Even and Odd Functions:

Even Function:

A function f is said to be even if $f(-x)=f(x)$, for every number x in the domain of f.

For Example:

$f(x)=x^2$ and $f(x)=\cos x$, $(\forall\ x\in R)$ are even functions.

Odd Function:

A function f is said to be odd if $f(-x)=-f(x)$, for every number x in the domain of f.

For Example:

$f(x)=x^3$ and $f(x)=\sin x$, $(\forall\ x\in R)$ are odd functions.

Q.1 Determine whether the following functions are even, odd or neither odd nor even.

(i) $\sin^2 x$

Solution:

$$\text{Let } f(x)=\sin^2 x$$

Replace x by $-x$

$$f(-x)=\left[\sin(-x)\right]^2$$

$$f(-x)=\left[-\sin x\right]^2$$

$$\because \sin(-\theta)=-\sin\theta$$

$$f(-x)=\sin^2 x$$

$$\Rightarrow f(-x)=f(x)$$

So, $f(x)$ is an even function.

(ii) $\sin x+\cos x$

Solution:

$$\text{Let } f(x)=\sin x+\cos x$$

Replace x by $-x$

$$f(-x)=\sin(-x)+\cos(-x)$$

$$f(-x)=-\sin x+\cos x$$

$$\because \sin(-\theta) = -\sin\theta \qquad \because \cos(-\theta) = \cos\theta$$

$$f(-x) = -[\sin x - \cos x]$$

$$\Rightarrow f(-x) \neq f(x) \text{ and}$$

$$f(-x) \neq -f(x)$$

So, $f(x)$ is neither even nor odd function

(iii) $\sin^4 x + \cos^4 x$

Solution:

Let $f(x) = \sin^4 x + \cos^4 x$

Replace *x* by –*x*

$$f(-x) = [\sin(-x)]^4 + [\cos(-x)]^4$$

$$f(-x) = [-\sin(x)]^4 + [\cos x]^4$$

$$\because \sin(-\theta) = -\sin\theta \qquad \because \cos(-\theta) = \cos\theta$$

$$f(-x) = \sin^4 x + \cos^4 x$$

$$\Rightarrow f(-x) = f(x)$$

So, $f(x)$ is an even function

(iv) $\tan x + \sec x$

Solution:

Let $f(x) = \tan x + \sec x$

Replace *x* by –*x*

$$f(-x) = \tan(-x) + \sec(-x)$$

$$f(-x) = -\tan x + \sec x$$

$$\because \tan(-\theta) = -\tan\theta \qquad \because \sec(-\theta) = \sec\theta$$

$$f(-x) = -[\tan x - \sec x]$$

$$\Rightarrow f(-x) \neq f(x) \text{ and}$$

$$f(-x) \neq -f(x)$$

So, $f(x)$ is neither even nor odd function.

(v) $\dfrac{1}{\operatorname{cosec}^3 x}$

Solution:

$$\text{Let } f(x)=\frac{1}{\operatorname{cosec}^3 x}$$

Replace *x* by –*x*

$$f(-x)=\frac{1}{\left[\operatorname{cosec}(-x)\right]^3}$$

$$\because \operatorname{cosec}(-\theta)=-\operatorname{cosec}\theta$$

$$f(-x)=\frac{1}{\left[-\operatorname{cosec}x\right]^3}$$

$$f(-x)=-\frac{1}{\operatorname{cosec}^3 x}$$

$$\Rightarrow f(-x)=-f(x)$$

So, $f(x)$ is an odd function

(vi) $\dfrac{\sin x+\sin 3x}{\cos x+\cos 3x}$

Solution:

$$\text{Let } f(x)=\frac{\sin x+\sin 3x}{\cos x+\cos 3x}$$

Replace *x* by –*x*

$$f(-x)=\frac{\sin(-x)+\sin 3(-x)}{\cos(-x)+\cos 3(-x)}$$

$$f(-x)=\frac{-\sin x-\sin 3x}{\cos x+\cos 3x}$$

$$\because \sin(-\theta)=-\sin\theta \qquad\qquad \because \cos(-\theta)=\cos\theta$$

$$f(-x)=-\left[\frac{\sin x+\sin 3x}{\cos x+\cos 3x}\right]$$

$$\Rightarrow f(-x)=-f(x)$$

So, $f(x)$ is an odd function.

(vii) $\dfrac{1}{\sec x+\sec^3 x}$

Solution:

$$\text{Let } f(x)=\frac{1}{\sec x+\sec^3 x}$$

Replace *x* by –*x*

$$f(-x)=\frac{1}{\sec(-x)+\left[\sec(-x)\right]^3}$$

$$f(-x)=\frac{1}{\sec x+\sec^3 x}$$

$$\because \sec(-\theta)=\sec\theta$$

$$\Rightarrow f(-x)=f(x)$$

So, $f(x)$ is an even function

(viii) $\dfrac{1}{\sec x+\cot^2 x}$

Solution:

$$\text{Let } f(x)=\frac{1}{\sec x+\cot^2 x}$$

Replace *x* by –*x*

$$f(-x)=\frac{1}{\sec(-x)+\left[\cot(-x)\right]^2}$$

$$f(-x)=\frac{1}{\sec x+\left[-\cot(x)\right]^2}$$

$$\because \sec(-\theta)=\sec\theta \qquad \because \cot(-\theta)=-\cot\theta$$

$$f(-x)=\frac{1}{\sec x+\cot^2 x}$$

$$\Rightarrow f(-x)=f(x)$$

So, $f(x)$ is an even function

Periodicity:

All the six trigonometric functions repeat their values for each **increase or decrease** of 2π in θ i.e. the values of trigonometric functions for θ and $\theta \pm 2n\pi$, where $\theta \in \mathbb{R}$ and $n \in Z$, are the same.

This behavior of trigonometric functions is called **periodicity**.

Period of Trigonometric Functions:

Period of a trigonometric function is the **smallest positive number** which, when **added** to the original circular measure of the angle, gives the **same** value of the function.

A function is periodic, if $f(\theta + p) = f(\theta)$, for all θ in domain of function and the least positive value of p is called the period of the function.

The following table shows the periods of trigonometric functions.

Trigonometric Function	Period
sine	2π
cosine	2π
secant	2π
cosecant	2π
tangent	π
cotangent	π

Q.2 Find the periods of the following functions:

(i) $\sin 5x$

Solution:

We know that the period of sine is 2π

$$\therefore \sin(5x + 2\pi) = \sin 5x$$

$$\Rightarrow \sin 5\left(x + \frac{2\pi}{5}\right) = \sin 5x$$

It means that the value of $\sin 5x$ repeats when *x* is increased by $\frac{2\pi}{5}$.

Hence $\frac{2\pi}{5}$ is the period of $\sin 5x$

(ii) $\cos 7x$

Solution:

We know that the period of cosine is 2π.

$$\therefore \cos(7x+2\pi)=\cos 7x$$

$$\Rightarrow \cos 7\left(x+\frac{2\pi}{7}\right)=\cos 7x$$

It means that the value of $\cos 7x$ repeats when *x* is increased by $\frac{2\pi}{7}$.

Hence $\frac{2\pi}{7}$ is the period of $\cos 7x$

(iii) $\tan 3x$

Solution:

We know that the period of tangent is π.

$$\therefore \tan(3x+\pi)=\tan 3x$$

$$\Rightarrow \tan 3\left(x+\frac{\pi}{3}\right)=\tan 3x$$

It means that the value of $\tan 3x$ repeats when *x* is increased by $\frac{\pi}{3}$.

Hence $\frac{\pi}{3}$ is the period of $\tan 3x$

(iv) $\cot \frac{x}{2}$

Solution:

We know that the period of cotangent is π.

$$\therefore \cot\left(\frac{x}{2}+\pi\right)=\cot\frac{x}{2}$$

$$\Rightarrow \cot\frac{1}{2}(x+2\pi)=\cot\frac{x}{2}$$

It means that the value of $\cot\frac{x}{2}$ repeats when *x* is increased by 2π.

Hence 2π is the period of $\cot\frac{x}{2}$.

(v) $19\sin\left(\frac{\pi}{20}x\right)$

Solution:

We know that the period of sine is 2π.

$$\therefore 19\sin\left(\frac{\pi}{20}x+2\pi\right)=19\sin\left(\frac{\pi}{20}x\right)$$

$$\Rightarrow 19\sin\frac{\pi}{20}(x+40)=19\sin\left(\frac{\pi}{20}x\right)$$

It means that the value of $19\sin\left(\frac{\pi}{20}x\right)$ repeats when x is increased by 40.

Hence 40 is the period of $19\sin\left(\frac{\pi}{20}x\right)$.

(vi) $\operatorname{cosec}\left(\frac{2x}{5}\right)$

Solution:

We know that the period of cosecant is 2π.

$$\therefore \operatorname{cosec}\left(\frac{2x}{5}+2\pi\right)=\operatorname{cosec}\left(\frac{2x}{5}\right)$$

$$\Rightarrow \operatorname{cosec}\frac{2}{5}(x+5\pi)=\operatorname{cosec}\left(\frac{2x}{5}\right)$$

It means that the value of $\operatorname{cosec}\left(\frac{2x}{5}\right)$ repeats when x is increased by 5π.

Hence 5π is the period of $\operatorname{cosec}\left(\frac{2x}{5}\right)$.

(vii) $\frac{1}{2}\sin\left(\frac{3x}{2}-\frac{\pi}{2}\right)$

Solution:

We know that the period of sine is 2π .

$$\therefore \frac{1}{2}\sin\left[\left(\frac{3x}{2}+2\pi\right)-\frac{\pi}{2}\right]=\frac{1}{2}\sin\left(\frac{3x}{2}-\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{1}{2}\sin\left[\frac{3}{2}\left(x+\frac{4\pi}{3}\right)-\frac{\pi}{2}\right]=\frac{1}{2}\sin\left(\frac{3x}{2}-\frac{\pi}{2}\right)$$

It means that the value of $\frac{1}{2}\sin\left(\frac{3x}{2}-\frac{\pi}{2}\right)$ repeats when x is increased by $\frac{4\pi}{3}$.

Hence $\frac{4\pi}{3}$ is the period of $\frac{1}{2}\sin\left(\frac{3x}{2}-\frac{\pi}{2}\right)$.

Alternate method:

$$\frac{1}{2}\sin\left(\frac{3x}{2}-\frac{\pi}{2}\right)=\frac{1}{2}\sin\left[-\left(\frac{\pi}{2}-\frac{3x}{2}\right)\right]$$

$$\because \sin(-\theta)=-\sin\theta \qquad \because \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$

$$=-\frac{1}{2}\cos\left(\frac{3x}{2}\right)$$

We know that the period of cosine is 2π.

$$\therefore -\frac{1}{2}\cos\left(\frac{3x}{2}+2\pi\right)=-\frac{1}{2}\cos\left(\frac{3x}{2}\right)$$

$$-\frac{1}{2}\cos\frac{3}{2}\left(x+\frac{4\pi}{3}\right)=-\frac{1}{2}\cos\left(\frac{3x}{2}\right)$$

It means that the value of $-\frac{1}{2}\cos\left(\frac{3x}{2}\right)$ repeats when x is increased by $\frac{4\pi}{3}$. Hence $\frac{4\pi}{3}$ is the period of $\frac{1}{2}\sin\left(\frac{3x}{2}-\frac{\pi}{2}\right)$

(viii) $-5-3\sec\left(7\pi x+\frac{\pi}{4}\right)$

Solution:

To find the period of $-5-3\sec\left(7\pi x+\frac{\pi}{4}\right)$,

consider only $-3\sec\left(7\pi x+\frac{\pi}{4}\right)$

We know that the period of secant is 2π.

$$\therefore -3\sec\left[(7\pi x+2\pi)+\frac{\pi}{4}\right]=-3\sec\left(7\pi x+\frac{\pi}{4}\right) \Rightarrow -3\sec\left[7\pi\left(x+\frac{2}{7}\right)+\frac{\pi}{4}\right]=-3\sec\left(7x+\frac{\pi}{4}\right)$$

It means that the value of $-3\sec\left(7x+\frac{\pi}{4}\right)$ repeats when x is increased by $\frac{2}{7}$.

Hence $\frac{2}{7}$ is the period of $-5-3\sec\left(7\pi x+\frac{\pi}{4}\right)$.

The addition of constant number –5 to the secant function does not affect the period.

(ix) $12+10\tan\left(\frac{\pi}{30}x\right)$

Solution:

To find the period of $12+10\tan\left(\frac{\pi}{30}x\right)$,

consider only $10\tan\left(\frac{\pi}{30}x\right)$

We know that the period of tangent is π.

$$\therefore 10\tan\left(\frac{\pi}{30}x+\pi\right)=10\tan\left(\frac{\pi}{30}x\right)$$

$$\Rightarrow 10\tan\frac{\pi}{30}(x+30)=10\tan\left(\frac{\pi}{30}x\right)$$

It means that the value of $10\tan\left(\frac{\pi}{30}x\right)$ repeats when x is increased by 30. Hence 30 is the period of $12+10\tan\left(\frac{\pi}{30}x\right)$.

The addition of constant number 12 to the tangent function does not affect the period.

(x) $6-4\cot\left(\frac{7x}{4}+\frac{\pi}{4}\right)$

Solution:

To find the period of $6-4\cot\left(\frac{7x}{4}+\frac{\pi}{4}\right)$,

consider only $-4\cot\left(\frac{7x}{4}+\frac{\pi}{4}\right)$.

We know that the period of cotangent is π.

$$\therefore -4\cot\left[\left(\frac{7x}{4}+\pi\right)+\frac{\pi}{4}\right]=-4\cot\left(\frac{7x}{4}+\frac{\pi}{4}\right)$$

$$\Rightarrow -4\cot\left[\frac{7}{4}\left(x+\frac{4\pi}{7}\right)+\frac{\pi}{4}\right]=-4\cot\left(\frac{7x}{4}+\frac{\pi}{4}\right)$$

It means that the value of $-4\cot\left(\frac{7x}{4}+\frac{\pi}{4}\right)$ repeats when x is increased by $\frac{4\pi}{7}$. Hence $\frac{4\pi}{7}$ is the period of $6-4\cot\left(\frac{7x}{4}+\frac{\pi}{4}\right)$

The addition of constant number 6 to the cotangent function does not affect the period.

(xi) $9+30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right)$

Solution:

To find the period of $9+30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right)$,

consider only $30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right)$

We know that the period of secant is 2π .

$\therefore 30\sec\left[\left(\frac{x}{15}+2\pi\right)+\frac{2\pi}{15}\right]=30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right) \Rightarrow 30\sec\left[\frac{1}{15}(x+30\pi)+\frac{2\pi}{15}\right]=30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right)$ It means that the value of $30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right)$ repeats when x is increase by 30π . Hence 30π is the period of $9+30\sec\left(\frac{x}{15}+\frac{2\pi}{15}\right)$. The addition of constant number 9 to the secant function does not affect the period