# Unit # 12, Limit and Continuity

## Limit of a Function

Let a function $f(x)$ be defined in an open interval near the number "$a$" (need not be at a). If, as $x$ approaches "$a$" from both left and right side of "$a$" $f(x)$ approaches a specific number "$L$" then "$L$", is called the Limit of $f(x)$ as $x$ approaches $a$. Symbolically it is written as:

$$\lim_{x\to a} f(x) = L \text{ read as "limit of } f(x) \text{ as } x\to a \text{ is } L\text{"}$$

## Theorems on Limits of a Functions

Let $f$ and $g$ be two functions for which $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

**(a) The limit of sum of two functions is equal to the sum of their limits.**

$$\lim_{x\to a}\left[f(x)+g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$

**(b) The limit of the difference of two functions is equal to the difference of their limits.**

$$\lim_{x\to a}\left[f(x)-g(x)\right] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$$

**(c) if $k$ is any real number, then**

$$\lim_{x\to a}\left[kf(x)\right] = k\lim_{x\to a} f(x) = kL$$

**(d) The limit of the product of the functions is equal to the product of their limits.**

$$\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x).\lim_{x\to a} g(x) = LM$$

**(e) The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of denominator is non-zero.**

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$$

**(f) Limit of $\left[f(x)\right]^n$, where $n$ is an integer**

$$\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n$$

**(g)** $\lim_{x\to a} x^p = a^p$, **where** $p > 0$ and $p \in R$

$$\lim_{x\to a} c = c$$

## Limit of a Sequence

Let $\{a_n\}$ be a sequence, the limit of a sequence $\{a_n\}$ is the value L that the terms of the sequence approach as $n\to\infty$, that is,

$$\text{Lim}_{n\to\infty} a_n = \text{L}$$

If such an L exists, the sequence is said to converge to L and $\{a_n\}$ is called convergent sequence. If no such L exists, the sequence is said to diverge.

## Limits of Important Functions

If by substituting the number that $x$ approaches into the function, we get $\left(\frac{0}{0}\right)$, then we evaluate the limits by using:

- $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$ where $n$ is non-zero integer and $a>0$.
- $\lim\limits_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$ where $n$ is an integer and $a>0$.
- $\lim\limits_{n\to +\infty}\left(1+\frac{1}{n}\right)^n=e$ Deduction: $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$
- $\lim\limits_{x\to 0}\frac{a^x-1}{x}=\log_e a$ Deduction: $\lim\limits_{x\to 0}\left(\frac{e^x-1}{x}\right)=\log_e e=1$
- Let $p$ be a positive rational number. If $x^p$ is defined, then

  $\lim\limits_{x\to\infty}\frac{a}{x^p}=0$ and $\lim\limits_{x\to-\infty}\frac{a}{x^p}=0$, where $a$ is any real number.
- $\lim\limits_{x\to\infty}e^x=\infty$ and $\lim\limits_{x\to-\infty}e^x=\lim\limits_{x\to\infty}\left(\frac{1}{e^x}\right)=0$
- If $\theta$ is measured in radian, then $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ also this result holds for $-\frac{\pi}{2}<\theta<0$.

## EXERCISE 12.1

**Q.1 Find the limit of the following sequences if exists:**

**(i)** $a_n = \dfrac{2n+3}{n+1}$

**Solution:**

$$a_n = \frac{2n+3}{n+1}$$

Applying limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+3}{n+1}$$

$$= \lim_{n\to\infty} \frac{n\left(2+\frac{3}{n}\right)}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{1+\frac{1}{n}}$$

As $n \to \infty, \frac{3}{n}, \frac{1}{n} \to 0$

$$= \frac{2+0}{1+0} = \frac{2}{1}$$

$$\lim_{n\to\infty} a_n = 2$$

**(ii)** $b_n = \dfrac{2n+3}{n^2+1}$

**Solution:**

$$b_n = \frac{2n+3}{n^2+1}$$

Applying limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2n+3}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{n^2\left(\frac{2}{n}+\frac{3}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\frac{2}{n}+\frac{3}{n^2}}{1+\frac{1}{n^2}}$$

As $n \to \infty, \frac{2}{n}, \frac{3}{n^2}, \frac{1}{n^2} \to 0$

$$= \frac{0+0}{1+0}$$

$$= \frac{0}{1} = 0$$

**(iii)** $c_n = \dfrac{5n^2}{2n+3}$

**Solution:**

$$c_n = \frac{5n^2}{2n+3}$$

Applying limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{5n^2}{2n+3}$$

$$= \lim_{n\to\infty} \frac{5n^2}{n^2\left(\frac{2}{n}+\frac{3}{n^2}\right)} = \lim_{n\to\infty} \frac{5}{\frac{2}{n}+\frac{3}{n^2}}$$

As $n \to \infty, \frac{2}{n}, \frac{3}{n^2} \to 0$

$$= \frac{5}{0+0} = \frac{5}{0} = \infty$$

Since, $\lim\limits_{n\to\infty} c_n$ does not exist, so $\{c_n\}$ is divergent sequence.

**(iv)** $d_n = \dfrac{n^2 - 3n + 1}{2n^2 + n + 4}$

**Solution:**

$$d_n = \frac{n^2 - 3n + 1}{2n^2 + n + 4}$$

Applying limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{n^2 - 3n + 1}{2n^2 + n + 4}$$

$$= \lim_{n\to\infty} \frac{n^2\left(1 - \frac{3}{n} + \frac{1}{n^2}\right)}{n^2\left(2 + \frac{1}{n} + \frac{4}{n^2}\right)} = \lim_{n\to\infty} \frac{1 - \frac{3}{n} + \frac{1}{n^2}}{2 + \frac{1}{n} + \frac{4}{n^2}}$$

As $n \to \infty, \frac{1}{n}, \frac{3}{n}, \frac{1}{n^2}, \frac{4}{n^2} \to 0$

$$\lim_{n\to\infty} d_n = \frac{1 - 0 + 0}{2 + 0 + 0} = \frac{1}{2}$$

**Q.2 Evaluate each limit by using theorems of limits:**

**(i)** $\underset{x\to 3}{Lim}(2x+4)$

**Solution:**

$$\underset{x\to 3}{Lim}(2x+4)$$

$$= \underset{x\to 3}{Lim}(2x) + \underset{x\to 3}{Lim}(4)$$

$$= 2\underset{x\to 3}{Lim}(x) + 4 = 2(3) + 4 = 6 + 4 = 10$$

$$\underset{x\to 3}{Lim}(2x+4) = 10$$

**(ii)** $\underset{x\to 1}{Lim}\left(3x^2 - 2x + 4\right)$

**Solution:**

$$\underset{x\to 1}{Lim}\left(3x^2 - 2x + 4\right)$$

$$= \underset{x\to 1}{Lim}\left(3x^2\right) - \underset{x\to 1}{Lim}(2x) + \underset{x\to 1}{Lim}(4)$$

$$= 3\underset{x\to 1}{Lim}\left(x^2\right) - 2\underset{x\to 1}{Lim}(x) + 4$$

$$= 3(1)^2 - 2(1) + 4 = 3 - 2 + 4 = 5$$

$$\underset{x\to 1}{Lim}\left(3x^2 - 2x + 4\right) = 5$$

**(iii)** $\underset{x\to 3}{Lim}\sqrt{x^2 + x + 4}$

**Solution:**

$$\underset{x\to 3}{Lim}\sqrt{x^2 + x + 4} = \sqrt{\underset{x\to 3}{Lim}\left(x^2 + x + 4\right)}$$

$$= \sqrt{\underset{x\to 3}{Lim}\, x^2 + \underset{x\to 3}{Lim}\, x + \underset{x\to 3}{Lim}\, 4}$$

$$=\sqrt{3^2+3+4}=\sqrt{9+3+4}=\sqrt{16}=4$$

$$\lim_{x\to 3}\sqrt{x^2+x+4}=4$$

**(iv)** $\lim\limits_{x\to 2}\sqrt{x^2+4}$

**Solution:**

$$=\sqrt{\lim_{x\to 2}\left(x^2+4\right)}=\sqrt{\lim_{x\to 2}\left(x^2\right)+\lim_{x\to 2}(4)}$$

$$\lim_{x\to 2}\sqrt{x^2+4}=\sqrt{2^2+4}$$

$$=\sqrt{8}=2\sqrt{2}$$

**(v)** $\lim\limits_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right)$

**Solution:**

$$\lim_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right)$$

$$=\sqrt{\lim_{x\to 2}\left(x^3+1\right)}-\sqrt{\lim_{x\to 2}\left(x^2+5\right)}$$

$$=\sqrt{2^3+1}-\sqrt{2^2+5}=\sqrt{9}-\sqrt{9}$$

$$\lim_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right)=0$$

**(vi)** $\lim\limits_{x\to -2}\dfrac{2x^3+5x}{3x-2}$

**Solution:**

$$\lim_{x\to -2}\frac{2x^3+5x}{3x-2}=\frac{2\lim\limits_{x\to -2}\left(x^3\right)+5\lim\limits_{x\to -2}(x)}{3\lim\limits_{x\to -2}(x)-\lim\limits_{x\to -2}(2)}$$

$$=\frac{2(-2)^3+5(-2)}{3(-2)-2}=\frac{2(-8)+5(-2)}{3(-2)-2}$$

$$=\frac{-16-10}{-6-2}=\frac{-26}{-8}=\frac{13}{4}$$

**Q.3 Evaluate each limit by using algebraic techniques.**

**(i)** $\lim\limits_{x\to -1}\dfrac{x^3-x}{x+1}$

**Solution:**

$$\lim_{x\to -1}\frac{x^3-x}{x+1}\qquad \because a^2-b^2=(a+b)(a-b)$$

$$=\lim_{x\to -1}\frac{x\left(x^2-1\right)}{x+1}=\lim_{x\to -1}\frac{x(x-1)(x+1)}{x+1}$$

$$=\lim_{x\to -1}x(x-1)=-1(-1-1)=-1(-2)=2$$

**(ii)** $\lim_{x\to 3}\left(\dfrac{x^2-5x+6}{x^2-2x-3}\right)$

**Solution:**

$$=\lim_{x\to 3}\frac{x^2-5x+6}{x^2-2x-3}=\lim_{x\to 3}\frac{x^2-3x-2x+6}{x^2-3x+x-3}$$

$$=\lim_{x\to 3}\frac{x(x-3)-2(x-3)}{x(x-3)+1(x-3)}$$

$$=\lim_{x\to 3}\frac{(x-2)(x-3)}{(x+1)(x-3)}=\lim_{x\to 3}\frac{x-2}{x+1}$$

$$=\frac{3-2}{3+1}=\frac{1}{4}$$

So, $\lim_{x\to 3}\dfrac{x^2-5x+6}{x^2-2x-3}=\dfrac{1}{4}$

**(iii)** $\lim_{x\to 2}\left(\dfrac{x^3-8}{x^2-5x+6}\right)$

**Solution:**

$$=\lim_{x\to 2}\frac{x^3-8}{x^2-5x+6}$$

$$\because a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$=\lim_{x\to 2}\frac{x^3-2^3}{x^2-5x+6}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2x+4)}{x^2-2x-3x+6}$$

$$=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x(x-2)-3(x-2)}$$

$$=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-3)(x-2)}$$

$$=\lim_{x\to 2}\frac{x^2+2x+4}{x-3}=\frac{4+4+4}{-1}=-12$$

So, $\lim_{x\to 2}\dfrac{x^3-8}{x^2-5x+6}=-12$

**(iv)** $\mathop{Lim}_{x\to 1}\dfrac{x^3-3x^2+3x-1}{x^3-x}$

**Solution:**

$$\mathop{Lim}_{x\to 1}\frac{x^3-3x^2+3x-1}{x^3-x}$$

$$=\mathop{Lim}_{x\to 1}\frac{(x^3)-3(x)^2(1)+3(x)(1)^2-(1)^3}{x(x^2-1)}$$

$$=\mathop{Lim}_{x\to 1}\frac{(x-1)^3}{x(x^2-1)}=\mathop{Lim}_{x\to 1}\frac{(x-1)^3}{x(x-1)(x+1)}$$



$$=\lim_{x\to 1}\frac{(x-1)^2}{x(x+1)}=\frac{(1-1)^2}{1(1+1)}=\frac{0}{2}=0$$

**(v)** $\lim_{x\to 2}\left(\frac{x^3-6x^2+12x-8}{x^3-4x}\right)$

**Solution:**

$$=\lim_{x\to 2}\frac{x^3-6x^2+12x-8}{x^3-4x}$$

$$=\lim_{x\to 2}\frac{x^3-8-6x^2+12x}{x(x^2-4)}$$

$$=\lim_{x\to 2}\frac{x^3-2^3-6x(x-2)}{x(x^2-4)}$$

$$=\lim_{x\to 2}\frac{x^3-2^3-6x(x-2)}{x(x^2-2^2)}$$

$$=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)-6x(x-2)}{x(x-2)(x+2)}$$

$$=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4-6x)}{x(x-2)(x+2)}$$

$$=\lim_{x\to 2}\frac{x^2-4x+4}{x(x+2)}=\frac{2^2-4(2)+4}{2(2+2)}$$

$$=\frac{4-8+4}{2(4)}=\frac{8-8}{8}=\frac{0}{8}=0$$

So, $\lim_{x\to 2}\frac{x^3-6x^2+12x-8}{x^3-4x}=0$

**(vi)** $\lim_{x\to 1}\left(\frac{x^4-1}{x^2-3x+2}\right)$

**Solution:**

$$=\lim_{x\to 1}\frac{x^4-1}{x^2-3x+2}=\lim_{x\to 1}\frac{(x^2)^2-(1)^2}{x^2-2x-x+2}$$

$\because a^2-b^2=(a+b)(a-b)$

$$=\lim_{x\to 1}\frac{(x^2+1)(x^2-1)}{x(x-2)-1(x-2)}$$

$$=\lim_{x\to 1}\frac{(x^2+1)(x^2-1^2)}{(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{(x^2+1)(x+1)(x-1)}{(x-1)(x-2)}$$



$$=\lim_{x\to1}\frac{\left(x^2+1\right)\left(x+1\right)}{x-2}=\frac{\left(1^2+1\right)\left(1+1\right)}{1-2}$$

$$=\frac{(1+1)(2)}{-1}=-(2)(2)=-4$$

$$\text{So, }\lim_{x\to1}\frac{x^4-1}{x^2-3x+2}=-4$$

**(vii)** $\lim_{x\to2}\frac{x-2}{\sqrt{x+2}-\sqrt{6-x}}$

**Solution:**

$$=\lim_{x\to2}\frac{x-2}{\sqrt{x+2}-\sqrt{6-x}}$$

Rationalizing by 'Denominator'

$$=\lim_{x\to2}\frac{x-2}{\sqrt{x+2}-\sqrt{6-x}}\times\frac{\sqrt{x+2}+\sqrt{6-x}}{\sqrt{x+2}+\sqrt{6-x}}$$

$$=\lim_{x\to2}\frac{(x-2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}{\left(\sqrt{x+2}\right)^2-\left(\sqrt{6-x}\right)^2}$$

$$=\lim_{x\to2}\frac{(x-2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}{x+2-6+x}$$

$$=\lim_{x\to2}\frac{(x-2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}{2x-4}$$

$$=\lim_{x\to2}\frac{(x-2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}{2(x-2)}$$

$$=\lim_{x\to2}\frac{\left(\sqrt{x+2}+\sqrt{6-x}\right)}{2}$$

$$=\frac{\sqrt{2+2}+\sqrt{6-2}}{2}=\frac{\sqrt{4}+\sqrt{4}}{2}$$

$$=\frac{2+2}{2}=\frac{4}{2}=2$$

$$\text{So, }\lim_{x\to2}\frac{x-2}{\sqrt{x+2}-\sqrt{6-x}}=2$$

**(viii)** $\underset{h\to0}{Lim}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

**Solution:**

$$\underset{h\to0}{Lim}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

By rationalizing with numerator

$$=\underset{h\to0}{Lim}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\underset{h\to0}{Lim}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left[\sqrt{x+h}+\sqrt{x}\right]}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h\left[\sqrt{x+h}+\sqrt{x}\right]}=\lim_{h\to 0}\frac{h}{h\left[\sqrt{x+h}+\sqrt{x}\right]}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

**(ix)** $\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}$

**Solution:**

$$\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}$$

$$=\lim_{x\to a}\frac{(x-a)\left(x^{n-1}+ax^{n-2}+a^2x^{n-3}+...+a^{n-1}\right)}{(x-a)\left(x^{m-1}+ax^{m-2}+a^2x^{m-3}+...+a^{m-1}\right)}=\lim_{x\to a}\frac{x^{n-1}+ax^{n-2}+a^2x^{n-3}+......+a^{n-1}}{x^{m-1}+ax^{m-2}+a^2x^{m-3}+......+a^{m-1}}$$

$$=\frac{a^{n-1}+aa^{n-2}+a^2a^{n-3}+......+a^{n-1}}{a^{m-1}+aa^{m-2}+a^2a^{m-3}+.....+a^{m-1}}$$

$$=\frac{a^{n-1}+a^{n-1}+a^{n-1}+......+a^{n-1}}{a^{m-1}+a^{m-1}+a^{m-1}+......+a^{m-1}}$$

$$=\frac{na^{n-1}}{ma^{m-1}}=\frac{na^{n-1-(m-1)}}{m}=\frac{na^{n-1-m+1}}{m}=\frac{n}{m}a^{n-m}$$

**Q.4 Evaluate the following limits:**

**(i)** $\lim_{x\to 0}\frac{\sin 5x}{x}$

**Solution:**

$$\lim_{x\to 0}\frac{\sin(5x)}{x}$$

Multiply and divide by 5

$$=\lim_{x\to 0}5\times\frac{\sin 5x}{5x}=5\lim_{x\to 0}\frac{\sin 5x}{5x}$$

As $x\to 0$, then $5x\to 0$

$$=5\lim_{5x\to 0}\frac{\sin 5x}{5x}=5(1)=5$$

**(ii)** $\lim_{x\to 0}\frac{\sin x^\circ}{x}$

**Solution:**

$$\lim_{x\to 0}\frac{\sin x^\circ}{x}=\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{180}x\right)}{x}\quad\because x^\circ=\frac{\pi}{180}x$$

Multiply and divide by $\frac{\pi}{180}$

$$=\lim_{x\to 0}\frac{\pi}{180}\times\frac{\sin\left(\frac{\pi}{180}x\right)}{\frac{\pi}{180}x}=\frac{\pi}{180}\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{180}x\right)}{\frac{\pi}{180}x}$$

When $x\to 0$, then $\frac{\pi}{180}x\to 0$

$$=\frac{\pi}{180}\lim_{\frac{\pi}{180}x\to 0}\frac{\sin\left(\frac{\pi}{180}x\right)}{\left(\frac{\pi}{180}x\right)}=\frac{\pi}{180}(1)=\frac{\pi}{180}$$



**(iii)** $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\sin\theta}$

**Solution:**

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}$$

By rationalizing with numerator

$$=\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}\times\frac{1+\cos\theta}{1+\cos\theta}$$

$$=\lim_{\theta\to 0}\frac{1^2-\cos^2\theta}{\sin\theta(1+\cos\theta)}=\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}\quad\because \sin^2\theta+\cos^2\theta=1$$

$$=\lim_{\theta\to 0}\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}=\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\quad=\frac{\sin 0}{1+\cos 0}=\frac{0}{1+1}=\frac{0}{2}=0$$

**(iv)** $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin x-\cos x}{x-\frac{\pi}{4}}$

**Solution:**

$$=\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\cos x\right)}{x-\frac{\pi}{4}}$$

$$\therefore \sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}\quad,\quad \cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$=\sqrt{2}\lim_{x\to\frac{\pi}{4}}\frac{\cos\frac{\pi}{4}\sin x-\sin\frac{\pi}{4}\cos x}{x-\frac{\pi}{4}}$$

$$=\sqrt{2}\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\frac{\pi}{4}}{x-\frac{\pi}{4}}$$

$$=\sqrt{2}(1)$$

$$=\sqrt{2}$$

**(v)** $\lim\limits_{x\to 0}\dfrac{\cos ax-\cos bx}{x^2}$

**Solution:**

$$\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2}$$

$$\because \cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$=\lim_{x\to 0}\frac{-2\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{ax-bx}{2}\right)}{x^2}=-2\lim_{x\to 0}\frac{\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{ax-bx}{2}\right)}{x^2}=-2\lim_{x\to 0}\frac{\left[\sin\left(\frac{a+b}{2}\right)x\right]}{x}\frac{\left[\sin\left(\frac{a-b}{2}\right)x\right]}{x}$$

Multiply and divide by $\left(\dfrac{a+b}{2}\right)$ and $\left(\dfrac{a-b}{2}\right)$

$$=-2\lim_{x\to 0}\left[\frac{\sin\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x}\right]\left[\frac{\sin\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x}\right]\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right)$$



$$= -2\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right)\left[\lim_{x\to 0}\frac{\sin\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x}\right]\left[\lim_{x\to 0}\frac{\sin\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x}\right]$$

As $x\to 0, \left(\frac{a+b}{2}\right)x, \left(\frac{a-b}{2}\right)x\to 0$

$$= -(a+b)\left(\frac{a-b}{2}\right)(1)(1) = -\frac{a^2-b^2}{2} = \frac{b^2-a^2}{2}$$

$$= \lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2} = \frac{b^2-a^2}{2}$$

**(vi)** $\lim_{x\to\frac{\pi}{4}}\frac{\tan x-1}{x-\frac{\pi}{4}}$

**Solution:**

$$\lim_{x\to\frac{\pi}{4}}\frac{\tan x-1}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}}\frac{\frac{\sin x}{\cos x}-1}{x-\frac{\pi}{4}}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\frac{\sin x-\cos x}{\cos x}}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\left(x-\frac{\pi}{4}\right)\cos x} \quad \text{(i)}$$

$$\because \sin x-\cos x = \sqrt{2}\left[\left(\frac{1}{\sqrt{2}}\right)\sin x-\left(\frac{1}{\sqrt{2}}\right)\cos x\right] \quad \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = 1$$

$$\sin x-\cos x = \sin x\cos\frac{\pi}{4}-\cos x\sin\frac{\pi}{4}$$

$$\sin x-\cos x = \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$$

Equation (i) becomes

$$= \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)}{\left(x-\frac{\pi}{4}\right)\cos x}$$

Put $h = x-\frac{\pi}{4}$

=As $x\to\frac{\pi}{4}, h\to 0$

$$= \lim_{h\to 0}\frac{\sqrt{2}\sin h}{h\cos\left(h+\frac{\pi}{4}\right)}$$

$$= \sqrt{2}\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\times\lim_{h\to 0}\frac{1}{\cos\left(h+\frac{\pi}{4}\right)}$$

$$= \sqrt{2}(1)\times\frac{1}{\cos\left(0+\frac{\pi}{4}\right)} = \frac{\sqrt{2}}{\cos\frac{\pi}{4}} = \sqrt{2}.\sqrt{2}$$

So, $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\tan x-1}{x-\frac{\pi}{4}}=2$

**(vii)** $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}$

**Solution:**

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$

$$=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}\qquad \because \cos 2x=1-2\sin^2 x$$

$$=2\lim_{x\to 0}\frac{\sin^2 x}{x^2}=2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

$$=2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2=2(1)^2=2$$

**(viii)** $\lim\limits_{x\to 0}\dfrac{\cos ax-\cos bx}{\cos cx-\cos dx}$

**Solution:**

$$\lim_{x\to 0}\frac{\cos ax-\cos bx}{\cos cx-\cos dx}$$

$$\because \cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$=\lim_{x\to 0}\frac{-2\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{ax-bx}{2}\right)}{-2\sin\left(\frac{cx+dx}{2}\right)\sin\left(\frac{cx-dx}{2}\right)}=\lim_{x\to 0}\frac{\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{ax-bx}{2}\right)}{\sin\left(\frac{cx+dx}{2}\right)\sin\left(\frac{cx-dx}{2}\right)}$$

Multiply and divide by $\left(\frac{ax+bx}{2}\right)$, $\left(\frac{ax-bx}{2}\right)$, $\left(\frac{cx+dx}{2}\right)$ and $\left(\frac{cx-dx}{2}\right)$.

$$=\lim_{x\to 0}\frac{\sin\left(\frac{ax+bx}{2}\right)}{\left(\frac{ax+bx}{2}\right)}\times\frac{\sin\left(\frac{ax-bx}{2}\right)}{\left(\frac{ax-bx}{2}\right)}\times\frac{\left(\frac{cx+dx}{2}\right)}{\sin\left(\frac{cx+dx}{2}\right)}\times\frac{\left(\frac{cx-dx}{2}\right)}{\sin\left(\frac{cx-dx}{2}\right)}\times\frac{\left(\frac{ax+bx}{2}\right)\left(\frac{ax-bx}{2}\right)}{\left(\frac{cx+dx}{2}\right)\left(\frac{cx-dx}{2}\right)}$$

$$=\lim_{x\to 0}\frac{\sin\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x}\times\lim_{x\to 0}\frac{\sin\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x}\times\lim_{x\to 0}\frac{\left(\frac{c+d}{2}\right)x}{\sin\left(\frac{c+d}{2}\right)x}\times\lim_{x\to 0}\frac{\left(\frac{c-d}{2}\right)x}{\sin\left(\frac{c-d}{2}\right)x}\times\frac{\frac{x}{2}(a+b).\frac{x}{2}(a-b)}{\frac{x}{2}(c+d)\frac{x}{2}(c-d)}$$

As, $x\to 0,\left(\frac{a+b}{2}\right)x,\left(\frac{a-b}{2}\right)x,\left(\frac{c+d}{2}\right)x,\left(\frac{c-d}{2}\right)x\to 0$

$$=(1)(1)(1)(1)\frac{(a+b)(a-b)}{(c+d)(c-d)}=\frac{a^2-b^2}{c^2-d^2}$$

So,

$$\lim_{x\to 0}\frac{\cos ax-\cos bx}{\cos cx-\cos dx}=\frac{a^2-b^2}{c^2-d^2}$$

**(ix)** $\lim_{x\to 1}\dfrac{x^3-1}{x^2-1}$

**Solution:**

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}$$

$$=\lim_{x\to 1}\frac{x^3-1^3}{x^2-1^2}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x+1)(x-1)}$$

$$=\lim_{x\to 1}\frac{x^2+x+1}{x+1}=\frac{1^2+1+1}{1+1}=\frac{3}{2}$$

So, $\lim_{x\to 1}\dfrac{x^3-1}{x^2-1}=\dfrac{3}{2}$

**(x)** $\lim_{x\to 3}\dfrac{x^2-x\log x+3\log x-9}{x-3}$

**Solution:**

$$\lim_{x\to 3}\frac{x^2-x\log x+3\log x-9}{x-3}$$

$$=\lim_{x\to 3}\frac{x^2-9-x\log x+3\log x}{x-3}$$

$$=\lim_{x\to 3}\frac{x^2-3^2-(\log x)(x-3)}{x-3}$$

$$=\lim_{x\to 3}\frac{(x-3)(x+3)-(\log x)(x-3)}{x-3}$$

$$=\lim_{x\to 3}\frac{(x-3)(x+3-\log x)}{x-3}$$

$$=\lim_{x\to 3}(x+3-\log x)$$

$$=3+3-\log 3=6-\log 3$$

So, $\lim_{x\to 3}\dfrac{x^2-x\log x+3\log x-9}{x-3}=6-\log 3$

**(xi)** $\lim_{x\to 0}\dfrac{x(2^x-1)}{1-\cos x}$

**Solution:**

$$\lim_{x\to 0}\frac{x(2^x-1)}{1-\cos x}$$

Multiply and divide by $x$

$$=\lim_{x\to 0}\frac{x(2^x-1)}{1-\cos x}\times\frac{x}{x}$$

$$=\lim_{x\to 0}\frac{x^2(2^x-1)}{x(1-\cos x)}$$



Rationalizing by '$1-\cos x$'

$$=\lim_{x\to 0}\frac{x^2\left(2^x-1\right)}{x\left(1-\cos x\right)}\times\frac{1+\cos x}{1+\cos x}$$

$$=\lim_{x\to 0}\frac{x^2\left(2^x-1\right)\left(1+\cos x\right)}{x\left(1-\cos^2 x\right)}$$

$\because \sin^2 x+\cos^2 x=1$

$$=\lim_{x\to 0}\frac{x^2\left(2^x-1\right)\left(1+\cos x\right)}{x\sin^2 x}$$

$$=\lim_{x\to 0}\left(\frac{x^2}{\sin^2 x}\right)\times\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)\times\lim_{x\to 0}\left(1+\cos x\right)$$

$$=\left(\lim_{x\to 0}\frac{x}{\sin x}\right)^2\times\left(\lim_{x\to 0}\frac{2^x-1}{x}\right)\times\left(1+\cos 0\right)$$

$$=\left(1\right)^2\times\log_e 2\times\left(1+1\right)=2\log_e 2$$

So, $\lim_{x\to 0}\dfrac{x\left(2^x-1\right)}{1-\cos x}=2\log_e 2$

**Q.5 Express each limit in terms of $e$:**

**(i)** $\underset{n\to+\infty}{Lim}\left(1+\dfrac{1}{n}\right)^{2n}$

**Solution:**

$$\underset{n\to+\infty}{Lim}\left(1+\frac{1}{n}\right)^{2n}$$

$$=\underset{n\to+\infty}{Lim}\left[\left(1+\frac{1}{n}\right)^n\right]^2=\left[\underset{n\to+\infty}{Lim}\left(1+\frac{1}{n}\right)^n\right]^2=e^2$$

**(ii)** $\underset{n\to+\infty}{Lim}\left(1+\dfrac{1}{n}\right)^{\frac{n}{2}}$

**Solution:**

$$\underset{n\to+\infty}{Lim}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}$$

$$=\underset{n\to+\infty}{Lim}\left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{1}{2}}=\left[\underset{n\to+\infty}{Lim}\left(1+\frac{1}{n}\right)^n\right]^{\frac{1}{2}}=e^{\frac{1}{2}}$$

**(iii)** $\lim\limits_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n}$

**Solution:**

$$\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n}=\lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$=\left[\lim_{n\to\infty}\left(1+\left(-\frac{1}{n}\right)\right)^{-n}\right]^{-1}=e^{-1}=\frac{1}{e}$$

**(iv)** $\lim\limits_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{n}$

**Solution:**

$$\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{n}$$

$$=\lim_{n\to+\infty}\left[\left(1+\frac{1}{3n}\right)^{n}\right]^{\frac{3}{3}}=\left[\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{n}=e^{\frac{1}{3}}$$

**(v)** $\lim\limits_{n\to+\infty}\left(1+\frac{4}{n}\right)^{n}$

**Solution:**

$$\lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{n}=\lim_{n\to\infty}\left[\left(1+\frac{4}{n}\right)^{n}\right]^{\frac{4}{4}}$$

$$=\left(\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{\frac{n}{4}}\right)^{4}=e^{4}$$

**(vi)** $\lim\limits_{x\to0}(1+3x)^{\frac{2}{x}}$

**Solution:**

$$\lim_{x\to0}(1+3x)^{\frac{2}{x}}=\lim_{x\to0}\left[(1+3x)^{\frac{2}{x}}\right]^{\frac{3}{3}}$$

$$=\left[\lim_{x\to0}(1+3x)^{\frac{1}{3x}}\right]^{6}=e^{6}$$

**(vii)** $\lim\limits_{x\to0}(1+2x^{2})^{\frac{1}{x^{2}}}$

**Solution:**

$$\lim_{x\to0}(1+2x^{2})^{\frac{1}{x^{2}}}=\lim_{x\to0}\left[(1+2x^{2})^{\frac{1}{x^{2}}}\right]^{\frac{2}{2}}$$

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$$\left[\underset{x\to 0}{Lim}\left(1+2x^2\right)^{\frac{2}{x^2}}\right]^2 = e^2$$

**(viii)** $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^{bx}}{abx}$

**Solution:**

We can rewrite the given expression as:

$$=\lim_{x\to 0}\frac{e^{ax}-1-e^{bx}+1}{abx}$$

$$=\lim_{x\to 0}\frac{\left(e^{ax}-1\right)-\left(e^{bx}-1\right)}{abx}=\lim_{x\to 0}\left(\frac{e^{ax}-1}{abx}-\frac{e^{bx}-1}{abx}\right)$$

$$=\lim_{x\to 0}\frac{e^{ax}-1}{abx}-\lim_{x\to 0}\frac{e^{bx}-1}{abx}$$

$$=\frac{1}{b}\left(\lim_{x\to 0}\frac{e^{ax}-1}{ax}\right)-\frac{1}{a}\left(\lim_{x\to 0}\frac{e^{bx}-1}{bx}\right)$$

$$\because \lim_{x\to 0}\frac{e^{x}-1}{x}=1$$

$$=\frac{1}{b}(1)-\frac{1}{a}(1)=\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}$$

So, $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^{bx}}{abx}=\dfrac{a-b}{ab}$

**(ix)** $\underset{x\to\infty}{Lim}\left(\dfrac{x}{1+x}\right)^{x}$

**Solution:**

$$\underset{x\to\infty}{Lim}\left(\frac{x}{1+x}\right)^{x}=\underset{x\to\infty}{Lim}\left(\frac{1+x}{x}\right)^{-x}$$

$$=\underset{x\to\infty}{Lim}\left(\frac{1}{x}+1\right)^{-x}=\underset{x\to\infty}{Lim}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{-1}$$

$$=\left[\underset{x\to\infty}{Lim}\left(1+\frac{1}{x}\right)^{x}\right]^{-1}=e^{-1}=\frac{1}{e}$$

**(x)** $\underset{x\to 0}{Lim}\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, x<0$

**Solution:**

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, x<0$$

Let $t=e^{\frac{1}{x}}$

As $x<0,$ when $x\to 0,$ then $t\to 0.$

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}=\lim_{t\to 0}\frac{t-1}{t+1}=\frac{0-1}{0+1}=-1$$

**(xi)** $\underset{x\to 0}{Lim}\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, x>0$

**Solution:**

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, x>0$$

Let $t=e^{\frac{1}{x}}$, As $x>0$, when $x\to 0$ then $t\to\infty$

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}=\lim_{t\to\infty}\frac{t-1}{t+1}$$

$$=\lim_{t\to\infty}\frac{t\left(1-\frac{1}{t}\right)}{t\left(1+\frac{1}{t}\right)}=\lim_{t\to\infty}\frac{1-\frac{1}{t}}{1+\frac{1}{t}}=\frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}}=\frac{1-0}{1+0}=1$$

**(xii)** $\lim_{x\to 2}\dfrac{e^{x}-e^{2}}{x-2}$

**Solution:**

$$\lim_{x\to 2}\frac{e^{x}-e^{2}}{x-2}$$

$$=\lim_{x\to 2}\frac{e^{2}\left(\frac{e^{x}}{e^{2}}-\frac{e^{2}}{e^{2}}\right)}{x-2}$$

$$=\lim_{x\to 2}\frac{e^{2}\left(e^{x-2}-1\right)}{x-2}$$

Put $x-2=t$

As $x\to 2, t\to 0$

$$=\lim_{t\to 0}e^{2}\left(\frac{e^{t}-1}{t}\right)$$

$$=e^{2}(1)$$

$$=e^{2}$$