Unit # 13, Differentiation

Theorems on differentiation:

(1) $\frac{d}{dx}(c)=0$ $(c \text{ is constant})$

(2) $\frac{d}{dx}(x^n)=nx^{n-1},$ $n\in R$ **(The Power rule)**

(3) $\frac{d}{dx}[cf(x)]=c.f'(x)$

(4) $\frac{d}{dx}(f(x)+g(x))=f'(x)+g'(x)$

(5) **The Product Rule:**

$$\frac{d}{dx}[f(x)g(x)]=f'(x)g(x)+f(x)g'(x)$$

(6) **The Quotient Rule:**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2},g(x)\neq 0$$

EXERCISE 13.2

1. Differentiate w.r.t "x"

(i) $y=x^4+2x^3+x^2$

Solution:

Let $y=x^4+2x^3+x^2$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}x^4+2\frac{d}{dx}x^3+\frac{d}{dx}x^2$$

$$\frac{dy}{dx}=4x^3+6x^2+2x$$

(ii) $y=x^{-3}+2x^{-\frac{3}{2}}+3$

Solution:

Let $y=x^{-3}+2x^{-\frac{3}{2}}+3$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dv}x^{-3}+2\frac{d}{dx}x^{-\frac{3}{2}}+\frac{d}{dx}(3)$$

$$\frac{dy}{dx}=-3.x^{-3-1}+2.\frac{-3}{2}x^{-\frac{3}{2}-1}+0$$

$$\frac{dy}{dx}=-3x^{-4}-3x^{-\frac{5}{2}}$$

$$\frac{dy}{dx}=-3\left[x^{-4}+x^{-\frac{5}{2}}\right]$$

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$$\frac{dy}{dx}=-3\left[\frac{1}{x^4}+\frac{1}{x^{\frac{5}{2}}}\right]$$

(iii) $y=\dfrac{2x-3}{2x+1}$

Solution:

Let $y=\dfrac{2x-3}{2x+1}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x-3}{2x+1}\right)$$

$$\frac{dy}{dx}=\frac{(2x+1)\frac{d}{dx}(2x-3)-(2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$\frac{dy}{dx}=\frac{(2x+1)2-(2x-3)2}{(2x+1)^2}$$

$$\frac{dy}{dx}=\frac{4x+2-4x+6}{(2x+1)^2}$$

$$\frac{dy}{dx}=\frac{8}{(2x+1)^2}$$

(iv) $y=\dfrac{\left(1+\sqrt{x}\right)\left(x-x^{\frac{3}{2}}\right)}{\sqrt{x}}$

Solution:

Let $y=\dfrac{\left(1+\sqrt{x}\right)\left(x-x^{\frac{3}{2}}\right)}{\sqrt{x}}$

First of all simplify the given function

$$y=\frac{x-x^{\frac{3}{2}}+x^{\frac{3}{2}}-x^2}{\sqrt{x}} \qquad \because a^m.a^n=a^{m+n}$$

$$y=\frac{x}{\sqrt{x}}-\frac{x^2}{\sqrt{x}} \qquad \because \frac{a^m}{a^n}=a^{m-n}$$

$$y=\sqrt{x}-x^{2-\frac{1}{2}}$$

$$y=x^{\frac{1}{2}}-x^{\frac{3}{2}}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}x^{\frac{1}{2}}-\frac{d}{dx}x^{\frac{3}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}x^{\frac{+1}{2}-1}-\frac{3}{2}x^{\frac{3}{2}-1}$$

$$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}-\frac{3}{2}\sqrt{x}$$

$$\frac{dy}{dx}=\frac{1-3x}{2\sqrt{x}}$$

(v) $y=\left(\sqrt{x}-\frac{1}{x}\right)^2$

Solution:

Let $y=\left(\sqrt{x}-\frac{1}{x}\right)^2$

First of all simplify the given function

$$y=x+\frac{1}{x^2}-2\sqrt{x}.\frac{1}{x}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}(x)+\frac{d}{dx}(x)^{-2}-2\frac{d}{dx}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=1-2x^{-3}+2.\frac{1}{2}x^{-\frac{1}{2}-1}$$

$$\frac{dy}{dx}=1-\frac{2}{x^3}+\frac{1}{x.\sqrt{x}} \qquad \because x^{\frac{3}{2}}=x.\sqrt{x}$$

$$\frac{dy}{dx}=1+2x^{-3}+x^{\frac{-3}{2}}$$

(vi) $y=(x-5).(3-x)$

Solution:

Let $y=(x-5).(3-x)$

Differentiate w.r.t "x"

$$\frac{dy}{dx}\left[(x-5).(3-x)\right]$$

$$\frac{dy}{dx}=(x-5)\frac{d}{dx}(3-x)+(3-x)\frac{d}{dx}(x-5)\ \frac{dy}{dx}=(x-5)(0-1)+(3-x)(1-0)$$

$$\frac{dy}{dx}=-x+5+3-x$$

$$\frac{dy}{dx}=8-2x$$

(vii) $y=\frac{(x^2+1)^2}{x^2-1}$

Solution:

Let $y=\frac{(x^2+1)^2}{x^2-1}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\frac{(x^2+1)^2}{x^2-1}$$

$$\frac{dy}{dx}=\frac{(x^2-1)\frac{d}{dx}(x^2+1)^2-(x^2+1)^2\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{\left(x^2-1\right)2\left(x^2+1\right).2x-\left(x^2+1\right)^2.(2x)}{\left(x^2-1\right)^2}$$

$$\frac{dy}{dx}=\frac{2x\left(x^2+1\right)\left[2x^2-2-x^2-1\right]}{\left(x^2-1\right)^2}$$

$$\frac{dy}{dx}=\frac{2x\left(x^2+1\right)\left(x^2-3\right)}{\left(x^2-1\right)^2}$$

$$\because \left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}$$

$$\because \frac{d}{dx}\left[f(x)\right]^n=n.\left[f(x)\right]^{n-1}.f'(x)$$

(viii) $y=\dfrac{x^2+1}{x^2-3}$

Solution:

Let $y=\dfrac{x^2+1}{x^2-3}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$$

$$\frac{dy}{dx}=\frac{\left(x^2-3\right)\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)\frac{d}{dx}\left(x^2-3\right)}{\left(x^2-3\right)^2} \quad \frac{dy}{dx}=\frac{\left(x^2-3\right)2x-\left(x^2+1\right)2x}{\left(x^2-3\right)^2}$$

$$\frac{dy}{dx}=2x\frac{\left[x^2-3-x^2-1\right]}{\left(x^2-3\right)^2}$$

$$\frac{dy}{dx}=\frac{-8x}{\left(x^2-3\right)^2}$$

(ix) $y=\dfrac{2x-1}{\sqrt{x^2+1}}$

Solution:

Let $y=\dfrac{2x-1}{\sqrt{x^2+1}}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x-1}{\sqrt{x^2+1}}\right)$$

$$\frac{dy}{dx}=\frac{\sqrt{x^2+1}\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}\sqrt{x^2+1}}{\left(\sqrt{x^2+1}\right)^2}$$

$$\frac{dy}{dx}=\frac{\sqrt{x^2+1}(2)-(2x-1)\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}2x}{\left(x^2+1\right)}$$

$$\frac{dy}{dx}=2\sqrt{x^2+1}-\frac{\left(2x^2-x\right)}{\sqrt{x^2+1}}$$

Taking L.C.M of numerator

$$\frac{dy}{dx}=\frac{2\left(x^2+1\right)-2x^2+x/\sqrt{x^2+1}}{\left(x^2+1\right)}$$

$$\frac{dy}{dx}=\frac{2x^2+2-2x^2+x}{\left(x^2+1\right)\sqrt{x^2+1}}$$

$$\frac{dy}{dx}=\frac{2+x}{\left(x^2+1\right)\left(x^2+1\right)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=\frac{2+x}{\left(x^2+1\right)^{\frac{3}{2}}}$$

(x) $y=\sqrt{\dfrac{a-x}{a+x}}$

Solution:

Let $y=\sqrt{\dfrac{a-x}{a+x}}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\sqrt{\frac{a-x}{a+x}} \qquad \because \frac{d}{dx}\sqrt{f(x)}=\frac{1}{2\sqrt{f(x)}}f'(x)$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\frac{a-x}{a+x}}}.\frac{d}{dx}\left(\frac{a-x}{a+x}\right)$$

$$\frac{dy}{dx}=\sqrt{\frac{a+x}{2\sqrt{a-x}}}\left[\frac{(a+x)(-1)-(a-x)(1)}{(a+x)^2}\right]$$

$$\frac{dy}{dx}=\frac{\sqrt{a+x}}{2\sqrt{a-x}}\left[\frac{-a-x-a+x}{(a+x)^2}\right]$$

$$\frac{dy}{dx}=\frac{-2a.\sqrt{a+x}}{2\sqrt{a-x}\,(a+x)^2}$$

$$\frac{dy}{dx}=\frac{-a}{\sqrt{a-x}\,(a+x)^{\frac{3}{2}}}$$

(xi) $y=\sqrt{\dfrac{x^2+1}{x^2-1}}$

Solution:

Let $y=\sqrt{\dfrac{x^2+1}{x^2-1}}$ $\qquad \because \dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$ $\qquad \because \dfrac{d}{dx}\sqrt{f(x)}=\dfrac{1}{2\sqrt{f(x)}}f'(x)$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{\frac{x^2+1}{x^2-1}}\right)$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\frac{x^2+1}{x^2-1}}}.\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx}=\frac{\sqrt{x^2-1}}{2\sqrt{x^2+1}}.\frac{(x^2-1)(x^2+1)'-(x^2+1)(x^2-1)'}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{\sqrt{x^2-1}}{2\sqrt{x^2+1}}\frac{(x^2-1)(2x)-(x^2+1)2x}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{2x(x^2-1-x^2-1).\sqrt{x^2-1}}{2\sqrt{x^2+1}.(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{-2x}{\sqrt{x^2+1}(x^2-1)^{\frac{3}{2}}}$$

2. **Find** $\frac{dy}{dx}$ **if** $y=\frac{(\sqrt{x}+1)\left(x^{\frac{3}{2}}-1\right)}{\sqrt{x}-1}, x\neq 1$

Solution:

$$y=\frac{(\sqrt{x}+1)\left(x^{\frac{3}{2}}-1\right)}{\sqrt{x}-1} \qquad \because x\neq 1$$

$$=\frac{(\sqrt{x}+1)\left[(\sqrt{x})^3-(1)^3\right]}{\sqrt{x}-1}$$

$$=\frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+1+\sqrt{x})}{\sqrt{x}-1}$$

$$=(\sqrt{x}+1)(x+1+\sqrt{x})$$

$$=(\sqrt{x}+1)(\sqrt{x}+1+x)$$

$$=(\sqrt{x}+1)^2+(\sqrt{x}+1)x$$

$$=x+1+2\sqrt{x}+x\sqrt{x}+x$$

$$=x^{\frac{3}{2}}+2x+2x^{\frac{1}{2}}+1$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{\frac{3}{2}}+2x+2x^{\frac{1}{2}}+1\right)$$

$$=\frac{d}{dx}\left(x^{\frac{3}{2}}\right)+\frac{d}{dx}(2x)+\frac{d}{dx}\left(2x^{\frac{1}{2}}\right)+\frac{d}{dx}(1)$$

$$=\frac{3}{2}x^{\frac{1}{2}}+2(1)+2.\frac{1}{2\sqrt{x}}+0$$

$$=\frac{3}{2}\sqrt{x}+2+\frac{1}{\sqrt{x}}$$

(Alternative)

Find $\frac{dy}{dx}$ **if** $y=\frac{\left(\sqrt{x}+1\right)\left(x^{\frac{3}{2}}-1\right)}{\sqrt{x}-1}, x\neq 1$

Solution:

First of all simplify the function

$$y=\frac{\left(\sqrt{x}+1\right)\left[\left(x^{\frac{1}{2}}\right)^3-1\right]}{\sqrt{x}-1} \qquad \because a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$y=\frac{\left(\sqrt{x}+1\right)\left[\left(\sqrt{x}-1\right)\left(x+\sqrt{x}+1\right)\right]}{\left(\sqrt{x-1}\right)}$$

$$y=\left(\sqrt{x}+1\right)\left(x+\sqrt{x}+1\right)$$

$$y=x^{\frac{3}{2}}+x+\sqrt{x}+x+\sqrt{x}+1$$

$$y=x^{\frac{3}{2}}+2\sqrt{x}+2x+1$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}x^{\frac{3}{2}}+2\frac{d}{dx}\sqrt{x}+2\frac{d}{dx}x+\frac{d}{dx}(1)$$

$$\frac{dy}{dx}=\frac{3}{2}x^{\frac{3}{2}-1}+2.\frac{1}{2\sqrt{x}}+2+0$$

$$\frac{dy}{dx}=\frac{3\sqrt{x}}{2}+\frac{1}{\sqrt{x}}+2=\frac{3x+2+4\sqrt{x}}{2\sqrt{x}}$$

$$\frac{dy}{dx}=\frac{3x+2+4\sqrt{x}}{2\sqrt{x}}$$

3. Differentiate $\frac{\left(\sqrt{x}+1\right)\left(x^{\frac{3}{2}}-1\right)}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}$ **with respect to** x.

Solution:

Let $y=\frac{\left(\sqrt{x}+1\right)\left(x^{\frac{3}{2}}-1\right)}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}$

$$=\frac{\left(\sqrt{x}+1\right)\left(\left(\sqrt{x}\right)^3-1\right)}{\sqrt{x}(x-1)}$$

$$=\frac{\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)\left(x+\sqrt{x}+1\right)}{\sqrt{x}(x-1)}$$

$$=\frac{(x-1)\left(x+\sqrt{x}+1\right)}{\sqrt{x}(x-1)}$$

$$= \frac{x+\sqrt{x}+1}{\sqrt{x}}$$

Differentiate w.r.t "x" we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x+\sqrt{x}+1}{\sqrt{x}}\right]$$

$$= \frac{\sqrt{x}\frac{d}{dx}\left(x+\sqrt{x}+1\right)-\left(x+\sqrt{x}+1\right)\frac{d}{dx}\left(\sqrt{x}\right)}{\left(\sqrt{x}\right)^2} = \frac{\sqrt{x}\left(1+\frac{1}{2}x^{-\frac{1}{2}}+0\right)-\left(x+\sqrt{x}+1\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{x}$$

$$= \frac{\sqrt{x}\left(1+\frac{1}{2\sqrt{x}}\right)-\left(x+\sqrt{x}+1\right)\frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{\sqrt{x}\left(\frac{2\sqrt{x}+1}{2\sqrt{x}}\right)-\frac{x+\sqrt{x}+1}{2\sqrt{x}}}{x}$$

$$= \frac{2x+\sqrt{x}-x-\sqrt{x}-1}{\sqrt{x}.2\sqrt{x}} = \frac{x-1}{2x^{\frac{3}{2}}}$$

4. **If** $y=\sqrt{x}-\frac{1}{\sqrt{x}}$ **show that** $2x\frac{dy}{dx}+y=2\sqrt{x}$

Solution:

If $y=\sqrt{x}-\frac{1}{x}$

Multiplying by $\sqrt{x}$

$$\sqrt{x}.y = x-1$$

Differentiate w.r.t "x"

$$\frac{d}{dx}\left(\sqrt{x}.y\right) = \frac{d}{dx}(x-1)$$

Using product rule

$$\sqrt{x}\frac{d}{dx}(y)+y\frac{d}{dx}\sqrt{x} = 1$$

$$\sqrt{x}\frac{dy}{dx}+y.\frac{1}{2\sqrt{x}} = 1$$

Multiplying by $2\sqrt{x}$

$$2x\frac{dy}{dx}+y=2\sqrt{x}$$

5. **If** $y = x^4 + 2x^2 + 2$ **prove that** $\frac{dy}{dx} = 4x\sqrt{y-1}$

Solution:

Let $y = x^4 + 2x^2 + 2$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}x^4 + 2\frac{d}{dx}x^2 + \frac{d}{dx}(2)$$

$$\frac{dy}{dx} = 4x^3 + 2.(2x) + 0$$

$$\frac{dy}{dx} = 4x(x^2+1) = 4x\sqrt{(x^2+1)^2}$$

$\because$ if $x > 0$ $\quad\because \sqrt{x^2} = x$

$$\frac{dy}{dx} = 4x\sqrt{x^4 + 2x^2 + 1}$$

$$\frac{dy}{dx} = 4x\sqrt{x^4 + 2x^2 + 2 - 1}$$

$\because y = x^4 + 2x^2 + 2$

$$\frac{dy}{dx} = 4x\sqrt{y-1}$$

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