Complex Polynomials as a Product of Linear Factors:

A complex polynomial $P(z)$ is a polynomial function of the complex variable z with complex coefficients. It is expressed in the general form as:

$$P(z)=a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$$

Where $a_n, a_{n-1}, ..., a_1, a_0$ are complex numbers $(a_n \neq 0)$, and $n \geq 0$ is an integer representing the degree of the polynomial.

According to the **Fundamental theorem of algebra**, a polynomial of degree $n \geq 1$ has exactly n roots in complex numbers system C.

REMEMBER

If $n=0$, then $P(z)$ becomes constant polynomial.

Note:

*If $P(z)$ is a polynomial function, the values of z that satisfy $P(z)=0$ are called the **zeros** of the function $P(z)$ and roots of the polynomial equation $P(z)=0$.*

Solution of Quadratic Equations by Completing the Square:

This technique involves rewriting a quadratic equation in the form $ax^2+bx+c=0$ into a perfect square trinomial, which can then be solved by taking the square root of both sides. This method is especially valuable when the quadratic equation does not factor easily.

EXERCISE 1.3

1. Factorize the following:

(i) a^2+4b^2

Solution:

a^2+4b^2

$=a^2-4b^2i^2$

$=(a)^2-(2bi)^2 \quad =(a-2bi)(a+2bi)$

(ii) $9a^2+16b^2$

Solution:

$9a^2+16b^2$

$=9a^2-16b^2i^2 \quad =(3a)^2-(4bi)^2$

$=(3a-4bi)(3a+4bi)$

(iii) $3x^2+3y^2$

Solution:

$3x^2+3y^2$

$3(x^2+y^2)$

$=3(x^2-i^2y^2) \quad =3\left[(x)^2-(iy)^2\right]$

$=3(x+iy)(x-iy)$

(iv) $144x^2+225y^2$

Solution:

$=144x^2-225y^2i^2 \quad =(12x)^2-(15iy)^2$

$=(12x-15iy)(12x+15iy)$

$=9(4x-5iy)(4x+5iy)$

(v) $z^2-2iz-1$

Solution:

$z^2-2iz-1$

$=z^2-2iz+i^2 \quad =z^2-iz-iz+i^2$

$=z(z-i)-i(z-i) \quad =(z-i)(z-i)$

(vi) $z^2+6z+13$

Solution:

$z^2+6z+13$

$=z^2+6z+9+4 \quad =(z+3)^2-4i^2$

$=(z+3)^2-(2i)^2$

$=(z+3+2i)(z+3-2i)$

(vii) z^2+4z+5

Solution:

z^2+4z+5

$=z^2+4z+4+1 \quad =(z+2)^2-i^2$

$=(z+2+i)(z+2-i)$

(viii) $2z^2-22z+65$

Solution:

Using quadratic formula:

$$z=\frac{22\pm\sqrt{(-22)^2-4(2)(65)}}{2(2)}$$

$$z=\frac{22\pm\sqrt{484-520}}{4}=\frac{22\pm 6i}{4}=\frac{11\pm 3i}{2}$$

So, factors of $2z^2-22z+65$ are

$$\left(z-\frac{11+3i}{2}\right)\left(z-\frac{11-3i}{2}\right)$$

2. Factorize the following polynomial into its linear factors:

(i) z^3+8

Solution:

$$=(z)^3+(2)^3$$

$$\Rightarrow(z+2)(z^2-2z+4) \qquad (i)$$

Next to find factors of z^2-2z+4, using quadratic formula:

$$z=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$$

$$z=\frac{2\pm\sqrt{4-16}}{2}=\frac{2+2\sqrt{3}i}{2}$$

$$z=1\pm\sqrt{3}i$$

So quadratic factors are

$$z^2-2z+4=\left(z-\left(1+\sqrt{3}i\right)\right)\left(z-\left(1-\sqrt{3}i\right)\right)$$

$$=\left(z-1-\sqrt{3}i\right)\left(z-1+\sqrt{3}i\right)$$

Substitutes in (i), we have

$$(z+2)\left(z-1-\sqrt{3}i\right)\left(z-1+\sqrt{3}i\right)$$

(ii) z^3+27

Solution:

$$z^3+27$$

$$=(z)^3+(3)^3$$

$$=(z+3)(z^2-3z+9) \qquad (i)$$

To find factors of z^2-3z+9, using quadratic formula:

$$z=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(9)}}{2(1)}$$

$$z=\frac{3\pm\sqrt{9-36}}{2}=\frac{3\pm\sqrt{27}i}{2}=\frac{3\pm3\sqrt{3}i}{2}$$

So, quadratic factors of z^2-3z+9 are

$$\left(z-\frac{3+3\sqrt{3}i}{2}\right) \text{ and } \left(z-\frac{3-3\sqrt{3}i}{2}\right)$$

Substitute in (i) we have,

$$(z+3)\left(z-\frac{3}{2}+\frac{3\sqrt{3}i}{2}\right)\left(z-\frac{3}{2}-\frac{3\sqrt{3}i}{2}\right)$$

(iii) $z^3-2z^2+16z-32$

Solution:

$$z^3-2z^2+16z-32$$

$$=z^2(z-2)+16(z-2)$$

$$=(z-2)(z^2+16)$$

$$=(z-2)(z^2-16i^2)$$

$$=(z-2)\left[(z)^2-(4i)^2\right]$$

$$=(z-2)(z-4i)(z+4i)$$

(iv) z^4+21z^2-100

Solution:

$$z^4+25z^2-4z^2-100$$

$$=z^2(z^2+25)-4(z^2+25)$$

$$=(z^2+25)(z^2-4)$$

$$=(z^2-2^2)(z^2-25i^2)$$

$$=(z-2)(z+2)(z-5i)(z+5i)$$

(v) z^4-16

Solution:

$$z^4-16$$

$$=(z^2)^2-(4)^2$$

$$=(z^2-4)(z^2+4)$$

$$=\left[(z)^2-(2)^2\right]\left[(z)^2-(2i)^2\right]$$

$$=(z+2)(z-2)(z+2i)(z-2i)$$

(vi) z^4+3z^2-4

Solution:

$$=z^4+4z^2-z^2-4$$

$$=z^2(z^2+4)-1(z^2+4)$$

$$=(z^2-1)(z^2+4)$$

$$=\left[(z)^2-(1)^2\right]\left[(z)^2-(2i)^2\right]$$

$$=(z-1)(z+1)(z+2i)(z-2i)$$

(vii) z^4+5z^2+6

Solution:

$$=z^4+3z^2+2z^2+6$$

$$=z^2(z^2+3)+2(z^2+3)$$

$$=(z^2+2)(z^2+3)$$

$$=\left[(z)^2-\left(\sqrt{2}i\right)^2\right]\left[(z)^2-\left(\sqrt{3}i\right)^2\right]$$

$$=\left(z+\sqrt{2}i\right)\left(z-\sqrt{2}i\right)\left(z+\sqrt{3}i\right)\left(z-\sqrt{3}i\right)$$

(viii) z^4-32z^2-3969

Solution:

$$z^4-32z^2-3969$$
$$=z^4-81z^2+49z^2-3969$$
$$=z^2\left(z^2-81\right)+49\left(z^2-81\right)$$
$$=\left(z^2-81\right)\left(z^2+49\right)$$
$$=\left(z^2-9^2\right)\left(z^2-(7i)^2\right)$$
$$=(z-9)(z+9)(z-7i)(z+7i)$$

3. Find the roots of $z^4+7z^2-144=0$ and hence express it as a product of linear factors.

Solution:

$$z^4+16z^2-9z^2-14=0$$
$$z^2\left(z^2+16\right)-9\left(z^2+16\right)=0$$
$$\left(z^2-9\right)\left(z^2+16\right)=0$$
$$\left[(z)^2-(3)^2\right]\left[(z)^2-(4i)^2\right]=0$$
$$(z+3)(z-3)(z+4i)(z-4i)=0$$

Roots are $3,-3,4i,-4i$

The linear factors are

$(z+3)(z-3)(z+4i)(z-4i)$

4. Solve the following complex quadratic equation by completing square method:

(i) $2z^2-3z+4=0$

Solution:

$$\Rightarrow 2z^2-3z=-4 \quad \Rightarrow z^2-\frac{3}{2}z=-2$$
$$\Rightarrow z^2-\frac{3}{2}z+\frac{9}{16}=-2+\frac{9}{16}$$
$$\Rightarrow \left(z-\frac{3}{4}\right)^2=-\frac{23}{16} \quad \Rightarrow z-\frac{3}{4}=\pm\sqrt{-\frac{23}{16}}$$
$$z=\frac{3}{4}\pm\frac{\sqrt{23}i}{4}$$
$$z=\frac{3\pm\sqrt{23}i}{4}$$

(ii) $z^2-6z+30=0$

Solution:

$$\Rightarrow z^2-6z=-30$$
$$\Rightarrow z^2-6z+9=-30+9$$
$$\Rightarrow (z-3)^2=-21 \quad \Rightarrow z-3=\pm\sqrt{-21}$$
$$z=3\pm\sqrt{21}i$$

(iii) $3z^2-18z+50=0$

Solution:

$$3z^2-18z=-50$$
$$\Rightarrow z^2-6z=-\frac{50}{3}$$
$$\Rightarrow z^2-6z+9=-\frac{50}{3}+9$$
$$\Rightarrow (z-3)^2=-\frac{23}{3}$$
$$\Rightarrow (z-3)^2=\sqrt{-\frac{23}{3}}$$
$$\Rightarrow (z-3)=\pm\sqrt{\frac{23}{3}}i$$
$$\Rightarrow z=3\pm\sqrt{\frac{23}{3}}i$$

(iv) $z^2+4z+13=0$

Solution:

$$z^2+4z=-13$$
$$\Rightarrow z^2+4z+4=-13+4$$
$$\Rightarrow (z+2)^2=-9$$
$$\Rightarrow z+2=\pm 3i$$
$$\Rightarrow z=-2\pm 3i$$

(v) $2z^2+6z+9=0$

Solution:

$$\Rightarrow 2z^2+6z=-9$$
$$\Rightarrow z^2+3z=-\frac{9}{2}$$
$$\Rightarrow z^2+3z+\frac{9}{4}=-\frac{9}{2}+\frac{9}{4}$$
$$\Rightarrow \left(z+\frac{3}{2}\right)^2=-\frac{9}{4} \quad \Rightarrow z+\frac{3}{2}=\pm\frac{3}{2}i$$
$$\Rightarrow z=\frac{-3}{2}\pm\frac{3}{2}i$$

(vi) $3z^2-5z+7=0$

Solution:

$$\Rightarrow 3z^2-5z=-7 \quad \Rightarrow z^2-\frac{5}{3}z=-\frac{7}{3}$$
$$\Rightarrow z^2-\frac{5}{3}z+\frac{25}{36}=-\frac{7}{3}+\frac{25}{36}$$
$$\Rightarrow \left(z-\frac{5}{6}\right)^2=-\frac{59}{36} \quad \Rightarrow z-\frac{5}{6}=\pm\frac{\sqrt{59}i}{6}$$
$$\Rightarrow z=\frac{5}{6}\pm\frac{\sqrt{59}i}{6}$$

5. Solve the following equations:

(i) $2z^4 - 32 = 0$

Solution:

$\Rightarrow 2\left(z^4 - 16\right) = 0$

$\left(z^2\right)^2 - (4)^2 = 0$

$\left(z^2 - 4\right)\left(z^2 + 4\right) = 0$

$\left[(z)^2 - (2)^2\right]\left[(z)^2 - (2i)^2\right] = 0$

$(z-2)(z+2)(z-2i)(z+2i) = 0$

So, $z = 2, -2, 2i, -2i$

(ii) $3z^5 - 243z = 0$

Solution:

$3z\left(z^4 - 81\right) = 0$

$\Rightarrow z\left[\left(z^2\right)^2 - (9)^2\right] = 0$

$\Rightarrow z\left[\left(z^2 - 9\right)\left(z^2 + 9\right)\right] = 0$

$\Rightarrow z\left[(z)^2 - (3)^2\right]\left[(z)^2 - (3i)^2\right] = 0$

$\Rightarrow z(z+3)(z-3)(z+3i)(z-3i) = 0$

$\Rightarrow z = 0, -3, 3, 3i, -3i$

(iii) $5z^5 - 5z = 0$

Solution:

$5z\left(z^4 - 1\right) = 0$

$\Rightarrow z\left(z^4 - 1\right) = 0$

$\Rightarrow z\left[\left(z^2\right)^2 - \left(1^2\right)^2\right] = 0$

$\Rightarrow z\left[\left(z^2 - 1\right)\left(z^2 + 1\right)\right] = 0$

$\Rightarrow z(z-1)(z+1)(z-i)(z+i) = 0$

So, $z = 0, \pm 1, \pm i$

(iv) $z^3 - 5z^2 + z - 5 = 0$

Solution:

$z^2(z-5) + 1(z-5) = 0$

$\Rightarrow (z-5)\left(z^2 + 1\right) = 0$

$\Rightarrow (z-5)\left(z^2 - i^2\right) = 0$

$\Rightarrow (z-5)(z+i)(z-i) = 0$

So, $z = 5, \pm i$

(v) $4z^4 - 25z^2 - 21 = 0$

Solution:

Let $z^2 = t$, so

$4t^2 - 25t - 21 = 0$

Using quadratic formula:

$$t = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(4)(-21)}}{2(4)}$$

$$t = \frac{25 \pm \sqrt{625 + 336}}{8} = \frac{25 \pm 31}{8}$$

$\Rightarrow t = 7$ or $t = \frac{-3}{4}$

Replacing the value of t,

$z^2 = 7$ or $z^2 = -\frac{3}{4}$

$\Rightarrow z = \pm\sqrt{7}$ or $z = \pm\frac{\sqrt{3}}{2}i$

So, $z = \sqrt{7}, -\sqrt{7}, \frac{\sqrt{3}}{2}i, -\frac{\sqrt{3}}{2}i$

(vi) $z^3 + z^2 + z + 1 = 0$

Solution:

$\Rightarrow z^2(z+1) + 1(z+1) = 0$

$\Rightarrow (z+1)\left(z^2 + 1\right) = 0$

$\Rightarrow (z+1)\left(z^2 - i^2\right) = 0$

$\Rightarrow (z+1)(z+i)(z-i) = 0$

Roots are: $z = -1, i, -i$

6. Find a polynomial $P(z)$ of degree 3 with zeros $3, -2i, 2i$ and satisfying $P(1) = 20$.

Solution:

Given $3, -2i, 2i$ are zeroes of polynomial. $P(z)$ then:

$x = P(z) = a(z-3)(z+2i)(z-2i)$

$= a(z-3)\left(z^2 - 4i^2\right)$

$= a(z-3)\left(z^2 + 4\right)$

$= a\left(z^3 + 4z - 3z^2 - 12\right)$

$P(z) = a\left(z^3 - 3z^2 + 4z - 12\right)$ (i)

Also $P(1) = 20$

$P(1) = a[1 - 3 + 4 - 12]$

$\Rightarrow 20 = a(-10)$

$a = -2$ put in (i)

$P(z) = x = -2\left(z^3 - 3z^2 + 4z - 12\right)$

$x = -2z^3 + 6z^2 - 8z + 24$

7. Find a polynomial $P(z)$ of degree 4 with zeros $2i,-2i,1,-1,$ and satisfying $P(2)=240.$

Solution:

Given $2i,-2i,1,-1$ are zeros of polynomial $P(z).$

So,

$$x=P(z)=a(z+1)(z-1)(z+2i)(z-2i)$$

$$x=P(z)=a(z^2-1)(z^2-4i^2)$$

$$x=P(z)=a(z^2-1)(z^2+4)$$

$$=a(z^4+4z^2-z^2-4)$$

$$P(z)=a(z^4+3z^2-4) \qquad \text{(i)}$$

Given $P(2)=240$

$$P(2)=a\left[(2)^4+3(2)^2-4\right]$$

$$240=a[16+12-4]\Rightarrow a=10 \text{ put in (i)}$$

$$x=10(z^4+3z^2-4)$$

$$x=10z^4+30z^2-40$$

8. Find a polynomial $P(z)$ of degree 4 with zeros $4,-4,1+i,1-i$ and satisfying $P(2)=72.$

Solution:

Given $4,-4,1+i,1-i$ are zeros of a polynomial $P(z)$ so

$$x=P(z)=a(z-4)(z+4)(z-(1+i))(z-(1-i))$$

$$P(z)=a(z^2-16)(z-1-i)(z-1+i)$$

$$=a(z^2-16)\left[(z-1)^2-i^2\right]$$

$$=a(z^2-16)\left[z^2-2z+1+1\right]$$

$$=a(z^2-16)\left[z^2-2z+2\right]$$

$$=a(z^4-2z^3+2z^2-16z^2+32z-32)$$

$$=a(z^4-2z^3-14z^2+32z-32) \quad \text{(i)}$$

Also $P(2)=72$ put in (i)

$$72=a\left[2^4-2(2)^3-14(2)^2+32(2)-32\right]$$

$$72=a[-24]\Rightarrow a=-3 \text{ put in (i)}$$

$$P(z)=x=-3(z^4-2z^3-14z^2+32z-32)$$

$$=-3z^4+6z^3+42z^2-96z+96$$