Unit # 14, VECTORS

Scalar Quantity:

A scalar is a quantity that has only magnitude or size, such as mass, time, density, temperature, length, volume, speed and work etc.

Vector Quantity:

A vector is a quantity that has both magnitude and direction, for example displacement, velocity, acceleration, weight, force, momentum, electric and magnetic fields, etc.

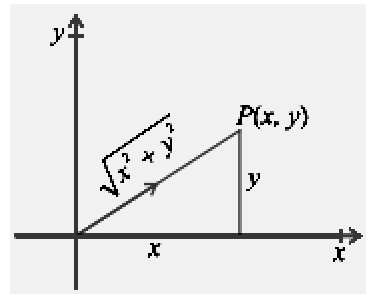
Geometric Interpretation of Vector:

Geometrically, a vector is represented by a directed line segment $\overrightarrow{AB}$ with A its initial point and B its terminal point.

Magnitude/Length/Norm of a Vector:

The magnitude (or norm or length) of a vector in 2D represents the length of the vector from the origin to the point represented by the vector. For any vector $\underline{u}=[x,y]$ in R^2, we define the magnitude, as the distance of the point $P(x,y)$ from the origin O.

$$|\underline{u}|=\sqrt{x^2+y^2}$$



Unit Vector:

A unit vector $\hat{\underline{v}}$ (read as $\underline{v}$ hat) of a given vector $\underline{v}$ is a vector with magnitude one and direction same as vector $\underline{v}$.

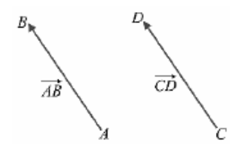
Mathematically i.e. $\hat{\underline{v}}=\dfrac{\underline{v}}{|\underline{v}|}$

Null Vector:

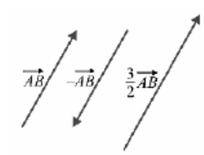
A vector whose terminal point coincides with its initial point is called **null** or **zero vector.**

Equal Vectors:

Two vectors $\underline{u}$ and $\underline{v}$ are said to be equal, if they have same magnitude and same direction.

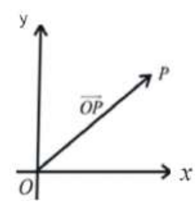


Parallel Vectors: Two vectors are parallel if an only If they are non-zero scalar multiple of each other. For example, $\overrightarrow{AB}$, $-\overrightarrow{AB}$ and $\dfrac{3}{2}\overrightarrow{AB}$ are parallel



Position Vector:

A vector which describes the location of a point w.r.t origin is called position vector. The vector, whose initial point is the origin **O** and whose terminal point **P** is called the position vector of the point **P** and is written as $\overrightarrow{OP}$



Distance between two points in space:

If $\overrightarrow{OP}_1$ and $\overrightarrow{OP}_2$ are the position vectors of the points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, the vector $\overrightarrow{P_1P_2}$ is given by $\overrightarrow{P_1P_2}=\overrightarrow{OP_2}-\overrightarrow{OP_1}=[x_2-x_1,y_2-y_1,z_2-z_1]$

Distance between P_1 and P_2 $=\left|\overrightarrow{P_1P_2}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

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Direction Angles and Direction Cosines of a Vector:

If $\underline{r}=\overrightarrow{OP}=x\underline{i}+y\underline{j}+z\underline{k}$ be a non-zero vector. If α,β,γ are direction angles of a single vector such that $0\le\alpha\le\pi, 0\le\beta\le\pi, 0\le\gamma\le\pi$ then $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ where the numbers $\cos\alpha,\cos\beta$ and $\cos\gamma$ are called direction cosines of the vector $\underline{r}$

EXERCISE 14.1

Q.1 Let $\underline{u}=3\underline{i}+2\underline{j}-5\underline{k}$, $\underline{v}=\underline{i}-5\underline{j}-\underline{k}$ and $\underline{w}=-4\underline{i}-\underline{j}+7\underline{k}$. Find the following.

(i) $\underline{u}+2\underline{v}+\underline{w}$

Solution:

$\underline{u}+2\underline{v}+\underline{w}= 3\underline{i}+2\underline{j}-5\underline{k}+2\left(\underline{i}-5\underline{j}-\underline{k}\right)+\left(-4\underline{i}-\underline{j}+7\underline{k}\right)$

$=\underline{i}-9\underline{j}$

(ii) $\underline{v}-3\underline{w}$

Solution:

$\underline{v}-3\underline{w}=\underline{i}-5\underline{j}-\underline{k}-3\left(-4\underline{i}-\underline{j}+7\underline{k}\right)$

$=\underline{i}-5\underline{j}-\underline{k}+12\underline{i}+3\underline{j}-21\underline{k}$

$=13\underline{i}-2\underline{j}-22\underline{k}$

(iii) $\left|3\underline{v}+\underline{w}\right|$

Solution:

$\left|3\underline{v}+\underline{w}\right|$

$=\left|3\left(\underline{i}-5\underline{j}-\underline{k}\right)+\left(-4\underline{i}-\underline{j}+7\underline{k}\right)\right|$

$=\left|3\underline{i}-15\underline{j}-3\underline{k}-4\underline{i}-\underline{j}+7\underline{k}\right|$

$=\left|-\underline{i}-16\underline{j}+4\underline{k}\right|$

$=\sqrt{(-1)^2+(-16)^2+(4)^2}$

$=\sqrt{273}$

Q.2 Find the magnitude of the vector $\underline{v}$ and write the direction cosines of $\underline{v}$:

(i) $\underline{v}=3\underline{i}-2\underline{j}+6\underline{k}$

Solution:

$\left|\underline{v}\right|=\sqrt{9+4+36}=7$

Unit vector of $\underline{v}$ is $=\dfrac{\underline{v}}{\left|\underline{v}\right|}=\dfrac{3\underline{i}-2\underline{j}+6\underline{k}}{7}$

$\hat{v}=\dfrac{3}{7}\underline{i}-\dfrac{2}{7}\underline{j}+\dfrac{6}{7}\underline{k}$

So direction cosines are $\dfrac{3}{7},-\dfrac{2}{7},\dfrac{6}{7}$

(ii) $\underline{v}=-4\underline{i}+4\underline{j}+2\underline{k}$

Solution:

$\left|\underline{v}\right|=\sqrt{(-4)^2+4^2+2^2}=6$

Unit vector of $\underline{v}$ is,

$$\hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{-4\underline{i} + 4\underline{j} + 2\underline{k}}{6} = -\frac{2}{3}\underline{i} + \frac{2}{3}\underline{j} + \frac{1}{3}\underline{k}$$

$$\hat{v} = -\frac{2}{3}\underline{i} + \frac{2}{3}\underline{j} + \frac{1}{3}\underline{k}$$

So,

Direction cosines are, $-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$

(iii) $\underline{v} = -6\underline{i} + 8\underline{j}$

Solution:

$$|\underline{v}| = |-6\underline{i} + 8\underline{j}|$$

$$= \sqrt{(-6)^2 + (8)^2} = \sqrt{36 + 48} = \sqrt{100}$$

$$|\underline{v}| = 10$$

Unit vector of $\underline{v}$ is,

$$\hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{-6\underline{i} + 8\underline{j}}{10} = -\frac{3}{5}\underline{i} + \frac{4}{5}\underline{j}$$

So, direction cosines are, $-\frac{3}{5}, \frac{4}{5}, 0$

Q.3 Find t so that $|2\underline{i} + (t-1)\underline{j} + t\underline{k}| = \sqrt{13}$

Solution:

We have, $|2\underline{i} + (t-1)\underline{j} + t\underline{k}| = \sqrt{13}$

$$\Rightarrow \sqrt{(2)^2 + (t-1)^2 + (t)^2} = \sqrt{13}$$

$$\Rightarrow \sqrt{4 + t^2 - 2t + 1 + t^2} = \sqrt{13}$$

Squaring both sides, we have,

$$2t^2 - 2t + 5 = 13$$

$$\Rightarrow 2t^2 - 2t - 8 = 0$$

Gives $t^2 - t - 4 = 0$

Using quadratic formula, we have,

$$t = \frac{(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)} = \frac{1 \pm \sqrt{17}}{2}$$

Q.4 Find a unit vector in the direction of $\underline{v} = -\underline{i} + 4\underline{j} - 8\underline{k}$

Solution:

We have,

$$|\underline{v}| = \sqrt{(-1)^2 + (4)^2 + (-8)^2}$$

$$|\underline{v}| = \sqrt{1 + 16 + 64} \Rightarrow |\underline{v}| = 9$$

So unit vector of $\underline{v}$ is,

$$\hat{v} = \frac{-\underline{i} + 4\underline{j} - 8\underline{k}}{9}$$

So, $\hat{v} = -\frac{1}{9}\underline{i} + \frac{4}{9}\underline{j} - \frac{8}{9}\underline{k}$

Q.5 **If** $\underline{u}=2\underline{i}+\underline{j}-3\underline{k}$, $\underline{v}=-\underline{i}+4\underline{j}+2\underline{k}$ **and** $\underline{w}=3\underline{i}-2\underline{j}+\underline{k}$ **, find a unit vector parallel to** $4\underline{u}-3\underline{v}+2\underline{w}$

Solution:

Given vector is $4\underline{u}-3\underline{v}+2\underline{w}$, so we have,

$$4\underline{u}-3\underline{v}+2\underline{w}=4\left(2\underline{i}+\underline{j}-3\underline{k}\right)-3\left(-\underline{i}+4\underline{j}+2\underline{k}\right)+2\left(3\underline{i}-2\underline{j}+\underline{k}\right)$$

$$=17\underline{i}-12\underline{j}-16\underline{k}$$

So it's magnitude is

$$\left|4\underline{u}-3\underline{v}+2\underline{w}\right|=\sqrt{(17)^2+(-12)^2+(-16)^2}$$

$$=\sqrt{289+144+256}$$

$$=\sqrt{689}$$

Hence required unit vector is

$$=\frac{17}{\sqrt{689}}\underline{i}-\frac{12}{\sqrt{689}}\underline{j}-\frac{16}{\sqrt{689}}\underline{k}$$

Q.6 **Find a vector whose**

(i) **Magnitude is 5 and is parallel to** $3\underline{i}+4\underline{j}-\underline{k}$ **.**

Solution:

Let $\underline{u}=3\underline{i}+4\underline{j}-\underline{k}$

so it's unit vector will be,

$$\hat{u}=\frac{3\underline{i}+4\underline{j}-\underline{k}}{\sqrt{(3)^2+(4)^2+(-1)^2}}=\frac{3\underline{i}+4\underline{j}-\underline{k}}{\sqrt{9+16+1}}$$

$$=\frac{3\underline{i}+4\underline{j}-\underline{k}}{\sqrt{26}}$$

Hence the required vector will be $5\hat{u}$ i.e; $5\hat{u}=\dfrac{15\underline{i}+20\underline{j}-5\underline{k}}{\sqrt{26}}$

(ii) **Magnitude is 7 and is parallel to** $-\underline{i}+\underline{j}+\underline{k}$.

Solution:

Let $\underline{v}=-\underline{i}+\underline{j}+\underline{k}$

So its unit vector will be,

$$\hat{v}=\frac{\underline{v}}{|\underline{v}|}=\frac{-\underline{i}+\underline{j}+\underline{k}}{\sqrt{(-1)^2+(1)^2+(1)^2}}$$

$$\hat{v}=\frac{-\underline{i}+\underline{j}+\underline{k}}{\sqrt{1+1+1}}$$

$$\hat{v}=\frac{-\underline{i}+\underline{j}+\underline{k}}{\sqrt{3}}$$

So the required vector will be $7\hat{v}$ i.e;

$$7\hat{v}=\frac{-7\underline{i}+7\underline{j}+7\underline{k}}{\sqrt{3}}$$

Q.7 **If** $\underline{u}=x\underline{i}+2\underline{j}+3\underline{k}$, $\underline{v}=\underline{i}+y\underline{j}-3\underline{k}$ **and** $\underline{w}=-2\underline{i}-3\underline{j}$ **represent the sides of a triangle, find the values of** x **and** y**.**

Solution:

If $\underline{u}=x\underline{i}+2\underline{j}+3\underline{k}$, $\underline{v}=\underline{i}+y\underline{j}-3\underline{k}$ and $\underline{w}=-2\underline{i}-3\underline{j}$ represent the sides of a triangle then we must have $\underline{u}+\underline{v}=\underline{w}$

i.e; $(x+1)\underline{i}+(2+y)\underline{j}=-2\underline{i}-3\underline{j}$

Equating co-efficient of $\underline{i}$ and $\underline{j}$, we have $x+1=-2 \Rightarrow x=-3$ and $2+y=-3 \Rightarrow y=-5$.

Q.8 **The position vectors of points** A, B, C **and** D **are** $\underline{u}=\underline{i}+2\underline{j}+\underline{k}$, $\underline{v}=7\underline{i}+8\underline{j}+4\underline{k}$, $\underline{w}=-\underline{i}+\underline{k}$, $\underline{z}=\underline{i}+2\underline{j}+2\underline{k}$ **respectively. Show that** $\overrightarrow{AB}$ **is parallel to** $\overrightarrow{CD}$.

Solution:

Here we have

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\underline{v}-\underline{u}$

$=(7\underline{i}+8\underline{j}+4\underline{k})-(\underline{i}+2\underline{j}+\underline{k})$

So $\overrightarrow{AB}=6\underline{i}+6\underline{j}+3\underline{k}$ and similarly

$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=\underline{z}-\underline{w}$

$=(\underline{i}+2\underline{j}+2\underline{k})-(-\underline{i}+\underline{k})$

$\overrightarrow{CD}=2\underline{i}+2\underline{j}+\underline{k}$

$\overrightarrow{AB}$ and $\overrightarrow{CD}$ will be parallel if $\overrightarrow{AB}=\lambda\overrightarrow{CD}$ $\quad(\lambda\neq 0)$

So we have $\overrightarrow{AB}=6\underline{i}+6\underline{j}+3\underline{k}$

$\overrightarrow{AB}=3(2\underline{i}+2\underline{j}+\underline{k})$

Gives $\overrightarrow{AB}=3\overrightarrow{CD}$

Which shows that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

Q.9 **We say that two vectors** $\underline{v}$ **and** $\underline{w}$ **in space are parallel if there is a scalar** c **such that** $\underline{v}=c\underline{w}$ **the vectors point in the same direction if** $c>0$, **and the vectors point in the opposite direction if** $c<0$.

Solution:

(a) **Find two vectors of length 2 parallel to the vector** $\underline{v}=2\underline{i}-4\underline{j}+4\underline{k}$

Solution:

$\underline{v}=2\underline{i}-4\underline{j}+4\underline{k}$

$|\underline{v}|=\sqrt{2^2+(-4)^2+(4)^2}=\sqrt{36}=6$

If it is unit vector parallel to $\underline{v}$ then,

$\hat{\underline{v}}=\dfrac{\underline{v}}{|\underline{v}|}=\dfrac{2\underline{i}-4\underline{j}+4\underline{k}}{6}$

$=\dfrac{2}{6}\underline{i}-\dfrac{4}{6}\underline{j}+\dfrac{4}{6}\underline{k}=\dfrac{1}{3}\underline{i}-\dfrac{2}{3}\underline{j}+\dfrac{2}{3}\underline{k}$

The two vectors of length 2 and parallel to $\hat{v}$ are $2\hat{v}$ and $-2\hat{v}$

$2\hat{\underline{v}}=2\left(\dfrac{1}{3}\underline{i}-\dfrac{2}{3}\underline{j}+\dfrac{2}{3}\underline{k}\right)$

and $-2\hat{v}=-2\left(\dfrac{1}{3}\underline{i}-\dfrac{2}{3}\underline{j}+\dfrac{2}{3}\underline{k}\right)$

$2\underline{\hat{v}}=\frac{2}{3}\underline{i}-\frac{4}{3}\underline{j}+\frac{4}{3}\underline{k}$

and $-2\underline{\hat{v}}=-\frac{2}{3}\underline{i}+\frac{4}{3}\underline{j}-\frac{4}{3}\underline{k}$

(b) Find the constant α so that the vectors $\underline{v}=\underline{i}-3\underline{j}+4\underline{k}$ and $\underline{w}=\alpha\underline{i}+9\underline{j}-12\underline{k}$ are parallel.

Solution:

If $\underline{v}$ and $\underline{w}$ are parallel then, $\underline{v}=c\underline{w}$

$\underline{i}-3\underline{j}+4\underline{k}=c\left(\alpha\underline{i}+9\underline{j}-12\underline{k}\right)$

$\underline{i}-3\underline{j}+4\underline{k}=c\alpha\underline{i}+9c\underline{j}-12c\underline{k}$

Comparing both side

$\alpha c=1\ldots\text{(i)},\ 9c=-3$

$c=-\frac{1}{3}$, Put in (i), $\alpha=\left(-\frac{1}{3}\right)=1$

$\alpha=-3$

(c) Find a vector of length 5 in the direction opposite that of $\underline{v}=\underline{i}-2\underline{j}+3\underline{k}$.

Solution:

$\underline{v}=\underline{i}-2\underline{j}+3\underline{k}$ $|\underline{v}|=\sqrt{(1)^2+(-2)^2+(3)^2}=\sqrt{1+4+9}=\sqrt{14}$ If $\hat{v}$ be unit vector in the direction of $\underline{v}$ then $\hat{v}=\frac{\underline{v}}{|\underline{v}|}=\frac{\underline{i}-2\underline{j}+3\underline{k}}{\sqrt{14}}$

Thus the required vector of length 5 and direction opposite is:

$-5\underline{\hat{v}}=-5\left(\frac{\underline{i}-2\underline{j}+3\underline{k}}{\sqrt{14}}\right)$

$=\frac{-5}{\sqrt{14}}\underline{i}+\frac{10}{\sqrt{14}}\underline{j}-\frac{15}{\sqrt{14}}\underline{k}$

(d) Find *a* and *b* so that the vectors $3\underline{i}-\underline{j}+4\underline{k}$ and $a\underline{i}+b\underline{j}-2\underline{k}$ are Parallel.

Solution:

Let $\underline{v}=3\underline{i}-\underline{j}+4\underline{k}$ and $\underline{w}=a\underline{i}+b\underline{j}-2\underline{k}$

$\underline{v}$ and $\underline{w}$ are parallel if $\underline{w}=c\underline{v}$

$a\underline{i}+b\underline{j}-2\underline{k}=c\left(3\underline{i}-\underline{j}+4\underline{k}\right)$

$a\underline{i}+b\underline{j}-2\underline{k}=3c\underline{i}-c\underline{j}+4c\underline{k}$

Comparing both sides

$a=3c\ \ldots\text{(i)},\ b=-c\ \ldots\text{(ii)},$

$-2=4c\ldots\text{(iii)}\Rightarrow c=-\frac{1}{2}$

Put value of c in equation (i) and (ii), we have

$a=3\left(\frac{-1}{2}\right)$ and $b=-\left(\frac{-1}{2}\right)$

$a=\frac{-3}{2}$ and $b=\frac{1}{2}$

Q.10 A spacecraft moves from point $(120,240,-50)$ to point $(130,210,80)$ in kilometers. What is the magnitude of the displacement vector in kilometers?

Solution:

Let $A(120,240,-50)$ and $B(130,210,80)$ then displacement vector will be $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=(130,210,80)-(120,240,-50)$

So, $\overrightarrow{AB}=10\underline{i}-30\underline{j}+130\underline{k}$

So magnitude of displacement will be $\left|\overrightarrow{AB}\right|=\sqrt{(10)^2+(-30)^2+(130)^2}$

$\left|\overrightarrow{AB}\right|=10\sqrt{179}$ kilometers

Q.11 Find the direction cosines for the given vector:

Solution:

(i) $\underline{u}=-6\underline{i}+3\underline{j}+2\underline{k}$

$|\underline{u}|=|-6\underline{i}+3\underline{j}+2\underline{k}|$

$=\sqrt{(-6)^2+(3)^2+(2)^2}$

$=7$

$u=\frac{\underline{u}}{|\underline{u}|}=\frac{-6\underline{i}+3\underline{j}+2\underline{k}}{7}$

So, direction cosines of $\underline{u}$ are: $-\frac{6}{7},\frac{3}{7},\frac{2}{7}$

(ii) $\underline{v}=4\underline{i}+2\underline{j}-5\underline{k}$

Solution:

$\underline{v}=4\underline{i}+2\underline{j}-5\underline{k}$

$|\underline{v}|=|4\underline{i}+2\underline{j}-5\underline{k}|=\sqrt{(4)^2+(2)^2+(-5)^2}=\sqrt{45}=3\sqrt{5}$

The direction cosines of $\underline{v}$ are:

$\frac{4}{3\sqrt{5}},\frac{2}{3\sqrt{5}},-\frac{\sqrt{5}}{3}$

(iii) $\overrightarrow{PQ}$**, where** $P=(9,3,13)$ **and** $Q=(11,6,19)$

Solution:

$\overrightarrow{PQ}=\overline{OQ}-\overline{OP}=(11,6,19)-(9,3,13)$

$\overrightarrow{PQ}=2\hat{i}+3\hat{j}+6\hat{k}$

$\left|\overrightarrow{PQ}\right|=\sqrt{(2)^2+(3)^2+(6)^2}=7$

The direction cosines of $\overrightarrow{PQ}$ are:

$\frac{2}{7},\frac{3}{7},\frac{6}{7}$

Q.12 Which of the following triples can be the direction angles of a single vector:

(i) 45°,45°,60°

Solution:

45°,45°,60°

Let $\alpha=45°$, $\beta=45°$, $\gamma=60°$

We are to show that

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$$\text{L.H.S} = \cos^2(45^\circ) + \cos^2(45^\circ) + \cos^2(60^\circ)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} + \frac{1}{4}$$

$$= \frac{2+2+1}{4} = \frac{5}{4} \neq 1$$

Thus the given angles are not direction angles of a vector.

(ii) 30°,45°,60°

Solution:

Let $\alpha = 30^\circ$, $\beta = 45^\circ$, $\gamma = 60^\circ$

We are to show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

L.H.S =

$$\cos^2(30^\circ) + \cos^2(45^\circ) + \cos^2(60^\circ)$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{2} + \frac{1}{4}$$

$$= \frac{3+2+1}{4} = \frac{6}{4} \neq 1$$

Thus the given angles are not direction angle of the vector.

(iii) 45°,60°,60°

Solution:

Let $\alpha = 45^\circ$, $\beta = 60^\circ$, $\gamma = 60^\circ$

We are to show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2(45^\circ) + \cos^2(60^\circ) + \cos^2(60^\circ)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{2+1+1}{4} = \frac{4}{4} = 1$$

Thus given angles are direction angles of a single vector.