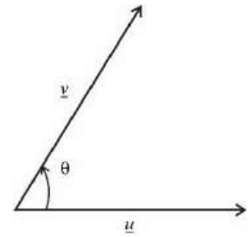
THE DOT PRODUCT OF TWO VECTORS:

Definition: 1



Let two non-zero vectors $\underline{u}$ and $\underline{v}$, in the plane or in space, have some initial point. The dot product of $\underline{u}$ and $\underline{v}$, written as $\underline{u}.\underline{v}$, is defined by:

$$\underline{u}.\underline{v} = |\underline{u}|\,|\underline{v}|\cos\theta$$

where θ is the angle between $\underline{u}$ and $\underline{v}$ and $0 \le \theta \le \pi$.

The unit vectors

$\underline{i}, \underline{j}, \underline{k}$ **:**

(a) $\underline{i}.\underline{i} = 1,\ \underline{j}.\underline{j} = 1,\ \underline{k}.\underline{k} = 1$

(b) $\underline{i}.\underline{j} = 0,\ \underline{j}.\underline{k} = 0,\ \underline{k}.\underline{i} = 0$

(c) $\underline{u}.\underline{v} = \underline{v}.\underline{u}$ (Dot product of two vectors is commutative)

Definition: 2

(a) If $\underline{u} = a_1\underline{i} + b_1\underline{j}$ *and* $\underline{v} = a_2\underline{i} + b_2\underline{j}$ are two vectors in a plane, then the dot product of $\underline{u}$ and $\underline{v}$ is

$$\underline{u}.\underline{v} = \left(a_1\underline{i} + b_1\underline{j}\right).\left(a_2\underline{i} + b_2\underline{j}\right) = a_1a_2 + b_1b_2$$

(b) If $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ are two vectors in space, then the dot product of $\underline{u}$ and $\underline{v}$ is

$$\underline{u}.\underline{v} = \left(a_1\underline{i} + b_1\underline{j} + c_1\underline{k}\right).\left(a_2\underline{i} + b_2\underline{j} + c_2\underline{k}\right) = a_1a_2 + b_1b_2 + c_1c_2$$

Perpendicular (Orthogonal) Vectors:

Two non-zero vectors $\underline{u}$ and $\underline{v}$ are perpendicular if and only if $\underline{u}.\underline{v} = 0$

Angle between two vectors:

The angles between two vectors $\underline{u}$ and $\underline{v}$ is

$$\underline{u}.\underline{v} = |\underline{u}|\,|\underline{v}|\cos\theta \text{ where } 0 \le \theta \le \pi \quad \therefore \quad \cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}|\,|\underline{v}|}$$

Work Done by a Constant Force:

If a constant force F, applied to a body, acts at an angle θ to the direction of motion, then the work done by $\underline{F}$ is defined to be the product of the component of $\underline{F}$ in the direction of the displacement and the distance that the body moves.

Work done = (component of $\underline{F}$ along *AB*) (displacement)

$$= \left(F\cos\theta\right)\left(AB\right) = \underline{F}.\underline{AB} = \underline{F}.\underline{d}$$

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EXERCISE 14.2

Q.1 Find the cosines of the angle θ between $\underline{u}$ and $\underline{v}$

(i) $\underline{u}=2\underline{i}+3\underline{j}+\underline{k}$, $\underline{v}=-\underline{i}+2\underline{j}+2\underline{k}$

Solution:

$$\underline{u}=2\underline{i}+3\underline{j}+\underline{k}, \underline{v}=-\underline{i}+2\underline{j}+2\underline{k}$$

$$\cos\theta=\frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\cos\theta=\frac{\left(2\underline{i}+3\underline{j}+\underline{k}\right).\left(-\underline{i}+2\underline{j}+2\underline{k}\right)}{\sqrt{(2)^2+(3)^2+(1)^2}\sqrt{(-1)^2+(2)^2+(2)^2}}=\frac{(2)(-1)+(3)(2)+(1)(2)}{\sqrt{4+9+1}\sqrt{1+4+4}}$$

$$=\frac{6}{3\sqrt{14}}, \cos\theta=\frac{2}{\sqrt{14}}=\frac{\sqrt{14}}{7}$$

(ii) $\underline{u}=[-3,2,5]$, $\underline{v}=[1,6,-2]$

Solution:

$$\underline{u}=-3\underline{i}+2\underline{j}+5\underline{k}, \underline{v}=\underline{i}+6\underline{j}-2\underline{k}$$

$$\cos\theta=\frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\cos\theta=\frac{\left(-3\underline{i}+2\underline{j}+5\underline{j}\right).\left(\underline{i}+6\underline{j}-2\underline{k}\right)}{\sqrt{(-3)^2+(2)^2+(5)^2}\sqrt{(1)^2+(6)^2+(-2)^2}}$$

$$=\frac{-3+12-10}{\sqrt{9+4+25}\sqrt{1+36+4}}=\frac{-1}{\sqrt{38\times41}}$$

$$\cos\theta=\frac{-1}{\sqrt{1558}}$$

Q.2 If $\underline{a}+\underline{b}+\underline{c}=\underline{0}$ and $|\underline{a}|=3, |\underline{b}|=5$ and $|\underline{c}|=7$. Find the angle between $\underline{a}$ and $\underline{b}$.

Solution:

Given that $\underline{a}+\underline{b}+\underline{c}=\underline{0}$, so we have $\underline{a}+\underline{b}=-\underline{c}$

Squaring both sides, we get

$$\left(\underline{a}+\underline{b}\right)^2=\left(-\underline{c}\right)^2$$

$|\underline{a}|^2+|\underline{b}|^2+2|\underline{a}||\underline{b}|\cos\theta=|\underline{c}|^2$ where θ is the angle between them. Putting given values, we have

$$\Rightarrow(3)^2+(5)^2+2(3)(5)\cos\theta=(7)^2$$

$$\Rightarrow 9+25+30\cos\theta=49$$

$$\Rightarrow 30\cos\theta=49-9-25$$

$$\Rightarrow 30\cos\theta=15$$

So we have $\cos\theta=\frac{15}{30}$

Gives $\cos\theta=\frac{1}{2}\Rightarrow\theta=60^\circ$ or $\Rightarrow\theta=\frac{\pi}{3}$

Q.3 If $|\underline{a}|=3, |\underline{b}|=4$ and $|\underline{a}+\underline{b}|=5$. Find the angle between $\underline{a}$ and $\underline{b}$.

Solution:

Here we have $|\underline{a}+\underline{b}|=5$ and suppose θ is the angle between $\underline{a}$ and $\underline{b}$ then squaring the above result, we get

$$|\underline{a}+\underline{b}|^2=(5)^2$$

$$\Rightarrow|\underline{a}|^2+|\underline{b}|^2+2\underline{a}.\underline{b}=25$$

$\Rightarrow |\underline{a}|^2 + |\underline{b}|^2 + 2|\underline{a}||\underline{b}|\cos\theta = 25$

Putting given values, we have

$(3)^2 + (4)^2 + 2(3)(4)\cos\theta = 25$

$\Rightarrow 9 + 16 + 24\cos\theta = 25$

$\Rightarrow 24\cos\theta = 25 - 25$

$\Rightarrow 24\cos\theta = 0 \Rightarrow \cos\theta = 0$

Gives $\theta = 90°$ or $\theta = \dfrac{\pi}{2}$

Or we can say $\underline{a} \perp \underline{b}$

Q.4 Calculate the projection of $\underline{a}$ along $\underline{b}$ and projection of $\underline{b}$ along $\underline{a}$ when

(i) $\underline{a} = 2\underline{i} + 3\underline{j} - \underline{k}$, $\underline{b} = \underline{i} - 2\underline{j} + 4\underline{k}$

Solution:

$\underline{a} = 2\underline{i} + 3\underline{j} - \underline{k} \Rightarrow |\underline{a}| = \sqrt{4+9+1} = \sqrt{14}$

$\underline{b} = \underline{i} - 2\underline{j} + 4\underline{k} \Rightarrow |\underline{b}| = \sqrt{1+4+16} = \sqrt{21}$

Projection of $\underline{a}$ along $\underline{b}$ $= \dfrac{\underline{a}.\underline{b}}{|\underline{b}|}$

$= \dfrac{(2\underline{i} + 3\underline{j} - \underline{k}).(\underline{i} - 2\underline{j} + 4\underline{k})}{\sqrt{21}} = \dfrac{2-6-4}{\sqrt{21}}$

$= \dfrac{-8}{\sqrt{21}}$

Projection of $\underline{b}$ along $\underline{a}$ $= \dfrac{\underline{a}.\underline{b}}{|\underline{a}|} = \dfrac{-8}{\sqrt{14}}$

(ii) $\underline{a} = 4\underline{i} - 2\underline{j} + 3\underline{k}$, $\underline{b} = \underline{i} + \underline{j} + \underline{k}$

Solution:

$\underline{a} = 4\underline{i} - 2\underline{j} + 3\underline{k}$

$|\underline{a}| = \sqrt{(4)^2 + (-2)^2 + (3)^2} = \sqrt{29}$

$\underline{b} = \underline{i} + \underline{j} + \underline{k}$

$|\underline{b}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

Projection of $\underline{a}$ along $\underline{b}$ $= \dfrac{\underline{a}.\underline{b}}{|\underline{b}|}$

$= \dfrac{(4\underline{i} - 2\underline{j} + 3\underline{k}).(\underline{i} + \underline{j} + \underline{k})}{\sqrt{3}}$

$= \dfrac{(4)(1) + (-2)(1) + (3)(1)}{\sqrt{3}} = \dfrac{5}{\sqrt{3}}$

Projection of $\underline{b}$ along $\underline{a}$ $= \dfrac{\underline{a}.\underline{b}}{|\underline{a}|} = \dfrac{5}{\sqrt{29}}$

Q.5 Find a real number α so that the vectors $\underline{u}$ and $\underline{v}$ are perpendicular.

(i) $\underline{u} = \alpha \underline{i} + 3\underline{j} + \underline{k}$, $\underline{v} = \underline{i} - 2\underline{j} + \alpha \underline{k}$

Solution:

The vectors $\underline{u}$ and $\underline{v}$ are perpendicular

So $\underline{u}.\underline{v} = 0$

$$\left(\alpha \underline{i} + 3\underline{j} + \underline{k}\right).\left(\underline{i} - 2\underline{j} + \alpha \underline{k}\right) = 0$$

$$\alpha - 6 + \alpha = 0 \Rightarrow 2\alpha - 6 = 0$$

$$\alpha = 3$$

(ii) $\underline{u} = \alpha \underline{i} + 2\alpha \underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha \underline{j} + 3\underline{k}$

Solution:

The vectors $\underline{u}$ and $\underline{v}$ are perpendicular

So $\underline{u}.\underline{v} = 0$

$$\left(\alpha \underline{i} + 2\alpha \underline{j} - \underline{k}\right).\left(\underline{i} + \alpha \underline{j} + 3\underline{k}\right) = 0$$

$$\alpha + 2\alpha^2 - 3 = 0 \Rightarrow 2\alpha^2 + \alpha - 3 = 0$$

$$2\alpha^2 + 3\alpha - 2\alpha - 3 = 0$$

$$\alpha(2\alpha + 3) - 1(2\alpha + 3) = 0$$

$$(\alpha - 1)(2\alpha + 3) = 0$$

Either $\alpha - 1 = 0$ or $2\alpha + 3 = 0$

$\alpha = 1$ or $\alpha = \dfrac{-3}{2}$

Q.6 Find the number z so that the triangle with vertices $A(3,0,-2)$, $B(0,3,1)$ and $C(1,1,z)$ is right triangle with right angle at C.

Solution:

$$\overrightarrow{AC} = \left(\underline{i} + \underline{j} + z\underline{k}\right) - \left(3\underline{i} + 0\underline{j} - 2\underline{k}\right)$$

$$= -2\underline{i} + \underline{j} + (z+2)\underline{k}$$

$$\overrightarrow{BC} = \left(\underline{i} + \underline{j} + z\underline{k}\right) - \left(0\underline{i} + 3\underline{j} + \underline{k}\right)$$

$$= \underline{i} + -2\underline{j} + (z-1)\underline{k}$$

As $\overrightarrow{AC}.\overrightarrow{BC} = 0$

$$\left(-2\underline{i} + \underline{j} + (z+2)\underline{k}\right).\left(\underline{i} - 2\underline{j} + (z-1)\underline{k}\right) = 0$$

$$-2 - 2 + (z+2)(z-1) = 0$$

$$-4 + z^2 + z - 2 = 0$$

$$z^2 + z - 6 = 0$$

$$z^2 + 3z - 2z - 6 = 0$$

$$z(z+3) - 2(z+3) = 0$$

$$(z-2)(z+3) = 0$$

$$z = 2,\ z = -3$$

Q.7 **If $\hat{a}$ and $\hat{b}$ are unit vectors and 2θ is the angle between them, show that** $\sin\theta = \frac{1}{2}\left|\hat{a}-\hat{b}\right|$.

Solution:

Since $\hat{a}$ and $\hat{b}$ are unit vectors, so we must have $|\hat{a}| = 1,\ |\hat{b}| = 1$

Considering $\left|\hat{a}-\hat{b}\right|^2 = |\hat{a}|^2 + |\hat{b}|^2 - 2\hat{a}.\hat{b}$

$\Rightarrow \left|\hat{a}-\hat{b}\right|^2 = (1)^2 + (1)^2 - 2|\hat{a}|.|\hat{b}|\cos 2\theta$ since angle between given unit vectors is 2θ.

Now, simplifying the above expression, we get

$\left|\hat{a}-\hat{b}\right|^2 = 2 - 2(1)(1)\cos 2\theta$

$\Rightarrow \left|\hat{a}-\hat{b}\right|^2 = 2 - 2\cos 2\theta$

$\Rightarrow \left|\hat{a}-\hat{b}\right|^2 = 2(1-\cos 2\theta)$

Using the trigonometric identity $1-\cos 2\theta = 2\sin^2\theta$ we have

$\left|\hat{a}-\hat{b}\right|^2 = 2(2\sin^2\theta)$

$\Rightarrow \left|\hat{a}-\hat{b}\right|^2 = 4\sin^2\theta \Rightarrow 4\sin^2\theta = \left|\hat{a}-\hat{b}\right|^2 \Rightarrow \sin^2\theta = \frac{1}{4}\left|\hat{a}-\hat{b}\right|^2$

Taking square root on both sides, we get

$\sin\theta = \frac{1}{2}\left|\hat{a}-\hat{b}\right|$

Hence the proof.

Q.8 **If $|\underline{a}+\underline{b}| = |\underline{a}-\underline{b}|$ then show that $\underline{a}$ and $\underline{b}$ are perpendicular.**

Solution:

Given that $|\underline{a}+\underline{b}| = |\underline{a}-\underline{b}|$, squaring both sides, we get,

$|\underline{a}+\underline{b}|^2 = |\underline{a}-\underline{b}|^2$

$\Rightarrow (\underline{a}+\underline{b}).(\underline{a}+\underline{b}) = (\underline{a}-\underline{b}).(\underline{a}-\underline{b})$

$\Rightarrow \underline{a}.(\underline{a}+\underline{b}) + \underline{b}.(\underline{a}+\underline{b}) = \underline{a}.(\underline{a}-\underline{b}) - \underline{b}.(\underline{a}-\underline{b}) \Rightarrow \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{b}.\underline{a} + \underline{b}.\underline{b} = \underline{a}.\underline{a} - \underline{a}.\underline{b} - \underline{b}.\underline{a} + \underline{b}.\underline{b}$ As we know that

$\underline{a}.\underline{b} = \underline{b}.\underline{a}$ and $\underline{a}.\underline{a} = |\underline{a}|^2$ similarly $\underline{b}.\underline{b} = |\underline{b}|^2$ so we have $|\underline{a}|^2 + 2\underline{a}.\underline{b} + |\underline{b}|^2 = |\underline{a}|^2 - 2\underline{a}.\underline{b} + |\underline{b}|^2$

$\Rightarrow 2\underline{a}.\underline{b} = -2\underline{a}.\underline{b} \Rightarrow 4\underline{a}.\underline{b} = 0$

$\Rightarrow \underline{a}.\underline{b} = 0$

Since dot product of both vectors is 0, therefore we must have $\underline{a} \perp \underline{b}$ which completes the proof.

Q.9

(i) **Show that the vectors** $3\underline{i} - 2\underline{j} + \underline{k}$, $\underline{i} - 3\underline{j} + 5\underline{k}$ **and** $2\underline{i} + \underline{j} - 4\underline{k}$ **form a right triangle.**

Solution:

Let $\underline{u}(3\underline{i} - 2\underline{j} + \underline{k}), \underline{v}(\underline{i} - 3\underline{j} + 5\underline{k})$ and $\underline{w}(2\underline{i} + \underline{j} - 4\underline{k})$ are vectors along sides of the triangle,

$\underline{u}.\underline{w} = (3\underline{i} - 2\underline{j} + \underline{k}).(2\underline{i} + \underline{j} - 4\underline{k})$

$= 3(2) - 2(1) + 1(-4) = 6 - 2 - 4$

$\underline{u}.\underline{w} = 0 \Rightarrow \underline{u} \perp \underline{w}$

So $\underline{u}, \underline{v}$ and $\underline{w}$ form a right triangle

(ii) Show that the set of points $P=(4,-1,2)$, $Q\ (1,3,-1)$ and $R(-2,4,6)$ form a right triangle.

Solution:

$\overrightarrow{PQ}=(\underline{i}+3\underline{j}-\underline{k})-(4\underline{i}-\underline{j}+2\underline{k})$

$=-3\underline{i}+4\underline{j}-3\underline{k}$

$\overrightarrow{QR}=(-2\underline{i}+4\underline{j}+6\underline{k})-(\underline{i}+3\underline{j}-\underline{k})$

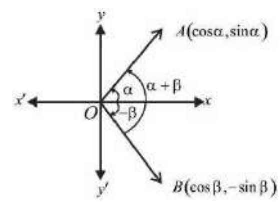
$=-3\underline{i}+\underline{j}+7\underline{k}$

$\overrightarrow{PQ}.\overrightarrow{QR}=(-3\underline{i}+4\underline{j}-3\underline{k}).(-3\underline{i}+\underline{j}+7\underline{k})$ $+9+4-21=13-21=-7\neq 0$

So the points do not form a right triangle.

Q.10 Prove that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Solution:



Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are unit vectors in $xy-$plane making angles α and $-\beta$ with the positive $x-axis$ respectively.

so that $\angle AOB=\alpha+\beta$

then $\overrightarrow{OA}=\cos\alpha\,\underline{i}+\sin\alpha\,\underline{j}$

and $\overrightarrow{OB}=\cos\beta\,\underline{i}-\sin\beta\,\underline{j}$

$\overrightarrow{OB}.\overrightarrow{OA}=(\cos\beta\,\underline{i}-\sin\beta\,\underline{j}).(\cos\alpha\,\underline{i}+\sin\alpha\,\underline{j})$

$|\overrightarrow{OA}||\overrightarrow{OB}|\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Q.11 Prove that in any triangle ABC.

(i) $b=c\cos A+a\cos C$

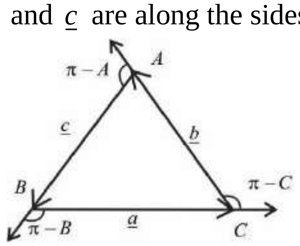
Solution:

Let the vector $\underline{a},\underline{b}$ and $\underline{c}$ are along the sides $\overrightarrow{BC},\overrightarrow{CA}$ and $\overrightarrow{AB}$ of the triangle ABC

$\therefore \underline{a}+\underline{b}+\underline{c}=0$

$\underline{b}=-(\underline{a}+\underline{c})$

Now taking dot product with $\underline{b}$



$\underline{b}.\underline{b}=-(\underline{a}+\underline{c}).\underline{b}$

$b^2=-(a.\underline{b}+\underline{c}.\underline{b})=-\underline{a}.\underline{b}-\underline{c}.\underline{b}$

$=-ab\cos(\pi-C)-bc\cos(\pi-A)$

$=-ab(-\cos C)-bc(-\cos A)$

$b^2=ab\cos C+bc\cos A$

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$b = a\cos C + c\cos A$

(ii) $c = a\cos B + b\cos A$

Solution:

Let the vectors $\underline{a}, \underline{b}$ and $\underline{c}$ are along the sides $\overrightarrow{BC}, \overrightarrow{CA}$ and $\overrightarrow{AB}$ of the triangle ABC, *then,*

$\therefore \underline{a} + \underline{b} + \underline{c} = 0 \Rightarrow \underline{c} = -\underline{a} - \underline{b}$

Now taking dot product with $\underline{c}$

$\underline{c}.\underline{c} = -(\underline{a} + \underline{b}).\underline{c}$

$c^2 = -(\underline{a}.c + \underline{b}.\underline{c}) = -\underline{a}.\underline{c} - \underline{b}.\underline{c}$

$= -ac\cos(\pi - B) - bc\cos(\pi - A)$

$= -ac\cos(\pi - B) - bc\cos(\pi - A)$

$= -ac(-\cos B) - bc(-\cos A)$

$c^2 = ac\cos B + bc\cos A$

$c = a\cos B + b\cos A$

(iii) $b^2 = c^2 + a^2 - 2ca\cos B$

Solution:

Let the vectors $\underline{a}, \underline{b}$ and $\underline{c}$ along the sides $\overrightarrow{BC}, \overrightarrow{CA}$ and $\overrightarrow{AB}$ of triangle ABC, then,

$\Rightarrow \underline{a} + \underline{b} + \underline{c} = 0 \Rightarrow \underline{b} = -(\underline{a} + \underline{c})$

$\underline{b}.\underline{b} = -(\underline{a} + \underline{c}).\underline{b}$

$b^2 = (-(\underline{a} + \underline{c}).(-(\underline{a} + \underline{c}))$

$= (\underline{a} + \underline{c}).(\underline{a} + \underline{c})$

$= \underline{a}.\underline{a} + \underline{a}.\underline{c} + \underline{c}.\underline{a} + \underline{c}.\underline{c}$

$= a^2 + 2\underline{a}.\underline{c} + c^2$

$= a^2 + c^2 + 2\underline{c}.\underline{a}$

$= a^2 + c^2 + 2ca\cos(\pi - B)$

$b^2 = a^2 + c^2 - 2ca\cos B$

(iv) $c^2 = a^2 + b^2 - 2ab\cos C$

Solution:

Let $\underline{a}, \underline{b}$ and $\underline{c}$ are vectors along sides $\overrightarrow{BC}, \overrightarrow{CA}$ and $\overrightarrow{AB}$ of triangle ABC

$\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{c} = -\underline{a} - \underline{b}$

$\underline{c} = -(\underline{a} + \underline{b})$

$\underline{c}.\underline{c} = (-(\underline{a} + \underline{b})).(-(\underline{a} + \underline{b}))$

$c^2 = (\underline{a} + \underline{b}).(\underline{a} + \underline{b})$

$= \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{b}.\underline{a} + \underline{b}.\underline{b} = a^2 + 2\underline{a}.\underline{b} + b^2$

$= a^2 + 2\underline{a}.\underline{b} + b^2 = a^2 + b^2 + 2ab\cos(\pi - C)$

$c^2 = a^2 + b^2 - 2ab\cos C$

Q.12 Show that for any vectors $\underline{a}$ and $\underline{b}$, $\left||\underline{a}| - |\underline{b}|\right| \le |\underline{a} + \underline{b}| \le |\underline{a}| + |\underline{b}|$

Solution:

We know from the previous step that

$|\underline{x} + \underline{y}| \le |\underline{x}| + |\underline{y}|$

Let $x = \underline{a} + \underline{b}$ and $y = -\underline{b}$

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Then $|\underline{a}| = |(\underline{a}+\underline{b})+(-\underline{b})| \leq |\underline{a}+\underline{b}| + |-\underline{b}|$

Since $|-\underline{b}| = |\underline{b}|$

$|\underline{a}| \leq |\underline{a}+\underline{b}| + |\underline{b}|$

Rearrange the inequality:

$|\underline{a}| - |\underline{b}| \leq |\underline{a}+\underline{b}|$

Similarly, Let $x = \underline{a}+\underline{b}$ and $y = -\underline{a}$

Then $|\underline{b}| = |(\underline{a}+\underline{b})+(-\underline{a})| \leq |\underline{a}+\underline{b}| + |-\underline{a}|$

Since $|-\underline{a}| = |\underline{a}|$

$|\underline{b}| \leq |\underline{a}+\underline{b}| + |\underline{a}|$

Rearrange the inequality

$|\underline{b}| - |\underline{a}| \leq |\underline{a}+\underline{b}|$

We have two inequalities:

$|\underline{a}| - |\underline{b}| \leq |\underline{a}+\underline{b}|$

$-(|\underline{a}| - |\underline{b}|) \leq |\underline{a}+\underline{b}|$

These two inequalities together imply:

$||\underline{a}| - |\underline{b}|| \leq |\underline{a}+\underline{b}|$

Q.13 Find the work done, if the point at which the constant force $\underline{F} = 2\underline{i} + 5\underline{j} + 3\underline{k}$ is applied to an object, moves it from $P_1(2,-3,1)$ to $P_2(7,5,3)$.

Solution:

$\underline{d} = \overrightarrow{P_1P_2} = (7\underline{i} + 5\underline{j} + 3\underline{k}) - (2\underline{i} - 3\underline{j} + \underline{k}) = 5\underline{i} + 8\underline{j} + 2\underline{k}$

Work done = $\underline{F}.\underline{d}$

$= (2\underline{i} + 5\underline{j} + 3\underline{k}) \cdot (5\underline{i} + 8\underline{j} + 2\underline{k})$

$= 10 + 40 + 6 = 56$ units.

Q.14 A particle, acted by constant forces $\underline{F}_1 = 3\underline{i} + 4\underline{j} - 3\underline{k}$ and $\underline{F}_2 = \underline{i} + 4\underline{j} - \underline{k}$ is displaced from $A(2,1,3)$ to $B(5,4,4)$. Find the work done.

Solution: Here,

$\underline{d} = \overrightarrow{AB} = (5-2)\underline{i} + (4-1)\underline{j} + (4-3)\underline{k}$

$\underline{d} = 3\underline{i} + 3\underline{j} + \underline{k}$

The net force will be $\underline{F} = \underline{F}_1 + \underline{F}_2$

$\underline{F} = 4\underline{i} + 8\underline{j} - 4\underline{k}$

So work done = $\underline{F}.\underline{d}$

$= (4\underline{i} + 8\underline{j} - 4\underline{k}) \cdot (3\underline{i} + 3\underline{j} + \underline{k})$

$= 12 + 24 - 4 = 32$ units

Q.15 **A particle is displaced from the point** $A(5,-5,-7)$ **to the point** $B(6,2,-2)$ **under the action of constant forces defined by** $10\underline{i}-\underline{j}+11\underline{k}$, $4\underline{i}+5\underline{j}+9\underline{k}$ **and** $-2\underline{i}+\underline{j}-9\underline{k}$ **. Show that the total work done by the force is** 102 **units.**

Solution:

Here

$$\underline{d}=\overrightarrow{AB}=(6-5)\underline{i}+(2+5)\underline{j}+(-2+7)\underline{k}$$

$$\underline{d}=\underline{i}+7\underline{j}+5\underline{k}$$

And net force is

$$\underline{F}=\underline{F}_1+\underline{F}_2+\underline{F}_3$$

$$\underline{F}=12\underline{i}+5\underline{j}+11\underline{k}$$

So work done = $\underline{F}.\underline{d}=(12\underline{i}+5\underline{j}+11\underline{k})\bullet(\underline{i}+7\underline{j}+5\underline{k})$

$$=(12)(1)+(5)(7)+(11)(5)$$

$$=12+35+55=102 \text{ units}$$

Q.16 **A force of magnitude** 6 **acting parallel to** $4\underline{i}+3\underline{j}-\underline{k}$ **displaces the point of application from** $A(2,-1,3)$ **to** $B(7,3,2)$**. Find the work done.**

Solution:

Let the given force be $\underline{F}=4\underline{i}+3\underline{j}-\underline{k}$ then $|\underline{F}|=\sqrt{(4)^2+(3)^2+(-1)^2}=\sqrt{26}$

So required force will be $\underline{F}'=6\hat{F}$

$$\underline{F}'=6\left(\frac{4\underline{i}+3\underline{j}-\underline{k}}{\sqrt{26}}\right)$$

$$\underline{F}'=\frac{24}{\sqrt{26}}\underline{i}+\frac{18}{\sqrt{26}}\underline{j}-\frac{6}{\sqrt{26}}\underline{k}$$

And displacement is

$$\underline{d}=\overrightarrow{AB}=(7-2)\underline{i}+(3+1)\underline{j}+(2-3)\underline{k}$$

$$\underline{d}=5\underline{i}+4\underline{j}-\underline{k}$$

So work done will be

$$\underline{F}.\underline{d}=\left(\frac{24}{\sqrt{26}}\underline{i}+\frac{18}{\sqrt{26}}\underline{j}-\frac{6}{\sqrt{26}}\underline{k}\right)\bullet(5\underline{i}+4\underline{j}-\underline{k})$$

$$=\frac{120}{\sqrt{26}}+\frac{72}{\sqrt{26}}+\frac{6}{\sqrt{26}}$$

$$=\frac{198}{\sqrt{26}} \text{ units}$$