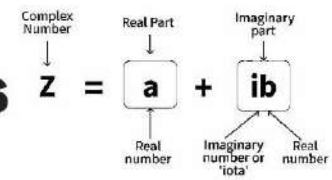
1 UNIT COMPLEX NUMBERS



Mathematics 11 (PECTAA)

Complex Numbers:

The numbers of the form $z = a + ib$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, are called **complex numbers**. For example, $3 + 4i, 2 - \frac{5}{7}i, -7 - 2i$etc. are complex numbers and the set of all complex numbers is denoted by $\mathbb{C}$. $\sqrt{-1}$ does not belong to the set of real numbers. We, therefore, for convenience call it **imaginary number** and denote it by i (read as iota). a is called the **real part** and b is called **imaginary part** for convenient, real part is denoted by Re z and imaginary part by Im z of a complex number z.

> **NOTE**
> ***Every real number is a complex number with 0 as its imaginary part.***

Remember that, the product of a non-zero real number and i is also an **imaginary number**.

Conjugate Complex Numbers:

Let $z = a + ib$ be a *complex number*, then $a - ib$ is called the complex conjugate of $a + ib$. It is denoted by $\bar{z}$. Thus $5 - 4i$ is complex conjugate of $5 + 4i$.

> **REMEMBER**
> ***Geometrically, conjugate of a complex number is its mirror image about x-axis***

A real number is self-conjugate.

Operations on Complex numbers:

1. Addition of Two complex numbers.

Let $z_1 = a + ib = (a, b)$ and $z_2 = c + id = (c, d)$ then their sum is:

$z_1 + z_2 = (a, b) + (c, d) = a + ib + c + id = (a + c) + i(b + d) = (a + c, b + d)$

2. Scalar Multiplication:

Let $z = a + ib = (a, b)$ and $k \neq 0$ be any real number. Then,

$$kz = k(a + bi) = ka + ikb = (ka, kb)$$

3. Subtraction of Two complex numbers.

Let $z_1 = a + ib = (a, b)$ and $z_2 = c + id = (c, d)$ then their difference is:

$z_1 - z_2 = (a, b) - (c, d) = a + ib - c - id = (a - c) + i(b - d) = (a - c, b - d)$

4. Multiplication of Two complex numbers

Let $z_1 = a + ib = (a, b)$ and $z_2 = c + id = (c, d)$ then their product is:

$z_1 z_2 = (a, b)(c, d) = (a + ib)(c + id) = (ac - bd) + i(ad + cb) = (ac - bd, ad + cb)$

Properties of the Fundamental Operations on Complex Numbers:

It can be easily verified that the set $\mathbb{C}$ satisfies all the field axioms i.e., it possesses the properties of real numbers.

By way of explanation of some points we observe as follows:

(i) The additive identity in $\mathbb{C}$ is $(0,0)$.

(ii) Every complex number (a,b) has the additive inverse $(-a,-b)$ i.e., $(a,b)+(-a,-b)=(0,0)$

(iii) The multiplicative identity is $(1,0)$ i.e.,

$$(a,b)(1,0)=(a1-b0,b1+a0)=(a,b) \quad =(1,0)(a,b)$$

(iv) Every non-zero complex number {i.e., number not equal to $(0,0)$ } has a multiplicative inverse. The multiplicative inverse of (a,b) is $\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$

$$\because (a,b)\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=(1,0),\ \text{the identity element}$$

$$=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)(a,b)$$

(v) $(a,b)\left[(c,d)\pm(e,f)\right]=(a,b)(c,d)\pm(a,b)(e,f)$

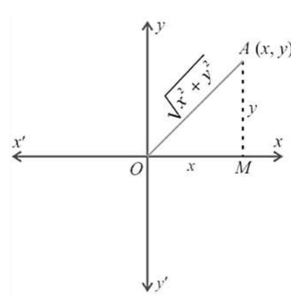
> **NOTE**
>
> ***The set $\mathbb{C}$ of complex numbers does not satisfy the order axioms. In fact, there is no sense in saying that one complex number is greater or less than the other.***

Argand Diagram:

Every complex number will be represented by one and only one point of the coordinate plane and every point of the plane will represent one and only one complex number. In this representation the $x-axis$ is called the real axis and the $y-axis$ is called the imaginary axis. The coordinate plane itself is called the complex plane or z-plane. The figure representing one or more complex numbers on the complex plane is called an Argand Diagram.

Modulus of Complex Number:

The real number $\sqrt{x^2+y^2}$ .is called the modulus of the complex number $x+iy$ and it is denoted by $|x+iy|$. The modulus of a complex number is the distance from the origin to the point representing the number.



REMEMBER

For any $z=x+iy,\ |z|=|-z|=|\bar{z}|=|-\bar{z}|, |z|\geq 0$

EXERCISE 1.1

1. Find the multiplicative inverse of each of the following numbers:

(i) $(-4,7)$

Solution:

Using $(a,b)^{-1}=\left(\dfrac{a}{a^2+b^2},\dfrac{-b}{a^2+b^2}\right)$

Putting the values

$$(-4,7)^{-1}=\left(\frac{-4}{(-4)^2+(7)^2},\frac{-7}{(-4)^2+(7)^2}\right)$$

$$=\left(\frac{-4}{16+49},\frac{-7}{16+49}\right)=\left(\frac{-4}{65},\frac{-7}{65}\right)$$

(ii) $(\sqrt{2},-\sqrt{5})$

Solution:

$$(\sqrt{2},-\sqrt{5})^{-1}=\left(\frac{\sqrt{2}}{(\sqrt{2})^2+(-\sqrt{5})^2},\frac{\sqrt{5}}{(\sqrt{2})^2+(-\sqrt{5})^2}\right)$$

$$=\left(\frac{\sqrt{2}}{7},\frac{\sqrt{5}}{7}\right)$$

(iii) $(1,0)$

Solution:

$$(1,0)^{-1}=\left(\frac{1}{1^2+0^2},\frac{0}{1^2+0^2}\right)$$

$$=(1,0)$$

2. Separate into real and imaginary parts (write as a simple complex number):

(i) $\dfrac{2-7i}{4+5i}$

Solution:

$$\frac{2-7i}{4+5i}$$

Multiply and divide by $4-5i$

$$=\frac{2-7i}{4+5i}\times\frac{4-5i}{4-5i}=\frac{(2-7i)(4-5i)}{(4)^2-(5i)^2}$$

$$=\frac{8-10i-28i+35i^2}{16-25(-1)}$$

$$=\frac{8-38i+35(-1)}{16+25}=\frac{-27-38i}{41}$$

$$=\frac{-27}{41}-\frac{38}{41}i$$

Real Part $=\dfrac{-27}{41}$, Imaginary Part $=\dfrac{-38}{41}$

(ii) $\dfrac{(-2+3i)^2}{1+i}$

Solution:

$$\frac{(-2+3i)^2}{1+i}=\frac{(-2)^2+(3i)^2+2(-2)(3i)}{1+i}$$

$$=\frac{4+9i^2-12i}{1+i}$$

$$=\frac{4-9-12i}{1+i}=\frac{-5-12i}{1+i}=\frac{-5-12i}{1+i}\times\frac{1-i}{1-i}$$

$$=\frac{(-5-12i)(1-i)}{(1+i)(1-i)}=\frac{-5+5i-12i+12i^2}{1^2-i^2}$$

$$=\frac{-5-7i+12(-1)}{1-(-1)}=\frac{-5-7i-12}{1+1}$$

$$=\frac{-17-7i}{2}=\frac{-17}{2}-\frac{7}{2}i$$

Real Part $=\dfrac{-17}{2}$, Imaginary Part $=\dfrac{-7}{2}$

(iii) $\dfrac{i}{1+i}$

Solution:

$$\frac{i}{1+i}=\frac{i}{1+i}\times\frac{1-i}{1-i}=\frac{i(1-i)}{1^2-i^2}$$

$$=\frac{i-i^2}{1-(-1)}=\frac{i-(-1)}{2}=\frac{1+i}{2}=\frac{1}{2}+\frac{1}{2}i$$

Real Part $=\dfrac{1}{2}$, Imaginary Part $=\dfrac{1}{2}$

(iv) $\dfrac{(4+3i)^2}{4-3i}$

Solution:

$$=\frac{(4)^2+(3i)^2+2(4)(3i)}{4-3i}$$

$$=\frac{16+9i^2+24i}{4-3i}=\frac{16-9+24i}{4-3i}=\frac{7+24i}{4-3i}$$

$$=\frac{7+24i}{4-3i}\times\frac{4+3i}{4+3i}=\frac{(7+24i)(4+3i)}{(4-3i)(4+3i)}$$

$$=\frac{28+21i+96i+72i^2}{(4)^2-(3i)^2}=\frac{28+117i-72}{16-9i^2}$$

$$=\frac{-44+117i}{16+9}=\frac{-44+117i}{25}=-\frac{44}{25}+\frac{117}{25}i$$

Real part $=-\dfrac{44}{25}$, Imaginary part $\dfrac{117}{25}$

3. Prove that $\overline{z}=z$ iff z is real.

Solution:

Let $z=x+iy\ldots$(i)

Then $\overline{z}=x-iy$

If $z=\overline{z}$

Now we have to prove z is real

As $z=\overline{z}$

$x+iy=x-iy$

$x+iy-x+iy=0$

$2iy=0$

As $2i\neq 0,\ y=0$

Put in a equation (i)

$z=x+i(0)\Rightarrow z=x$

Hence z is a real number.

Conversely, suppose z is a real number

Now we have to prove that $z=\overline{z}$

Let $z=x$

$z=x+0i$

$\overline{z}=x-0i=x$

Hence $\overline{z}=z$

4. For $z\in C$, show that:

(i) $\dfrac{z+\overline{z}}{2}=\operatorname{Re}(z)$

Solution:

Let $z=a+bi,\ a,b\in R$

Then $\overline{z}=\overline{a+ib}$

$\Rightarrow\overline{z}=a-ib$

Now $\dfrac{z+\overline{z}}{2}=\dfrac{a+ib+a-ib}{2}$

$=\dfrac{2a}{2}=a=\operatorname{Re}(z)$

(ii) $\dfrac{z-\overline{z}}{2i}=\operatorname{Im}(z)$

Solution:

Let $z=a+ib$

Then $\overline{z}=a-ib$

Now, $\dfrac{z-\overline{z}}{2i}=\dfrac{(a+ib)-(a-ib)}{2i}$

$=\dfrac{a+ib-a+ib}{2i}=\dfrac{2ib}{2i}=b=\operatorname{Im}(z)$

(iii) $|z|^2=z.\overline{z}$

Solution:

Let $z=a+ib$

Then $|z|=\sqrt{a^2+b^2}$

$\Rightarrow|z|^2=a^2+b^2$ (i)

Now, $\overline{z}=a-ib$

$z.\overline{z}=(a+ib)(a-ib)$

$=a^2-(ib)^2$

$=a^2+b^2$ (ii)

From (i) and (ii) $|z|^2=z.\overline{z}$

5. If $z_1=2+i$, $z_2=3-2i, z_3=1+3i$ then express $\dfrac{\overline{z_1}\,\overline{z_3}}{z_2}$ in the form of $a+ib$.

Solution:

$$\frac{\overline{z_1}\,\overline{z_3}}{z_2}=\frac{\left(\overline{2+i}\right)\left(\overline{1+3i}\right)}{3-2i}=\frac{(2-i)(1-3i)}{3-2i}$$

$$=\frac{(2-3)+(-6-1)i}{3-2i}=\frac{-1-7i}{3-2i}$$

$$=\frac{(-1-7i)(3+2i)}{(3-2i)(3+2i)}$$

$$=\frac{(-3+14)+(-2-21)i}{3^2+2^2}=\frac{11}{13}-\frac{23}{13}i$$

6. If $z_1=2+7i$ and $z_2=-5+3i$, then evaluate the following:

(i) $|2z_1-4z_2|$

Solution:

Firstly, $2z_1-4z_2$

$=2(2+7i)-4(-5+3i)$
$=4+14i+20-12i$
$=24+2i$

Now, $|2z_1-4z_2|=\sqrt{(24)^2+(2)^2}$
$=\sqrt{576+4}=\sqrt{580}=2\sqrt{145}$

(ii) $|3z_1+2\bar{z}_1|$

Solution:

Firstly, $3z_1+2\bar{z}_1$
$=3(2+7i)+2(2-7i)$
$=6+21i+4-14i=10+7i$

Now, $|3z_1+2\bar{z}_1|=\sqrt{(10)^2+(7)^2}$
$=\sqrt{100+49}=\sqrt{149}$

(iii) $|-7z_2+2\bar{z}_2|$

Solution:

Firstly, $-7z_2+2\bar{z}_2$
$=-7(-5+3i)+2(-5-3i)$
$=35-21i-10-6i$
$=25-27i$

Now, $|-7z_2+2\bar{z}_2|=\sqrt{(25)^2+(-27)^2}$
$=\sqrt{625+729}=\sqrt{1354}$

(iv) $|(z_1+z_2)^3|$

Solution:

Firstly: $(z_1+z_2)^3=(2+7i-5+3i)^3$
$=(-3+10i)^3$
$=(-3)^3+(10i)^3+3(-3)^2(10i)+3(-3)(10i)^2$
$=-27+1000i^3+270i-9(100i^2)$
$=-27-1000i+270i+900$
$=873-730i$

Now, $|(z_1+z_2)^3|=\sqrt{(873)^2+(-730)^2}$
$=\sqrt{762129+532900}$
$\sqrt{1295029}=109\sqrt{109}$

7. **Show that:** $i^{n+1}+i^{n+2}+i^{n+3}+i^{n+4}=0$, **for all** $n\in N$.

Solution:

$\text{L.H.S}=i^{n+1}+i^{n+2}+i^{n+3}+i^{n+4}$
$=i^{n+1}(1+i+i^2+i^3)$
$=i^{n+1}(1+i-1-i)$
(As $i^2=-1$ and $i^3=-i$)
$=i^{n+1}(0)=0=\text{R.H.S}$

8. **Find the least positive value of** n, **if** $\left(\frac{1+i}{1-i}\right)^{2n}=1$

Solution:

$\frac{1+i}{1-i}=\frac{1+i}{1-i}\times\frac{1+i}{1+i}$
$=\frac{(1+i)^2}{1^2-i^2}=\frac{1^2+i^2+2i}{1+1}$
$=\frac{1-1+2i}{2}=\frac{2i}{2}=i$

So, $\frac{1+i}{1-i}=i$

Now, $\left(\frac{1+i}{1-i}\right)^{2n}=1$

$\Rightarrow i^{2n}=1\Rightarrow i^{2n}=i^4$ (As $i^4=1$)

By comparing $2n=4\Rightarrow n=2$

Least positive value of n is 2.

9. **Show that, the value of** i^n **for** $n\in N$ **and** $n>4$ **is** i^r, **where** r **is the remainder when** n **is divided by** 4.

Solution:

By division algorithm,

$n = 4q + r$, where $n \in N$ and $n > 4$, r is remainder.

$\Rightarrow i^n = i^{4q+r} = i^{4q}.i^r$

$= \left(i^4\right)^q.i^r = (1)^q.i^r$ (As $i^4 = 1$)

$i^n = i^r$ Hence proved