Unit # 13, Differentiation

Differentiation:

Differentiation of a function $y = f(x)$ is the process of finding the derivative $f'(x)$, which represents the rate of change of $'y'$ with respect to $'x'$.

Mathematically, it is defined as:

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Where

- δx is a very small increment in x.
- $f(x+\delta x) - f(x)$ is the corresponding change in y.

In simple words:

Differentiation measure how a small change in x affects y.

Example: Speed of a car

Derivative as the Limit of a Difference Quotient:

Let f be a real valued function continuous in the interval $(x, x_1) \subseteq D_f$ then $\dfrac{f(x_1) - f(x)}{x_1 - x}$ is called average rate of change of the function.

If x_1 approaches to x then $\lim\limits_{x_1 \to x} \dfrac{f(x_1) - f(x)}{x_1 - x}$ is called the instantaneous rate of change of function with respect to x and is written as $f'(x)$ (read as "f prime of x")

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Finding $f'(x)$ from definition of derivative:

If $y = f(x)$ (i)

Step 1 $y + \delta y = f(x + \delta x)$ (ii)

Subtracting equation (i) from equation (ii)

Step 2 $\delta y = f(x + \delta x) - f(x)$

Step 3 $\dfrac{\delta y}{\delta x} = \dfrac{f(x + \delta x) - f(x)}{\delta x}$

Step 4 $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x) - f(x)}{\delta x}$ and

$\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ is denoted by $\dfrac{dy}{dx}$, so

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

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EXERCISE 13.1

1. Find by definition, the derivatives w.r.t 'x' of the following functions defined as:

(i) $y = 2x^2 + 1$

Solution:

Let $y = 2x^2 + 1$

Step I: Taking small increment

$$y + \delta y = 2(x + \delta x)^2 + 1$$

Step II:

$$\delta y = 2(x + \delta x)^2 + 1 - y$$

$$\delta y = 2\left(x^2 + (\delta x)^2 + 2x.\delta x\right) + 1 - 2x^2 - 1$$

$$\delta y = 2x^2 + 2(\delta x)^2 + 4x\delta x + 1 - 2x^2 - 1$$

$$\delta y = \delta x[2\delta x + 4x]$$

Step III: Divide by δx on both sides,

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x}[2\delta x + 4x]$$

Step IV: Taking limit as δx approaches to 0,

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}(2\delta x + 4x)$$

$$\frac{dy}{dx} = 2(0) + 4x$$

$$\frac{dy}{dx} = 4x$$

(ii) $y = 2 - \sqrt{x}$

Solution:

Let $y = 2 - \sqrt{x}$

Taking small increment,

$$y + \delta y = 2 - \sqrt{x + \delta x}$$

$$\delta y = 2 - \sqrt{x + \delta x} - y$$

$$\delta y = 2 - \sqrt{x + \delta x} - \left(2 - \sqrt{x}\right)$$

$$\delta y = 2 - \sqrt{x + \delta x} - 2 + \sqrt{x}$$

$$\delta y = \sqrt{x} - \sqrt{x + \delta x}$$

$$\delta y = \left(\sqrt{x} - \sqrt{x + \delta x}\right).\left[\frac{\sqrt{x} + \sqrt{x + \delta x}}{\sqrt{x} + \sqrt{x + \delta x}}\right]$$

$$\delta y = \frac{\left(\sqrt{x}\right)^2 - \left(\sqrt{x + \delta x}\right)^2}{\left(\sqrt{x} + \sqrt{x + \delta x}\right)}$$

$$\delta y = \frac{x - x - \delta x}{\sqrt{x} + \sqrt{x + \delta x}}$$

$$\delta y = \delta x\left[\frac{-1}{\sqrt{x} + \sqrt{x + \delta x}}\right]$$

Dividing by δx on both sides

$$\frac{\delta y}{\delta x}=\frac{\delta x}{\delta x}\left[\frac{-1}{\sqrt{x}+\sqrt{x+\delta x}}\right]$$

Taking limit as δx approaches to 0,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{-1}{\left(\sqrt{x}+\sqrt{x+\delta x}\right)}$$

$$\frac{dy}{dx}=\frac{-1}{\left(\sqrt{x}+\sqrt{x+0}\right)}$$

$$\frac{dy}{dx}=-\frac{1}{2\sqrt{x}} \qquad \because y=2-\sqrt{x}$$

$$\frac{d}{dx}\left(2-\sqrt{x}\right)=-\frac{1}{2\sqrt{x}}$$

(iii) $y=\dfrac{1}{\sqrt{x}}$

Solution:

Let $y=\dfrac{1}{\sqrt{x}}$

Taking small increment,

$$y+\delta y=\frac{1}{\sqrt{x+\delta x}}$$

$$\delta y=\frac{1}{\sqrt{x+\delta x}}-y$$

$$\delta y=\frac{1}{\sqrt{x+\delta x}}-\frac{1}{\sqrt{x}}$$

$$\delta y=\frac{\sqrt{x}-\sqrt{x+\delta x}}{\sqrt{x}.\sqrt{x+\delta x}}$$

Rationalizing by numerator

$$\delta y=\frac{\sqrt{x}-\sqrt{x+\delta x}}{\sqrt{x}.\sqrt{x+\delta x}}.\left[\frac{\sqrt{x}+\sqrt{x+\delta x}}{\sqrt{x}+\sqrt{x+\delta x}}\right]$$

$$\delta y=\frac{\left(\sqrt{x}\right)^2-\left(\sqrt{x+\delta x}\right)^2}{\sqrt{x}.\sqrt{x+\delta x}\left[\sqrt{x}+\sqrt{x+\delta x}\right]}$$

$$\delta y=\frac{x-x-\delta x}{\sqrt{x}.\sqrt{x+\delta x}\left[\sqrt{x}+\sqrt{x+\delta x}\right]}$$

$$\delta y=\delta x\left[\frac{-1}{\sqrt{x}.\sqrt{x+\delta x}\left[\sqrt{x}+\sqrt{x+\delta x}\right]}\right]$$

Dividing by δx on both sides

$$\frac{\delta y}{\delta x}=\frac{\delta x}{\delta x}\left[\frac{-1}{\sqrt{x}.\sqrt{x+\delta x}\left[\sqrt{x}+\sqrt{x+\delta x}\right]}\right]$$

Taking limit as δx approaches to 0,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\frac{-1}{\sqrt{x}.\sqrt{x+\delta x}\left[\sqrt{x}+\sqrt{x+\delta x}\right]}\right] \quad \frac{dy}{dx}=\frac{1}{\sqrt{x}.\sqrt{x+0}\left[\sqrt{x}+\sqrt{x+0}\right]}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)=\frac{-1}{x\left(2\sqrt{x}\right)}$$

$\because a^m.a^n=a^{m+n}$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)=\frac{-1}{2x^{\frac{3}{2}}}$$

(iv) $y=x(x-3)$

Solution:

Let $y=x(x-3)$

$y=x^2-3x$

Taking small increment,

$y+\delta y=(x+\delta x)^2-3(x+\delta x)$

$\delta y=x^2+(\delta x)^2+2x\delta x-3x-3\delta x-y$

$\delta y=x^2+(\delta x)^2+2x\delta x-3x-3\delta x+3x-x^2\ \delta y=8x[\delta x+2x-3]$

Dividing by δx

$$\frac{\delta y}{\delta x}=\frac{\delta x}{\delta x}[\delta x+2x-3]$$

Applying $\lim\limits_{\delta x\to 0}$ on both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(\delta x+2x-3)$$

$$\because \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{dy}{dx}$$

$$\frac{dy}{dx}=0+2x-3$$

$$\frac{d}{dx}\left[x(x-3)\right]=2x-3$$

2. Find $\frac{dy}{dx}$ from first principle and find gradient of curve at the given point.

(i) $\sqrt{x+2}$ at $x=6$

Solution:

Let $y=\sqrt{x+2}$

Taking small increment,

$y+\delta y=\sqrt{x+\delta x+2}$

$\delta y=\sqrt{x+\delta x+2}-y$

$\delta y=\sqrt{x+\delta x+2}-\sqrt{x+2}$

By rationalizing, we get

$$\delta y=\sqrt{x+\delta x+2}-\sqrt{x+2}.\left[\frac{\sqrt{x+\delta x+2}+\sqrt{x+2}}{\sqrt{x+\delta x+2}+\sqrt{x+2}}\right]\ \delta y=\frac{\left(\sqrt{x+\delta x+2}\right)^2-\left(\sqrt{x+2}\right)^2}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\delta y=\frac{x+\delta x+2-x-2}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\delta y=\delta x\left[\frac{1}{\sqrt{x+\delta x+2}+\sqrt{x+2}}\right]$$

Dividing by δx on both sides,

$$\frac{\delta y}{\delta x}=\frac{\delta x}{\delta x}\left[\frac{1}{\sqrt{x+\delta x+2}+\sqrt{x+2}}\right]$$

Taking limit as δx approaches to 0,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{1}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{x+0+2}+\sqrt{x+2}}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x+2}}$$

At $x=6$

$$\left.\frac{dy}{dx}\right|_{x=6}=\frac{1}{2\sqrt{6+2}}$$

$$\left.\frac{dy}{dx}\right|_{x=6}=\frac{1}{4\sqrt{2}}$$

(ii) **Let** $y=\frac{1}{\sqrt{x+a}}$ at $x=a$

Solution:

$$y=\frac{1}{\sqrt{x+a}}$$

Taking small increment,

$$y+\delta y=\frac{1}{\sqrt{x+a+\delta x}}$$

$$\delta y=\frac{1}{x+a+\delta x}-y$$

$$\delta y=\frac{1}{\sqrt{x+a+\delta x}}-\frac{1}{\sqrt{x+a}}$$

$$\delta y\frac{\sqrt{x+a}-\sqrt{x+a+\delta x}}{\sqrt{x+a+\delta x}.\sqrt{x+a}}$$

Rationalizing by numerator

$$\delta y=\frac{\sqrt{x+a}-\sqrt{x+a+\delta x}}{\sqrt{x+a+\delta x}.\sqrt{x+a}}.\left[\frac{\sqrt{x+a}+\sqrt{x+a+\delta x}}{\sqrt{x+a}+\sqrt{x+a+\delta x}}\right]$$

$$\delta y=\frac{\left(\sqrt{x+a}\right)^2-\left(\sqrt{x+a+\delta x}\right)^2}{\left(\sqrt{x+a+\delta x}.\sqrt{x+a}\right)\left(\sqrt{x+a+\delta x}+\sqrt{x+a}\right)}$$

$$\delta y=\frac{x+a-x-a-\delta x}{\left(\sqrt{x+a+\delta x}.\sqrt{x+a}\right)\left[\sqrt{x+a+\delta x}+\sqrt{x+a}\right]}$$

$$\delta y=\delta x\left[\frac{-1}{\left(\sqrt{x+a+\delta x}\sqrt{x+a}\right)\left(x+a+\delta x+\sqrt{x+a}\right)}\right]$$

Dividing by δx on both sides,

$$\frac{\delta y}{\delta x}=\frac{\delta x}{\delta x}\left[\frac{-1}{\sqrt{x+a+\delta x}.\sqrt{x+a}\left(\sqrt{x+a+\delta x}+\sqrt{x+a}\right)}\right]$$

Taking limit as δx approaches to 0,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\frac{-1}{\left(\sqrt{x+a+\delta x}\sqrt{x+a}\right)\left(\sqrt{x+a+\delta x}+\sqrt{x}\right)}\right]$$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{x+a+0}\sqrt{x+a}\left[\sqrt{x+a+0}+\sqrt{x+a}\right]}$$

$$\frac{dy}{dx}=\frac{-1}{\left(x+a\right)\left[2\sqrt{x+a}\right]}$$

$$\frac{dy}{dx}=\frac{-1}{2(x+a)^{\frac{3}{2}}}$$

At $x=a$

$$\left.\frac{dy}{dx}\right|_{x=0}=\frac{-1}{2(a+a)^{\frac{3}{2}}}$$

$$\left.\frac{dy}{dx}\right|_{x=a}=\frac{-1}{2(2a)^{\frac{3}{2}}}$$

Gradient $=\dfrac{-1}{2^{\frac{5}{2}}.a^{\frac{3}{2}}}$

3.

(i) Find the derivative of $x^{\frac{2}{3}}$ at $x=8$ from the first principle.

Solution:

Let $y=x^{\frac{2}{3}}$

Taking small increment,

$$y+\delta y=(x+\delta x)^{\frac{2}{3}}$$

Step II:

$$\delta y=(x+\delta x)^{\frac{2}{3}}-y$$

$$\delta y=(x+\delta x)^{\frac{2}{3}}-x^{\frac{2}{3}}$$

$$\delta y=x^{\frac{2}{3}}\left[1+\frac{\delta x}{x}\right]^{\frac{2}{3}}-x^{\frac{2}{3}}$$

By using binomial series expansion $\because (1+x)^n=1+nx+\text{involving terms } x\ldots)$

$$\delta y=x^{\frac{2}{3}}\left[1+\frac{2}{3}\left(\frac{\delta x}{x}\right)+\ldots\text{involving terms..}\left(\frac{\delta x}{x}\right)\ldots-1\right]$$

$$\delta y=x^{\frac{2}{3}}\delta x\left[\frac{2}{3}.\frac{1}{x}+\ldots\text{involving terms...}\left(\frac{\delta x}{x}\right)\right]$$ Dividing by δx on both sides,

$$\frac{\delta y}{\delta x}=\frac{x^{\frac{2}{3}}\delta x}{\delta x}\left[\frac{2}{3}.\frac{1}{x}+\text{involving terms}\frac{\delta x}{x}\right]$$

Taking limit as δx approaches to 0,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{\frac{2}{3}}\left[\frac{2}{3}.\frac{1}{x}+\text{involving terms}\frac{\delta x}{x}\right]\ \frac{dy}{dx}=x^{\frac{2}{3}}\left[\frac{2}{3x}+0+0+....+0...\right]$$

$$\frac{dy}{dx}=\frac{2}{3}x^{\frac{2}{3}-1}$$

$$\frac{dy}{dx}=\frac{2}{3}x^{\frac{-1}{3}}$$

At $x=8$

$$\left.\frac{dy}{dx}\right|_{x=8}=\frac{2}{3}(8)^{\frac{-1}{3}}$$

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$$\left.\frac{dy}{dx}\right|_{x=8}=\frac{2}{3}\times 2^{3\times\frac{-1}{3}}$$

$$\left.\frac{dy}{dx}\right|_{x=8}=\frac{2}{3}2^{-1}$$

$$\left.\frac{dy}{dx}\right|_{x=8}=\frac{1}{3}$$

(ii) Find the derivative of x^2+2x+3 by definition.

Solution:

Let $y=x^2+2x+3$

Taking small increment,

$$y+\delta y=(x+\delta x)^2+2(x+\delta x)+3$$

Step II:

$$\delta y=x^2+(\delta x)^2+2x\delta x+2x+2\delta x+3-y \quad \delta y=x^2+(\delta x)^2+2x\delta x+2\delta x+3-x^2-2x-3+2x \quad \delta y=\delta x[\delta x+2x+2]$$

Dividing by δx on both sides,

$$\frac{\delta y}{\delta x}=\frac{\delta x}{\delta x}[\delta x+2x+2]$$

Taking limit as δx approaches to 0,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}[\delta x+2x+2]$$

$$\frac{dy}{dx}=0+2x+2$$

$$\frac{d}{dx}(y)=2x+2$$

$$\frac{d}{dx}\left(x^2+2x+3\right)=2x+2$$

4. Find from first principle, the derivatives of the following expressions w.r.t their respective independent variables:

(i) $(3x-2)^{-2}$

Solution:

Let $y=(3x-2)^{-2}$

Taking small increment,

$$y+\delta y=[3(x+\delta x)-2]^{-2}$$

Step II:

$$\delta y=(3x+3\delta x-2)^{-2}-y$$

$$\delta y=(3x-2+3\delta x)^{-2}-(3x-2)^{-2}$$

$$\delta y=(3x-2)^{-2}\left[1+\frac{3\delta x}{3x-2}\right]^{-2}-(3x-2)^{-2} \quad \delta y=(3x-2)^{-2}\left[\left(1+\frac{3\delta x}{3x-2}\right)^{-2}-1\right]$$

By using Binomial series expansion

$$\because (1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

$\because (1+x)^n = 1+nx+\text{involving terms } x$

$\delta y = (3x-2)^{-2}\left[1-2\dfrac{3\delta x}{3x-2}+\text{involving terms}-1\right]$ $\delta y = (3x-2)^{-2}\delta x\left[\dfrac{-6}{3x-2}+\text{involving terms }\delta x\right]$ Dividing by δx on both sides,

$\dfrac{\delta y}{\delta x} = \dfrac{(3x-2)^{-2}\delta x}{\delta x}\left[\dfrac{-6}{3x-2}+\text{involving terms }\delta x\right]$ Taking limit as δx approaches to 0,

$\lim\limits_{\delta x\to 0}\dfrac{\delta y}{\delta x} = \lim\limits_{\delta x\to 0}(3x-2)^{-2}\left[\dfrac{-6}{3x-2}+\text{involving terms }\delta x\right]$ $\dfrac{dy}{dx} = (3x-2)^{-2}\left[\dfrac{-6}{3x-2}+0+0+\ldots\right]$

$$\frac{dy}{dx} = -6(3x-2)^{-3}$$

$$\frac{d}{dx}(y) = -6(3x-2)^{-3}$$

$$\frac{d}{dx}(3x-2)^{-2} = -6(3x-2)^{-3}$$

(ii) $y=(2t+3)^5$

Solution:

Let $y=(2t+3)^5$

Taking small increment,

$$y+\delta y = \left[2(t+\delta t)+3\right]^5$$

Step II:

$$\delta y = (2t+3+2\delta t)^5 - y$$

$$\delta y = \left[(2t+3)+2\delta t\right]^5 - (2t+3)^5$$

By using Binomial Theorem i-e

$$(a+b)^n = a^n + n.a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + b^n$$

$$\delta y = (2t+3)^5 + 5(2t+3)^4(2\delta t) + \ldots$$

$$+(2\delta t)^5 - (2t+3)^5$$

$$\delta y = \delta t\left[10(2t+3)^4 + \ldots + 2^5(\delta t)^4\right]$$

Dividing by δx on both sides,

$$\frac{\delta y}{\delta t} = \frac{\delta t}{\delta t}\left[10(2t+3)^4 + \ldots + 32(\delta t)^4\right]$$

Taking limit as δx approaches to 0,

$\lim\limits_{\delta t\to 0}\dfrac{\delta y}{\delta t} = \lim\limits_{\delta t\to 0}\left[10(2t+3)^4 + \ldots + 32(\delta t)^4\right]$ $\dfrac{dy}{dt} = 10(2t+3)^4 + 0 + 0 + \ldots + 0$

$$\frac{d}{dt}(2t+3)^5 = 10(2t+3)^4$$

(iii) $y=(aw+b)^7$

Solution:

Let $y=(aw+b)^7$

Taking small increment,

$y+\delta y=\left(aw+b+a\delta w\right)^7$

Step II:

$\delta y=\left(aw+b+a\delta w\right)^7-y$

$\delta y=\left(aw+b+a\delta w\right)^7-\left(aw+b\right)^7$

By using binomial theorem from $n\in N$

$\left(a+b\right)^n=a^n+n.a^{n-1}b+...+b^n$

$\delta y=\left(aw+b\right)^7+7.\left(aw+b\right)^6a\delta w+...+\left(a\delta w\right)^7-\left(aw+b\right)^7$ $\delta y=\delta w\left[70\left(aw+b\right)^6+...+a^7.\left(\delta w\right)^6\right]$

Dividing by δx on both sides,

$\frac{\delta y}{\delta w}=\frac{\delta w}{\delta w}\left[7a\left(aw+b\right)^6+...+a^7\left(\delta w\right)^6\right]$

Taking limit as δx approaches to 0,

$\lim_{\delta w\to 0}\frac{\delta y}{\delta w}=\lim_{\delta w\to 0}\left[7a\left(aw+b\right)^6+...+a^7\left(\delta w\right)^6\right]$ $\frac{dy}{dw}=7a\left(aw+b\right)^6+0+0+...+0^7\left(0\right)$

$\frac{dy}{dw}=7a\left(aw+b\right)^6$

$\frac{d}{dw}\left(aw+b\right)^7=7a\left(aw+b\right)^6$

5. Find the gradient and equation of tangent line to $y=3x^2-4x+1$ at $x=2$

Solution:

Gradient $=m\Rightarrow\lim_{x\to a}\frac{f\left(x\right)-f\left(a\right)}{x-a}$

$m=\lim_{x\to 2}\frac{f\left(x\right)-f\left(2\right)}{x-2}$

$m=\lim_{x\to 2}\left[\frac{3x^2-4x+1-\left(3\left(2\right)^2-4\left(2\right)+1\right)}{x-2}\right]$ $m=\lim_{x\to 2}\left[\frac{3x^2-4x+1-5}{x-2}\right]$

$m=\lim_{x\to 2}\left(\frac{3x^2-4x-4}{x-2}\right)$

$m=\lim_{x\to 2}\left(\frac{3x^2-6x+2x-4}{x-2}\right)$

$m=\lim_{x\to 2}\frac{\left(3x+2\right)\left(x-2\right)}{\left(x-2\right)}$

$m=3\left(2\right)+2$

$m=8$

Hence gradient =8

If $x=2$ then $y=3\left(2\right)^2-4\left(2\right)+1$

$y=12-8+1$

$y=5$

So point is $\left(2,5\right)=\left(x_1,y_1\right)$

Equation of tangent is at $(2,5)$

$y-y_1=m(x-x_1)$

$y-5=8(x-2)$

$y-5=8x-16$

$8x-y-11=0$

$y=8x-11$

6. For the function $f(x)=2x^3+x$, calculate the equation of the tangent line at $x=-1$.

Solution:

Step I: To find point (x_1,y_1)

If $x_1=-1$ then

$y_1=2(-1)^3+(-1)$

$y_1=-3 \Rightarrow f(-1)=-3$

$P(x_1,y_1)=(-1,-3)$

Step II:

To find slope (m) or gradient

$m=\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$

$m=\frac{d}{dx}(2x^3+x)$ at $(-1,-3)$

$m=2\frac{d}{dx}x^3+\frac{d}{dx}(x)$ at $(-1,-3)$

$\because \frac{d}{dx}x^n=n.x^{n-1}$

$m=2(3)x^2+1$ at $(-1,-3)$

$m=6(-1)^2+1$

$m=7$

Required Eq. of tangent is

$y-y_1=m(x-x_1)$

$y-(-3)=7(x-(-1))$

$y+3=7x+7$

$y=7x+4$

7. Find the coordinates of the point of tangency and the equation of tangent line for $f(x)=x^3-2x+1$ at $x=1$

Solution:

For point of tangency

If $x=1$ then $y=(1)^3-2(1)+1$

$y=1-2+1$

$y=0$

Point of tangency is $(1,0)$

Slope of tangent line

$m=\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ $\qquad \because y=f(x)$

$m=\frac{d}{dx}x^3-2\frac{d}{dx}(x)+\frac{d}{dx}(1)$

$m=3x^2-2$ at $(1,0)$

$m=3(1)^2-2$

$m=1$ and $P(1,0)$

Required equation of tangent is

$y-y_1=m(x-x_1)$

$y-0=1(x-1)$

$y=x-1$

8. Find the gradient of the curve $f(x)=3x^2+2x$ at $x=1$

Solution:

Gradient = m

$m=\frac{dy}{dx}$ at (x_1,y_1)

If $x_1=1$ then $y_1=3(1)^2+2(1)$

$x_1=1$ then $y_1=5$

$(x_1,y_1)=(1,5)$

$m=\frac{d}{dx}(3x^2+2x)$ at $(1,5)$

$m=3\frac{d}{dx}x^2+2\frac{d}{dx}(x)$ at $(1,5)$

$\because \frac{d}{dx}x^n=nx^{n-1}$

$m=3(2)(x)+2(1)$ at $(1,5)$

$m=6x+2$ at $(1,5)$

$m=6(1)+2$

$m=8$

9. Find the gradient and an equation of tangent line to the graph of $f(x)=\sqrt{x}$ at $x=9$.

Solution:

$f(x)=\sqrt{x} \Rightarrow y=\sqrt{x}$

If $x=9$ then $y=\sqrt{9}$

$x=9$ and $y=3$

So point of tangency is $(9,3)$

Gradient $\Rightarrow m=\frac{dy}{dx}$ at (x_1,y_1)

Here $y=\sqrt{x}$

$(x_1,y_1)=(9,3)$

$\frac{dy}{dx}=\frac{d}{dx}(x)^{\frac{1}{2}}$ at $(9,3)$

By using theorem

$\frac{d}{dx}x^n=nx^{n-1}$

$m=\frac{1}{2\sqrt{x}}$ at $(9,3)$

$m=\frac{1}{2\sqrt{9}} \Rightarrow m=\frac{1}{6}$

Equation of tangent line is:

$y-y_1=m(x-x_1)$

$y-3=\frac{1}{6}(x-9)$

$6y-18=x-9$

$x-6y+9=0$

is required equation of tangent

10. The position of a car after t hours is given by: $s(t)=2t^3-3t^2+t$ **(in kilometres)**

(i) Find average velocity over the interval $[1,4]$

Solution:

$s(t)=2t^3-3t^2+t$

$[1,4]=(a,b)$

Let $s(t)$ distance equation by using formula.

Average velocity $=\frac{s(b)-s(a)}{b-a}$

$s(b)=s(4) \Rightarrow 2(4)^3-3(4)^2+4$

$s(4)=128-48+4$

$s(4)=80+4$

$s(4)=84$

$s(a)=s(1) \Rightarrow 2(1)^3-3(1)^2+4$

$s(1)=2-3+4$

$s(1)=3$

Average velocity $=\frac{s(4)-s(1)}{4-1}$

Average velocity $=\frac{84-3}{3}$

Average velocity = 27 km/h

(ii) Find the instantaneous velocity at $t=2$

Solution:

Instantaneous Velocity at $t=2$

Instantaneous Velocity $\lim_{t\to 2}\frac{s(t)-s(2)}{t-2}$

$\because \lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$

So,

$s'(t)=2\frac{d}{dt}(t)^3-3\frac{d}{dt}(t)^2+\frac{d}{dt}(t)$

Using $\frac{d}{dx}x^n=nx^{n-1}$

$s'(t)=6t^2-6t+1$

$s'(2)=24-12+1$

$a'(2)=13$

So Instantaneous Velocity is 13km/h

$s(t)=-16t^2+32t+10$

11. A stone is thrown upwards and its height after t seconds is given by: $s(t)=-16t^2+32t+10$ (in feet), find the instantaneous velocity at $t=1$.

Solution:

$s(t)=-16t^2+32t+10$

$v(t)=s'(t)=\frac{d}{dt}\left(-16t^2+32t+10\right)$

$\because \frac{d}{dt}\left(at^n\right)=nat^{n-1}:$

$v(t)=-16(2)t^{2-1}+32(1)t^{1-1}+0$

$v(t)=-32t+32$

Instantaneous velocity at $t=1$

Substitute $t=1$ into the velocity function $v(t)$

$v(1)=-32(1)+32$

$v(1)=-32+32$

$v(1)=0$

The instantaneous velocity at $t=1$ is 0 feet per second.

12. The outdoor temperature $(\text{in } ^\circ\text{C})$ over time is modeled by: $T(t)=-t^2+12t+10$, where t is the time in hours. Find the instantaneous rate of change at $t=2$.

Solution:

By using formula

Instantaneous Rate of change $=\lim_{t\to a}\frac{T(t)-T(a)}{t-a}$

$$\because \lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

$$T'(a)=\lim_{t\to a}\frac{T(t)-T(a)}{t-a}$$

$$T(t)=-t^2+12t+10$$

$$\frac{d}{dt}\left[T(t)\right]=-\frac{d}{dt}t^2+12\frac{d}{dt}(t)+\frac{d}{dt}(10)$$

$$T(t)=-2t+12$$

$$T'(2)=-2(2)+12$$

$$T'(2)=8$$