Equality of Two Complex Numbers:

The two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are said to be equal iff their real and imaginary parts are equal i.e., $a + bi = c + di \Leftrightarrow a = c$ and $b = d$.

Square root of a Complex Number:

The square root of a complex number is another complex number that, when squared, give the original complex number.

Let $w = p + qi$ is a square root of a complex number $z = x + iy$, where $p, q, x, y \in R$, then $w = \sqrt{z}$ … (i), taking square on both sides, we get

$$w^2 = z$$

$$(p + iq)^2 = x + iy$$

$$p^2 + 2pqi - q^2 = x + iy$$

Equating real and imaginary part, we have

$$x = p^2 - q^2 \qquad \text{(ii)}$$

$$y = 2pq \qquad \text{(iii)}$$

$$\Rightarrow p = \pm\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} \text{ and } q = \pm\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}$$

From equation (iii): $y = 2pq$, thus we have

- $y > 0$, if p and q have the same sign
- $y < 0$, if p and q have opposite sign
- $y = 0$, if $p = 0$ or $q = 0$

Therefore, the square root of the complex number $z = x + iy$ is given by

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z| + x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z| - x}{2}}\right) \ldots\text{(v), where } |z| = \sqrt{x^2 + y^2} \geq 0 \text{ is modulus of } z.$$

EXERCISE 1.2

1. Find the values of x and y in each of the following.

(i) $x+iy+2-3i=i(5-i)(3+4i)$

Solution:

$x+iy+2-3i=i(5-i)(3+4i)$

$\Rightarrow (x+2)+i(y-3)=i\left[15+20i-3i-4i^2\right]$

$\Rightarrow (x+2)+i(y-3)=i[15+4+17i]$

$(x+2)+i(y-3)=-17+19i$

Comparing real and imaginary parts, we have

$x+2=-17$; $y-3=19$

$\Rightarrow x=-19$; $y=22$

(ii) $(x+iy)(1-i)=(2-3i)(-5+5i)\left(-i\dfrac{3}{5}\right)$

Solution:

$(x+iy)(1-i)=(2-3i)(-5+5i)\left(-i\dfrac{3}{5}\right)$

$=(2-3i)5(-1+i)\left(-i\dfrac{3}{5}\right)$

$\Rightarrow x-xi+yi-i^2y=(2-3i)\left(3i-3i^2\right)$

$\Rightarrow x-xi+yi+y=(2-3i)(3i+3)$

$\Rightarrow (x+y)+i(y-x)=6i+6-9i^2-9i$

$\Rightarrow (x+y)+i(y-x)=15-3i$

Comparing real and imaginary parts, we have

$x+y=15$ (i)

$-x+y=-3$ (ii)

Adding (i) and (ii)

$2y=12$

$\Rightarrow y=6$ put in (i)

$x+6=15 \Rightarrow x=15-6 \Rightarrow x=9$

So, $x=9,\ y=6$

(iii) $\dfrac{x}{2+i}+\dfrac{y}{3-i}=4+5i$

Solution:

$\dfrac{x}{2+i}+\dfrac{y}{3-i}=4+5i$

$\Rightarrow \dfrac{x(3-i)+y(2+i)}{(2+i)(3-i)}=4+5i$

$\Rightarrow 3x-xi+2y+yi=(4+5i)\left(6-2i+3i-i^2\right)$

$\Rightarrow (3x+2y)+(-x+y)i=(4+5i)(6+1+i)$

$\Rightarrow (3x+2y)+(-x+y)i=(4+5i)(7+i)$

$\Rightarrow (3x+2y)+(y-x)i=28+4i+35i+5i^2$

$\Rightarrow (3x+2y)+i(y-x)=23+39i$

Comparing real and imaginary parts, we get

$3x+2y=23$ (i)

$-x+y=39$ (ii)

From (ii)

$y=39+x$, put in (i)

$3x+2(39+x)=23$

$\Rightarrow 3x+78+2x=23$

$5x=23-78 \Rightarrow x=-11$

Put in (ii)

$-(-11)+y=39$

$y=39-11$

$y=28$

So, $x=-11$ and $y=28$

2. If $z_1=-13+24i$ and $z_2=x+yi$, find the values of x and y such that $z_1-z_2=-27+15i$

Solution:

Given that: $z_1-z_2=-27+15i$

Replacing the values of z_1 and z_2 we get;

$(-13+24i)-(x+iy)=-27+15i$

Comparing real and imaginary parts, we get

$-13-x=-27$; $24-y=15$

$\Rightarrow x=-13+27$; $-y=15-24$

$x=14$; $y=9$

So, values of x and y are 14 and 9 respectively.

3. Find the value of x and y if:

(i) $(x+iy)^2=25+60i$

Solution:

Given: $(x+iy)^2=25+60i$

$\Rightarrow x^2+(iy)^2+2xyi=25+60i$

$\Rightarrow x^2-y^2+2xyi=25+60i$

By comparing real and imaginary parts,

$x^2-y^2=25$ (i)

$2xy=60$ (ii)

From (ii) $y=\dfrac{30}{x}$ put in (i)

$$x^2-\left(\frac{30}{x}\right)^2=25\Rightarrow x^2-\frac{900}{x^2}=25$$

$$\Rightarrow x^4-900=25x^2\Rightarrow x^4-25x^2-900=0$$

$$x^4-45x^2+20x^2-900=0$$

$$\Rightarrow x^2\left(x^2-45\right)+20\left(x^2-45\right)=0$$

$$\Rightarrow x^2=45,\text{ or } x^2=-20\quad \text{(Not Possible)}$$

So, $x^2=45$

Put in (i)

$$45-y^2=25$$

$$y^2=20$$

$$y=2\sqrt{5}$$

$$x=3\sqrt{5},\ y=2\sqrt{5}$$

Or $x=-3\sqrt{5},\ y=-2\sqrt{5}$

(ii) $\left(x+iy\right)^2=64+48i$

Solution:

Given: $\left(x+iy\right)^2=64+48i$

$$\Rightarrow x^2+\left(iy\right)^2+2ixy=64+48i$$

$$\Rightarrow x^2-y^2+2xyi=64+48i$$

By comparing real and imaginary parts, we get,

$$x^2-y^2=64 \qquad \text{(i)}$$

$$2xy=48 \qquad \text{(ii)}$$

From (ii) $y=\dfrac{24}{x}$ Put in (i)

$$x^2-\left(\frac{24}{x}\right)^2=64\Rightarrow x^2-\frac{576}{x^2}=64$$

$$\Rightarrow x^4-576-64x^2=0$$

$$\Rightarrow x^4-64x^2-576=0$$

$$x^4-72x^2+8x^2-576=0$$

$$x^2\left(x^2-72\right)+8\left(x^2-72\right)=0$$

$$\Rightarrow\left(x^2-72\right)\left(x^2+8\right)=0$$

$$\Rightarrow x^2=72\ \text{ or }\ x^2=-8\ \text{(Not possible)}$$

So, $x^2=72$ Put in (i)

$$72-y^2=64\Rightarrow y^2=72-64\Rightarrow y^2=8$$

$$x=6\sqrt{2},\ y=2\sqrt{2}$$

or $x=-6\sqrt{2},\ y=-2\sqrt{2}$

(iii) $\left(x+iy\right)^2=\dfrac{2i-3}{3+i}$

Solution:

Given: $\left(x+iy\right)^2=\dfrac{2i-3}{3+i}$

$$x^2+\left(iy\right)^2+2ixy=\frac{2i-3}{3+i}\times\frac{3-i}{3-i}$$

$$x^2-y^2+2ixy=\frac{6i-2i^2-9+3i}{3^2-i^2}$$

$$=\frac{9i+2-9}{10}$$

$$x^2-y^2+2ixy=-\frac{7}{10}+\frac{9}{10}i$$

Comparing real and imaginary parts, we get:

$$x^2-y^2=-\frac{7}{10} \qquad \text{(i)}$$

$$2xy=\frac{9}{10}$$

$$xy=\frac{9}{20}$$

$$y=\frac{9}{20x} \qquad \text{(ii)}$$

Put in (i)

$$x^2-\left(\frac{9}{20x}\right)^2=-\frac{7}{10}$$

$$x^2-\frac{81}{400x^2}=-\frac{7}{10}$$

$$x^2=\frac{81}{400x^2}-\frac{7}{10}$$

$$400x^4+280x^2-81=0$$

Using quadratic formula:

$$x^2=\frac{-280\pm\sqrt{\left(280\right)^2-4\left(400\right)\left(-81\right)}}{2\left(400\right)}$$

$$=\frac{-280\pm\sqrt{78400+129600}}{800}$$

$$=\frac{-280\pm\sqrt{208000}}{800}$$

$$=\frac{-280\pm40\sqrt{130}}{800}$$

$$x^2=\frac{-7\pm\sqrt{130}}{20}$$

Take $x^2=\dfrac{-7-\sqrt{130}}{20}$

This gives complex roots

Take $x^2 = \frac{-7+\sqrt{130}}{20}$

$= \frac{-7+11.4}{20}$

$x^2 = 0.22$

$x = \pm 0.46$

When $x = 0.46$, $y = \frac{9}{20x}$ becomes

$y = \frac{9}{20(0.46)} = \frac{9}{9.2} = 0.97$

When $x = -0.46$, $y = \frac{9}{20x}$ becomes

$y = \frac{9}{20(-0.46)} = -\frac{9}{9.2} = -0.97$

4. If $z_1 = 2+3i$ and $z_2 = 1-\alpha$, find the value of α such that $\text{Im}(z_1.z_2) = 7$.

Solution:

Given that: $\text{Im}(z_1.z_2) = 7$

$\Rightarrow \text{Im}[(2+3i)(1-\alpha)] = 7$

$\Rightarrow \text{Im}[2(1-\alpha)+3i(1-\alpha)] = 7$

$\Rightarrow 3(1-\alpha) = 7$

$1-\alpha = \frac{7}{3}$

$1-\frac{7}{3} = \alpha \Rightarrow \alpha = -\frac{4}{3}$

5. If $z_1 = x+yi$ and $z_2 = a+bi$, find x, y, a and b such that $z_1 + z_2 = 10+4i$ and $z_1 - z_2 = 6+2i$.

Solution:

Firstly, $z_1 + z_2 = 10+4i$

$\Rightarrow x+iy+a+ib = 10+4i$

$\Rightarrow (x+a)+(y+b)i = 10+4i$

By comparing real and imaginary parts:

$x+a = 10$ (i)

$y+b = 4$ (ii)

Also, $z_1 - z_2 = 6+2i$

$\Rightarrow (x+iy)-(a+ib) = 6+2i$

$\Rightarrow (x-a)+(y-b)i = 6+2i$

By comparing real and imaginary parts;

$x-a = 6$ (iii)

$y-b = 2$ (iv)

Adding (i) and (iii)

$x+a = 10$

$x-a = 6$

$2x = 16$

$x = 8$ put in (i)

$8+a = 10$

$a = 2$

Also, solving (ii) and (iv)

$y+b = 4$

$y-b = 2$ adding

$2y = 6$

$y = 3$ Put in (ii)

$3+b = 4 \Rightarrow b = 1$

So, values of x, y, a and b respectively are $8, 3, 2$ and 1.

6. Show that $\forall\ z_1, z_2 \in C, \overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$

Solution:

Let $z_1 = a+bi$ and $z_2 = c+di$

Now $z_1.z_2 = (a+bi)(c+di)$

$= (ac-bd)+(ad+bc)i$

L.H.S:

$\overline{z_1 z_2} = (ac-bd)-(ad+bc)i$

$ac-adi-bci+bdi^2$

$= a(c-di)-ib(c-di)$

$= (a-ib)(c-di)$

$\overline{z_1}.\overline{z_2} = \text{R.H.S}$

SHORT QUESTION

Show that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

7. Find the square root of the following complex numbers:

(i) $-7-24i$

Solution:

Let $z = x+iy = -7-24i$

$\Rightarrow x = -7, y = -24 < 0$

$|z| = \sqrt{49+576} = 25$

Applying the square root formula for complex numbers, we get

$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$

$$\Rightarrow \sqrt{-7-24i} = \pm\left(\sqrt{\frac{25-7}{2}} + \frac{i(-24)}{24}\sqrt{\frac{25+7}{2}}\right)$$

$$\Rightarrow \pm(3-i4)$$

As $y<0$, so square root of $-7-24i$ are $3-4i$ or $-3+4i$

(ii) $8-6i$

Solution:

Let $z = x+iy = 8-6i$

$\Rightarrow x=8,\ y=-6<0$

$|z| = \sqrt{64+36} = 10$

Applying square root formula for complex numbers, we get:

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+x}{2}} + \frac{iy}{|y|}\sqrt{\frac{|z|-x}{2}}\right)$$

$$\Rightarrow \sqrt{8-6i} = \pm\left(\sqrt{\frac{10+8}{2}} + \frac{(-6)i}{|-6|}\sqrt{\frac{10-8}{2}}\right)$$

$$= \pm(3-i)$$

As $y<0$, so square root of complex number $8-6i$ are $3-i$ or $-3+i$.

(iii) $-15-36i$

Solution:

Let $z = x+iy = -15-36i$

$\Rightarrow x=-15,\ y=-36<0$

$|z| = \sqrt{(-15)^2 + (-36)^2} = 39$

Applying square root formula, we get:

$$\sqrt{-15-36i} = \pm\left(\sqrt{\frac{39-15}{2}} + \frac{(-36)i}{|-36|}\sqrt{\frac{39+15}{2}}\right)$$

$$\Rightarrow \pm\left(2\sqrt{3} - 3\sqrt{3}i\right)$$

As $y<0$, so square root of complex number $-15-36i$ are $2\sqrt{3}-3\sqrt{3}i$ or $-2\sqrt{3}+3\sqrt{3}i$

(iv) $119+120i$

Solution:

Let $z = x+iy = 119+120i$

Where $x=119,\ y=120>0$

$|z| = \sqrt{119^2+120^2} = 169,$

Applying square root formula for complex numbers, we get,

$$\sqrt{119+120i} = \pm\left(\sqrt{\frac{169+119}{2}} + \frac{i(120)}{|120|}\sqrt{\frac{169-119}{2}}\right)$$

$$= \pm(12+5i)$$

As $y>0$, so square root of $119+120i$ are $12+5i$ or $-12-5i$.

8. Find the square root of $13-20\sqrt{3}i$ and represent them on an Argand diagram.

Solution:

Let $z = x+iy = 13-20\sqrt{3}i$

$\Rightarrow x=13,\ \ y=-20\sqrt{3}<0$

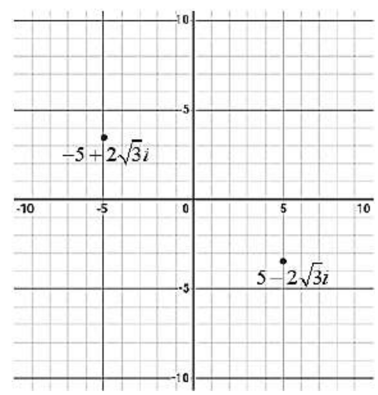
$|z| = \sqrt{(13)^2 + \left(-20\sqrt{3}\right)^2} = 37$

Applying the square root formula for complex number, we get

$$\sqrt{13-20\sqrt{3}i} = \pm\left(\sqrt{\frac{37+13}{2}} + \frac{i\left(-20\sqrt{3}\right)}{\left|-20\sqrt{3}\right|}\sqrt{\frac{37-13}{2}}\right)$$

$$= \pm\left(5-2\sqrt{3}i\right)$$

As $y<0$, so square root of $13-20\sqrt{3}i$ are $5-2\sqrt{3}i$ or $-5+2\sqrt{3}i$



9. Find the value of x and y if $(-7+i)(x+iy)+(-1-5i) = i(11-i)$

Solution:

Given:

$(-7+i)(x+iy)+(-1-5i) = i(11-i)$

$\Rightarrow -7x-7yi+xi+yi^2-1-5i = 11i-i^2$

$\Rightarrow -7x-7yi+xi-y-1-5i = 1+11i$

$\Rightarrow (-7x-y-1)+(-7y+x-5)i = 1+11i$

Comparing real and imaginary parts.

$-7x-y-1 = 1 \qquad \text{(i)}$

$x-7y-5=11 \qquad \text{(ii)}$

From (i) $y=-7x-2$ put in (ii)

$x-7(-7x-2)=16$

$x+49x+14=16 \Rightarrow 50x=2$

$x=\dfrac{1}{25}$ put in (ii)

$\dfrac{1}{25}-5-11=7y$

$7y=\dfrac{1}{25}-16$

$7y=\dfrac{399}{25}$

$y=-\dfrac{57}{25}$

So, $x=\dfrac{1}{25},\ y=-\dfrac{57}{25}$

10. Find the value of x and y if $(5-2i)(x+iy)+3=i(11-i)-4i$

Solution:

Given:

$(5-2i)(x+iy)+3=i(11-i)-4i$

$\Rightarrow 5x+5yi-2xi-2yi^2+3=11i-i^2-4i$

$\Rightarrow 5x+5yi-2xi+2y+3=7i+1$

$\Rightarrow (5x+2y+3)+(5y-2x)i=1+7i$

Comparing real and imaginary parts:

$5x+2y+3=1 \qquad \text{(i)}$

$5y-2x=7 \qquad \text{(ii)}$

Multiplying equation (i) with 2 and equation (ii) with 5, we get:

$$\begin{array}{r} 10x+4y=-4 \\ -10x+25y=35 \\ \hline 29y=31 \end{array} \quad \text{Adding}$$

$\Rightarrow y=\dfrac{31}{29}$ put in (i)

$5x+2\left(\dfrac{31}{29}\right)=-2 \Rightarrow 5x=-2-\dfrac{62}{29}$

$5x=-\dfrac{120}{29} \Rightarrow x=-\dfrac{24}{29}$

So, $x=-\dfrac{24}{29}$ and $y=\dfrac{31}{29}$

11. Find the real values of u and v if $\dfrac{u-2}{2+i}+\dfrac{v-3}{2-i}=4i$

Solution:

Given: $\dfrac{u-2}{2+i}+\dfrac{v-3}{2-i}=4i$

$\Rightarrow \dfrac{(u-2)(2-i)+(v-3)(2+i)}{(2+i)(2-i)}=4i$

$\Rightarrow 2u-iu-4+2i+2v+iv-6-3i=4i(4-i^2)$

$\Rightarrow 2u+2v-10-iu+iv-i=4i(4+1)$

$\Rightarrow 2u+2v-10-iu+iv-i=20i$

$\Rightarrow 2u+2v-10+i(v-u-1)=20i$

Comparing real and imaginary parts:

$2u+2v-10=0$	$-u+v-1=20$
$\Rightarrow u+v-5=0 \quad \text{(i)}$	$-u+v-21=0 \quad \text{(ii)}$

Adding (i) and (ii)

$$\begin{array}{r} u+v-5=0 \\ -u+v-21=0 \\ \hline 2v-26=0 \end{array}$$

$\Rightarrow v=13$ put in (i)

$u+13-5=0$

$u=-8$

So values of u and v are –8 and 13 respectively.

12. If $z_1=4+5i$ and $z_2=\alpha-2i$, find the real values of α such that $\text{Re}(z_1z_2)=20.$

Solution:

Given: $\text{Re}(z_1z_2)=20$

$\Rightarrow \text{Re}\left[(4+5i)(\alpha-2i)\right]=20$

$\text{Re}\left[4\alpha-8i+5\alpha i+10\right]=20$

$\Rightarrow 4\alpha+10=20$

$4\alpha=10$

$\alpha=\dfrac{5}{2}$