

**Maximum and Minimum Values of Trigonometric Functions:**

Following are the trigonometric functions whose maximum and minimum values to be calculated:

- $a + b \sin \theta$
- $a + b \sin(c \theta + d)$
- $a + b \cos \theta$
- $a + b \cos(c \theta + d)$

**Important Points:**

- Consider the function,  $f(\theta) = a + b \sin(c \theta + d)$ .....(i)  
Here 'a' represent the **vertical shift** refers to the vertical translation of the graph of a periodic function, achieved by shifting the entire graph upward or downward. This shift, also known as the **vertical displacement**, moves the function's position along the  $y$ -axis without altering its shape or period.
- **Amplitude**  $|b|$  is the maximum height of a wave measured from its midline.
- The period of (i) is equal to  $\frac{2\pi}{c}$ .
- **Phase shift** 'd' indicates the horizontal translation of the graph of a periodic function, determining how far the wave is shifted left or right along the  $x$ -axis. A positive  $d$  shifts the graph to the left, while a negative  $d$  shifts it to the right, altering the starting point of the wave without changing its shape or period.
- Thus the maximum value of  $f(\theta)$  is :  $M = a + |b|$ , whenever  $\sin \theta = 1$  or  $\cos \theta = 1$  where  $M$  denotes the maximum value of the function.
- The minimum value of a function is  $m = a - |b|$ , whenever  $\sin \theta = -1$  or  $\cos \theta = -1$  and  $m$  denotes the minimum value of the function.
- Amplitude =  $\frac{\text{Maximum value} - \text{Minimum value}}{2}$

**EXERCISE # 11.3**

1. Find the maximum and minimum values of the following functions:

(i)  $3 - \sin 3x$

**Solution:**

Let  $f(x) = 3 - \sin 3x$

Here  $a = 3$  and  $b = -1$ ,

Maximum value of the function:

$M = a + |b| = 3 + |-1| = 3 + 1 = 4$

Minimum value of the function:

$m = a - |b| = 3 - |-1| = 3 - 1 = 2$

Thus, maximum value of the function is 4 and the minimum value is 2.

(ii)  $3 + \sin 2x$

**Solution:**

Let  $f(x) = 3 + \sin 2x$

Here  $a = 3$  and  $b = 1$ .

Maximum value of the function:

$M = a + |b| = 3 + |1| = 3 + 1 = 4$

Minimum value of the function:  $m = a - |b| = 3 - |1| = 3 - 1 = 2$

Thus, maximum value of the function is 4 and the minimum value is 2.

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(iii)  $\frac{1}{2} + \sin(5x + \pi)$

**Solution:**

Let  $f(x) = \frac{1}{2} + \sin(5x + \pi)$

Here  $a = \frac{1}{2}$  and  $b = 1$ .

Maximum value of the function:

$$M = a + |b| = \frac{1}{2} + |1| = \frac{1}{2} + 1 = \frac{3}{2}$$

Minimum value of the function:  $m = a - |b| = \frac{1}{2} - 1 = -\frac{1}{2}$

Thus, the maximum value of the functions is  $\frac{3}{2}$  and minimum value of the function is  $-\frac{1}{2}$ .

(iv)  $\frac{3}{2} + \cos\left(x - \frac{\pi}{4}\right)$

**Solution:**

Let  $f(x) = \frac{3}{2} + \cos\left(x - \frac{\pi}{4}\right)$

Here  $a = \frac{3}{2}$  and  $b = 1$

Maximum value of the function:  $M = a + |b| = \frac{3}{2} + 1 = \frac{5}{2}$

And,

Minimum value of the function:  $m = a - |b| = \frac{3}{2} - 1 = \frac{1}{2}$

Thus, maximum value of the function is  $\frac{5}{2}$  and minimum value of the function is  $\frac{1}{2}$ .

(v)  $1 - 3\cos 2x$

**Solution:**

Let  $f(x) = 1 - 3\cos 2x$

Here  $a = 1$  and  $b = -3$

Maximum value of the function:

$$M = a + |b| = 1 + |-3| = 1 + 3 = 4$$

And,

Minimum value of the function:  $m = a - |b| = 1 - |-3| = 1 - 3 = -2$

Thus, maximum value of the function is 4 and minimum value of the function is -2.

(vi)  $1 + 2\sin\left(x + \frac{\pi}{6}\right)$

**Solution:**

Let  $f(x) = 1 + 2\sin\left(x + \frac{\pi}{6}\right)$

Here  $a = 1$  and  $b = 2$

Maximum value of the function:

$$M = a + |b| = 1 + |2| = 1 + 2 = 3$$

And,

Minimum value of the function:  $m = a - |b| = 1 - |2| = 1 - 2 = -1$

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Thus, maximum value of the function is 3 and minimum value of the function is  $-1$ .

(vii)  $\frac{1}{10 - 2\sin 3x}$

**Solution:**

Let  $g(x) = \frac{1}{10 - 2\sin 3x}$

To find the maximum and minimum values of  $g(x)$ , first we find the maximum and minimum values of  $10 - 2\sin 3x$

So, consider  $f(x) = 10 - 2\sin 3x$

Here  $a = 10$  and  $b = -2$ .

Maximum value of the function:

$$M = a + |b| = 10 + |-2| = 10 + 2 = 12$$

And,

Minimum value of the function:

$$m = a - |b| = 10 - |-2| = 10 - 2 = 8$$

Thus, maximum value of the function is 12 and the minimum value of the function is 8.

Now, the reciprocal of the maximum value is the minimum of  $g(x)$  and the reciprocal of the minimum value is the maximum of  $g(x)$ .

Maximum value of  $g(x) = \frac{1}{m} = \frac{1}{8}$

Minimum value of  $g(x) = \frac{1}{M} = \frac{1}{12}$

(viii)  $\frac{1}{7 + 3\cos(-2x)}$

**Solution:**

Let  $g(x) = \frac{1}{7 + 3\cos(-2x)}$

To find the maximum and minimum values of  $g(x)$ , first we find the maximum and minimum values of  $7 + 3\cos(-2x)$

So, consider  $f(x) = 7 + 3\cos(-2x)$

$$\Rightarrow f(x) = 7 + 3\cos(2x)$$

$$\because \cos(-\theta) = \cos \theta$$

Here  $a = 7$  and  $b = 3$ .

Maximum value of the function:  $M = a + |b| = 7 + |3| = 7 + 3 = 10$

And,

Minimum value of the function:  $m = a - |b| = 7 - |3| = 7 - 3 = 4$

Thus, maximum value of the function is 10 and minimum value of the function is 4.

Now, the reciprocal of the maximum value is the minimum of  $g(x)$  and the reciprocal of the minimum value is the maximum of  $g(x)$ .

Maximum value of  $g(x) = \frac{1}{m} = \frac{1}{4}$

Minimum value of  $g(x) = \frac{1}{M} = \frac{1}{10}$ .

(ix)  $\frac{1}{5 - 3\cos(3x - 1)}$

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**Solution:**

$$\text{Let } g(x) = \frac{1}{5 - 3\cos(3x-1)}$$

To find the maximum and minimum values of  $g(x)$ , first we find the maximum and minimum values of  $5 - 3\cos(3x-1)$

$$\text{So, consider } f(x) = 5 - 3\cos(3x-1)$$

Here  $a = 5$  and  $b = -3$

Maximum value of the function:

$$M = a + |b| = 5 + |-3| = 5 + 3 = 8$$

And,

Minimum value of the function:

$$m = a - |b| = 5 - |-3| = 5 - 3 = 2$$

Thus, maximum value of the function is 8 and the minimum value of the function is 2.

Now, the reciprocal of the maximum value is the minimum of  $g(x)$  and the reciprocal of the minimum value is the maximum of  $g(x)$ .

$$\text{Maximum value of } g(x) = \frac{1}{m} = \frac{1}{2}$$

$$\text{Minimum value of } g(x) = \frac{1}{M} = \frac{1}{8}$$

**2. The temperature  $T$  in a certain city varies throughout the day according to the equation**

$$T(t) = \frac{13}{2} \sin\left(\frac{\pi}{6}t - \frac{\pi}{9}\right) + 15, \text{ where } t \text{ is the time in hours, with } t = 0 \text{ corresponding to midnight.}$$

**(a) Find the maximum and minimum temperature during the day.**

**(b) Find the temperature at  $t = 9$  hours (9:00a.m.).**

**Solution:**

$$T(t) = \frac{13}{2} \sin\left(\frac{\pi}{6}t - \frac{\pi}{9}\right) + 15 \quad (i)$$

**(a) Find the maximum and minimum temperature during the day.**

For maximum temperature:  $T_{\max} = a + |b|$ , where  $a = 15$  and  $b = \frac{13}{2}$

$$\text{So, } T_{\max} = 15 + \left|\frac{13}{2}\right| = 15 + \frac{13}{2} = \frac{43}{2}$$

$$T_{\max} = 21.5^\circ$$

For minimum temperature:  $T_{\min} = a - |b| = 15 - \left|\frac{13}{2}\right| = 15 - \frac{13}{2} = \frac{17}{2}$

$$\text{So, } T_{\min} = 8.5^\circ$$

**(b) Find the temperature at  $t = 9$  hours (9:00a.m.).**

$$\therefore T(t) = \frac{13}{2} \sin\left(\frac{\pi}{6}t - \frac{\pi}{9}\right) + 15 \quad (i)$$

Put  $t = 9$  in equation (i)

$$T(9) = \frac{13}{2} \sin\left(\frac{\pi}{6} \times 9 - \frac{\pi}{9}\right) + 15$$

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$$= \frac{13}{2} \sin\left(\frac{3\pi}{2} - \frac{\pi}{9}\right) + 15$$

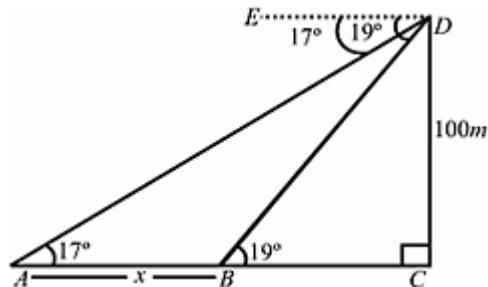
$$= \frac{13}{2} \left[-\cos\frac{\pi}{9}\right] + 15 = -\frac{13}{2} \cos\frac{\pi}{9} + 15$$

$$T(9) \approx 8.89^\circ$$

So, temperature at 9:00 a.m. is  $8.89^\circ$  approximately.

3. A man on the top of 100m high light house is in the line with two ships on the same side of it, whose angles of depression from the man are  $17^\circ$  and  $19^\circ$  respectively. Find the distance between the ships.

Solution:



Let  $A$  and  $B$  be the position of ships, and distance between ships =  $m\overline{AB} = x$

Height of light house =  $m\overline{CD} = 100m$

From figure,

$$m\angle ADE = m\angle CAD$$

[ $\because$  Alternate Angles]

$$m\angle BDE = m\angle CBD$$

In  $\Delta BDC$

$$\tan 19^\circ = \frac{100}{\overline{BC}}$$

$$\overline{BC} = \frac{100}{\tan 19^\circ} = 290.42m$$

Now In  $\Delta ACD$

$$\tan 17^\circ = \frac{100}{x + 290.42}$$

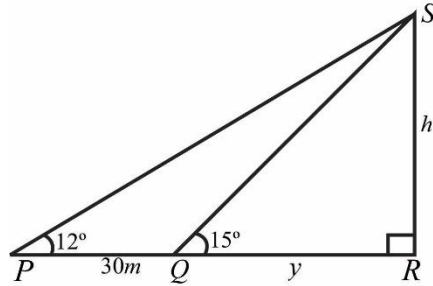
$$x + 290.42 = \frac{100}{\tan 17^\circ}$$

$$x = 327.08 - 290.42$$

Distance between Ships  $\Rightarrow x = 36.66m$

4. *P* and *Q* are two points in line with a tree. If the distance between '*P*' and '*Q*' be 30*m* and the angles of elevation of the top of the tree at *P* and *Q* be 12° and 15° respectively, find the height of the tree.

Solution:



Let height of tree is  $\overline{mRS} = h$

Distance between *P* and *Q* is  $\overline{mPQ} = 30m$

In  $\Delta QRS$

$$\tan 15^\circ = \frac{h}{y}$$

$$h = y \tan 15^\circ \dots\dots\dots(i)$$

In  $\Delta PRS$

$$\tan 12^\circ = \frac{h}{30 + y}$$

$$h = (30 + y) \tan 12^\circ \dots\dots\dots(ii)$$

Comparing (i) and (ii)

$$y \tan 15^\circ = (30 + y) \tan 12^\circ$$

$$y \tan 15^\circ = 30 \tan 12^\circ + y \tan 12^\circ$$

$$y \tan 15^\circ - y \tan 12^\circ = 6.3766$$

$$y(0.0553) = 6.3766$$

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$$y = 115.34m$$

Put in (i)

$$h = y \tan 15^\circ \Rightarrow h = (115.34)(0.268)$$

$$h = 30.91m$$

5. A giant Ferris wheel has a diameter of 60 feet. The lowest point of the wheel is located 6 feet above the ground. The wheel completes one full revolution every 80 seconds.

- (a) Model an equation that represent the height  $h(t)$  of a rider on the Ferris wheel at any given time  $t$ .
- (b) Find the maximum height of a rider.
- (c) Find the height of the rider from the ground after 35 seconds.

**Solution:**

Since it takes 80 seconds for the Ferris wheel to complete one full revolution (one cycle), which is the period of the Ferris wheel, so, period = 80.

$$\frac{2\pi}{c} = 80 \Rightarrow c = \frac{2\pi}{80} \Rightarrow c = \frac{\pi}{40} \qquad \therefore \text{Period} = \frac{2\pi}{c}$$

The amplitude " $b$ " which is equal to the radius of a ferris wheel (in this case  $b = \frac{60}{2} = 30 \text{ ft}$ )  $\therefore \text{Radius} = \frac{\text{Diameter}}{2}$

The vertical shift " $a$ " is the height of the center of the Ferris wheel above the ground.

Since the lowest point is 6 feet above the ground. So,

$$a = 6 + b = 6 + 30 = 36.$$

We can model the height of a rider using (sine or cosine), because it reflects the periodic nature of the motion.

We usually choose a cosine function if the rider starts at the maximum height.

Since the rider starts at the lowest point and goes up, we can easily model the required equation as a negative cosine function,

So,

$$h(t) = a - b \cos(ct), \text{ where } t \text{ is the time and } h \text{ is the height.}$$

- (a) Model an equation that represent the height  $h(t)$  of a rider on the Ferris wheel at any given time  $t$ .

$$h(t) = 36 - 30 \cos\left(\frac{\pi}{40}t\right)$$

- (b) Find the maximum height of a rider.

Maximum height of a rider:

$$= a + |b| = 36 + |30| = 36 + 30 = 66 \text{ ft}$$

- (c) Find the height of the rider from the ground after 35 seconds.

The height of the rider from the ground after 35 seconds is: put  $t = 35$  in

$$h(t) = -30 \cos\left(\frac{\pi}{40}t\right) + 36$$

$$h(35) = -30 \cos\left(\frac{\pi}{40} \times 35\right) + 36 = 63.72 \text{ ft}$$

6. A child is playing on a swing in a playground. The height  $h(t)$  of the swing seat above the ground (in meters) at time  $t$  (in seconds) is modeled by the function:

$$h(t) = 1.5 + 1.2 \sin(3\pi t)$$

- (a) What is the maximum height reached by the swing seat?
- (b) What is the minimum height reached by the swing seat?

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- (c) How long does it take for the swing to complete one full back-and-forth motion (period)?
- (d) At what times(s) does the swing seat first reach a height of 2.12 meters?

**Solution:**

The height of the swing is modeled by the function:

$$h(t) = 1.5 + 1.2\sin(3\pi t) \dots\dots\dots(i)$$

Where,  $h(t)$  is the height in meters and "t" is the time in seconds.

- (a) What is the maximum height reached by the swing seat?

The maximum value of  $\sin(3\pi t)$  is 1.

$$h_{\max} = 1.5 + 1.2(1) = 2.7m$$

- (b) What is the minimum height reached by the swing seat?

The minimum value of  $\sin(3\pi t)$  is -1.

$$h_{\min} = 1.5 + 1.2(-1) = 0.3m$$

- (c) How long does it take for the swing to complete one full back-and-forth motion (period)?

The period of  $\sin(3\pi t)$  is

$$\frac{2\pi}{3\pi} = \frac{2}{3} \text{ second} \quad \therefore \text{Period} = \frac{2\pi}{c}$$

- (d) At what times(s) does the swing seat first reach a height of 2.12 meters?

$$\therefore h(t) = 2.12 \text{ (put in (i))}$$

$$2.12 = 1.5 + 1.2\sin(3\pi t)$$

$$0.62 = 1.2\sin(3\pi t)$$

$$\sin(3\pi t) = \frac{0.62}{1.2} = \frac{31}{60}$$

$$3\pi t = \sin^{-1}\left(\frac{31}{60}\right) \Rightarrow t = \frac{\sin^{-1}\left(\frac{31}{60}\right)}{3\pi}$$

$$t \approx 0.057$$

Time to reach 2.12 meters is 0.05 seconds approximately.

- 7. A carnival ride consists of a vertical wheel with a diameter of 40 feet. The centre of the wheel is 28 feet above the ground. The wheel rotates at a constant speed and takes 120 second to make one complete revolution. Model an equation that describes the height  $h(t)$  of a rider on the wheel as a function of time  $t$ . How high is the rider from the ground after 90 seconds? At what times will the rider be 36 feet above the ground?

**Solution:**

The general form of a function is:

$$h(t) = -A\cos(Bt - C) + D$$

The amplitude  $A$  is the radius of the wheel:

$$\Rightarrow A = \frac{40}{2} = 20$$

The vertical shift  $D$  is the height of the center:

$$\Rightarrow D = 28$$

The period  $T$  is 120 seconds, So  $B$  is:

$$B = \frac{2\pi}{120} = \frac{\pi}{60} \quad \therefore T = \frac{2\pi}{B}$$

Assuming the rider starts at the highest point,

$$\Rightarrow C = 0$$

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The height function is:

$$h(t) = -20 \cos\left(\frac{\pi}{60}t\right) + 28 \quad (i)$$

We put  $t = 90$  into the height function:

$$h(90) = -20 \cos\left(\frac{\pi}{60} \times 90\right) + 28$$

$$h(90) = -20 \cos\left(\frac{3\pi}{2}\right) + 28$$

$$h(90) = -20(0) + 28$$

$$\Rightarrow h(90) = 28 \text{ ft}$$

Put  $h(t) = 36$  in eq. (i) and solve for  $t$ :

$$36 = -20 \cos\left(\frac{\pi}{60}t\right) + 28$$

$$8 = -20 \cos\left(\frac{\pi}{60}t\right)$$

$$-\frac{8}{20} = \cos\left(\frac{\pi}{60}t\right)$$

$$\cos^{-1}\left(-\frac{2}{5}\right) = \frac{\pi}{60}t$$

$$\frac{1.98 \times 60}{\pi} = t$$

$$\Rightarrow t \approx 37.87 \text{ seconds}$$

Now, to calculate "t" within the period :

$$t = 120 - \frac{60}{\pi} \cos^{-1}\left(-\frac{2}{5}\right)$$

$$\Rightarrow t \approx 82.13 \text{ seconds}$$

8. Suppose the temperature  $T$  in degrees Fahrenheit of Lahore city in a month of December throughout the day can be modeled by the equation:  $T = 64 + 8 \sin\left(\frac{\pi}{12}(t-8)\right)$ , where  $t$  is the time in hours. The temperature oscillates 8 degrees above and below an average temperature of 64 degrees.

- (a) Find the temperature at  $t = 9$  hours?  
 (b) At what time the temperature will be maximum?  
 (c) Calculate the maximum temperature.

Solution:

$$\therefore T = 64 + 8 \sin\left(\frac{\pi}{12}(t-8)\right) \quad (i)$$

Where "T" is the temperature in degrees and "t" is the time in hours.

- (a) Find the temperature at  $t = 9$  hours?

Put  $t = 9$  in equation (i), we get:

$$T = 64 + 8 \sin\left(\frac{\pi}{12} \times (9-8)\right)$$

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$$T = 64 + 8\sin\left(\frac{\pi}{12}\right)$$

$$T = 64 + 2.07$$

$$\Rightarrow T = 66.07^\circ F$$

**(b) At what time the temperature will be maximum?**

For maximum temperature =  $64 + |8| = 64 + 8 = 72$

So, put  $T = 72$  in equation (i), we get:

$$72 = 64 + 8\sin\left(\frac{\pi}{12}(t-8)\right)$$

$$72 - 64 = 8\sin\left(\frac{\pi}{12}(t-8)\right)$$

$$\frac{8}{8} = \sin\left(\frac{\pi}{12}(t-8)\right)$$

$$\Rightarrow \sin^{-1}(1) = \frac{\pi}{12}(t-8)$$

$$\frac{\pi}{2} \times \frac{12}{\pi} = t - 8$$

$$\Rightarrow t = 14 \text{ hours or } 2 \text{ p.m.}$$

**(c) Calculate the maximum temperature.**

To calculate maximum temperature,

put  $\sin\left(\frac{\pi}{12}(t-8)\right) = 1$  in eq (i)

So, equation (i) becomes:

$$T = 64 + 8(1)$$

$$\Rightarrow T = 72^\circ F$$

**9. Suppose the population of a coastal city follows a sinusoidal pattern due to seasonal migration. The population of the city over the course of a year can be modeled by the equation:**

$$P(t) = 70000 + 10000\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right), P(t) \text{ is the population at time } t \text{ ( } t \text{ is the time in months, with } t = 0$$

corresponding to January 1<sup>st</sup> ) where  $t$  denoted the months in a year.

**(a) Find the population of a city at  $t = 7$  months**

**(b) Find the maximum population**

**Solution:**

The population is modeled by:  $P(t) = 70000 + 10000\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right)$  (i)

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Where "t" is the time in months, with  $t = 0$  corresponding to January 1<sup>st</sup>.

$\therefore$  The cosine function has a maximum value of 1.

**(a) Find the population of a city at  $t = 7$  months**

Put  $t = 7$  into equation (i), we get:

$$P(7) = 70000 + 10000 \cos\left(\frac{\pi}{6}(7) - \frac{\pi}{2}\right)$$

$$P(7) = 70000 + 10000 \cos\left(\frac{7\pi}{6} - \frac{3\pi}{6}\right)$$

$$P(7) = 70000 + 10000 \cos\left(\frac{4\pi}{6}\right)$$

$$P(7) = 70000 + 10000 \cos\left(\frac{2\pi}{3}\right)$$

$$\therefore \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$P(7) = 70000 + 10000\left(-\frac{1}{2}\right)$$

$$P(7) = 70000 - 5000 = 65000$$

**(b) Find the maximum population**

The maximum value of  $\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right)$  is 1.

So, we put the maximum value of cosine into equation (i)  $P_{\max} = 70000 + 10000(1)$

$$\Rightarrow P_{\max} = 80000$$