Exercise 14.1

1. Let $\underline{u} = 3i + 2j - 5k$, $\underline{v} = i - 5j - \underline{k}$ and $w = -4i - j + 7k$. Find the following.

(i) $\underline{u} + 2\underline{v} + \underline{w}$

$$\underline{u} + 2\underline{v} + \underline{w} = (3i + 2j - 5k) + 2(i - 5j - k) + (-4i - j + 7k)$$

$$= 3i + 2j - 5k + 2i - 10j - 2k - 4i - j + 7k$$

$$= 5i - 4i - 8j - j - 7k + 7k$$

$$= i - 9j$$

(ii) $\underline{v} - 3\underline{w}$

$$\underline{v} - 3\underline{w} = (\underline{i} - 5\underline{j} - \underline{k}) - 3(-4\underline{i} - \underline{j} + 7k)$$

$$= \underline{i} - 5\underline{j} - \underline{k} + 12\underline{i} + 3\underline{j} - 21\underline{k}$$

$$= 13i - 2j - 22k$$

(iii) $|3\underline{v} + \underline{w}|$

$$|3\underline{v} + \underline{w}| = 3(\underline{i} - 5\underline{j} - \underline{k}) + (-4\underline{i} - \underline{j} + 7\underline{k})$$

$$= 3\underline{i} - 15\underline{j} - 3k - 4\underline{i} - \underline{j} + 7\underline{k}$$

$$= -\underline{i} - 16\underline{j} + 4\underline{k}$$

$$= \sqrt{(-1)^2 + (-16)^2 + (4)^2}$$

$$= \sqrt{1 + 256 + 16}$$

$$= \sqrt{257 + 16}$$

$$= \sqrt{273}$$

2. Find the magnitude of the vector $\underline{v}$ and write the direction cosines of $\underline{v}$.

(i) $\underline{v} = 3\underline{i} - 2\underline{j} + 6\underline{k}$

$= \sqrt{(3)^2 + (-2)^2 + (6)^2}$

$= \sqrt{9 + 4 + 36}$

$= \sqrt{49}$

$= 7$

$\cancel{\cos\alpha} = \dfrac{x}{|\underline{v}|} = \dfrac{3}{7}$

$\cos\beta = \dfrac{y}{|\underline{v}|} = -\dfrac{2}{7}$

$\cos\gamma = \dfrac{z}{|\underline{v}|} = \dfrac{6}{7}$

(ii) $\underline{v} = -4\underline{i} + 4\underline{j} + 2\underline{k}$

$= \sqrt{(-4)^2 + (4)^2 + (2)^2}$

$= \sqrt{16 + 16 + 4}$

$= \sqrt{36}$

$= 6$

$\cos\alpha = \dfrac{x}{|\underline{v}|} = \dfrac{-4}{6} = -\dfrac{2}{3}$

$\cos\beta = \dfrac{y}{|\underline{v}|} = \dfrac{4}{6} = \dfrac{2}{3}$

$\cos\gamma = \dfrac{z}{|\underline{v}|} = \dfrac{2}{6} = \dfrac{1}{3}$

(iii) $V = -6i + 8j$

$= \sqrt{(-6)^2 + (8)^2}$

$= \sqrt{36 + 64}$

$= \sqrt{100}$

$= 10$

$\cos\alpha = \frac{x}{|V|} = \frac{-6}{10} = \frac{-3}{5}$

$\cos\beta = \frac{y}{|V|} = \frac{8}{10} = \frac{4}{5}$

$\cos\gamma = \frac{z}{|V|} = \frac{0}{10} = 0$

3. Find t, so that $|2\underline{i} + (t-1)\underline{j} + t\underline{k}| = \sqrt{13}$

$|2\underline{i} + (t-1)\underline{j} + t\underline{k}| = \sqrt{13}$

$\sqrt{(2)^2 + (t-1)^2 + (t)^2} = \sqrt{13}$

taking square on both sides

$\left(\sqrt{4 + t^2 + 1 - 2t + t^2}\right)^2 = (\sqrt{13})^2$

$2t^2 - 2t + 5 - 13 = 0$

$2t^2 - 2t - 8 = 0$

$2(t^2 - t - 4) = 0$

$t^2 - t - 4 = 0$

$t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2}$

$$t = \frac{1 \pm \sqrt{1+16}}{2}$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

4. Find a unit vector in the direction of $\underline{v} = -\underline{i} + 4\underline{j} - 8\underline{k}$

$$|\underline{v}| = \sqrt{(-1)^2 + (4)^2 + (8)^2}$$
$$= \sqrt{1+16+64}$$
$$= \sqrt{17+64}$$
$$= \sqrt{81}$$
$$= 9$$

$$\hat{\underline{v}} = \frac{-\underline{i} + 4\underline{j} - 8\underline{k}}{9}$$

$$\hat{\underline{v}} = -\frac{1}{9}\underline{i} + \frac{4}{9}\underline{j} - \frac{8}{9}\underline{k}$$

5. If $\underline{u} = 2\underline{i} + \underline{j} - 3\underline{k}$, $\underline{v} = -\underline{i} + 4\underline{j} + 2\underline{k}$ and $\underline{w} = 3\underline{i} - 2\underline{j} + \underline{k}$, Find a unit vector parallel to $4\underline{u} - 3\underline{v} + 2\underline{w}$.

Let

$$A = 4\underline{u} - 3\underline{v} + 2\underline{w}$$

$$A = 4(2\underline{i} + \underline{j} - 3\underline{k}) - 3(-\underline{i} + 4\underline{j} + 2\underline{k}) + 2(3\underline{i} - 2\underline{j} + \underline{k})$$

$$A = 8\underline{i} + 4\underline{j} - 12\underline{k} + 3\underline{i} - 12\underline{j} - 6\underline{k} + 6\underline{i} - 4\underline{j} + 2\underline{k}$$

$A = 17i - 12j - 16k$

$|A| = \sqrt{(17)^2 + (-12)^2 + (-16)^2}$

$= \sqrt{289 + 144 + 256}$

$= \sqrt{689}$

$\hat{A} = \frac{A}{|A|} = \frac{17i - 12j - 16k}{\sqrt{689}}$

$=$

6. Find a vector whose

(i) magnitude is 5 and parallel to $3i + 4j - k$

Let be the required vector is v.

$|\vec{V}| = 5$

$u = 3i + 4j - k$

$\hat{u} = \frac{3i + 4j - k}{\sqrt{(3)^2 + (4)^2 + (-1)^2}}$

$\hat{u} = \frac{3i + 4j - k}{\sqrt{9 + 16 + 1}}$

$\hat{u} = \frac{3i + 4j - k}{\sqrt{26}}$

$\vec{V} = |\vec{V}|\,\hat{u}$

$= 7\left(\frac{3i + 4j - k}{\sqrt{26}}\right)$

$$\vec{v} = \frac{21}{\sqrt{26}}\underline{i} + \frac{28}{\sqrt{26}}\underline{j} - \frac{7}{\sqrt{26}}\underline{k}.$$

(ii) magnitude is 7 and is parallel to $-\underline{i}+\underline{j}+\underline{k}$.

Let be required vector is $\vec{v}$.

$|\vec{v}| = 7$.

$u = -\underline{i} + \underline{j} + \underline{k}$

$|u| = \sqrt{(-1)^2 + (1)^2 + (1)^2}$

$= \sqrt{1+1+1}$

$|u| = \sqrt{3}$

$$\hat{u} = \frac{-\underline{i} + \underline{j} + \underline{k}}{\sqrt{3}}$$

$\vec{v} = |\vec{v}|\,\hat{u}$

$$= 7\left(\frac{-\underline{i} + \underline{j} + \underline{k}}{\sqrt{3}}\right)$$

$$= -\frac{7}{\sqrt{3}}\underline{i} + \frac{7}{\sqrt{3}}\underline{j} + \frac{7}{\sqrt{3}}\underline{k}$$

7. If $\underline{u} = x\underline{i} + 2\underline{j} + 3\underline{k}$, $v = \underline{i} + y\underline{j} - 3\underline{k}$ and $\underline{w} = -2\underline{i} - 3\underline{j}$ represent the sides of a triangle. Find the value of x and y.

$\underline{u} = x\underline{i} + 2\underline{j} + 3\underline{k}$

$\underline{v} = \underline{i} + y\underline{j} - 3\underline{k}$

$\underline{w} = -2\underline{i} - 3\underline{j}$

$u + v = w$

$xi + 2j + \cancel{3k} + i + yj - \cancel{3k} = -2\underline{i} - 3j$

$(x+1)i + (y+2)j = -2\underline{i} - 3\underline{j}$

$x + 1 = -2$ $\qquad$ $y + 2 = -3$

$\boxed{x = -3}$ $\qquad$ $\boxed{y = -5}$

8. The position vectors of points A, B, C and D are $\underline{u} = \underline{i} + 2\underline{j} + \underline{k}$, $\underline{v} = 7\underline{i} + 8\underline{j} + 4\underline{k}$, $\underline{w} = -\underline{i} + \underline{k}$ and $\underline{z} = i + 2\underline{j} + 2k$ respectively. Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

$\overline{OA} = u$

$\overline{OB} = v$

$\overrightarrow{AB} = \overline{OB} - \overline{OA}$

$\overrightarrow{AB} = 7\underline{i} + 8\underline{j} + 4\underline{k} - (\underline{i} + 2\underline{j} + \underline{k})$

$= 7\underline{i} + 8\underline{j} + 4\underline{k} - \underline{i} - 2\underline{j} - \underline{k}$

$= 6\underline{i} + 6\underline{j} + 3\underline{k}$

$= 3(2\underline{i} + 2\underline{j} + \underline{k})$

$\overrightarrow{CD} = \underline{OD} - \underline{OC}$

$= z - w$

$= i + 2j + 2k - (-i + k)$

$= i + 2j + 2k + i - k$

$\vec{CD} = 2i + 2j + k$

$\vec{AB} = 3\vec{CD}$

$AB \parallel CD$

9. We say that two vectors $\underline{v}$ and $\underline{w}$ in space are parallel if there is a scalar c such that $\underline{v} = c\underline{w}$. The vectors point in the same direction if $c > 0$ and the vector points in the opposite direction if $c < 0$

(a) Find two vectors of length 2 parallel to the vector $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$.

Let be the required vector is w.

$$v = 2\underline{i} - 4\underline{j} + 4\underline{k}$$

$$\hat{v} = \hat{w} = \frac{2i - 4j + 4k}{\sqrt{(2)^2 + (-4)^2 + (4)^2}}$$

$$= \frac{2i - 4j + 4k}{\sqrt{4+16+16}}$$

$$= \frac{2i - 4j + 4k}{6}$$

$$\hat{v} = \hat{w} = \frac{1}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}$$

$$\underline{w} = 2\hat{w}$$

$$\underline{w} = 2\left(\tfrac{1}{3}\underline{i} - \tfrac{2}{3}\underline{j} + \tfrac{2}{3}\underline{k}\right)$$

$$= \frac{2}{3}\underline{i} - \frac{4}{3}\underline{j} + \frac{4}{3}\underline{k}$$

Another vector

$$\underline{w} = -2\hat{w}$$

$$\underline{w} = -2\left(\tfrac{1}{3}\underline{i} - \tfrac{2}{3}\underline{j} + \tfrac{2}{3}\underline{k}\right)$$

$$= -\frac{2}{3}\underline{i} + \frac{4}{3}\underline{j} - \frac{4}{3}\underline{k}$$

(b)

$$V = cw$$

$$i - 3j + 4k = c(ai + 9j - 12k)$$

$$i - 3j + 4k = aci + 9cj - 12ck$$

$ac = 1$	$9c = -3$	$-12c = 4$
$a(-\frac{1}{3}) = 1$	$c = -\frac{3}{9}$	
$a = -1 \times 3$		
$a = -3$	$c = -\frac{1}{3}$	

(c) Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$

$$|\underline{v}| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$$

$$|\underline{v}| = \sqrt{1+4+9}$$

$$|v| = \sqrt{14}$$

$$\hat{v} = \frac{v}{|v|}$$

$$\hat{v} = \frac{\underline{i} - 2\underline{j} + 3\underline{k}}{\sqrt{14}}$$

$$\hat{v} = \frac{1}{\sqrt{14}}\underline{i} - \frac{2}{\sqrt{14}}\underline{j} + \frac{3}{\sqrt{14}}\underline{k}$$

according to the condition

$$-5\hat{v} = -5\left(\frac{1}{\sqrt{14}}i - \frac{2}{\sqrt{14}}\underline{j} + \frac{3}{\sqrt{14}}k\right)$$

$$= \frac{-5}{\sqrt{14}}\underline{i} + \frac{10}{\sqrt{14}}\underline{j} - \frac{15}{\sqrt{14}}k$$

(d) Find a and b so that the vectors $3\underline{i} - \underline{j} + 4\underline{k}$ and $ai + b\underline{j} - 2k$ are parallel.

Let $v = 3i - j + 4k$, $w = ai + bj - 2k$

$$V = cw$$

$3i - j + 4k = c(ai + bj - 2k)$

$3i - j + 4k = cai + cbj - 2ck$

$3 = ca \qquad -1 = cb \qquad 4 = -2c$

$3 = (-2) \times a \qquad -1 = -2 \times b \qquad \frac{4^2}{-\cancel{2}} = c$

$\boxed{a = -\frac{3}{2}} \qquad \boxed{b = \frac{1}{2}} \qquad \boxed{c = -2}$

10. A Spacecraft moves from point (120, 240, -50) to point (130, 210, 80) in kilometres. What is the magnitude of the displacement vector in kelometres?

$OA = (120, 240, -50)$

$OB(130, 210, 80)$

$AB = OB - OA$

$= 130i + 210j + 80k - (120i + 240j - 50k)$

$= 130i + 210j + 80k - 120i - 240j + 50k$

$= 10i - 30j + 130k$

$|AB| = \sqrt{(10)^2 + (-30)^2 + (130)^2}$

$= \sqrt{100 + 900 + 16900}$

$= \sqrt{17900} = \sqrt{179 \times 100}$

$= 10\sqrt{179}$ kilometres

11. Find the direction cosines for the given vector:

(i) $\underline{u} = -6\underline{i} + 3\underline{j} + 2\underline{k}$

$|\underline{u}| = \sqrt{(-6)^2 + (3)^2 + (2)^2}$

$= \sqrt{36 + 9 + 4}$

$|\underline{u}| = \sqrt{49} = 7$

$\cos\alpha = \frac{x}{|\underline{u}|} = \frac{-6}{7}$

$\cos\beta = \frac{y}{|\underline{u}|} = \frac{3}{7}$

$\cos\gamma = \frac{z}{|u|} = \frac{2}{7}$

(ii) $\underline{v} = 4\underline{i} + 2\underline{j} - 5\underline{k}$

$|\underline{v}| = \sqrt{(4)^2 + (2)^2 + (-5)^2}$

$= \sqrt{16 + 4 + 25}$

$|\underline{v}| = \sqrt{45} = 3\sqrt{5}$

$\cos\alpha = \frac{x}{|v|} = \frac{4}{3\sqrt{5}}$

$\cos\beta = \frac{y}{|\underline{v}|} = \frac{2}{3\sqrt{5}}$

$\cos\gamma = \frac{z}{|\underline{v}|} = \frac{-5}{3\sqrt{5}}$

$$= \frac{-\sqrt{5}\cdot\sqrt{\cancel{5}}}{3\cancel{\sqrt{5}}} = \frac{-\sqrt{5}}{3}.$$

(iii) $\overrightarrow{PQ}$ $\cancel{(x_2-x_1)}$, where $P(9,3,13)$ and $Q(11,6,19)$.

$$\overrightarrow{PQ} = (x_2-x_1)i+(y_2-y_1)j+(z_2-z_1)k$$

$$= (11-9)i+(6-3)j+(19-13)k$$

$$= 2i+3j+6k$$

$$|PQ| = \sqrt{(2)^2+(3)^2+(6)^2}$$

$$= \sqrt{4+9+36}$$

$$|PQ| = \sqrt{49} = 7$$

$$\cos\alpha = \frac{x}{|PQ|} = \frac{2}{7}$$

$$\cos\beta = \frac{y}{|PQ|} = \frac{3}{7}$$

$$\cos\gamma = \frac{z}{|PQ|} = \frac{6}{7}$$

12. Which of the following triple can be the direction angles of a single vector?

(i) $45°, 45°, 60°$

As $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Let $\alpha = 45°, \beta = 45°, \gamma = 60°$

$= (\cos 45)^2 + (\cos 45)^2 + (\cos 60)^2$

$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$

$= \frac{1}{2} + \frac{1}{2} + \frac{1}{4}$

$= \frac{2 + 2 + 1}{4} = \frac{5}{4}$

$\frac{5}{4} \neq 1$

Cannot be direction angles.

(ii) $30°, 45°, 60°$

As $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Let $\alpha = 30°$, $\beta = 45°$, $\gamma = 60°$

$= (\cos^2\alpha) + (\cos^2\beta) + (\cos^2\gamma)$

$= (\cos 30)^2 + (\cos 45)^2 + (\cos 60)^2$

$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$= \frac{3}{4} + \frac{2}{4} + \frac{1}{4}$

$= \frac{3 + 2 + 1}{4}$

$= \frac{6}{4} = \frac{3}{2}$

cannot be direction angles.

(iii) $45°, 60°, 60°$

As $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Let $\alpha = 45°$, $\beta = 60°$, $\gamma = 60°$

$= (\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2$

$= (\cos 45°)^2 + (\cos 60)^2 + (\cos 60)^2$

$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$

$= \frac{2 + 1 + 1}{4} = \frac{4}{4}$

$1 = 1$

So, Satisfy the Condition for direction angles of a single vector.

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