10.2

1. Without using table find the values of the following:

(i) $\sin 15^\circ$

$= \sin(45^\circ - 30^\circ)$

$= \sin 45 \cos 30 - \cos 45 \sin 30$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

(ii) $\cos 15^\circ$

$= \cos(45^\circ - 30^\circ)$

$= \cos 45 \cos 30 + \sin 45 \sin 30$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{3}}$

(iii) $\tan 15^\circ$

$\tan(45^\circ - 30^\circ)$

$= \dfrac{\tan 45 - \tan 30^\circ}{1 + \tan 45 \tan 30}$

$= \dfrac{1 - 1/\sqrt{3}}{1 + 1 \times 1/\sqrt{3}}$

$= \dfrac{\frac{\sqrt{3} - 1}{\cancel{\sqrt{3}}}}{\frac{\sqrt{3} + 1}{\cancel{\sqrt{3}}}}$

$= \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \dfrac{\sqrt{3} - 1}{\sqrt{3} - 1}$

$= \dfrac{(\sqrt{3} - 1)^2}{(\sqrt{3})^2 - (1)^2}$

$= \dfrac{(\sqrt{3})^2 + (1)^2 - 2(1)(\sqrt{3})}{3 - 1}$

$= \dfrac{3 + 1 - 2\sqrt{3}}{2}$

$= \dfrac{4 - 2\sqrt{3}}{2}$

$= \dfrac{\cancel{2}(2 - \sqrt{3})}{\cancel{2}}$

$= 2 - \sqrt{3}$

(iv) $\sin 105^\circ$

$= \sin(60^\circ + 45^\circ)$

$= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$

$= \dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}} + \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{3}}{2\sqrt{2}} + \dfrac{1}{2\sqrt{2}}$

$= \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$

(v) $\cos 105^\circ$

$= \cos(60^\circ + 45^\circ)$

$= \cos 60^\circ \cos 45 - \sin 60 \sin 45$

$= \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}} - \dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}}$

$= \dfrac{1}{2\sqrt{2}} - \dfrac{\sqrt{3}}{2\sqrt{2}}$

$= \dfrac{1 - \sqrt{3}}{2\sqrt{2}}$

(vi) $\tan 105^\circ$

$\tan(60^\circ + 45^\circ)$

$= \dfrac{\tan 60 + \tan 45}{1 - \tan 60 \tan 45}$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}\times 1}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{(\sqrt{3}+1)(1+\sqrt{3})}{(1)^2-(\sqrt{3})^2}$$

$$= \frac{\sqrt{3}+3+1+\sqrt{3}}{1-3}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

$$= \frac{\cancel{2}(2+\sqrt{3})}{-\cancel{2}}$$

$$= -(2+\sqrt{3})$$

$$= -2-\sqrt{3}$$

2. Prove that:

(i) $\sin(45^\circ+\alpha) = \frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$

L.H.S

$= \sin 45^\circ \cos\alpha + \cos 45^\circ \sin\alpha$

$= \frac{1}{\sqrt{2}} \times \cos\alpha + \frac{1}{\sqrt{2}} \times \sin\alpha$

$= \frac{1}{\sqrt{2}}(\cos\alpha + \sin\alpha)$

$= \frac{1}{\sqrt{2}}(\sin\alpha + \cos\alpha) \Rightarrow R.H.S$

Proved

(ii) $\cos(\alpha + 45^\circ) = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$

L.H.S

$= \cos\alpha \cos 45^\circ - \sin\alpha \sin 45^\circ$

$= \cos\alpha \times \frac{1}{\sqrt{2}} - \sin\alpha \times \frac{1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$

Proved

3. Prove that:

(i) $\tan(45^\circ + A)\tan(45^\circ - A) = 1$

$\tan(45^\circ + A)\tan(45^\circ - A)$

$= \frac{\tan 45 + \tan A}{1 - \tan 45^\circ \tan A} \times \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A}$

$= \frac{1 + \tan A}{1 - 1 \times \tan A} \times \frac{1 - \tan A}{1 + 1 \times \tan A}$

$$= \frac{1+\cancel{\tan A}}{1-\cancel{\tan A}} \times \frac{1-\cancel{\tan A}}{1+\cancel{\tan A}}$$

$$= 1$$

Proved

(ii) $\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right) = 0$

$$= \tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\theta}{1+\tan\frac{\pi}{4}.\tan\theta} + \frac{\tan\frac{3\pi}{4} + \tan\theta}{1-\tan\frac{3\pi}{4}\cdot\tan\theta}$$

$$= \frac{1-\tan\theta}{1+1.\tan\theta} + \frac{-1+\tan\theta}{1-(-1)\tan\theta}$$

$$= \frac{1-\tan\theta}{1+\tan\theta} - \frac{1+\tan\theta}{1+\tan\theta}$$

$$= \frac{\cancel{1}-\tan\theta-\cancel{1}+\tan\theta}{1+\tan\theta}$$

$$= \frac{0}{1+\tan\theta}$$

$$= 0$$

Proved

(iii) $\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right)=\cos\theta$

$=\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}+\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}$

$=\sin\theta\times\frac{\sqrt{3}}{2}+\cos\theta\,\frac{1}{2}+\cos\theta\,\frac{1}{2}-\sin\theta\,\frac{\sqrt{3}}{2}$

$=\frac{\cancel{\sin\theta\sqrt{3}}+\cos\theta+\cos\theta-\cancel{\sin\theta\sqrt{3}}}{2}$

$=\frac{\cancel{2}\cos\theta}{\cancel{2}}$

$=\cos\theta$

(iv) $\frac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}}=\tan\frac{\theta}{2}$

L.H.S

$\frac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\theta/2}$

$=\frac{\sin\theta-\cos\theta\cdot\frac{\sin\theta/2}{\cos\theta/2}}{\cos\theta+\sin\theta\times\frac{\sin\theta/2}{\cos\theta/2}}$

$=\frac{\frac{\sin\theta\cos\theta/2-\cos\theta\sin\theta/2}{\cancel{\cos\theta/2}}}{\frac{\cos\theta\cos\theta/2+\sin\theta\sin\theta/2}{\cancel{\cos\theta/2}}}$

$$= \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\cos\left(\theta - \frac{\theta}{2}\right)}$$

$$= \frac{\sin\left(\frac{2\theta - \theta}{2}\right)}{\cos\left(\frac{2\theta - \theta}{2}\right)}$$

$$= \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

$$= \tan \theta/2 \Rightarrow R.H.S$$

Proved.

(v) $$\frac{1 - \tan\theta \tan\phi}{1 + \tan\theta \tan\phi} = \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)}$$

L.H.S

$$= \frac{1 - \tan\theta \tan\phi}{1 + \tan\theta \tan\phi}$$

$$= \frac{1 - \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin\phi}{\cos\phi}}{1 + \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin\phi}{\cos\phi}}$$

$$= \frac{\frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\cancel{\cos\theta\cos\phi}}}{\frac{\cos\theta\cos\phi + \sin\theta\sin\phi}{\cancel{\cos\theta\cos\phi}}}$$

$$= \frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\cos\theta\cos\phi + \sin\theta\sin\phi}$$

$$= \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)} \Rightarrow R.H.S$$

Proved

4. Show that: $\cos(\alpha+\beta)\cos(\alpha-\beta)$

$= \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$

$\cos(\alpha+\beta)\cos(\alpha-\beta) \rightarrow (i)$

$\cos\alpha\cos\beta - \sin\alpha\sin\beta \cdot \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$(\cos\alpha\cos\beta)^2 - (\sin\alpha\sin\beta)^2$

$\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$

$\cancel{\cos^2\alpha(1-\sin^2\beta) - \sin^2\alpha(1-\cos^2\beta)}$

$\cancel{\cos^2\alpha - \cos^2\alpha\sin^2\beta - \sin^2\alpha + \cos^2\beta}\sin^2\alpha$

$\cos^2\alpha(1-\sin^2\beta) - (1-\cos^2\alpha)\sin^2\beta$

$\cos^2\alpha - \cancel{\cos^2\alpha\sin^2\beta} - \sin^2\beta + \cancel{\cos^2\alpha\sin^2\beta}$

$\cos^2\alpha - \sin^2\beta \longrightarrow (ii)$

$1-\sin^2\alpha - (1-\cos^2\beta)$

$\cancel{1} - \sin^2\alpha - \cancel{1} + \cos^2\beta$

$\cos^2\beta - \sin^2\alpha \rightarrow (iii)$

from (i), (ii), and (iii)

$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$

5. Show that: $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$

L.H.S

$$\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$

$$\frac{\sin\alpha\cos\beta + \cancel{\cos\alpha\sin\beta} + \sin\alpha\cos\beta - \cancel{\cos\alpha\sin\beta}}{\cos\alpha\cos\beta - \cancel{\sin\alpha\sin\beta} + \cos\alpha\cos\beta + \cancel{\sin\alpha\sin\beta}}$$

$$\frac{\cancel{2}\sin\alpha\cancel{\cos\beta}}{\cancel{2}\cos\alpha\cancel{\cos\beta}}$$

$$\frac{\sin\alpha}{\cos\alpha}$$

$$\tan\alpha \Rightarrow R.H.S$$

Proved.

6. Show that:

6. Show that:

(i) $\sin^2\left(\alpha+\frac{\beta}{2}\right)-\sin^2\left(\alpha-\frac{\beta}{2}\right)=\sin 2\alpha\cdot\sin\beta$

L.H.S

$$(\sin(\alpha+\tfrac{\beta}{2})+\sin(\alpha-\tfrac{\beta}{2}))(\sin(\alpha+\tfrac{\beta}{2})-\sin(\alpha-\tfrac{\beta}{2})$$

$$(\sin\alpha\cos\tfrac{\beta}{2}+\cancel{\cos\alpha\sin\tfrac{\beta}{2}}+\sin\alpha\cos\tfrac{\beta}{2}-\cancel{\cos\alpha\sin\tfrac{\beta}{2}})$$

$$(\cancel{\sin\alpha\cos\tfrac{\beta}{2}}+\cos\alpha\sin\tfrac{\beta}{2}-\cancel{\sin\alpha\cos\tfrac{\beta}{2}}+\cos\alpha\sin\tfrac{\beta}{2})$$

$$(2\sin\alpha\cos\tfrac{\beta}{2})(2\cos\alpha\sin\tfrac{\beta}{2})$$

$$(2\sin\alpha\cos\alpha)(2\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2})$$

$$\sin2\alpha\,\sin\cancel{2}(\tfrac{\beta}{\cancel{2}})$$

$$\sin2\alpha\,.\,\sin\beta \Rightarrow R.H.S$$

Proved

L.H.S = R.H.S

(ii) $\sin^2\alpha + \sin^2\beta + \cos^2(\alpha+\beta) + 2\sin\alpha\sin\beta \cdot \cos(\alpha+\beta) = 1$

L.H.S

$= \sin^2\alpha + \sin^2\beta + \cos^2(\alpha+\beta) + 2\sin\alpha.\sin\beta.\cos(\alpha+\beta)$

$= \sin^2\alpha + \sin^2\beta + \cos(\alpha+\beta)[\cos(\alpha+\beta) + 2\sin\alpha.\sin\beta]$

$= \text{''} + \text{''} + \cos(\alpha+\beta)[\cos\alpha\cos\beta - \sin\alpha\sin\beta + 2\sin\alpha\sin\beta]$

$= \text{''} + \text{''} + \cos(\alpha+\beta)[\cos\alpha\cos\beta + \sin\alpha\sin\beta]$

$= \sin^2\alpha + \sin^2\beta + \cos(\alpha+\beta)[\cos\alpha\cos\beta + \sin\alpha\sin\beta]$

$= \sin^2\alpha + \sin^2\beta + \cos\alpha\cos\beta - \sin\alpha\sin\beta(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= \sin^2\alpha + \sin^2\beta + \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$

$= \sin^2\alpha + \sin^2\beta + (1-\sin^2\alpha)\cos^2\beta - \sin^2\alpha(1-\cos^2\beta)$

$= \cancel{\sin^2\alpha} + \sin^2\beta + \cos^2\beta - \cancel{\sin^2\alpha\cos^2\beta} - \cancel{\sin^2\alpha} + \cancel{\sin^2\alpha\cos^2\beta}$

$= \sin^2\beta + \cos^2\beta$

$= 1 \Rightarrow$ R.H.S

Proved

7. Show that

(i) $\cos(\alpha-\beta) = \dfrac{1+\tan\alpha\tan\beta}{\sec\alpha\sec\beta}$

R.HS

$= \dfrac{1+\tan\alpha\tan\beta}{\sec\alpha\sec\beta}$

$$= \frac{1 + \frac{\sin\alpha}{\cos\alpha} \times \frac{\sin\beta}{\cos\beta}}{\frac{1}{\cos\alpha} \times \frac{1}{\cos\beta}}$$

$$= \frac{\frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}}$$

$$= \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{1}$$

$$= \cos(\alpha - \beta) \Rightarrow L.H.S$$

Proved

(ii) $\sin(\alpha+\beta) = \frac{1 + \cot\alpha\tan\beta}{\operatorname{cosec}\alpha\sec\beta}$

L.H.S

$$= \frac{1 + \cot\alpha\tan\beta}{\operatorname{cosec}\alpha\sec\beta}$$

$$= \frac{1 + \frac{\cos\alpha}{\sin\alpha} \times \frac{\sin\beta}{\cos\beta}}{\frac{1}{\sin\alpha} \times \frac{1}{\cos\beta}}$$

$$= \frac{\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta}}{\frac{1}{\sin\alpha\cos\beta}}$$

$$= \sin(\alpha+\beta) \Rightarrow R.H.S$$

Proved

(iii) $\cot(\alpha-\beta) = \dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

R.H.S

$= \dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

$= \dfrac{\dfrac{\cos\alpha}{\sin\alpha} \times \dfrac{\cos\beta}{\sin\beta} + 1}{\dfrac{\cos\beta}{\sin\beta} - \dfrac{\cos\alpha}{\sin\alpha}}$

$= \dfrac{\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$

$= \dfrac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)}$

$= \cot(\alpha-\beta) \Rightarrow$ R.H.S

Proved

(iv) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

L.H.S

$= \dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$

$$= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}}$$

$$= \frac{\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cancel{\cos\alpha\cos\beta}}}{\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cancel{\cos\alpha\cos\beta}}}$$

$$= \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} \Rightarrow R.H.S$$

Proved

(V) $\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$

R.H.S

$$= \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

$$= \frac{\frac{\cos\alpha}{\sin\alpha} \times \frac{\cos\beta}{\sin\beta} - 1}{\frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta}}$$

$$= \frac{\frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cancel{\sin\alpha\sin\beta}}}{\frac{\cos\alpha\sin\beta + \sin\alpha\cos\beta}{\cancel{\sin\alpha\sin\beta}}}$$

$$= \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}$$

$\cot(\alpha+\beta) \Rightarrow$ L.H.S

Proved

8. If $\sin\alpha = \frac{24}{25}$ and $\cos\beta = \frac{20}{29}$, where $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$. Show that $\sin(\alpha-\beta) = \frac{333}{725}$.

$$\sin(\alpha-\beta) = \frac{333}{725}$$

L.H.S

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \rightarrow (i)$

Now

$$\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$$

$$= \sqrt{1-\left(\frac{24}{25}\right)^2}$$

$$= \sqrt{1-\frac{576}{625}}$$

$$= \sqrt{\frac{625-576}{625}}$$

$$= \sqrt{\frac{49}{625}}$$

$$\cos\alpha = \frac{7}{25}$$

again

$$\sin\beta = \pm\sqrt{1-\cos^2\beta}$$

$$= \sqrt{1-\left(\frac{20}{29}\right)^2}$$

$$= \sqrt{1-\frac{400}{841}}$$

$$= \sqrt{\frac{841-400}{841}}$$

$$= \sqrt{\frac{441}{841}}$$

$$\sin\beta = \frac{21}{29}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{24}{25}\cdot\frac{20}{29} - \frac{7}{25}\cdot\frac{21}{29}$$

$$= \frac{480}{725} - \frac{147}{725}$$

$$= \frac{480-147}{725}$$

$$= \frac{333}{725}$$

9. If $\sin\alpha = -\frac{8}{17}$ and $\cos\beta = -\frac{4}{5}$

where $\frac{3\pi}{2} < \alpha < 2\pi$ and $\pi < \beta < \frac{3\pi}{2}$.

Find In which quadrants do

(a) the terminal sides of

the angles of measures $(\alpha+\beta)$ and

$(\alpha-\beta)$ lie?

(a) Part

$$\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$$

$$= \sqrt{1-\left(\frac{8}{17}\right)^2}$$

$$= \sqrt{1-\frac{64}{289}}$$

$$= \sqrt{\frac{289-64}{289}}$$

$$= \sqrt{\frac{225}{289}}$$

$$\cos\alpha = \frac{15}{17}$$

(α in IV quadrent)

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-8/17}{15/17}$$

$\tan\alpha = \frac{-8}{15}$

$\sin\beta = \pm\sqrt{1-\cos^2\beta}$

as β lies in III quad.

$= -\sqrt{1-\left(-\frac{4}{5}\right)^2}$

$= -\sqrt{1-\frac{16}{25}}$

$= -\sqrt{\frac{25-16}{25}}$

$= -\sqrt{\frac{9}{25}}$

$\sin\beta = -\frac{3}{5}$

$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{+3/\cancel{5}}{+4/\cancel{5}} = \frac{3}{4}$

(i) $\sin(\alpha+\beta)$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{-8}{17}\right)\left(-\frac{4}{5}\right) + \left(\frac{15}{17}\right)\left(\frac{-3}{5}\right)$

$= \frac{32}{85} - \frac{45}{85}$

$$= \frac{32-45}{85}$$

$$= \frac{-13}{85}$$

(ii) $\cos(\alpha+\beta)$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \left(\frac{15}{17}\right)\left(\frac{-4}{5}\right) - \left(\frac{-8}{17}\right)\left(-\frac{3}{5}\right)$$

$$= \frac{-60}{85} - \frac{24}{85}$$

$$= \frac{-60-24}{85}$$

$$= \frac{-84}{85}$$

(iii) $\tan(\alpha+\beta)$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{-8}{15} + \frac{3}{4}}{1 - \left(-\frac{8}{15}\right)\left(\frac{3}{4}\right)}$$

$$= \frac{\frac{-32+45}{60}}{1 + \frac{24}{60}}$$

$$= \frac{\frac{13}{60}}{\frac{60+24}{60}}$$

$$= \frac{13}{84}$$

(iv) $\sin(\alpha-\beta)$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \left(\frac{-8}{17}\right)\left(\frac{-4}{5}\right) - \left(\frac{15}{17}\right)\left(\frac{-3}{5}\right)$$

$$= \frac{32}{85} + \frac{45}{85}$$

$$= \frac{32+45}{85}$$

$$= \frac{77}{85}$$

(v) $\cos(\alpha-\beta)$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \left(\frac{15}{17}\right)\left(\frac{-4}{5}\right) + \left(\frac{-8}{17}\right)\left(\frac{-3}{5}\right)$$

$$= \frac{-60}{85} + \frac{24}{85}$$

$$= \frac{-60+24}{85} = \frac{-36}{85}$$

$$= \frac{32-45}{85}$$

$$= \frac{-13}{85}$$

(ii) $\cos(\alpha+\beta)$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \left(\frac{15}{17}\right)\left(\frac{-4}{5}\right) - \left(\frac{-8}{17}\right)\left(\frac{-3}{5}\right)$$

$$= \frac{-60}{85} - \frac{24}{85}$$

$$= \frac{-60-24}{85}$$

$$= \frac{-84}{85}$$

(iii) $\tan(\alpha+\beta)$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{\frac{-8}{15} + \frac{3}{4}}{1-\left(-\frac{8}{15}\right)\left(\frac{3}{4}\right)}$$

$$= \frac{\frac{-32+45}{60}}{1 + \frac{24}{60}}$$

$$= \frac{\frac{13}{\cancel{60}}}{\frac{60+24}{\cancel{60}}}$$

$$= \frac{13}{84}$$

(iv) $\sin(\alpha-\beta)$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \left(\frac{-8}{17}\right)\left(\frac{-4}{5}\right) - \left(\frac{15}{17}\right)\left(\frac{-3}{5}\right)$$

$$= \frac{32}{85} + \frac{45}{85}$$

$$= \frac{32+45}{85}$$

$$= \frac{77}{85}$$

(v) $\cos(\alpha-\beta)$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \left(\frac{15}{17}\right)\left(\frac{-4}{5}\right) + \left(\frac{-8}{17}\right)\left(\frac{-3}{5}\right)$$

$$= \frac{-60}{85} + \frac{24}{85}$$

$$= \frac{-60+24}{85} = \frac{-36}{85}$$

(vi) $\tan(\alpha - \beta)$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

$$= \frac{-\frac{8}{15} - \frac{3}{4}}{1 + \left(-\frac{8}{15}\right)\left(\frac{3}{4}\right)}$$

$$= \frac{\frac{-32-45}{60}}{1 - \frac{24}{60}}$$

$$= \frac{\frac{-77}{\cancel{60}}}{\frac{60-24}{\cancel{60}}}$$

$$= \frac{-77}{36}$$

Part (b)

So, $(\alpha + \beta)$ lie in III quadrent and $(\alpha - \beta)$ lie in II quadrent.

10. Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, given that

(i) $\tan\alpha = 3/4$, $\cos\beta = 5/13$ and neither the terminal side of the angle of measure α nor that of β is in the quadrent I.

$$\sec\alpha = \pm\sqrt{1+\tan^2\alpha}$$

as α lies in II quad

$$= -\sqrt{1+\left(\frac{3}{4}\right)^2}$$

$$= -\sqrt{1+\frac{9}{16}}$$

$$= -\sqrt{\frac{16+9}{16}}$$

$$= -\sqrt{\frac{25}{16}}$$

$$= -5/4$$

$$\sin^2\alpha = 1-\cos^2\alpha$$

$$= 1-\left(-4/5\right)^2$$

$$= 1-\frac{16}{25}$$

$$= \frac{25-16}{25}$$

$$\sqrt{\sin^2\alpha} = \sqrt{\frac{9}{25}}$$

$$\sin\alpha = -\frac{3}{5}$$

α lies in III quadrent

$$\sin\beta = \pm\sqrt{1-\cos^2\beta}$$

as β lies in IV quadrent.

$$= -\sqrt{1-\left(\frac{5}{13}\right)^2}$$

$$= -\sqrt{1-\frac{25}{169}}$$

$$= -\sqrt{\frac{169-25}{169}}$$

$$= -\sqrt{\frac{144}{169}}$$

$$\sin\beta = \frac{-12}{13}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{-3}{5} \times \frac{5}{13} + \frac{(-4)}{5} \times \frac{-1}{1}$$

$$= \frac{-15}{65} + \frac{48}{65}$$

$$= \frac{-15+48}{65}$$

$$\sin(\alpha+\beta) = \frac{33}{65}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{-4}{5} \times \frac{5}{13} - \frac{(-3)}{5} \times \frac{-12}{13}$$

$$= \frac{-20}{65} - \frac{36}{65}$$

$$= \frac{-20-36}{65}$$

$$\cos(\alpha+\beta) = \frac{-56}{65}$$

(ii) $\tan\alpha = -\frac{15}{8}$ and $\sin\beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor that of β is in the quadrant IV.

So, α lies in II quadrant and β lies in III quadrant.

$$\sec\alpha = \pm\sqrt{1+\tan^2\alpha}$$

As α in II quad

$$\sec\alpha = -\sqrt{1+\tan^2\alpha}$$

$$= -\sqrt{1+\left(\frac{-15}{8}\right)^2}$$

$$= -\sqrt{1+\frac{225}{64}}$$

$$= -\sqrt{\frac{64+225}{64}}$$

$$= -\sqrt{\frac{289}{64}}$$

$$\sec\alpha = -\frac{17}{8}$$

$$\cos\alpha = -8/17$$

$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$$

As α lies in II quadrant

$$\sin\alpha = \sqrt{1-\left(\frac{-8}{17}\right)^2}$$

$$= \sqrt{1-\frac{64}{289}}$$

$$= \sqrt{\frac{289-64}{289}}$$

$$= \sqrt{\frac{225}{289}}$$

$$= \frac{15}{17}$$

$$\cos\beta = \pm\sqrt{1-\sin^2\beta}$$

As β lies in III quadrant

$$\cos\beta = -\sqrt{1-\left(\frac{-7}{25}\right)^2}$$

$$= -\sqrt{1-\frac{49}{625}}$$

$$= -\sqrt{\frac{625-49}{625}}$$

$$= -\sqrt{\frac{576}{625}}$$

$$\cos\beta = -\frac{24}{25}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{15}{17} \times \frac{-24}{25} + \frac{(-8)}{17} \times \frac{-7}{25}$$

$$= \frac{-360}{425} + \frac{56}{425}$$

$$= \frac{-360+56}{425}$$

$$\sin(\alpha+\beta) = -304/425$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{-8}{17} \times \frac{-24}{25} - \frac{15}{17} \times \frac{-7}{25}$

$= \frac{192}{425} + \frac{105}{425}$

$= \frac{192+105}{425}$

$\cos(\alpha+\beta) = \frac{297}{425}$

So $(\alpha+\beta)$ lies in IV quadrant.

11. Prove that: $\frac{\cos 19 + \sin 19}{\cos 19 - \sin 19} = \tan 64°$.

R.H.S

$= \tan 64°$

$= \tan(45° + 19°)$

$= \frac{\sin(45° + 19°)}{\cos(45° + 19°)}$

$= \frac{\sin 45 \cos 19 + \cos 45 \sin 19}{\cos 45 \cos 19 - \sin 45 \sin 19}$

$= \frac{1/\sqrt{2} \cos 19 + 1/\sqrt{2} \sin 19}{1/\sqrt{2} \cos 19 - 1/\sqrt{2} \sin 19}$

$= \dfrac{\frac{1}{\sqrt{2}}(\cos 19 + \sin 19)}{\frac{1}{\sqrt{2}}(\cos 19 - \sin 19)}$

$= \dfrac{\cos 19 + \sin 19}{\cos 19 - \sin 19} \Rightarrow$ R.H.S

Proved

12. Prove that:

$\cos(60° + \theta)\cos(60° - \theta) + \sin(60° + \theta)\sin(60° - \theta) = \cos 2\theta$

L.H.S

$\cos(60° + \theta - (60° - \theta)$

$\cos(\cancel{60°} + \theta - \cancel{60°} + \theta)$

$\cos(2\theta)$

$\cos 2\theta \Rightarrow$ R.H.S Proved

15. Express the following in the form $r\sin(\theta + \phi)$ or $r\sin(\theta - \phi)$ where terminal side of the angles of measures θ and ϕ are in the first quadrant:

(i) $24\sin\theta + 7\cos\theta$

$a = 24, \; b = 7$ $\qquad \phi = \tan^{-1}(b/a)$

$r = \sqrt{(24)^2 + (7)^2}$ $\qquad \phi = \tan^{-1}(7/24)$

$r = \sqrt{576 + 49}$ $\qquad r\sin(\theta + \phi)$

$r = \sqrt{625} = 25$ $\qquad 25\sin(\theta + 7/24)$

(ii) $12\sin\theta - 5\cos\theta$

$a = 12, \quad b = 5$

$r = \sqrt{(12)^2 + (5)^2}$

$r = \sqrt{144 + 25}$

$r = \sqrt{169}$

$r = 13$

$\phi = \tan^{-1}(5/12)$

$r\sin(\theta - \phi)$

$13\sin(\theta - 5/12)$

(iii) $\sin\theta - \cos\theta$

$a = 1, \quad b = 1$

$r = \sqrt{(1)^2 + (1)^2}$

$r = \sqrt{1+1}$

$r = \sqrt{2}$

$\tan\phi = \frac{1}{1}$

$\tan\phi = 1$

$r\sin(\theta + \phi)$

$\sqrt{2}\sin(\theta + \phi)$

(iv) $8\sin\theta - 6\cos\theta$

$a = 8, \quad b = 6$

$r = \sqrt{(8)^2 + (6)^2}$

$r = \sqrt{64 + 36}$

$r = \sqrt{100}$

$r = 10$

$\tan\phi = \frac{\cancel{6}}{4} \ \frac{3}{4}$

$r\sin(\theta - \phi)$

$10\sin(\theta - \phi)$

(v) $\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta$

$a = \frac{1}{2}, \quad b = \frac{\sqrt{3}}{2}$

$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$	$\tan\phi = \dfrac{\frac{\sqrt{3}}{\cancel{2}}}{\frac{1}{\cancel{2}}}$
$= \sqrt{\frac{1}{4} + \frac{3}{4}}$	$\tan\phi = \frac{\sqrt{3}}{1}$
$= \sqrt{\frac{1+3}{4}}$	$\tan\phi = \sqrt{3}$
$= \sqrt{\frac{4}{4}}$	$1 \sin(\theta + \phi)$
$r = 1$	

(vi) $13 \sin\theta - 84 \cos\theta$

$a = 13, \; b = 84$	$\tan\phi = \frac{84}{13}$
$r = \sqrt{(13)^2 + (84)^2}$	
$= \sqrt{169 + 7056}$	$85 \sin(\theta - \phi)$
$= \sqrt{7225}$	
$r = 85.$	

13. If α, β, γ are the angles of a triangle ABC, Show that

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$$

$\alpha + \beta + \gamma = 180$

$\alpha + \beta = 180^\circ - \gamma$

$$\frac{\alpha+\beta}{2} = \frac{180-\gamma}{2}$$

$$\frac{\alpha}{2} + \frac{\beta}{2} = 90^\circ - \frac{\gamma}{2}$$

Taking tan on both sides

$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(90 - \frac{\gamma}{2}\right)$$

$$\frac{\tan \alpha/2 + \tan \beta/2}{1 - \tan \alpha/2 \tan \beta/2} = \cot\frac{\gamma}{2}$$

$$\frac{\frac{1}{\cot \alpha/2} + \frac{1}{\cot \beta/2}}{1 - \frac{1}{\cot \alpha/2} \cdot \frac{1}{\cot \beta/2}} = \cot\frac{\gamma}{2}$$

$$\frac{\frac{\cot \beta/2 + \cot \alpha/2}{\cancel{\cot \alpha/2 \cot \beta/2}}}{\frac{\cot \alpha/2 \cot \beta/2 - 1}{\cancel{\cot \alpha/2 \cot \beta/2}}} = \cot\frac{\gamma}{2}$$

$$\frac{\cot \beta/2 + \cot\frac{\alpha}{2}}{\cot \alpha/2 \cot \beta/2 - 1} = \cot\frac{\gamma}{2}$$

$$\cot \beta/2 + \cot\frac{\alpha}{2} = \cot\frac{\gamma}{2}\left(\cot\frac{\alpha}{2}\cot \beta/2 - 1\right)$$

$\cot \beta/2 + \cot \alpha/2 = \cot \gamma/2 . \cot \alpha/2 . \cot \beta/2 - \cot \gamma/2$

$\cot \alpha/2 + \cot \beta/2 + \cot \gamma/2 = \cot \alpha/2 \cot \beta/2 \cot \gamma/2$

Proved

14. If $\alpha+\beta+\gamma = 180°$, show that:

$\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$

$\alpha+\beta+\gamma = 180$

$\alpha+\beta = 180° - \gamma$

$\tan(\alpha+\beta) = \tan(180° - \gamma)$

$$\frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = -\tan\gamma$$

$$\frac{\frac{1}{\cot\alpha} + \frac{1}{\cot\beta}}{1 - \frac{1}{\cot\alpha}\frac{1}{\cot\beta}} = -\frac{1}{\cot\gamma}$$

$$\frac{\frac{\cot\beta + \cot\alpha}{\cancel{\cot\alpha\cot\beta}}}{\frac{\cot\alpha\cot\beta - 1}{\cancel{\cot\alpha\cot\beta}}} = -\frac{1}{\cot\gamma}$$

$$\frac{\cot\beta + \cot\alpha}{\cot\alpha\cot\beta - 1} = -\frac{1}{\cot\gamma}$$

Cross multiply

$\cot\gamma(\cot\beta + \cot\alpha) = -1(\cot\alpha\cot\beta - 1)$

$\cot\gamma\cot\beta + \cot\gamma\cot\alpha = -\cot\alpha\cot\beta + 1$

$\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$

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S.S.S (Math)