + \underline{k}$$
$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ -1 & 3 & 1 \end{vmatrix}$$
$$= \underline{i}(-1-3 - \underline{j}(2+1) + \underline{k}(6-1)$$
$$= -4\underline{i} - 3\underline{j} + 5\underline{k}$$
$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \left|-4\underline{i} - 3\underline{j} + 5\underline{k}\right| = \sqrt{16+9+25}$$
$$= \sqrt{50} = 5\sqrt{2}$$
$$= \sqrt{(-3)^2 + (-3)^2 + (3)^2} = \sqrt{9+9+9}$$
$$= \sqrt{27} = 3\sqrt{3}$$

Area of triangle PQR is

$$\Delta = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}(5\sqrt{2})\text{ sq. units}$$

Q.4 Find the area of parallelogram, whose vertices are:

(i) $A(1,1,1), B(4,2,3), C(5,6,7), D(2,5,5)$

Solution:

$$\overrightarrow{AB} = (4\underline{i} + 2\underline{j} + 3\underline{k}) - (\underline{i} + \underline{j} + \underline{k})$$
$$= 3\underline{i} + \underline{j} + 2\underline{k}$$
$$\overrightarrow{AC} = (5\underline{i} + 6\underline{j} + 7\underline{k}) - (\underline{i} + \underline{j} + \underline{k})$$
$$= 4\underline{i} + 5\underline{j} + 6\underline{k}$$
$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & 2 \\ 4 & 5 & 6 \end{vmatrix}$$
$$= \underline{i}(6-10) - \underline{j}(18-8) + \underline{k}(15-4)$$
$$= -4\underline{i} - 10\underline{j} + 11\underline{k}$$

Area of parallelogram $ABCD$ is $\Delta = \left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$

$$= \sqrt{(-4)^2 + (-10)^2 + (11)^2} = \sqrt{16+100+121}$$
$$= \sqrt{237}\text{ sq. units}$$

(ii) $A(4,5,6), B(1,3,2), C(-2,0,1), D(1,2,5)$

Solution:

$$\overrightarrow{AB} = (\underline{i} + 3\underline{j} + 2\underline{k}) - (4\underline{i} + 5\underline{j} + 6\underline{k})$$
$$= -3\underline{i} - 2\underline{j} - 4\underline{k}$$

$\overrightarrow{AC} = \left(-2\underline{i} + 0\underline{j} + \underline{k}\right) - \left(4\underline{i} + 5\underline{j} + 6\underline{k}\right)$

$= -6\underline{i} - 5\underline{j} - 5\underline{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -3 & -2 & -4 \\ -6 & -5 & -5 \end{vmatrix}$$

$= \underline{i}(10-20) - \underline{j}(15-24) + \underline{k}(15-12)$

$= -10\underline{i} + 9\underline{j} + 3\underline{k}$

Area of parallelogram $ABCD$ is

$\Delta = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-10)^2 + (9)^2 + (3)^2}$

$= \sqrt{100+81+9} = \sqrt{190}$ sq. units

Q.5 If the cross product of the vectors $\underline{u} = 7\underline{i} - 4\underline{j} + 5\underline{k}$ and $\underline{v} = a\underline{i} - b\underline{j} + 3\underline{k}$ is zero, then find the values of a and b.

Solution:

Given that $\underline{u} \times \underline{v} = 0$

$$\Rightarrow \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 7 & -4 & 5 \\ a & -b & 3 \end{vmatrix} = 0$$

$\underline{i}\left(-12+5b\right) - \underline{j}\left(21-5a\right) + \underline{k}\left(-7b+4a\right) = 0$

Gives $-12+5b = 0 \Rightarrow b = \dfrac{12}{5}$

And $-\left(21-5a\right) = 0 \Rightarrow a = \dfrac{21}{5}$

Q.6 Which vectors, if any, are perpendicular or parallel.

(i) $\underline{u} = 5\underline{i} - \underline{j} + \underline{k}$; $\underline{v} = \underline{j} - 5\underline{k}$,

$\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

Solution:

$\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k} = -3(5\underline{i} - \underline{j} + \underline{k}) = -3\underline{u}$

$\underline{u}$ and $\underline{w}$ are parallel

(ii) $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$, $\underline{v} = -\underline{i} + \underline{j} + \underline{k}$,

$\underline{w} = -\dfrac{\pi}{2}\underline{i} - \pi\underline{j} + \dfrac{\pi}{2}\underline{k}$

Solution:

$\underline{u}.\underline{v} = (\underline{i} + 2\underline{j} - \underline{k}).(-\underline{i} + \underline{j} + \underline{k})$

$= (1)(-1) + 2(1) - (1)(1) = -1 + 2 - 1 = 0$

The vectors $\underline{u}$ and $\underline{v}$ are perpendicular

$\underline{v}.\underline{w} = (-\underline{i} + \underline{j} + \underline{k}).\left(\dfrac{-\pi}{2}\underline{i} - \pi\underline{j} + \dfrac{\pi}{2}\underline{k}\right) = \dfrac{\pi}{2} - \pi + \dfrac{\pi}{2} = 0$

The vectors $\underline{v}$ and $\underline{w}$ are perpendicular

Q.7 Use definition of cross product, for any vectors $\underline{u}, \underline{v}, \underline{w}$ and scalar k, prove that

(i) $\underline{u} \times \left(-\underline{u}\right) = 0$

Solution:

Let $\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$ then $-\underline{u} = -u_1\underline{i} - u_2\underline{j} - u_3\underline{k}$. Taking cross product, we have

$$\underline{u} \times \left(-\underline{u}\right) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ -u_1 & -u_2 & -u_3 \end{vmatrix} = \underline{i}\left(-u_2u_3 + u_2u_3\right) - \underline{j}\left(-u_1u_3 + u_1u_3\right) + \underline{k}\left(-u_1u_2 + u_1u_2\right)$$

$= 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}$

(ii) $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$

Solution:

Let $\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$ and $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$ then

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \underline{i}(u_2v_3 - u_3v_2) - \underline{j}(u_1v_3 - u_3v_1) + \underline{k}(u_1v_2 - u_2v_1) \ldots\ldots(1)$$

$$\underline{v} \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \underline{i}(u_3v_2 - u_2v_3) - \underline{j}(u_3v_1 - u_1v_3) + \underline{k}(u_2v_1 - u_1v_2) \ldots\ldots(2)$$

Equation (2) can be written as

$$-\underline{v} \times \underline{u} = -\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \underline{i}(u_2v_3 - u_3v_2) - \underline{j}(u_1v_3 - u_3v_1) + \underline{k}(u_1v_2 - u_2v_1) \ldots\ldots(3)$$

From equations (1) and (3), we can conclude that $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$

(iii) $\underline{u} \times (k\underline{v}) = (k\underline{u}) \times \underline{v} = k(\underline{u} \times \underline{v})$

Solution:

Since k is a scalar, so it can't be the part of cross product since cross product can only be done when both are vectors. So it's clear by definition of cross product that $\underline{u} \times (k\underline{v}) = (k\underline{u}) \times \underline{v} = k(\underline{u} \times \underline{v})$

(iv) $\underline{u} \times (\underline{v} + \underline{w}) = (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w})$

Solution:

Using **distributive property of cross product**, we can deduce the above result.

Q.8 Prove that $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = \underline{0}$

Solution:

L.H.S = $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$

$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{b} \times \underline{a} + \underline{c} \times \underline{a} + \underline{c} \times \underline{b}$

$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + (-(\underline{a} \times \underline{b})) + (-(\underline{a} \times \underline{c})) + (-(\underline{b} \times \underline{c})) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} - \underline{a} \times \underline{b} - \underline{a} \times \underline{c} - \underline{b} \times \underline{c}$

= $\underline{0}$ = R.H.S

Q.9 If $\underline{a} + \underline{b} + \underline{c} = 0$**, then prove that** $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

$\underline{a} + \underline{b} + \underline{c} = 0$

Taking cross product with $\underline{a}$

$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = 0 \Rightarrow \underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0$ $\underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0$

$0 + \underline{a} \times \underline{b} - (\underline{c} \times \underline{a}) = 0$, $\underline{a} \times \underline{b} = \underline{c} \times \underline{a}$ …(i)

Taking cross product with $\underline{b}$

$\underline{b}\times(\underline{a}+\underline{b}+\underline{c})=0$, $\underline{b}\times\underline{a}+\underline{b}\times\underline{b}+\underline{b}\times\underline{c}=0$

$-\underline{a}\times\underline{b}+0+\underline{b}\times\underline{c}=0$, $\underline{b}\times\underline{c}=\underline{a}\times\underline{b}\ldots$(ii)

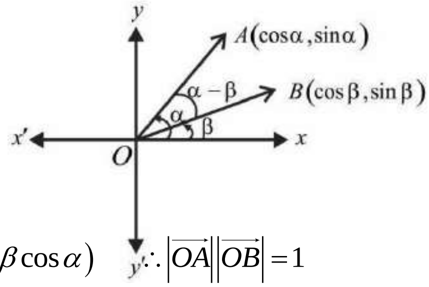
From (i) and (ii), $\underline{a}\times\underline{b}=\underline{b}\times\underline{c}=\underline{c}\times\underline{a}$

Q.10 Prove that $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

Solution:

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are unit vectors in xy-plane making angle α and β with the positive $x-axis$ respectively.

So that $m\angle BOA=\alpha-\beta$



Now $\overrightarrow{OA}=\cos\alpha\,\underline{i}+\sin\alpha\,\underline{j}$, $\overrightarrow{OB}=\cos\beta\,\underline{i}+\sin\beta\,\underline{j}$,

$$\overrightarrow{OB}\times\overrightarrow{OA}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \cos\beta & \sin\beta & 0\\ \cos\alpha & \sin\alpha & 0\end{vmatrix}$$

$\left|\overrightarrow{OB}\right|\left|\overrightarrow{OA}\right|\sin(\alpha-\beta)\underline{k}=\underline{i}(0-0)-\underline{j}(0-0)+\underline{k}(\cos\beta\sin\alpha-\sin\beta\cos\alpha)$ $\because\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|=1$

$\sin(\alpha-\beta)\hat{k}=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\hat{k}\Rightarrow\sin(\alpha-\beta)=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$

Q. 11 Show that $\left|\underline{a}\times\underline{b}\right|^2=\left|\underline{a}\right|^2\left|\underline{b}\right|^2-(\underline{a}.\underline{b})^2$

Solution:

We know that $\left|\underline{a}\times\underline{b}\right|=\left|\underline{a}\right|\left|\underline{b}\right|\sin\theta$ where θ is the angle between the given vectors.

LHS $=\left|\underline{a}\times\underline{b}\right|^2$

$=\left(\left|\underline{a}\right|\left|\underline{b}\right|\sin\theta\right)^2=\left|\underline{a}\right|^2\left|\underline{b}\right|^2\sin^2\theta$

$=\left|\underline{a}\right|^2\left|\underline{b}\right|^2\left(1-\cos^2\theta\right)\left(\because\sin^2\theta=1-\cos^2\theta\right)$ $=\left|\underline{a}\right|^2\left|\underline{b}\right|^2-\left|\underline{a}\right|^2\left|\underline{b}\right|^2\cos^2\theta$

$=\left|\underline{a}\right|^2\left|\underline{b}\right|^2-\left(\left|\underline{a}\right|\left|\underline{b}\right|\cos\theta\right)^2$

$=\left|\underline{a}\right|^2\left|\underline{b}\right|^2-(\underline{a}.\underline{b})^2\left(\because\underline{a}.\underline{b}=\left|\underline{a}\right|\left|\underline{b}\right|\cos\theta\right)$

= RHS

Q.12 Use definition of cross product, prove that for any vectors $\underline{u}$ and $\underline{v}$

$(\underline{u}+\underline{v})\times(\underline{u}-\underline{v})=-2(\underline{u}\times\underline{v})$

Solution:

Here LHS = $(\underline{u}+\underline{v})\times(\underline{u}-\underline{v})$

$=\underline{u}\times(\underline{u}-\underline{v})+\underline{v}\times(\underline{u}-\underline{v})$

$=\underline{u}\times\underline{u}-\underline{u}\times\underline{v}+\underline{v}\times\underline{u}-\underline{v}\times\underline{v}$ (Distributive property of cross product)

$\underline{0}-\underline{u}\times\underline{v}-\underline{u}\times\underline{v}-\underline{0}$ since $\underline{u}\times\underline{u}=0$ and $\underline{v}\times\underline{u}=-\underline{u}\times\underline{v}$

$\Rightarrow-2(\underline{u}\times\underline{v})=RHS$

Q.13 Find the moment about the point $M(1,-3,3)$ of the force represented by $\overrightarrow{AB}$ where the coordinates of points $A(4,3,-1)$ and $B(-1,3,7)$ are given.

Solution:

Here $\overrightarrow{AB}=(-1-4)\,\underline{i}+(3-3)\,\underline{j}+(7+1)\,\underline{k}$

$\overrightarrow{AB}=-5\underline{i}+8\underline{k}$

And given that $\underline{r}=\overrightarrow{MA}=\overrightarrow{OA}-\overrightarrow{OM}=(4-1)\,\underline{i}+(3+3)\,\underline{j}+(-1-3)\,\underline{k}$ $\underline{r}=3\underline{i}+6\underline{j}-4\underline{k}$

So moment of the force is given by $\underline{M} = \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 6 & -4 \\ -5 & 0 & 8 \end{vmatrix}$

$\underline{M} = \underline{i}(48+0) - \underline{j}(24-20) + \underline{k}(0+30)$

$\underline{M} = 48\underline{i} - 4\underline{j} + 30\underline{k}$

Q.14 A force $\vec{F} = 6\underline{i} + 4\underline{j} - 4\underline{k}$ is applied at the point $A(1,-1,2)$. Find the moment of the force about the point $B(3,-2,3)$.

Solution:

Here $\vec{F} = 6\underline{i} + 4\underline{j} - 4\underline{k}$ and $\underline{r} = \overrightarrow{BA} = (1-3)\underline{i} + (-1+2)\underline{j} + (2-3)\underline{k}$

$\underline{r} = -2\underline{i} + \underline{j} - \underline{k}$.

So moment of the force is given by $\underline{M} = \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2 & 1 & -1 \\ 6 & 4 & -4 \end{vmatrix}$

$\underline{M} = \underline{i}(-4+4) - \underline{j}(8+6) + \underline{k}(-8-6)$

$\underline{M} = -14\underline{j} - 14\underline{k}$

Q.15 Given a force $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ acting at a point $A(1,-2,1)$. Find the moment of $\vec{F}$ about the point $B(2,0,-2)$.

Solution:

Here $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ and $\underline{r} = \overrightarrow{BA} = (1-2)\underline{i} + (-2-0)\underline{j} + (1+2)\underline{k}$

$\underline{r} = -\underline{i} - 2\underline{j} + 3\underline{k}$

So moment of the force is given by $\underline{M} = \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$

$\underline{M} = \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(-1+4)$

$\underline{M} = 3\underline{i} + 3\underline{j} + 3\underline{k}$

Q.16 A force $\underline{F} = -2\underline{i} + \underline{j} - 3\underline{k}$ is applied at $P(-1,-3,2)$. Find its moment about the point $Q(4,2,2)$.

Solution:

Here $\underline{F} = -2\underline{i} + \underline{j} - 3\underline{k}$ and $\underline{r} = \overrightarrow{QP} = (-1-4)\underline{i} + (-3-2)\underline{j} + (2-2)\underline{k}$ $\underline{r} = -5\underline{i} - 5\underline{j}$

So moment of the force is given by $\underline{M} = \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -5 & -5 & 0 \\ -2 & 1 & -3 \end{vmatrix}$

$\underline{M} = \underline{i}(15-0) - \underline{j}(15+0) + \underline{k}(-5-10)$

$\underline{M} = 15\underline{i} - 15\underline{j} - 15\underline{k}$