EXERCISE 14.3

1. Compute the cross product $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$. Check your answer by showing that each $\underline{a}$ and $\underline{b}$ are perpendicular to $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$.

(i) $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\underline{a}\times\underline{b} = \underline{i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$\underline{a}\times\underline{b} = \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1)$

$\underline{a}\times\underline{b} = 0\underline{i} - 3\underline{j} - 3\underline{k}$

$\underline{b}\times\underline{a} = -(\underline{a}\times\underline{b})$

$\underline{b}\times\underline{a} = -(0\underline{i} - 3\underline{j} - 3\underline{k})$

$\underline{b}\times\underline{a} = 0\underline{i} + 3\underline{j} + 3\underline{k}$

$\underline{a}.(\underline{a}\times\underline{b}) = (2\underline{i} + \underline{j} - \underline{k}).(0\underline{i} - 3\underline{j} - 3\underline{k})$

$\underline{a}.(\underline{a}\times\underline{b}) = 0 - 3 + 3$

$\underline{a}.(\underline{a}\times\underline{b}) = 0$

$\underline{a} \perp \underline{a}\times\underline{b}$

$\underline{a}.(\underline{b}\times\underline{a}) = (2\underline{i} + \underline{j} - \underline{k}).(0\underline{i} + 3\underline{j} + 3\underline{k})$

$\underline{a} \cdot (\underline{b} \times \underline{a}) = 0 + 3 - 3$

$\underline{a} \cdot (\underline{b} \times \underline{a}) = 0$

$a \perp \underline{b} \times \underline{a}$

$\underline{b} \cdot (\underline{a} \times \underline{b}) = (\underline{i} - \underline{j} + \underline{k}) \cdot (0\underline{i} - 3\underline{j} - 3\underline{k})$

$\underline{b} \cdot (\underline{a} \times \underline{b}) = 0 + 3 - 3$

$\underline{b} \cdot (\underline{a} \times \underline{b}) = 0$

$b \perp \underline{a} \times \underline{b}$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = (\underline{i} - \underline{j} + \underline{k}) \cdot (0\underline{i} + 3\underline{j} + 3\underline{k})$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = 0 - 3 + 3$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = 0$

$b \perp \underline{b} \times \underline{a}$

(ii) $\underline{a} = \underline{i} + 3\underline{j} + 2\underline{k}$, $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{vmatrix}$$

$$\underline{a} \times \underline{b} = \underline{i} \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}$$

$\underline{a} \times \underline{b} = \underline{i}(3+2) - \underline{j}(1-4) + \underline{k}(-1-6)$

$\underline{a} \times \underline{b} = 5\underline{i} + 3\underline{j} - 7\underline{k}$

$\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$

$\underline{b} \times \underline{a} = -(5\underline{i} + 3\underline{j} - 7\underline{k})$

$\underline{b} \times \underline{a} = -5\underline{i} - 3\underline{j} + 7\underline{k}$

$\underline{a} \cdot (\underline{a} \times \underline{b}) = (\underline{i} + 3\underline{j} + 2\underline{k}) \cdot (+5\underline{i} + 3\underline{j} - 7\underline{k})$

$\underline{a} \cdot (\underline{a} \times \underline{b}) = 5 + 9 - 14$

$\underline{a} \cdot (\underline{a} \times \underline{b}) = 0$

$a \perp \underline{a} \times \underline{b}$

$\underline{a} \cdot (\underline{b} \times \underline{a}) = (\underline{i} + 3\underline{j} + 2\underline{k}) \cdot (-5\underline{i} - 3\underline{j} + 7\underline{k})$

$\underline{a} \cdot (\underline{b} \times \underline{a}) = -5 - 9 + 14$

$\underline{a} \cdot (\underline{b} \times \underline{a}) = 0$

$a \perp \underline{b} \times \underline{a}$

$\underline{b} \cdot (\underline{a} \times \underline{b}) = (2\underline{i} - \underline{j} + \underline{k}) \cdot (5\underline{i} + 3\underline{j} - 7\underline{k})$

$\underline{b} \cdot (\underline{a} \times \underline{b}) = 10 - 3 - 7$

$b \cdot (\underline{a} \times \underline{b}) = 0$

$b \perp a \times \underline{b}$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} - \underline{j} + \underline{k}) (-5\underline{i} - 3\underline{j} + 7\underline{k})$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = -10 + 3 + 7$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = 0$

(iii) $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$, $\underline{b} = -\underline{i} + \underline{j} + 3\underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 1 \\ -1 & 1 & 3 \end{vmatrix}$$

$$\underline{a} \times \underline{b} = \underline{i} \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix}$$

$\underline{a} \times \underline{b} = \underline{i}(-6-1) - \underline{j}(6+1) + \underline{k}(2-2)$

$\underline{a} \times \underline{b} = 7\underline{i} + 7\underline{j} + 0.\underline{k}$

$\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$

$\underline{b} \times \underline{a} = -(7\underline{i} + 7\underline{j} + 0\underline{k})$

$\underline{b} \times \underline{a} = -7\underline{i} - 7\underline{j} - 0k$

$\underline{a}.(\underline{a} \times \underline{b}) = (2\underline{i} - 2\underline{j} + \underline{k}).(7\underline{i} + 7\underline{j} + 0\underline{k})$

$\underline{a}.(\underline{a} \times \underline{b}) = 14 - 14 + 0$

$\underline{a}.(\underline{a} \times \underline{b}) = 0$

$\underline{a} \perp \underline{a} \times \underline{b}$

$\underline{a}.(\underline{b} \times \underline{a}) = (-\underline{i} + \underline{j} + 3\underline{k}).(-7\underline{i} - 7\underline{j} - 0k)$

$\underline{a}.(\underline{b} \times \underline{a}) = 7 + 7 + 0$

$\underline{a}.(\underline{b} \times \underline{a}) = 0$

$a \perp \underline{b} \times \underline{a}$

$\underline{b}.(\underline{a} \times \underline{b}) = (-\underline{i} + \underline{j} + 3\underline{k}).(7\underline{i} + 7\underline{j} + 0\underline{k})$

$\underline{b}.(\underline{a} \times \underline{b}) = 7 + 7 + 0$

$\underline{b}.(\underline{a} \times b) = 0$

$b \perp \underline{a} \times \underline{b}$

$\underline{b}.(\underline{b} \times \underline{a}) = (-\underline{i} + \underline{j} + 3\underline{k}).(-7\underline{i} - 7\underline{j} - 0k)$

$\underline{b}.(\underline{b} \times \underline{a}) = 7 - 7 - 0$

$\underline{b}.(\underline{b} \times \underline{a}) = 0$

$b \perp \underline{b} \times \underline{a}$

(iv) $\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix}$$

$$\underline{a} \times \underline{b} = \underline{i} \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} -4 & -2 \\ 2 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} -4 & 1 \\ 2 & 1 \end{vmatrix}$$

$\underline{a} \times \underline{b} = \underline{i}(1+2) - \underline{j}(-4+4) + k(-4-2)$

$\underline{a} \times \underline{b} = 3\underline{i} - 0\underline{j} - 6\underline{k}$

$\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$

$\underline{b} \times \underline{a} = -(3\underline{i} - 0\underline{j} - 6\underline{k})$

$\underline{b} \times \underline{a} = -3\underline{i} + 0\underline{j} + 6\underline{k}$

$\underline{a}.(\underline{a} \times \underline{b}) = (-4\underline{i} + \underline{j} - 2\underline{k}).(3\underline{i} - 0\underline{j} - 6\underline{k})$

$\underline{a}.(\underline{a} \times \underline{b}) = 12 - 0 + 12$

$\underline{a}.(\underline{a} \times \underline{b}) = 0$

$\underline{a} \perp \underline{a} \times \underline{b}$

$\underline{a}.(\underline{b} \times \underline{a}) = (-4\underline{i} + \underline{j} - 2\underline{k}).(-3\underline{i} + 0\underline{j} + 6\underline{k})$

$\underline{a}.(\underline{b} \times \underline{a}) = 12 + 0 - 12$

$\underline{a}.(\underline{b} \times \underline{a}) = 0$

$\underline{a} \perp \underline{b} \times \underline{a}$

$\underline{b}.(\underline{a} \times \underline{b}) = (2\underline{i} + \underline{j} + \underline{k}).(3\underline{i} - 0\underline{j} - 6\underline{k})$

$\underline{b}.(\underline{a} \times \underline{b}) = 6 - 0 - 6$

$\underline{b} \perp \underline{a} \times \underline{b}$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} + \underline{j} + \underline{k}) \cdot (-3\underline{i} + 0\underline{j} + 6\underline{k})$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = -6 + 0 + 6$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = 0$

$\underline{b} \perp \underline{b} \times \underline{a}$

2. Find a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$. Also find sine of the angle between them.

(i) $\underline{a} = \underline{i} + 6\underline{j} - 3\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} + 3\underline{k}$

$$\hat{u} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$

$$\sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 1 & 6 & -3 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= i(18+3) - j(3+6) + k(1-12)$$

$$= 21\underline{i} - 9\underline{j} - 11\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(21)^2 + (-9)^2 + (-11)^2}$$

$$|\underline{a} \times \underline{b}| = \sqrt{441 + 81 + 121}$$

$$= \sqrt{643}$$

$$\hat{u} = \frac{21\underline{i} - 9\underline{j} - 11\underline{k}}{\sqrt{643}}$$

$$\sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|}$$

$$|a| = \sqrt{(1)^2 + (6)^2 + (-3)^2}$$

$$|a| = \sqrt{1 + 36 + 9}$$

$$|\underline{a}| = \sqrt{46}$$

$$|\underline{b}| = \sqrt{(2)^2 + (1)^2 + (3)^2}$$

$$= \sqrt{4 + 1 + 9}$$

$$|\underline{b}| = \sqrt{14}$$

$$\sin\theta = \frac{\sqrt{643}}{\sqrt{46}\sqrt{14}}$$

$$\sin\theta = \frac{\sqrt{643}}{\sqrt{644}}$$

(ii) $\underline{a} = -\underline{i} - \underline{j} - \underline{k}$, $\underline{b} = 2\underline{i} - 3\underline{j} + 4k$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

$$\underline{a} \times \underline{b} = \underline{i}\begin{vmatrix} -1 & -1 \\ -3 & 4 \end{vmatrix} - \underline{j}\begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} + \underline{k}\begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= \underline{i}(-4 - 3) - \underline{j}(-4 + 2) + \underline{k}(3 + 2)$$

$$= -7\underline{i} + 2\underline{j} + 5\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + (2)^2 + (5)^2}$$

$$= \sqrt{49 + 4 + 25}$$

$$= \sqrt{78}$$

$$a \quad \hat{u} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$

$$= \frac{7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}}$$

$$|\underline{a}| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2}$$

$$= \sqrt{1+1+1} = \sqrt{3}$$

$$|\underline{b}| = \sqrt{(2)^2 + (-3)^2 + (4)^2}$$

$$= \sqrt{4+9+16} = \sqrt{29}$$

$$\sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}|\,|\underline{b}|}$$

$$= \frac{\sqrt{78}}{\sqrt{3}\,\sqrt{29}}$$

$$= \frac{\sqrt{78}}{\sqrt{87}}$$

(iii) $\underline{a} = \underline{i} + \underline{j} + \underline{k}$, $\underline{b} = \underline{i} - \underline{j} - \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$\underline{a} \times \underline{b} = \underline{i}\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \underline{i}(-1+1) - \underline{j}(-1-1) + \underline{k}(-1-1)$$

$$= 0\underline{i} + 2\underline{j} - 2\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(0)^2 + (2)^2 + (-2)^2}$$

$$= \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\hat{u} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$

$$= \frac{0\underline{i} + 2\underline{j} - 2\underline{k}}{2\sqrt{2}} = \frac{\cancel{2}(\underline{j} - \underline{k})}{\cancel{2}\sqrt{2}}$$

$$|\underline{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2}$$

$$= \sqrt{1+1+1}$$

$$|\underline{a}| = \sqrt{3}$$

$$|\underline{b}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2}$$

$$|\underline{b}| = \sqrt{1+1+1} = \sqrt{3}$$

$$\sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|}$$

$$= \frac{2\sqrt{2}}{\sqrt{3}\sqrt{3}}$$

$$= \frac{2\sqrt{2}}{\sqrt{9}}$$

$$= \frac{2\sqrt{2}}{3}$$

(iv) $\underline{a} = 5\underline{i} + \underline{j} - 3\underline{k}$, $\underline{b} = -2\underline{i} + 4\underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 1 & -3 \\ -2 & 4 & 1 \end{vmatrix}$$

$$\underline{a} \times \underline{b} = \underline{i} \begin{vmatrix} 1 & -3 \\ 4 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} 5 & -3 \\ -2 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} 5 & 1 \\ -2 & 4 \end{vmatrix}$$

$$= \underline{i}(1+12) - \underline{j}(5-6) + \underline{k}(20+2)$$

$$= 13\underline{i} + \underline{j} + 22\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(13)^2 + (1)^2 + (22)^2}$$

$$= \sqrt{169 + 1 + 484}$$

$$= \sqrt{654}$$

$$\hat{u} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$

$$= \frac{13\underline{i} + \underline{j} + 22\underline{k}}{\sqrt{654}}$$

$$|\underline{a}| = \sqrt{(5)^2 + (1)^2 + (-3)^2}$$

$$= \sqrt{25 + 1 + 9}$$

$$= \sqrt{35}$$

$$|\underline{b}| = \sqrt{(-2)^2 + (4)^2 + (1)^2}$$

$$= \sqrt{4 + 16 + 1} = \sqrt{21}$$

$$\sin\theta = \frac{\sqrt{654}}{\sqrt{35}\sqrt{21}} = \frac{\sqrt{654}}{\sqrt{735}}$$

3. Find the area of the triangle, formed by the P, Q and R.

(i) $P(2,3,5)$; $Q(1,2,0)$; $R(4,1,2)$

$\overline{PQ} = \overline{OQ} - OP$

$= (1\underline{i}+2\underline{j}+0k) - (2\underline{i}+3\underline{j}+5k)$

$= -1\underline{i} - 1\underline{j} - 5\underline{k}$

$\overline{PR} = \overline{OR} - \overline{OP}$

$= (4\underline{i}+\underline{j}+2k) - (2\underline{i}+3\underline{j}+5\underline{k})$

$= 4\underline{i}+\underline{j}+2\underline{k}-2\underline{i}-3\underline{j}-5\underline{k}$

$= 2\underline{i} - 2\underline{j} - 3\underline{k}$

Area of triangle $= \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -1 & -1 & -5 \\ 2 & -2 & -3 \end{vmatrix}$$

$$= i\begin{vmatrix} -1 & -5 \\ -2 & -3 \end{vmatrix} - \underline{j}\begin{vmatrix} -1 & -5 \\ 2 & -3 \end{vmatrix} + \underline{k}\begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix}$$

$= \underline{i}(3-10) - \underline{j}(3+10) + \underline{k}(2+2)$

$= -7\underline{i} - 13\underline{j} + 4\underline{k}$

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-7)^2+(-13)^2+(4)^2}$

$= \sqrt{49+169+16}$

$= \sqrt{234}$

Area of triangle $= \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$$= \frac{1}{2}(\sqrt{234})$$

$$= \frac{\sqrt{234}}{2} = \frac{3\sqrt{26}}{2} \text{ square units}$$

(ii) $P(0,0,1)$; $Q(2,-1,2)$; $R(-1,3,2)$

$$\overline{PQ} = \overline{OQ} - \overline{OP}$$
$$= (2\underline{i} - \underline{j} + 2\underline{k}) - (0\underline{i} + 0\underline{j} + \underline{k})$$
$$= 2\underline{i} - \underline{j} + 2\underline{k} - 0\underline{i} - 0\underline{j} - \underline{k}$$
$$= 2\underline{i} - \underline{j} + \underline{k}$$

$$\overline{PR} = \overline{OR} - \overline{OP}$$
$$= (-\underline{i} + 3\underline{j} + 2\underline{k}) - (0\underline{i} + 0\underline{j} + \underline{k})$$
$$= -\underline{i} + 3\underline{j} + 2\underline{k} - 0\underline{i} - 0\underline{j} - \underline{k}$$
$$= -\underline{i} + 3\underline{j} + \underline{k}$$

$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ -1 & 3 & 1 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= \underline{i}(-1-3) - \underline{j}(2+1) + \underline{k}(6-1)$$

$$= -4\underline{i} - 3\underline{j} + 5\underline{k}$$

$|\overline{PQ} \times \overline{PR}| = \sqrt{(-4)^2 + (-3)^2 + (5)^2}$

$= \sqrt{16 + 9 + 25}$

$= \sqrt{50} = 5\sqrt{2}$

Area of triangle $= \frac{1}{2} |\overline{PQ} \times \overline{PR}|$

$= \frac{1}{2}(5\sqrt{2})$

$= \frac{5\sqrt{2}}{2}$ squar units

4. Find the area of a parallelogram, whose vertices are:

(i) $A(1,1,1); B(4,2,3); C(5,6,7); D(2,5,5)$

$\overline{AB} = \overline{OB} - OA$

$= 4\underline{i} + 2\underline{j} + 3\underline{k} - (\underline{i} + \underline{j} + \underline{k})$

$= 4\underline{i} + 2\underline{j} + 3\underline{k} - \underline{i} - \underline{j} - \underline{k}$

$\overline{AB} = 3\underline{i} + \underline{j} + 2\underline{k}$

$\overline{AC} = \overline{OC} - \overline{OA}$

$= 5\underline{i} + 6\underline{j} + 7\underline{k} - (\underline{i} + \underline{j} + \underline{k})$

$= 5\underline{i} + 6\underline{j} + 7\underline{k} - \underline{i} - \underline{j} - \underline{k}$

$\overline{AC} = 4\underline{i} + 5\underline{j} + 6\underline{k}$

$\overline{AD} = \overline{OD} - \overline{OA}$

$= 2\underline{i} + 5\underline{j} + 5\underline{k} - (\underline{i} + \underline{j} + \underline{k})$

$= 2\underline{i} + 5\underline{j} + 5\underline{k} - \underline{i} - \underline{j} - \underline{k}$

$\overline{AD} = \underline{i} + 4\underline{j} + 4\underline{k}$.

As, $\overline{AB} + \overline{AD} = \overline{AC}$

Area of parallelogram $= |\overline{AB} \times \overline{AD}|$

$$\overline{AB} \times \overline{AD} = \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 1 & 4 & 4 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} 1 & 2 \\ 4 & 4 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} + \underline{k}\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}$$

$= i(4-8) - \underline{j}(12-2) + \underline{k}(12-1)$

$= -4\underline{i} - 10\underline{j} + 11\underline{k}$

$|\overline{AB} \times \overline{AD}| = \sqrt{(-4)^2 + (-10)^2 + (11)^2}$

$= \sqrt{16 + 100 + 121}$

$= \cancel{\boxed{\sqrt{225} = 15}}\ \sqrt{237}$

Area of parallelogram $= \sqrt{237}$ Squar units

(ii) $A(4,5,6); B(1,3,2); C(-2,0,1); D(1,2,5)$

$\overline{AB} = \overline{OB} - \overline{OA}$

$= \underline{i} + 3\underline{j} + 2\underline{k} - (4\underline{i} + 5\underline{j} + 6\underline{k})$

$= \underline{i} + 3\underline{j} + 2\underline{k} - 4\underline{i} - 5\underline{j} - 6\underline{k}$

$= -3\underline{i} - 2\underline{j} - 4\underline{k}$

$\overline{AC} = \overline{OC} - \overline{OA}$

$= -2\underline{i} + 0\underline{j} + \underline{k} - (4\underline{i} + 5\underline{j} + 6\underline{k})$

$= -2\underline{i} + 0\underline{j} + \underline{k} - 4\underline{i} - 5\underline{j} - 6\underline{k}$

$= -6\underline{i} - 5\underline{j} - 5\underline{k}$

$\overline{AD} = \overline{OD} - \overline{OA}$

$= (\underline{i} + 2\underline{j} + 5\underline{k}) - (4\underline{i} + 5\underline{j} + 6\underline{k})$

$= \underline{i} + 2\underline{j} + 5\underline{k} - 4\underline{i} - 5\underline{j} - 6\underline{k}$

$= -3\underline{i} - 3\underline{j} - \underline{k}$

as, $\overline{AB} + \overline{AD} = \overline{AC}$

Area of parallelogram $= |\overline{AB} \times \overline{AD}|$

$$\overline{AB} \times \overline{AD} = \begin{vmatrix} i & j & k \\ -3 & -2 & -4 \\ -3 & -3 & -1 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} -2 & -4 \\ -3 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} -3 & -4 \\ -3 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} -3 & -2 \\ -3 & -3 \end{vmatrix}$$

$= \underline{i}(2 - 12) - \underline{j}(3 - 12) + \underline{k}(9 - 6)$

$= -10\underline{i} + 9\underline{j} + 3\underline{k}.$

$|\overline{AB} \times \overline{AD}| = \sqrt{(-10)^2 + (9)^2 + (3)^2}$

$= \sqrt{100 + 81 + 9}$

$= \sqrt{190}$

Area of parallelogram $= \sqrt{190}$ Squar units

5. If the cross product of two vectors $\underline{u} = 7\underline{i} - 4\underline{j} + 5\underline{k}$ and $\underline{v} = a\underline{i} - b\underline{j} + 3\underline{k}$ is zero, then find the value of a and b.

$\underline{u} \times \underline{v} = 0$

$$\begin{vmatrix} i & j & k \\ 7 & -4 & 5 \\ a & b & 3 \end{vmatrix} = 0$$

$$\underline{i}\begin{vmatrix} -4 & 5 \\ b & 3 \end{vmatrix} - \underline{j}\begin{vmatrix} 7 & 5 \\ a & 3 \end{vmatrix} + \underline{k}\begin{vmatrix} 7 & -4 \\ a & b \end{vmatrix} = 0$$

$i(-12 - 5b) - \underline{j}(21 - 5a) + \underline{k}(7b - 4a) = 0$

$-12 - 5b = 0$	$21 - 5a = 0$	$7b - 4a = 0$
$-5b = 12$	$+5a = +21$	
$\boxed{b = -\frac{12}{5}}$	$\boxed{a = \frac{21}{5}}$	

6. Which vectors, if any, are perpendicular or parallel.

(i) $\underline{u} = 5\underline{i} - \underline{j} + \underline{k}$; $\underline{v} = \underline{j} - 5\underline{k}$; $\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

$\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

$\underline{w} = -3(5\underline{i} - \underline{j} + \underline{k})$

$\underline{w} = -3\underline{u}$

$\underline{w} \parallel \underline{u}$

$\underline{u}.\underline{v} = \cancel{8} - 1 - 5 \neq 0$

$u \not\perp v$

$v.w = 0 + 3 + 15 \neq 0$

$v \not\perp w$

(ii) $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$; $\underline{v} = -\underline{i} + \underline{j} + \underline{k}$; $\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

$\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

$w = -\frac{\pi}{2}(\underline{i} + 2\underline{j} - \underline{k})$

$\underline{w} = -\pi/2\ \underline{u}$

$\underline{w} \parallel \underline{u}$

$\underline{u}.\underline{v} = -1 + 2 - 1 = -2 + 2 = 0$

$\underline{u} \perp \underline{v}$

$\underline{v}.\underline{w} = \frac{\pi}{2} - \pi + \frac{\pi}{2} = \frac{\pi - 2\pi + \pi}{2} = \frac{2\pi - 2\pi}{2} = \frac{0}{2}$

$\underline{v}.\underline{w} = \frac{\cancel{2}(\pi - \pi)}{\cancel{2}} = 0$

$\underline{v} \perp \underline{w}$

7. Use the definition of cross product, for any vectors $\underline{u}, \underline{v}, \underline{w}$ and scalar k, prove that

(i) $\underline{u} \times (-\underline{u}) = 0$

Let $u = [u_1 \ \ u_2 \ \ u_3]$

$$-u = [-u_1 \quad -u_2 \quad -u_3]$$

$$u \times (-u) = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ -u_1 & -u_2 & -u_3 \end{vmatrix}$$

$$= -\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

Since R_2 and R_3 are identical, so

$$\underline{u} \times (-\underline{u}) = 0$$

(ii) $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$

Let

$$\underline{u} = [u_1 \quad u_2 \quad u_3]$$

$$\underline{v} = [v_1 \quad v_2 \quad v_3]$$

$$\underline{u} \times \underline{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= -\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$u \times \underline{v} = -(\underline{v} \times u)$$

(iii) $\underline{u} \times (k\underline{v}) = (k\underline{u}) \times \underline{v} = k(\underline{u} \times \underline{v})$

$$\underline{u} \times (k\underline{v}) = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ kv_1 & kv_2 & kv_3 \end{vmatrix}$$

$$= k \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ ku_1 & ku_2 & ku_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= ku \times v$$

$$= k \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= k(\underline{u} \times \underline{v})$$

(iv.) $\underline{u} \times (\underline{v} + \underline{w}) = (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w})$

$\underline{u} = [u_1 \quad u_2 \quad u_3]$

$\underline{v} = [v_1 \quad v_2 \quad v_3]$

$\underline{w} = [w_1 \quad w_2 \quad w_3]$

$$\underline{u} \times (\underline{v} + \underline{w}) = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1+w_1 & v_2+w_2 & v_3+w_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$= (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w})$

8. Prove that

$\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = 0$

$= \underline{a} \times (\underline{b} + c) + \underline{b} \times (\underline{c} + \underline{a}) + c \times (a + b)$

$= \underline{a} \times b + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{b} \times \underline{a} + \underline{c} \times \underline{a} + c \times \underline{b}$

$= \underline{a} \times \underline{b} + \underline{b} \times \underline{a} + \underline{a} \times \underline{c} + \underline{c} \times \underline{a} + \underline{b} \times c + c \times \underline{b}$

$= \cancel{\underline{a} \times \underline{b}} - \cancel{\underline{a} \times \underline{b}} + \cancel{\underline{a} \times \underline{c}} - \cancel{\underline{a} \times \underline{c}} + \cancel{\underline{b} \times \underline{c}} - \cancel{\underline{b} \times c}$

$= 0$

It is proved.

9. If $\underline{a} + \underline{b} + \underline{c} = 0$, then prove that

$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

As

$\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{a} = -(\underline{b} + \underline{c}) \quad \rightarrow (i)$

$\underline{a} \times \underline{b} = -(\underline{b} + \underline{c}) \times \underline{b}$

$\underline{a} \times \underline{b} = -b \times \underline{b} - \underline{c} \times \underline{b}$

$\underline{a} \times \underline{b} = 0 - \underline{c} \times \underline{b}$

$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} \quad \rightarrow (ii)$

Now

From eq(i)

$\underline{c} \times \underline{a} = \cancel{(\underline{b}+\underline{c}) \times \underline{a}}\quad \underline{c} \times -(\underline{b}+\underline{c})$

$\underline{c} \times \underline{a} = \cancel{\underline{b} \times \underline{c} - \underline{c} \times \underline{c}}\; - \underline{c} \times \underline{b} - \underline{c} \times \underline{c}$

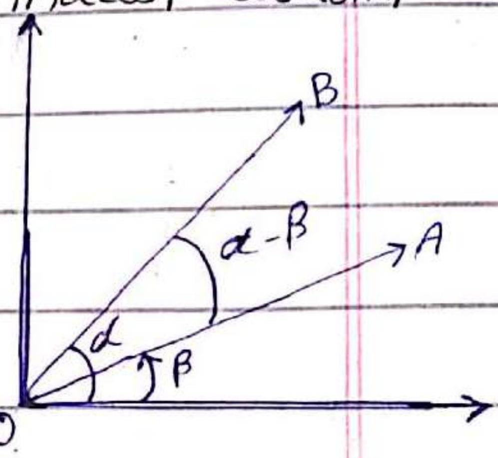
$\underline{c} \times \underline{a} = \cancel{\underline{b} \times \underline{c} - 0}\quad - \underline{c} \times \underline{b} - 0$

$\underline{c} \times \underline{a} = \cancel{\underline{c} \times \underline{b}}\quad \underline{b} \times \underline{c} \quad \rightarrow (iii)$

From (ii) and (iii)

$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

10. Prove that $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$



Let OA and OB are unit vectors with angle β and α

$\vec{OA} = \cos\beta\, i + \sin\beta\, j$

$\vec{OB} = \cos\alpha\, i + \sin\alpha\, j$

$$\vec{OA} \times \vec{OB} = \begin{vmatrix} i & j & k \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$|\vec{OA}||\vec{OB}| \sin(\alpha-\beta) k = \underline{i}(0-0) - \underline{j}(0-0) + k(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$\sin(\alpha-\beta) k = k(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(11) Show that $|\underline{a} \times \underline{b}|^2 = |\underline{a}|^2 |\underline{b}|^2 - (\underline{a} \cdot \underline{b})^2$

let $\underline{a}$ and b are unit vectors, then

$$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta \, \hat{n}$$

$$|\underline{a} \times \underline{b}| = ab \sin\theta$$

$$|\underline{a} \times \underline{b}|^2 = a^2 b^2 \sin^2\theta$$

$$= a^2 b^2 (1 - \cos^2\theta)$$

$$= a^2 b^2 - a^2 b^2 \cos^2\theta$$

$$= a^2 b^2 - (ab\cos\theta)^2$$

$$|\underline{a} \times \underline{b}|^2 = a^2 b^2 - (a \cdot b)^2$$

12. Use the definition of cross product, prove that for any vectors $\underline{u}$ and $\underline{v}$.

$$(\underline{u} + \underline{v}) \times (\underline{u} - \underline{v}) = -2(\underline{u} \times \underline{v})$$

L.H.S

$$= \underline{u} \times \underline{u} - \underline{u} \times \underline{v} + \underline{v} \times \underline{u} - \underline{v} \times \underline{v}$$

$$= 0 - \underline{u} \times \underline{v} + \underline{v} \times \underline{u} - 0$$

$$= 0 - \underline{u} \times \underline{v} - \underline{u} \times \underline{v} - 0$$

$$= -2(\underline{u} \times \underline{v})$$

= R.H.S

13. Find the moment about the point $M(1, -3, 3)$ of the force represen-

ted by $\vec{AB}$ where the cordination of points $A(4, 3, -1)$ and $B(-1, 3, 7)$ are given.

$$F = \vec{AB}$$

$$= (-1-4)\underline{i} + (3-3)\underline{j} + (7+1)\underline{k}$$

$$\vec{AB} = -5\underline{i} + 0\underline{j} + 8\underline{k}$$

Now

$$\overline{MA} = r = (4-1)\underline{i} + (3+3)\underline{j} + (-1-3)\underline{k}$$

$$= 3\underline{i} + 6\underline{j} - 4\underline{k}$$

Moment of $\vec{AB}$ about $M = \vec{r} \times \vec{F}$

$$= \vec{MA} \times \vec{AB}$$

$$\tau = \begin{vmatrix} i & j & k \\ 3 & 6 & -4 \\ -5 & 0 & 8 \end{vmatrix}$$

$$= i\begin{vmatrix} 6 & -4 \\ 0 & 8 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & -4 \\ -5 & 8 \end{vmatrix} + k\begin{vmatrix} 3 & 6 \\ -5 & 0 \end{vmatrix}$$

$$= i(48-0) - \underline{j}(24-20) + k(0+30)$$

$$\tau = 48\underline{i} - 4\underline{j} + 30\underline{k}.$$

14. A force $\vec{F} = 6\underline{i} + 4\underline{j} - 4\underline{k}$ is applied at the point $A(1, -1, 2)$. Find the moment of the force about the point $B(3, -2, 3)$

$$\vec{F} = 6\underline{i} + 4\underline{j} - 4\underline{k}$$

$\gamma = \overrightarrow{BA} = (1-3)\underline{i} + (-1+2)\underline{j} + (2-3)\underline{k}$

$\gamma = -2\underline{i} + \underline{j} - \underline{k}$

Moment $= \vec{\gamma} \times \vec{F}$

$$= \begin{vmatrix} i & j & k \\ -2 & 1 & -1 \\ 6 & 4 & -4 \end{vmatrix}$$

$= i(-4+4) - j(8+6) + k(-8-6)$

$= 0\underline{i} - 14\underline{j} - 14\underline{k}$

15. Give a force $F = 2\underline{i} + \underline{j} - 3\underline{k}$ acting at a point $A(1,-2,1)$. Find the moment of $\vec{F}$ about the point $B(2,0,-2)$

$\vec{F} = 2\underline{i} + \underline{j} - 3\underline{k}$

$\vec{\gamma} = \overrightarrow{BA} = (1-2)\underline{i} + (-2-0)\underline{j} + (1+2)\underline{k}$

$= -\underline{i} - 2\underline{j} + 3\underline{k}$

Moment $= \vec{\gamma} \times \vec{F} = \begin{vmatrix} i & j & k \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$

$= \underline{i}(6-3) - \underline{j}(3-6) + k(-1+4)$

$= 3\underline{i} + 3\underline{j} + 3\underline{k}$

16. A force $\vec{F} = -2\underline{i} + \underline{j} - 3\underline{k}$ is applied at $p(-1,-3,2)$. Find the moment

about the point $Q(4,2,2)$.

$$\vec{F} = -2\underline{i} + \underline{j} - 3\underline{k}$$

$$\vec{r} = \overrightarrow{QP} = (-1-4)\underline{i} + (-3-2)\underline{j} + (2-2)\underline{k}$$

$$= -5\underline{i} - 5\underline{j} + 0\underline{k}$$

$$\vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ -5 & -5 & 0 \\ -2 & 1 & -3 \end{vmatrix}$$

$$= i(15-0) - \underline{j}(15-0) + \underline{k}(-5-10)$$

$$= 15\underline{i} - 15\underline{j} - 15\underline{k}$$

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