Unit # 12, Limit and Continuity

Continuous Function:

A function f is said to be **continuous** at a number "c" if and only if the following three conditions are satisfied.

(i) $f(c)$ is defined

(ii) $\lim\limits_{x\to c} f(x)$ exists

(iii) $\lim\limits_{x\to c} f(x) = f(c)$

Discontinuous Function:

If one or more of these conditions fail to hold at "c", then the function f is said to be **discontinuous** at "c".

Existence of Limit of a Function

$\lim\limits_{x\to c} f(x) = L$ if and only if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = L$

EXERCISE 12.2

Q.1 Determine the left hand limit and the right hand limit and then, find the limit of the following functions when $x \to c$.

(i) $f(x) = 2x^2 + x - 5,\ c = 1$

Solution:

$f(x) = 2x^2 + x - 5$

L.H.L= $\underset{x\to 1^-}{Lim} f(x) = \underset{x\to 1^-}{Lim}(2x^2 + x - 5)$

$= 2(1)^2 + 1 - 5 = 2 + 1 - 5 = -2$

R.H.L = $\underset{x\to 1^+}{Lim} f(x) = \underset{x\to 1^+}{Lim}(2x^2 + x - 5)$

$= 2(1)^2 + 1 - 5 = 2 + 1 - 5 = -2$

As $\underset{x\to 1^-}{Lim} f(x) = \underset{x\to 1^+}{Lim} f(x)$

Hence $\underset{x\to 1}{Lim} f(x)$ exists and

$\underset{x\to 1}{Lim} f(x) = \underset{x\to 1}{Lim}(2x^2 + x - 5) = -2$

(ii) $f(x) = \dfrac{x^2 - 9}{x - 3},\quad c = -3$

Solution:

$f(x) = \dfrac{x^2 - 9}{x - 3},\quad c = -3$

L.H.L= $\underset{x\to -3^-}{Lim} f(x) = \underset{x\to -3^-}{Lim} \dfrac{x^2 - 9}{x - 3}$

$= \dfrac{(-3)^2 - 9}{-3 - 3} = \dfrac{9 - 9}{-6} = \dfrac{0}{-6} = 0$

R.H.L= $\underset{x\to -3^+}{Lim} f(x) = \underset{x\to -3^+}{Lim} \dfrac{x^2 - 9}{x - 3}$

$= \dfrac{(-3)^2 - 9}{-3 - 3} = \dfrac{9 - 9}{-6} = \dfrac{0}{-6} = 0$

As $\underset{x\to -3^-}{Lim} f(x) = \underset{x\to -3^+}{Lim} f(x)$

Hence, $\underset{x\to -3}{Lim} f(x)$ exists

and $\underset{x\to -3}{Lim} f(x) = \underset{x\to -3}{Lim} \dfrac{x^2 - 9}{x - 3} = 0$

(iii) $f(x)=|x-5|, \quad c=5$

Solution:

$f(x)=|x-5|, \quad c=5$

L.H.L= $\lim\limits_{x\to 5^-} f(x)=\lim\limits_{x\to 5^-}|x-5|$

$$|x-5|=\begin{cases}+(x-5) & \text{if } x\geq 5\\ -(x-5) & \text{if } x<5\end{cases}$$

$=\lim\limits_{x\to 5^-}[-(x-5)]=-(5-5)=0$

R.H.L= $\lim\limits_{x\to 5^+} f(x)=\lim\limits_{x\to 5^+}|x-5|$

$=\lim\limits_{x\to 5^+}(x-5) \;=5-5=0$

As, $\lim\limits_{x\to 5^-} f(x)=\lim\limits_{x\to 5^+} f(x)$

Hence, $\lim\limits_{x\to 5} f(x)$ exists and

$\lim\limits_{x\to 5} f(x)=\lim\limits_{x\to 5}|x-5|=0$

Q.2 Discuss the continuity of $f(x)$ at $x=c$:

(i) $f(x)=\begin{cases}2x+5 & \text{if} \quad x\leq 2\\ 4x+1 & \text{if} \quad x>2\end{cases}, \quad c=2$

Solution:

At $x=2$

$f(x)=2x+5 \Rightarrow f(2)=2(2)+5=9$

$f(2)=9$

Now,

L.H.L= $\lim\limits_{x\to 2^-} f(x)$

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}(2x+5)=2(2)+5=9$

R.H.L= $\lim\limits_{x\to 2^+} f(x)$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}(4x+1)=4(2)+1=9$

As,

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^+} f(x)$, so

$\lim\limits_{x\to 2} f(x)$ exists.

And, $f(2)=\lim\limits_{x\to 2} f(x)$

Hence, $f(x)$ is continuous at $x=2$.

(ii) $f(x)=\begin{cases}3x-1 & \text{if} \quad x<1\\ 4 & \text{if} \quad x=1,\\ 2x & \text{if} \quad x>1\end{cases} \quad c=1$

Solution:

At $x=1$

$f(x)=4 \Rightarrow f(1)=4$

Now,

L.H.L= $\lim\limits_{x\to 1^-} f(x)$

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(3x-1)=3(1)-1=2$

R.H.L= $\lim\limits_{x\to 1^+} f(x)$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(2x) = 2\lim\limits_{x\to 1^+}(x) = 2$

As,

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$,

so $\lim\limits_{x\to 1} f(x)$ exists.

And, $f(1) \neq \lim\limits_{x\to 1} f(x)$

$4 \neq 2$

Hence, $f(x)$ is discontinuous at $x=1$

Q.3 **If** $f(x)=\begin{cases} 3x & \text{if} \quad x\le -2 \\ x^2-1 & \text{if} \quad -2<x<2 \\ 3 & \text{if} \quad x\ge 2 \end{cases}$. **Discuss continuity at** $x=2$ **and** $x=-2$.

Solution:

(i) At $x=2$

$f(x)=3 \Rightarrow f(2)=3$

Now,

L.H.L= $\lim\limits_{x\to 2^-} f(x)$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-}(x^2-1) = 2^2-1 = 4-1 = 3$

R.H.L= $\lim\limits_{x\to 2^+} f(x)$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}(3) = 3$

As,

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$, So

$\lim\limits_{x\to 2} f(x)$ exists.

And $f(2) = \lim\limits_{x\to 2} f(x)$

Hence, $f(x)$ is continuous at $x=2$.

(ii) At $x=-2$

$f(x)=3x \Rightarrow f(-2)=3(-2)=-6$

Now,

L.H.L= $\lim\limits_{x\to -2^-} f(x)$

$\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-}(3x) = 3(-2) = -6$

R.H.L= $\lim\limits_{x\to -2^+} f(x)$

$\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+}(x^2-1)$

$=(-2)^2-1 = 4-1 = 3$

As,

$\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$,

So $\lim\limits_{x\to 2} f(x)$ does not exists.

Hence, $f(x)$ is discontinuous at $x=-2$.

Q.4 **If** $f(x)=\begin{cases} x+2 & , \quad x\le -1 \\ c+2 & , \quad x>-1 \end{cases}$

Find "c" **so that** $\lim\limits_{x\to -1} f(x)$ **exists.**

Solution:

As $\lim\limits_{x\to -1} f(x)$ exists.

So,

L.H.L = R.H.L

$\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^+} f(x)$

$\lim\limits_{x\to -1^-}(x+2)=\lim\limits_{x\to -1^+}(c+2)$

$-1+2=c+2\Rightarrow -1=c\Rightarrow c=-1$

Q.5 **Find the values** m **and** n**, so that given function** f **is continuous at** $x=3$**.**

(i) $f(x)=\begin{cases} mx & \text{if} \quad x<3 \\ n & \text{if} \quad x=3 \\ -2x+9 & \text{if} \quad x>3 \end{cases}$

Solution:

At $x=3$, $f(3)=n$

Now,

L.H.L= $\lim\limits_{x\to -3^-} f(x)$

$\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}(mx)=m(3)=3m$

And,

R.H.L= $\lim\limits_{x\to 3^+} f(x)$

$\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}(-2x+9)$

$=-2(3)+9=-6+9=3$

As $f(x)$ is continuous at $x=3$

So, $f(3)=\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^+} f(x)$

$n=3m=3$

So, $n=3$ and $3m=3\Rightarrow m=1$

(ii) $f(x)=\begin{cases} mx & \text{if} \quad x<3 \\ x^2 & \text{if} \quad x\ge 3 \end{cases}$

Solution:

At $x=3$

$f(x)=x^2\Rightarrow f(3)=3^2=9$

Now,

L.H.L= $\lim\limits_{x\to -3^-} f(x)$

$\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}(mx)=3m$

And,

R.H.L= $\lim\limits_{x\to 3^+} f(x)$

$\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}(x^2)=3^2=9$

As $f(x)$ is continuous at $x=3$

So, $f(3)=\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^+} f(x)$

$9 = 3m = 9 \Rightarrow 3m = 3$

$m = 1$

Q.6 **If** $f(x) = \begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} & , x \neq 2 \\ k & , x = 2 \end{cases}$ **find the value of** k **so that** f **is continuous at** $x = 2$.

Solution:

At $x = 2$

$f(x) = k \Rightarrow f(2) = k$

Now,

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$$

By rationalizing with numerator

$$= \lim_{x\to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} \times \frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$= \lim_{x\to 2} \frac{\left(\sqrt{2x+5}\right)^2 - \left(\sqrt{x+7}\right)^2}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x\to 2} \frac{2x+5-(x+7)}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x\to 2} \frac{2x+5-x-7}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x\to 2} \frac{x-2}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$= \frac{1}{\sqrt{2(2)+5}+\sqrt{2+7}} = \frac{1}{\sqrt{9}+\sqrt{9}} = \frac{1}{3+3}$$

$$\lim_{x\to 2} f(x) = \frac{1}{6}$$

As, $f(x)$ is continuous at $x = 2$

So, $f(2) = \lim_{x\to 2} f(x)$

$$k = \frac{1}{6}$$

Q.7 **Given the function** $f(x) = \begin{cases} 2x+3, & x \le 1 \\ -x+4, & x > 1 \end{cases}$ **Discuss the limit and continuity at** $x = 1$.

Solution:

For limit of $f(x)$ at $x = 1$

First, we find the left hand and right hand limit of $f(x)$ at $x = 1$

L.H.L= $\lim_{x\to -1^-} f(x)$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x+3)$

$= 2(1) + 3 = 2 + 3 = 5$

And,

R.H.L $= \lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(-x+4)$

$= -1+4$

$\lim_{x\to 1^+} f(x) = 3$

Since, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

So, $\lim_{x\to 1} f(x)$ does not exist and when $\lim_{x\to 1} f(x)$ does not exist, then $f(x)$ is discontinuous at $x=1$