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Unit # 3 Higher Order Derivative and Application

If we denote y' as the first derivative of y, then y'' is the derivative of y' and is called the 2nd derivative of y. y''' is the derivative of y'' and is called the third derivative of y

In general the nth derivative of y is written as $y^{(n)}$ for $n \geq 4$.

Notation

function	1st derivative	2nd derivative	3rd derivative	nth derivative
y	y'	y''	y'''	$y^{(n)}$
y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^ny}{dx^n}$
y	y_1	y_2	y_3	y_n
$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(n)}(x)$
$f(x)$	$D_x[f(x)]$	$D_x^2[f(x)]$	$D_x^3[f(x)]$	$D_x^n[f(x)]$

Note Leibnitz's notation for the second derivative of a function $y = f(x)$ is $\frac{d^2y}{dx^2}$ or $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ & third derivative is $\frac{d^3y}{dx^3}$ or $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and so on

Parametric equation

$x = f(t)$, $y = g(t)$

Diff: w.r.t t Diff: w.r.t t

$\frac{dx}{dt} = \square$ $\frac{dy}{dt} = \Delta$

$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$ chain Rule

$= \Delta \times \frac{1}{\square}$

$= \frac{\Delta}{\square}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left[\frac{dy}{dt}\,\frac{dt}{dx}\right]\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\cdot\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\left(\frac{d^2y}{dt^2}\right) - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2} \cdot \frac{dt}{dx}$$

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$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

Note Reciprocal rule

$$\frac{d}{dx}\frac{1}{x^n} = \frac{-n}{x^{n+1}}$$

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EXERCISE 3.1

1) Find the indicated higher derivatives of the following functions

a) $f(x) = 3x^3+4x+5$

Diff: w.r.t x

$$f'(x) = \frac{d}{dx}(3x^3+4x+5)$$

$$= 3(3x^2)+4.1+0$$

$$f'(x) = 9x^2+4$$

Diff: again w.r.t x

$$f''(x) = \frac{d}{dx}(9x^2+4)$$

$$= 9(2x)+0 = 18x$$

b) $f(x) = x+\frac{1}{x}$

Diff w.r.t x

$$f'(x) = \frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$= 1+\left(\frac{-1}{x^2}\right) = 1-\frac{1}{x^2}$$

Diff again w.r.t x

$$f''(x) = \frac{d}{dx}\left(1-\frac{1}{x^2}\right)$$

$$f''(x) = 0-\left(\frac{-2}{x^3}\right) = +2/x^3$$

Diff w.r.t x

$$f'''(x) = \frac{d}{dx}\left(\frac{+2}{x^3}\right)$$

$$= +2\left(\frac{-3}{x^4}\right) = \frac{-6}{x^4} \text{ Ans}$$

c) $f(x) = 1+\frac{2}{x}-\frac{3}{x^2}$

Diff w.r.t x

$$f'(x) = \frac{d}{dx}\left[1+\frac{2}{x}-\frac{3}{x^2}\right]$$

$$= 0+2\left(\frac{-1}{x^2}\right)-3\left(\frac{-2}{x^3}\right)$$

$$f'(x) = \frac{-2}{x^2}+\frac{6}{x^3}$$

Diff w.r.t x

$$f''(x) = \frac{d}{dx}\left(\frac{-2}{x^2}+\frac{6}{x^3}\right)$$

$$f''(x) = -2\left(\frac{-2}{x^3}\right)+6\left(\frac{-3}{x^4}\right)$$

$$= +\frac{4}{x^3}-\frac{18}{x^4}$$

Ans

d) $s(t) = \sqrt{5t+7}$

Diff: w.r.t. t

$$s'(t) = \frac{d}{dt}\sqrt{5t+7}$$

$$= \frac{1}{2}(5t+7)^{-1/2}\frac{d}{dt}(5t+7)$$

$$= \frac{1}{2}(5t+7)^{-1/2}(5\cdot 1+0)$$

$$s'(t) = \frac{5}{2}(5t+7)^{-1/2}$$

Diff. w.r.t. t

$$s''(t) = \frac{5}{2}\frac{d}{dt}(5t+7)^{-1/2}$$

$$= \frac{5}{2}\left\{-\frac{1}{2}(5t+7)^{-3/2}\frac{d}{dt}(5t+7)\right\}$$

$$= -\frac{5}{4}(5t+7)^{-3/2}(5\cdot 1+0)$$

$$= \frac{-25}{4}(5t+7)^{-3/2}$$

f) $y = (x+3)(x^2+7x+2)$ } use Product Rule or simplify then diff

$$y = x^3+7x^2+2x+3x^2+21x+6$$

$$y = x^3+10x^2+23x+6$$

Diff w.r.t. x

$$y' = \frac{d}{dx}(x^3+10x^2+23x+6)$$

$$= 3x^2+10\cdot 2x+23\cdot 1+0$$

$$y' = 3x^2+20x+23$$

Diff w.r.t. x

$$y'' = \frac{d}{dx}(3x^2+20x+23)$$

$$= 3(2x)+20\cdot 1+0$$

$$y'' = 6x+20$$

Ans

e) $y = \frac{x+1}{x-1}$

Diff w.r.t. x

$$y' = \frac{d}{dx}\left(\frac{x+1}{x-1}\right)$$

$$= \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(1+0) - (x+1)(1+0)}{(x-1)^2}$$

$$= \frac{\cancel{x}-1-\cancel{x}-1}{(x-1)^2}$$

$$y' = \frac{-2}{(x-1)^2}$$

Diff w.r.t. x

$$y'' = \frac{d}{dx}\left(\frac{-2}{(x-1)^2}\right)$$

$$y'' = \frac{(x-1)^2\frac{d}{dx}(-2) - (-2)\frac{d}{dx}(x-1)^2}{\left[(x-1)^2\right]^2}$$

$$= \frac{(x-1)^2(0)+2\left\{2(x-1)^1\frac{d}{dx}(x-1)\right\}}{(x-1)^4}$$

$$= \frac{0+4\cancel{(x-1)}(1)}{(x-1)^{\cancel{4}\,3}}$$

$$= \frac{4}{(x-1)^3} \quad \text{Ans}$$

Note: To find 2nd derivative in above case we can also write $y' = -2(x-1)^{-2}$

Diff w.r.t. x

$$y'' = -2(-2)(x-1)^{-3}\frac{d}{dx}(x-1)$$

$$= 4(x-1)^{-3}(1)$$

$$= \frac{4}{(x-1)^3} \quad \text{Ans}$$

2) Find the indicated derivative of the following trigonometric functions:

a) $y = \tan x$

Diff w.r.t. x

$y' = \frac{d}{dx}\tan x$

$y' = \sec^2 x$

Diff: w.r.t. x

$y'' = \frac{d}{dx}\sec^2 x$

$= 2\sec^1 x \frac{d}{dx}\sec x$

$= 2\sec x(\sec x.\tan x)$

$y'' = 2\sec^2 x.\tan x$

Diff w.r.t. x

$y''' = 2\frac{d}{dx}(\sec^2 x.\tan x)$

$= 2\left\{\sec^2 x\frac{d}{dx}\tan x + \tan x\frac{d}{dx}\sec^2 x\right\}$

$= 2\left\{\sec^2 x.\sec^2 x + \tan x \cdot 2\sec x.\frac{d}{dx}\sec x\right\}$

$= 2\left\{\sec^4 x + 2\tan x\sec x\{\sec x.\tan x\}\right\}$

$= 2\left\{\sec^4 x + 2\sec^2 x.\tan^2 x\right\}$ Ans

$= 2\sec^4 x + 4\sec^2 x(\sec^2 x - 1)$

$= 2\sec^4 x + 4\sec^4 x + 4\sec^2 x$

$= 6\sec^4 x + 4\sec^2 x$ Ans

b) $y = \ln\sin x$

Diff w.r.t. x

$y' = \frac{d}{dx}\ln\sin x$

$= \frac{1}{\sin x}\frac{d}{dx}\sin x$

$= \frac{1}{\sin x}.\cos x = \frac{\cos x}{\sin x}$

$y' = \cot x$

Diff w.r.t. x

$y'' = \frac{d}{dx}\cot x$

$y'' = -\operatorname{cosec}^2 x$

Diff: w.r.t. x

$y''' = -\frac{d}{dx}\operatorname{cosec}^2 x$

$= -\left\{2\operatorname{cosec}^1 x\frac{d}{dx}\operatorname{cosec} x\right\}$

$= -2\operatorname{cosec} x\{-\operatorname{cosec} x.\cot x\}$

$= +2\operatorname{cosec}^2 x.\cot x$

Ans

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c) $y = \sqrt{\sec 2x}$

Diff. w.r.t. x

$y' = \frac{d}{dx}\sqrt{\sec 2x}$

$= \frac{1}{2}(\sec 2x)^{-1/2}\frac{d}{dx}\sec 2x$

$= \frac{1}{2\sqrt{\sec 2x}}\sec 2x.\tan 2x\frac{d}{dx}2x$

$(\because x = \sqrt{x}\sqrt{x})$

$= \frac{1}{\cancel{2}\cancel{\sqrt{\sec 2x}}}\cancel{\sqrt{\sec 2x}}\sqrt{\sec 2x}.\tan 2x.\cancel{2}$

$|y' = \sqrt{\sec 2x}.\tan 2x.$

$y' = y\tan 2x$ $(\because y = \sqrt{\sec 2x}$

Diff w.r.t. x.

$y'' = y\frac{d}{dx}\tan 2x + \tan 2x.\frac{d}{dx}y$

$= y.\sec^2 2x\frac{d}{dx}2x + \tan 2x.y'$

$= y.\sec^2 2x.2.1 + \tan 2x\sqrt{\sec 2x}.\tan 2x$

$= \sqrt{\sec 2x}.\sec^2 2x.2 + \tan^2 x.\sqrt{\sec 2x}$

$= \sqrt{\sec 2x}\left[2\sec^2 2x + \tan^2 2x\right]$

$= \sqrt{\sec 2x}\left[2\sec^2 2x + \sec^2 2x - 1\right]$

$= \sqrt{\sec 2x}\left(3\sec^2 2x - 1\right).$ Ans.

$y''' = \frac{-10 - 16y^2 - 6y^4}{y^8}$

$= \frac{-2(5 + 8y^2 + 3y^4)}{y^8}$

Ans

d) $y = \tan(x+y)$

Diff w.r.t. x

$y' = \frac{d}{dx}\tan(x+y)$

$= \sec^2(x+y)\frac{d}{dx}(x+y)$

$= \sec^2(x+y)\left(1 + \frac{dy}{dx}\right)$

$= (1 + \tan^2(x+y))(1 + y')$

$y' = (1 + y^2)(1 + y')$

$\cancel{y'} = 1 + \cancel{y'} + y^2 + y^2 y'$

$-y^2 y' = 1 + y^2$

$\div$ ing by $-y^2$

$y' = \frac{1}{-y^2} + \frac{y^2}{-y^2}$

$\boxed{y' = -y^{-2} - 1}$

Diff w.r.t. x.

$y'' = \frac{d}{dx}(-y^{-2} - 1)$

$= -(-2y^{-3}y' - 0)$

$= 2y^{-3}y'$

$= 2y^{-3}(-y^{-2} - 1)$

$\boxed{y'' = -2y^{-5} - 2y^{-3}}$

Diff w.r.t. x

$y''' = -2\{-5y^{-6}y'\} - 2\{-3y^{-4}y'\}$

$= +10y^{-6}y' + 6y^{-4}y'$

$= +10y^{-6}(-y^{-2} - 1) + 6y^{-4}(-y^{-2} - 1)$

$= -10y^{-8} - 10y^{-6} - 6y^{-6} - 6y^{-4}$

$= -10y^{-8} - 16y^{-6} - 6y^{-4}$

$= \frac{-10}{y^8} - \frac{16}{y^6} - \frac{6}{y^4}$

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e) $y = \sin(\sin x)$

Diff : w.r.t. x

$$y' = \frac{d}{dx}\sin(\sin x)$$
$$= \cos(\sin x)\frac{d}{dx}\sin x$$
$$y' = \cos(\sin x)\cdot\cos x$$

Diff. w.r.t. x

$$y'' = \frac{d}{dx}\left[\cos(\sin x)\cdot\cos x\right]$$
$$= \cos(\sin x)\frac{d}{dx}\cos x + \cos x\frac{d}{dx}\cos(\sin x)$$
$$= \cos(\sin x)\left[-\sin x\right] + \cos x\left\{-\sin(\sin x)\frac{d}{dx}\sin x\right\}$$
$$= -\sin x\cos(\sin x) - \cos x\cdot\sin(\sin x)\cdot\cos x$$
$$y'' = -\sin x\cdot\cos(\sin x) - \cos^2 x\cdot\sin(\sin x) \quad \text{Ans}$$

Diff w.r.t. x.

$$y''' = -\frac{d}{dx}\left[\sin x\cdot\cos(\sin x)\right] - \frac{d}{dx}\left[\cos^2 x\cdot\sin(\sin x)\right]$$
$$= -\left\{\sin x\frac{d}{dx}\cos(\sin x) + \cos(\sin x)\frac{d}{dx}\sin x\right\} - \left\{\cos^2 x\frac{d}{dx}\sin(\sin x) + \sin(\sin x)\frac{d}{dx}\cos^2 x\right\}$$
$$= -\left\{\sin x\left(-\sin(\sin x)\frac{d}{dx}\sin x\right) + \cos(\sin x)\cdot\cos x\right\} - \left\{\cos^2 x\cdot\left(\cos(\sin x)\frac{d}{dx}\sin x\right) + \sin(\sin x)\cdot 2\cos x\frac{d}{dx}\cos x\right\}$$
$$= -\left\{-\sin x\cdot\sin(\sin x)\cdot\cos x + \cos x\cdot\cos(\sin x)\right\} - \left\{\cos^2 x\cdot\cos(\sin x)\cdot\cos x + 2\cos x\cdot\sin(\sin x)\cdot(-\sin x)\right\}$$
$$= +\sin x\cdot\cos x\cdot\sin(\sin x) - \cos x\cdot\cos(\sin x) - \cos^3 x\cdot\cos(\sin x) + 2\sin x\cos x\sin(\sin x)$$
$$y''' = 3\sin x\cdot\cos x\cdot\sin(\sin x) - \cos x\cdot\cos(\sin x) - \cos^3 x\cdot\cos(\sin x)$$

Ans

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3) Use implicit rule to find out the second derivative of the following functions.

a) $b^2x^2 + a^2y^2 = a^2b^2$

Diff: w.r.t x

$$\frac{d}{dx}[b^2x^2 + a^2y^2] = \frac{d}{dx}a^2b^2$$

$$b^2 \cdot 2x + a^2 \cdot 2y\frac{dy}{dx} = 0$$

$$\cancel{2}a^2y\frac{dy}{dx} = -\cancel{2}b^2x$$

$$y' \text{ or } \boxed{\frac{dy}{dx} = -\frac{b^2x}{a^2y}}$$

Diff w.r.t x

$$y'' = -\frac{b^2}{a^2}\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= \frac{-b^2}{a^2}\left\{\frac{y\cdot\frac{d}{dx}x - x\cdot\frac{dy}{dx}}{y^2}\right\}$$

$$= \frac{-b^2}{a^2y^2}\left\{y\cdot 1 - x\left(-\frac{b^2x}{a^2y}\right)\right\}$$

$$= -\frac{b^2}{a^2y^2}\left\{y + \frac{b^2x^2}{a^2y}\right\}$$

$$= \frac{-b^2}{a^2y^2}\left\{\frac{a^2y^2 + b^2x^2}{a^2y}\right\}$$

$$= -\frac{b^2}{a^2y^2}\left\{\frac{\cancel{a^2}b^2}{\cancel{a^2}y}\right\} \quad \because b^2x^2 + a^2y^2 = a^2b^2$$

$$\boxed{y'' = -\frac{b^4}{a^2y^3}}$$

Diff w.r.t x.

$$y''' = -\frac{b^4}{a^2}\frac{d}{dx}y^{-3}$$

$$y''' = -\frac{b^4}{a^2}\left\{-3y^{-4}\frac{dy}{dx}\right\}$$

$$y''' = +\frac{3b^4}{a^2}\cdot\frac{1}{y^4}\frac{dy}{dx}$$

$$= \frac{3b^4}{a^2y^4}\left\{\frac{-b^2x}{a^2y}\right\}$$

$$= \frac{-3b^6x}{a^4y^5}$$

Ans

b) $x^2 + y^2 = r^2$

Diff w.r.t x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}r^2$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\cancel{2}y\frac{dy}{dx} = -\cancel{2}x$$

$$\Rightarrow y' \text{ or } \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Diff w.r.t x

$$y'' = -\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= -\left\{\frac{y\frac{d}{dx}x - x\frac{dy}{dx}}{y^2}\right\}$$

$$= -\frac{1}{y^2}\left\{y\cdot 1 - x\left\{-\frac{x}{y}\right\}\right\}$$

$$= -\frac{1}{y^2}\left\{y + \frac{x^2}{y}\right\}$$

$$= -\frac{1}{y^2}\left\{\frac{y^2 + x^2}{y}\right\} \quad \because x^2 + y^2 = r^2$$

$$= -\frac{r^2}{y^3}$$

Ans

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c) $y^2 - 2xy = 0$

Diff: w.r.t. x

$$\frac{d}{dx}(y^2 - 2xy) = \frac{d}{dx}(0)$$

$$2y\frac{dy}{dx} - 2\frac{d}{dx}(xy) = 0$$

$\div$ ing by 2

$$y\frac{dy}{dx} - \left\{x\frac{dy}{dx} + y\cdot\frac{d}{dx}x\right\} = 0$$

$$\underline{y\frac{dy}{dx} - x\frac{dy}{dx}} - y\cdot 1 = 0$$

$$(y-x)\frac{dy}{dx} = y$$

$$\Rightarrow y' \text{ or } \boxed{\frac{dy}{dx} = \frac{y}{y-x}}$$

Diff: w.r.t. x

$$y'' = \frac{d}{dx}\left(\frac{y}{y-x}\right)$$

$$= \frac{(y-x)\frac{d}{dx}y - y\frac{d}{dx}(y-x)}{(y-x)^2}$$

$$= \frac{(y-x)\frac{dy}{dx} - y\left\{\frac{dy}{dx} - 1\right\}}{(y-x)^2}$$

$$= \frac{1}{(y-x)^2}\left[(y-x)\frac{dy}{dx} - y\frac{dy}{dx} + y\right]$$

$$= \frac{1}{(y-x)^2}\left[(\cancel{y} - x - \cancel{y})\frac{dy}{dx} + y\right]$$

$$= \frac{1}{(y-x)^2}\left[-x\cdot\frac{y}{y-x} + y\right]$$

$$= \frac{1}{(y-x)^2}\left(\frac{-x\cdot y + y(y-x)}{y-x}\right)$$

$$= \frac{-xy + y^2 - xy}{(y-x)^3} = \frac{y^2 - 2xy}{(y-x)^3} = \frac{0}{(y-x)^3} = 0$$

$\because y^2 - 2xy = 0$

d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

Diff w.r.t. x

$$\frac{d}{dx}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \frac{d}{dx}(0)$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} - 0 = 0$$

$$\frac{\cancel{2}y}{b^2}\frac{dy}{dx} = -\frac{\cancel{2}x}{a^2}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-b^2x}{a^2y}}$$

Diff: w.r.t. x

$$y'' = -\frac{b^2}{a^2}\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= -\frac{b^2}{a^2}\left\{\frac{y\frac{d}{dx}x - x\frac{dy}{dx}}{y^2}\right\}$$

$$= \frac{-b^2}{a^2y^2}\left\{y\cdot 1 - x\frac{dy}{dx}\right\}$$

$$= \frac{-b^2}{a^2y^2}\left\{y - x\cdot\left(\frac{-b^2x}{a^2y}\right)\right\}$$

$$= \frac{-b^2}{a^2y^2}\left\{y + \frac{b^2x^2}{a^2y}\right\}$$

$$= \frac{-b^2}{a^2y^2}\left\{\frac{a^2y^2 + b^2x^2}{a^2y}\right\}$$

$$= \frac{-b^2\{a^2y^2 + b^2x^2\}}{a^4y^3}$$

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e) $\sec x \cos y = c$

Diff w.r.t. x

$$\frac{d}{dx}(\sec x \cdot \cos y) = \frac{d}{dx} c$$

$$\sec x \cdot \frac{d}{dx}\cos y + \cos y \frac{d}{dx}\sec x = 0$$

$$\sec x \cdot \left(-\sin y \frac{dy}{dx}\right) + \cos y \cdot \sec x \cdot \tan x = 0$$

$$+ \cancel{\sec x} \cdot \sin y \frac{dy}{dx} = + \cos y \cdot \cancel{\sec x} \cdot \tan x$$

$$\frac{dy}{dx} = \frac{\cos y \cdot \tan x}{\sin y}$$

$$\frac{dy}{dx} = \cot y \cdot \tan x = \frac{1}{\tan y} \tan x$$

$$\boxed{\frac{dy}{dx} = \frac{\tan x}{\tan y}}$$

Diff w.r.t x

$$y'' = \frac{d}{dx}\left(\frac{\tan x}{\tan y}\right)$$

$$= \frac{\tan y \frac{d}{dx}\tan x - \tan x \frac{d}{dx}\tan y}{\tan^2 y}$$

$$= \frac{\tan y \cdot \sec^2 x - \tan x \cdot \sec^2 y \frac{dy}{dx}}{\tan^2 y}$$

$$= \frac{1}{\tan^2 y}\left[\tan y \cdot (1+\tan^2 x) - \tan x (1+\tan^2 y) \frac{\tan x}{\tan y}\right]$$

$$= \frac{1}{\tan^2 y}\left[\frac{\tan^2 y (1+\tan^2 x) - \tan^2 x (1+\tan^2 y)}{\tan y}\right]$$

$$= \frac{\tan^2 y + \cancel{\tan^2 x \tan^2 y} - \tan^2 x - \cancel{\tan^2 x \tan^2 y}}{\tan^3 y}$$

$$y'' = \frac{\tan^2 y - \tan^2 x}{\tan^3 y} \quad \text{Ans}$$

d) $e^x + x = e^y + y$

Diff w.r.t. x

$$\frac{d}{dx}(e^x + x) = \frac{d}{dx}(e^y + y)$$

$$e^x + 1 = e^y \frac{dy}{dx} + \frac{dy}{dx}$$

$$e^x + 1 = (e^y + 1)\frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{e^x + 1}{e^y + 1}}$$

Diff w.r.t. x

$$y'' = \frac{d}{dx}\left(\frac{e^x+1}{e^y+1}\right)$$

$$= \frac{(e^y+1)\frac{d}{dx}(e^x+1) - (e^x+1)\frac{d}{dx}(e^y+1)}{(e^y+1)^2}$$

$$= \frac{(e^y+1)(e^x+0) - (e^x+1)(e^y\frac{dy}{dx}+0)}{(e^y+1)^2}$$

$$= \frac{1}{(e^y+1)^2}\left[e^x(e^y+1) - e^y(e^x+1)\frac{dy}{dx}\right]$$

$$= \frac{1}{(e^y+1)^2}\left[e^x(e^y+1) - e^y(e^x+1)\frac{(e^x+1)}{e^y+1}\right]$$

$$= \frac{1}{(e^y+1)^2}\left[\frac{e^x(e^y+1)^2 - e^y(e^x+1)^2}{e^y+1}\right]$$

$$= \frac{e^x[e^{2y}+1+2e^y] - e^y(e^{2x}+1+2e^x)}{(e^y+1)^3}$$

$$= \frac{e^{x+2y} + e^x + \cancel{2e^{x+y}} - e^{2x+y} - e^y - \cancel{2e^{x+y}}}{(e^y+1)^3} \quad \text{(rearrange)}$$

$$= \frac{e^x - e^{2x+y} - e^y + e^{x+2y}}{(e^y+1)^3}$$

$$= \frac{e^x(1-e^{x+y}) - e^y(1-e^{x+y})}{(e^y+1)^3} = \frac{(1-e^{x+y})(e^x-e^y)}{(e^y+1)^3}$$

Ans

Note ① 1st derivative of Parametric function $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ (10

② 2nd derivative is $\frac{d^2y}{dx^2} = \dfrac{\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$

Q4 Use Parametric differentiation to find out $\frac{d^2y}{dx^2}$ for the following Parametric functions

a) $x = 4t^2 + 1$, $y = 6t^3 + 1$

Diff: w.r.t. t

$\frac{dx}{dt} = \frac{d}{dt}(4t^2+1)$

$= 4 \cdot 2t + 0$

$\boxed{\frac{dx}{dt} = 8t}$

Diff

$\frac{d^2x}{dt^2} = \frac{d}{dt}(8t)$

$\boxed{\frac{d^2x}{dt^2} = 8}$

$\frac{dy}{dt} = \frac{d}{dt}(6t^3+1)$

$= 6 \cdot 3t^2 + 0$

$\boxed{\frac{dy}{dt} = 18t^2}$

Diff

$\frac{d^2y}{dt^2} = \frac{d}{dt}(18t^2)$

$\boxed{\frac{d^2y}{dt^2} = 36t}$

Now

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{8t(36t) - 18t^2(8)}{(8t)^3}$$

$$= \frac{288t^2 - 144t^2}{512t^3}$$

$$= \frac{144\cancel{t^2}}{512t^{\cancel{3}1}} = \frac{9}{32t}$$

Ans.

b) $x = 3at^2 + 2$, $y = 6t^4 + 9$

Diff w.r.t. t

$\frac{dx}{dt} = \frac{d}{dt}(3at^2+2)$

$= 3a(2t) + 0$

$\boxed{\frac{dx}{dt} = 6at}$

Diff: w.r.t. t

$\frac{d^2x}{dt^2} = 6a.1$

$\boxed{\frac{d^2x}{dt^2} = 6a}$

Diff w.r.t. t

$\frac{dy}{dt} = \frac{d}{dt}(6t^4+9)$

$= 6(4t^3) + 0$

$\boxed{\frac{dy}{dt} = 24t^3}$

Diff w.r.t. t

$\frac{d^2y}{dt^2} = 24(3t^2)$

$\boxed{\frac{d^2y}{dt^2} = 72t^2}$

Now

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{6at(72t^2) - 24t^3(6a)}{(6at)^3}$$

$$= \frac{432at^3 - 144at^3}{216a^3t^3}$$

$$= \frac{288\cancel{a}\cancel{t^3}}{216a^{\cancel{3}2}\cancel{t^3}}$$

$$= \frac{4}{3a^2}$$

Ans.

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c) $x = a(t-\sin t)$, $y = a(1-\cos t)$

Diff w.r.t. t

$$\frac{dx}{dt} = a\frac{d}{dt}(t-\sin t)$$

$$\boxed{\frac{dx}{dt} = a(1-\cos t)}$$

Diff w.r.t. t

$$\frac{d^2x}{dt^2} = a\frac{d}{dt}(1-\cos t)$$

$$= a(0+\sin t)$$

$$\boxed{\frac{d^2x}{dt^2} = a\sin t}$$

Diff: w.r.t. t

$$\frac{dy}{dt} = a\frac{d}{dt}(1-\cos t)$$

$$= a(0+\sin t)$$

$$\boxed{\frac{dy}{dt} = a\sin t}$$

Diff: w.r.t. t

$$\frac{d^2y}{dt^2} = a\frac{d}{dt}\sin t$$

$$\boxed{\frac{d^2y}{dt^2} = a\cos t}$$

Now $$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$\frac{d^2y}{dx^2} = \frac{a(1-\cos t)\,a\cos t - a\sin t\,(a\sin t)}{[a(1-\cos t)]^3}$$

$$= \frac{a^2\cos t(1-\cos t) - a^2\sin^2 t}{a^3(1-\cos t)^3}$$

$$= \frac{a^2\cos t - a^2\cos^2 t - a^2\sin^2 t}{a^3(1-\cos t)^3}$$

$$= \frac{a^2\cos t - a^2(\cos^2 t + \sin^2 t)}{a^3(1-\cos t)^3}$$

$$= \frac{a^2\cos t - a^2(1)}{a^3(1-\cos t)^3}$$

$$= \frac{-\cancel{a^2}(\cancel{1-\cos t})}{\cancel{a^3}(1-\cos t)^{\cancel{3}\,2}}$$

$$= \frac{-1}{a(1-\cos t)^2} \quad \text{Ans}$$

d) $x = a\cos 2t$, $y = b\sin 2t$

Diff: w.r.t. t

$$\frac{dx}{dt} = a\frac{d}{dt}\cos 2t$$

$$= a(-\sin 2t\cdot 2\cdot 1)$$

$$\boxed{\frac{dx}{dt} = -2a\sin 2t}$$

Diff: w.r.t. t

$$\frac{d^2x}{dt^2} = -2a\frac{d}{dx}\sin 2t$$

$$= -2a(\cos 2t\cdot 2\cdot 1)$$

$$\boxed{\frac{d^2x}{dt^2} = -4a\cos 2t}$$

Diff: w.r.t. t

$$\frac{dy}{dt} = b\frac{d}{dt}\sin 2t$$

$$= b\cdot\cos 2t\cdot 2$$

$$\boxed{\frac{dy}{dt} = 2b\cos 2t}$$

Diff: w.r.t. t

$$\frac{d^2y}{dt^2} = 2b\frac{d}{dt}\cos 2t$$

$$= 2b(-\sin 2t\cdot 2)$$

$$\boxed{\frac{d^2y}{dt^2} = -4b\sin 2t}$$

Now $$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{-2a\sin 2t(-4b\sin 2t) - 2b\cos 2t(-4a\cos 2t)}{(-2a\sin 2t)^3}$$

$$= \frac{+8ab\sin^2 2t + 8ab\cos^2 2t}{-8a^3\sin^3 2t}$$

$$= \frac{\cancel{8}ab(\sin^2 2t + \cos^2 2t)}{-\cancel{8}a^{\cancel{3}\,2}\sin^3 2t}$$

$$= \frac{-b(1)}{a^2\sin^3 2t}$$

$$= \frac{-b}{a^2\sin^3 2t} \quad \text{Ans}$$

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d) <u>2nd method</u> To find 2nd derivative

$x = a\cos 2t$

Diff w.r.t. t

$$\frac{dx}{dt} = a\frac{d}{dt}\cos 2t$$

$$= a(-\sin 2t \cdot 2)$$

$$\boxed{\frac{dx}{dt} = -2a\sin 2t}$$

$y = b\sin 2t$

Diff: w.r.t. t

$$\frac{dy}{dt} = b\frac{d}{dt}\sin 2t$$

$$= b\cos 2t \cdot 2$$

$$\boxed{\frac{dy}{dt} = 2b\cos 2t}$$

Now $$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \cancel{2}b\cos 2t \cdot \frac{-1}{\cancel{2}a\sin 2t} = -\frac{b\cos 2t}{a\sin 2t}$$

Now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(-\frac{b}{a}\frac{\cos 2t}{\sin 2t}\right)\cdot\frac{-1}{2a\sin 2t}$$

$$= -\frac{b}{a}\left(\frac{\sin 2t\frac{d}{dt}\cos 2t - \cos 2t\frac{d}{dt}\sin 2t}{(\sin 2t)^2}\right) \times \frac{-1}{2a\sin 2t}$$

$$= -\frac{b}{a}\left(\frac{\sin 2t(-\sin 2t\cdot 2) - \cos 2t\cdot\cos 2t\cdot 2}{\sin^2 2t}\right) \times \frac{-1}{2a\sin 2t}$$

$$= -\frac{b}{a}\left(\frac{-2\sin^2 2t - 2\cos^2 2t}{\sin^2 2t}\right) \times \frac{-1}{2a\sin 2t}$$

$$= +\frac{\cancel{2}b}{a}\left(\frac{\sin^2 2t + \cos^2 2t}{\sin^2 2t}\right) \times \frac{-1}{\cancel{2}a\sin 2t}$$

$$= \frac{b}{a}\left(\frac{1}{\sin^2 2t}\right) \times \frac{-1}{a\sin 2t} = \frac{-b}{a^2\sin^3 2t}$$

Ans

e) $x = \dfrac{3at}{1+t^3}$

Diff: w.r.t. t

$$\frac{dx}{dt} = 3a\frac{d}{dt}\left(\frac{t}{1+t^3}\right)$$

$$= 3a\left[\frac{(1+t^3)\frac{d}{dt}t - t\frac{d}{dt}(1+t^3)}{(1+t^3)^2}\right]$$

$$= 3a\left(\frac{(1+t^3).1 - t(3t^2)}{(1+t^3)^2}\right)$$

$$= 3a\frac{[1+t^3-3t^3]}{(1+t^3)^2}$$

$$\boxed{\frac{dx}{dt} = \frac{3a(1-2t^3)}{(1+t^3)^2}}$$

$y = \dfrac{3at^2}{1+t^3}$

Diff. w.r.t. t

$$\frac{dy}{dt} = 3a\frac{d}{dt}\left(\frac{t^2}{1+t^3}\right)$$

$$= 3a\left[\frac{(1+t^3)\frac{d}{dt}t^2 - t^2\frac{d}{dt}(1+t^3)}{(1+t^3)^2}\right]$$

$$= 3a\left(\frac{(1+t^3)2t - t^2(3t^2)}{(1+t^3)^2}\right)$$

$$= 3a\frac{(2t+2t^4-3t^4)}{(1+t^3)^2}$$

$$\boxed{\frac{dy}{dt} = \frac{3a(2t-t^4)}{(1+t^3)^2}}$$

Now

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{\cancel{3a}(2t-t^4)}{\cancel{(1+t^3)^2}} \times \frac{\cancel{(1+t^3)^2}}{\cancel{3a}(1-2t^3)} = \frac{2t-t^4}{1-2t^3}$$

Now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{2t-t^4}{1-2t^3}\right) \times \frac{dt}{dx}$$

$$= \frac{(1-2t^3)\frac{d}{dt}(2t-t^4) - (2t-t^4)\frac{d}{dt}(1-2t^3)}{(1-2t^3)^2} \times \frac{dt}{dx}$$

$$= \frac{(1-2t^3)(2-4t^3) - (2t-t^4)(-6t^2)}{(1-2t^3)^2} \times \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{2-4t^3-4t^3+8t^6+12t^3-6t^6}{(1-2t^3)^2} \times \frac{dt}{dx}$$

$$= \frac{2t^6+4t^3+2}{(1-2t^3)^2} \times \frac{(1+t^3)^2}{3a(1-2t^3)}$$

$$\left(t^6+2t^2+1 = (t^3+1)^2 \right.$$

$$= \frac{2(t^6+2t^2+1)(1+t^3)^2}{3a(1-2t^3)^3}$$

$$= \frac{2(t^3+1)^2(1+t^3)^2}{3a(1-2t^3)^3} = \frac{2(1+t^3)^4}{3a(1-2t^3)^3} \quad \text{Ans}$$

---

f) $x = a\frac{(1-t^2)}{1+t^2}$

Diff: w.r.t. t

$$\frac{dx}{dt} = a\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= a\left\{\frac{(1+t^2)\frac{d}{dt}(1-t^2)-(1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right\}$$

$$= a\left\{\frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}\right\}$$

$$= a\left\{\frac{-2t-\cancel{2t^3}-2t+\cancel{2t^3}}{(1+t^2)^2}\right\}$$

$$= a\left\{\frac{-4t}{(1+t^2)^2}\right\}$$

$$\boxed{\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}}$$

$y = b\frac{2t}{1+t^2}$

Diff: w.r.t. t

$$\frac{dy}{dt} = 2b\frac{d}{dt}\left(\frac{t}{1+t^2}\right)$$

$$= 2b\left\{\frac{(1+t^2)\frac{d}{dt}t - t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right\}$$

$$= 2b\left\{\frac{(1+t^2).1-t(2t)}{(1+t^2)^2}\right\}$$

$$= 2b\left\{\frac{1+t^2-2t^2}{(1+t^2)^2}\right\}$$

$$\boxed{\frac{dy}{dt} = \frac{2b(1-t^2)}{(1+t^2)^2}}$$

Now $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

$$= \frac{\cancel{2}b(1-t^2)}{\cancel{(1+t^2)^2}} \times \frac{\cancel{(1+t^2)^2}}{-\cancel{4}_2 at} = \frac{-b(1-t^2)}{+2at}$$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(-\frac{b(1-t^2)}{2at}\right) \times \frac{dt}{dx}$$

$$= -\frac{b}{2a}\frac{d}{dt}\left(\frac{1-t^2}{t}\right) \times \frac{(1+t^2)^2}{-4at}$$

$$= \frac{b}{2a}\left\{\frac{t\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}t}{t^2}\right\} \times \frac{(1+t^2)^2}{4at}$$

$$= \frac{b}{2a}\left\{\frac{t(-2t) - (1-t^2)\cdot 1}{t^2}\right\} \times \frac{(1+t^2)^2}{4at}$$

$$= \frac{b}{2a}\left\{\frac{-2t^2 - 1 + t^2}{t^2}\right\} \times \frac{(1+t^2)^2}{4at}$$

$$= \frac{b(-t^2-1)}{2at^2} \times \frac{(1+t^2)^2}{4at}$$

$$= -\frac{b(t^2+1)}{2at^2} \times \frac{(1+t^2)^2}{4at} = -\frac{b(1+t^2)^3}{8a^2t^3} \quad \text{Ans}$$

Taylor's expansion

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \cdots$$

It is taylor's expansion at $x = x_0$

The nth order taylor's polynomial may be written as

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + f^n(x_0)\frac{(x-x_0)^n}{n!}$$

Note If we put $x - x_0 = h$ or $x = x_0 + h$ then Taylor's expansion becomes

$$f(x_0+h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2!} + f'''(x_0)\frac{h^3}{3!} + \cdots + f^n(x_0)\frac{h^n}{n!}$$

Maclaurin's expansion: It is a special case of taylor's expansion, when we put $x_0 = 0$ in taylor's expansion then the resulting expansion is known as Maclaurin's expansion. that is

$$f(x) = f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2!} + f'''(0) \cdot \frac{x^3}{3!} + \cdots\cdots$$

The nth order Maclaurin's polynomial may be written as

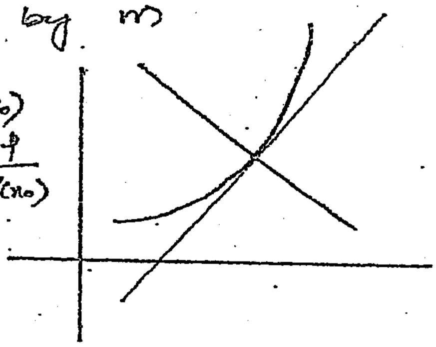
$$P_n(x) = f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2!} + \cdots\cdots + f^n(0)\, \frac{x^n}{n!}$$

Tangent line & Normal line

Generally we denoted slope by m

if slope of tangent line $= m = f'(x_0)$

then slope of Normal line $= -\frac{1}{m} = \frac{-1}{f'(x_0)}$



Eq: of tangent line line at (x_0, y_0) is $y - y_0 = m(x - x_0)$

& Eq: of Normal line is $y - y_0 = \frac{-1}{m}(x - x_0)$

The angle of intersection of two curves.

let (x_1, y_1) is the Point of intersection & m_1 is the slope of tangent line of the first curve & m_2 is the slope of tangent line of the 2nd curve at (x_1, y_1) then the angle of intersection of curves is $\tan\theta = \frac{m_1 - m_2}{1 + m_1 \cdot m_2}$

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Exercise 3.2

Note Eq: of tangent line at (x_1, y_1) is
$$y - y_1 = m(x - x_1), \quad m = f'(x_1)$$

① In each case, find the equation of the tangent line.

a) $y = \sqrt{x+1}$, $x = 3$

put $x = 3$

$y = \sqrt{3+1} = 2$

Point of Contact is $(3,2)$

$\because y = \sqrt{x+1}$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{x+1}$$

$$= \frac{1}{2}(x+1)^{-1/2}\frac{d}{dx}(x+1)$$

$$= \frac{1}{2(x+1)^{1/2}}(1)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}}$$

At $x = 3$

slope of tangent line is

$$m = \frac{1}{2\sqrt{3+1}} = \frac{1}{2(2)} = 1/4$$

Eq: of tangent line at Point $(\underset{x_1}{3}, \underset{y_1}{2})$ is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{4}(x - 3)$$

$\times$ ing by 4

$$4y - 8 = x - 3$$

$$4y = x + 5$$

$$y = \frac{x+5}{4} \quad \text{Ans}$$

b) $y = \sin(2x + \pi)$ — ①, $x = 0$

Diff w.r.t x

$$\frac{dy}{dx} = \cos(2x+\pi)\frac{d}{dx}(2x+\pi)$$

$$= \cos(2x+\pi)(2)$$

$$\frac{dy}{dx} = 2\cos(2x+\pi)$$

slope of tangent line at $x = 0$ is

$$m = 2\cos(2(0)+\pi)$$

$$= 2\cos(\pi) = 2(-1) = -2$$

To find Point of contact put $x = 0$ in ①

$$y = \sin(2(0)+\pi)$$

$$y = \sin\pi = 0$$

Point of Contact is $(x_1, y_1) = (0, 0)$

Eq: of tangent line at Point $(\underset{x_1}{0}, \underset{y_1}{0})$ is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -2(x - 0)$$

$$y = -2x \quad \text{Ans}$$

c) $y = x^2 e^{-x}$ — (1) $\boxed{x=1}$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{e^x}\right)$$

$$= \frac{e^x \frac{d}{dx}x^2 - x^2 \frac{d}{dx}e^x}{(e^x)^2}$$

$$= \frac{e^x \cdot 2x - x^2 e^x}{(e^x)^2}$$

$$= \frac{\cancel{e^x}(2x - x^2)}{(e^x)^{\cancel{2}}}$$

$$\frac{dy}{dx} = \frac{2x - x^2}{e^x}$$

slope of tangent line at $x=1$ is $m = \frac{2(1)-1^2}{e^1} = \frac{2-1}{e} = 1/e$

To find point of contact put $x=1$ in (1)

$y = 1^2 e^{-1} = e^{-1} = \frac{1}{e}$

Point of contact is $(x, y) = (1, \frac{1}{e})$

Eq: of tangent line at Point $(1, \frac{1}{e})$ is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{e} = \frac{1}{e}(x - 1)$$

X'ing by e

$$ey - \cancel{1} = x - \cancel{1}$$

$$ey = x$$

$$y = \frac{x}{e} \quad \text{Ans}$$

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d) $y = \frac{2x+1}{x+2}$ — (1) $x=2$

Diff w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x+1}{x+2}\right)$$

$$= \frac{(x+2)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$= \frac{(x+2)(2) - (2x+1)(1)}{(x+2)^2}$$

$$= \frac{\cancel{2x} + 4 - \cancel{2x} - 1}{(x+2)^2}$$

$$\frac{dy}{dx} = \frac{3}{(x+2)^2}$$

slope of tangent line at $x=2$ is

$$m = \frac{3}{(2+2)^2} = \frac{3}{16}$$

To find point of contact put $x=2$ in (1)

$$y = \frac{2(2)+1}{2+2} = 5/4$$

Point of contact is $(2, 5/4)$

Eq: of tangent line at $(\underset{x_1}{2}, \underset{y_1}{5/4})$ is

$$y - y_1 = m(x - x_1)$$

$$y - 5/4 = \frac{3}{16}(x - 2)$$

x'ing by 16

$$16y - 20 = 3(x - 2)$$

$$16y - 20 = 3x - 6$$

$$16y = 3x + 14 \quad \div\text{ing by } 16$$

$$y = \frac{3x}{16} + \frac{14}{16}$$

$$y = \frac{3x}{16} + \frac{7}{8} \quad \text{Ans}$$

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e) $y = \frac{x}{x^2+1}$ —— ① $x=1$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x^2+1}\right)$$

$$= \frac{(x^2+1)\frac{d}{dx}x - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{1-x^2}{(x^2+1)^2}$$

slope of tangent line at $x=1$ is $m = \frac{1-1^2}{(1+1)^2} = \frac{0}{4} = 0$

To find Point of contact put $x=1$ in ①

$$y = \frac{1}{1+1} = 1/2$$

Point of Contact is $(x,y) = (1, \frac{1}{2})$

Eq: of tangent line at Point $(\underset{x_1}{1}, \underset{y_1}{1/2})$ is

$$y - y_1 = m(x - x_1)$$

$$y - 1/2 = 0(x-1)$$

$$y - \frac{1}{2} = 0$$

$$y = \frac{1}{2}$$

$$y = 0.5 \quad \text{Ans}$$

g) $y = 2\ln x$ —— ① $x = e$

Diff w.r.t x

$$\frac{dy}{dx} = 2 \cdot \frac{1}{x} = 2/x$$

slope of tangent line at $x=e$ is $m = \frac{2}{e}$ see →

f) $y = 3\sin x - \cos x$ —— ① $(x = \pi$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(3\sin x - \cos x)$$

$$\frac{dy}{dx} = 3\cos x + \sin x$$

slope of tangent line at $x = \pi$ is

$$m = 3\cos\pi + \sin\pi$$

$$= 3(-1) + 0 = -3$$

To find Point of contact put $x = \pi$ in ①

$$y = 3\sin\pi - \cos\pi$$

$$= 3 \cdot 0 - (-1) = 0 + 1 = 1$$

Point of contact is $(\pi, 1)$

Eq: of tangent line at $(\underset{x_1}{\pi}, \underset{y_1}{1})$ is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -3(x - \pi)$$

$$y - 1 = -3x + 3\pi$$

$$y = -3x + 3\pi + 1 \quad \text{Ans}$$

To find Point of Contact:

put $x = e$ in ①

$$y = 2\ln e = 2(1) = 2$$

Point of contact is $(x,y) = (e, 2)$

Eq: of tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{2}{e}(x - e)$$

$\times$ by e

$$ey - \cancel{2e} = 2x - \cancel{2e}$$

$$ey = 2x \Rightarrow y = \frac{2x}{e} \quad \text{Ans}$$

h) $y = 3e^{x} + e^{-x}$ —① , $x=0$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}(3e^{x}+e^{-x})$

$= 3e^{x} + e^{-x}(-1)$

$\frac{dy}{dx} = 3e^{x} - e^{-x}$

slope of tangent line at $x=0$

is $m = 3e^{0} - e^{0} = 3-1 = 2$ ✓

To Find Point of contact

put $x=0$ in ①

$y = 3e^{0} + e^{0} = 3(1)+1 = 4$

Point of Contact is $(0,4)$

Eq: of tangent line at

Point $(\underset{x_1}{0}, \underset{y_1}{4})$ is

$y - y_1 = m(x - x_1)$

$y - 4 = 2(x - 0)$

$y - 4 = 2x$

$y = 2x + 4$

---

b) $y = 2\sin 3x$ —① $x = \pi$

Diff w.r.t. x

$\frac{dy}{dx} = 2\frac{d}{dx}\sin 3x$

$= 2\cos 3x \cdot 3$

$\frac{dy}{dx} = 6\cos 3x$

$\cos 3\pi = -1$

$\boxed{\cos odd\pi = -1}$

Slope of tangent line at $x=\pi$ is

$m = 6\cos 3\pi = 6(-1) = -6$

slope of Normal line is

$\boxed{-\frac{1}{m} = +\frac{1}{6}}$

To find Point of contact

put $x = \pi$ in ① see

$y = 2\sin 3\pi = 0$

Point of contact is $(\underset{x_1}{\pi}, \underset{y_1}{0})$

Eq: of Normal line is

$y - y_1 = -\frac{1}{m}(x - x_1)$

$y - 0 = \frac{1}{6}(x - \pi)$

$\Rightarrow y = \frac{x-\pi}{6}$ Ans

---

② Eq: of Normal line =? (20

a) $y = xe^{x}$ —① $x=1$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}(xe^{x})$

$= x\frac{d}{dx}e^{x} + e^{x}\frac{d}{dx}x$

$\frac{dy}{dx} = xe^{x} + e^{x} \cdot 1$

slope of tangent line at $x=1$

is $m = 1e^{1} + e^{1} = e + e = 2e$

$\Rightarrow$ slope of Normal line is

$-\frac{1}{m} = -\frac{1}{2e}$

To find Point of contact put $x=1$

in ① $y = 1e^{1} = e$ ✓

Point of contact is $(1,e)$

Eq: of Normal line

at Point $(\underset{x_1}{1}, \underset{y_1}{e})$ is

$y - y_1 = -\frac{1}{m}(x - x_1)$

$y - e = -\frac{1}{2e}(x - 1)$

X ing by $2e$

$2ey - 2e^{2} = -x + 1$

$2ey = -x + 2e^{2} + 1$

$y = -\frac{x + 2e^{2} + 1}{2e}$ Ans

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c) $y = 2\ln x$ —① $x=1$

Diff: w.r.t. x

$\frac{dy}{dx} = 2\frac{d}{dx}\ln x$

$= 2\cdot\frac{1}{x} = 2/x$

slope of tangent line at $x=1$

is $m = \frac{2}{1} = 2$

Now slope of Normal line

is $-\frac{1}{m} = -1/2$

To find Point of contact:

put $x=1$ in ①

$y = 2\ln 1 = 2(0) = 0$

Point of contact is $(1,0)$

Eq: of Normal line

at Point $(\underset{x_1}{1}, \underset{y_1}{0})$ is

$y - y_1 = \frac{-1}{m}(x - x_1)$

$y - 0 = \frac{-1}{2}(x-1)$

$y = \frac{1-x}{2}$ Ans

e) $y = \frac{e^x+1}{x}$ —① $x=1$

Diff: w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x+1}{x}\right)$

$= \frac{x\frac{d}{dx}(e^x+1) - (e^x+1)\frac{d}{dx}x}{x^2}$

$\frac{dy}{dx} = \frac{x(e^x+0)-(e^x+1)\cdot 1}{x^2} = \frac{xe^x - e^x - 1}{x^2}$

slope of tangent line at $x=1$

is $m = \frac{1\cancel{e^1} - \cancel{e^1} - 1}{1^2} = \frac{-1}{1} = -1$

To find Point of contact

put $x=1$ in ①. see

$y = \frac{e^1+1}{1} = e+1$

Point of contact is $(\underset{x_1}{1}, \underset{y_1}{e+1})$

Eq. of Normal line is

$y - y_1 = -\frac{1}{m}(x-x_1)$

$y - (e+1) = \frac{-1}{-1}(x-1)$

$y - e - \cancel{1} = +x - \cancel{1}$

$y = x + e$ ANS

d) $y = (2x+1)^6$ —① $x=0$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}(2x+1)^6$

$= 6(2x+1)^5\frac{d}{dx}(2x+1)$

$= 6(2x+1)^5(2)$

$\frac{dy}{dx} = 12(2x+1)^5$

slope of tangent line at $x=0$

is $m = 12(0+1)^5 = 12$

To find Point of contact

put $x=0$ in ①

$y = (2(0)+1)^6 = (0+1)^6 = 1$

Point of contact is $(0,1)$

Eq: of Normal line at

Point $(\underset{x_1}{0}, \underset{y_1}{1})$ is

$y - y_1 = -\frac{1}{m}(x-x_1)$

$y - 1 = -\frac{1}{12}(x-0)$

x by 12

$12y - 12 = -x$

$12y = 12 - x$

$y = \frac{12-x}{12}$ Ans

or

$y - 1 = -\frac{x}{12}$

$y = -\frac{x}{12} + 1$ Ans

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f) $y = \cos(x-\pi)$ — ① $\boxed{x = \pi/2}$

Diff: w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\cos(x-\pi)$

$= -\sin(x-\pi)\frac{d}{dx}(x-\pi)$

$\frac{dy}{dx} = -\sin(x-\pi)\cdot 1$

slope of tangent line at $x = \pi/2$

is $m = -\sin\left(\frac{\pi}{2}-\pi\right)$

$m = -\sin\left(-\frac{\pi}{2}\right) = -(-1) = 1$

To find Point of contact

put $x = \pi/2$ in ①

$y = \cos\left(\frac{\pi}{2}-\pi\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

Point of contact is $\left(\frac{\pi}{2}, 0\right)$.

Eq: of Normal line

at Point $\left(\underset{x_1}{\frac{\pi}{2}}, \underset{y_1}{0}\right)$ is

$y - y_1 = -\frac{1}{m}(x - x_1)$

$y - 0 = -\frac{1}{1}(x - \pi/2)$

$y = -x + \pi/2$ Ans

h) $y = \sqrt{x^2+1}$ — ① $x = 2$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\sqrt{x^2+1}$

$= \frac{1}{2}(x^2+1)^{-1/2}\frac{d}{dx}(x^2+1)$

$= \frac{1}{\cancel{2}(x^2+1)^{1/2}}(\cancel{2}x+0)$

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}$

slope of tangent line at $x=2$

is $m = \frac{2}{\sqrt{2^2+1}} = \frac{2}{\sqrt{4+1}} = 2/\sqrt{5}$

To find Point of contact

put $x=2$ in ① See

g) $y = x^3 \ln x$ — ① $x = 1$

Diff w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}(x^3 \ln x)$

$= x^3 \frac{d}{dx}\ln x + \ln x \cdot \frac{d}{dx}x^3$

$= x^{3-2}\cdot\frac{1}{x} + \ln x \cdot 3x^2$

$\frac{dy}{dx} = x^2 + 3x^2 \ln x$

slope of tangent line at $x=1$

is $m = 1^2 + 3(1)^2 \ln 1$

$= 1 + 3(0) = 1$

To find Point of contact

put $x = 1$ in ①

$y = 1^3 \ln 1 = 0$

Point of contact is $(1, 0)$.

Eq: of Normal line

at Point $(\overset{x_1}{1}, \overset{y_1}{0})$ is

$y - y_1 = -\frac{1}{m}(x - x_1)$

$y - 0 = -\frac{1}{1}(x - 1)$

$y = -x + 1$ Ans

① ⇒ $y = \sqrt{2^2+1} = \sqrt{4+1} = \sqrt{5}$

Point of contact is $(2, \sqrt{5})$

Eq: of Normal line at

Point $(\overset{x_1}{2}, \overset{y_1}{\sqrt{5}})$ is

$y - y_1 = -\frac{1}{m}(x - x_1)$

$y - \sqrt{5} = \frac{-1}{2/\sqrt{5}}(x - 2)$

$y - \sqrt{5} = -\frac{\sqrt{5}}{2}(x - 2)$

$y - \sqrt{5} = -\frac{\sqrt{5}}{2}x + \sqrt{5}$

$y = -\frac{\sqrt{5}}{2}x + 2\sqrt{5}$ Ans

3) a Eq: of T = ?

$x^2+y^2 = 13$ $(-2,3)$

Diff. w.r.t. x

$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx} 13$

$2x+2y\frac{dy}{dx} = 0$

$\cancel{2}y\frac{dy}{dx} = -\cancel{2}x$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

slope of tangent line at $(-2,3)$

is $m = -\frac{(-2)}{3} = +\frac{2}{3}$

Eq: of tangent line at $(-2,3)$

is $y-y_1 = m(x-x_1)$

$y-3 = \frac{2}{3}(x+2)$ Ans

x'ing by 3

$3y-9 = 2x+4$

$3y = 2x+13$

$y = \frac{2}{3}x + \frac{13}{3}$ Ans

---

c) $x^2+2xy = y^3$ $(1,-1)$

Diff: w.r.t. x

$\frac{d}{dx}(x^2+2xy) = \frac{d}{dx}y^3$

$2x+2\{x\frac{dy}{dx}+y\frac{d}{dx}x\} = 3y^2\frac{dy}{dx}$

$2x+2x\frac{dy}{dx}+2y\cdot 1 = 3y^2\frac{dy}{dx}$

$2x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -2x-2y$

$(2x-3y^2)\frac{dy}{dx} = -2(x+y)$

$\frac{dy}{dx} = \frac{-2(x+y)}{2x-3y^2}$

slope of tangent line at $(1,-1)$ is

$m = \frac{-2(1-1)}{2(1)-3(-1)^2} = \frac{-2(0)}{2-3} = \frac{-0}{-1} = 0$ see

b) $\sin(x-y) = xy$ $(0,\pi)$

Diff: w.r.t. x

$\frac{d}{dx}\sin(x-y) = \frac{d}{dx}(xy)$

$\cos(x-y)\frac{d}{dx}(x-y) = x\frac{d}{dx}y + y\frac{d}{dx}x$

$\cos(x-y)(1-\frac{dy}{dx}) = x\frac{dy}{dx} + y\cdot 1$

$\cos(x-y) - \cos(x-y)\cdot\frac{dy}{dx} = x\frac{dy}{dx}+y$

$\cos(x-y) - y = x\frac{dy}{dx} + \cos(x-y)\frac{dy}{dx}$

$\cos(x-y) - y = [x+\cos(x-y)]\frac{dy}{dx}$

or

$\Rightarrow \frac{dy}{dx} = \frac{\cos(x-y)-y}{x+\cos(x-y)}$

slope of tangent line at $(0,\pi)$ is

$m = \frac{\cos(0-\pi)-\pi}{0+\cos(0-\pi)} = \frac{-1-\pi}{-1} = \pi+1$

Eq: of tangent line at point $(0,\pi)$ is

$y-y_1 = m(x-x_1)$

$y-\pi = (\pi+1)(x-0)$

$y-\pi = (\pi+1)x$ Ans

Eq: of Normal line at $(1,-1)$ is

$y-y_1 = -\frac{1}{m}(x-x_1)$

$y+1 = \frac{-1}{0}(x-1)$ Ans

$(y+1)0 = -1(x-1)$

$0 = -x+1$

$\boxed{x=1}$

which is the req. Eq. of line.

Note Taylor's expansion of $f(x)$ at $x = x_0$ is

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \cdots$$

4) a) $f(x) = \tan x$ $\quad x = \pi/4 \quad x_0 = \pi/4$

From Taylor's expansion

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \cdots$$

put $x_0 = \pi/4$

$$f(x) = f(\tfrac{\pi}{4}) + f'(\tfrac{\pi}{4})(x-\pi/4) + f''(\tfrac{\pi}{4})\frac{(x-\pi/4)^2}{2!} + f'''(\pi/4)\frac{(x-\pi/4)^3}{3!} + \cdots \quad \text{--- (1)}$$

$\because$ $f(x) = \tan x \Rightarrow \boxed{f(\tfrac{\pi}{4}) = \tan\tfrac{\pi}{4} = 1}$

Diff w.r.t. x

$f'(x) = \sec^2 x \Rightarrow \boxed{f'(\pi/4) = \sec^2\tfrac{\pi}{4} = (\sqrt{2})^2 = 2}$ $\quad \left(\cos\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}} \Rightarrow \sec\tfrac{\pi}{4} = \sqrt{2}\right.$

Diff w.r.t. x

$f''(x) = 2\sec x \frac{d}{dx}\sec x = 2\sec x(\sec x \cdot \tan x)$

$f''(x) = 2\sec^2 x \cdot \tan x \Rightarrow f''(\pi/4) = 2\sec^2\tfrac{\pi}{4} \cdot \tan\tfrac{\pi}{4} = 2(\sqrt{2})^2 \cdot 1 = 4$

Diff w.r.t. x $\quad \Rightarrow \boxed{f''(\tfrac{\pi}{4}) = 4}$

$f'''(x) = 2\frac{d}{dx}(\sec^2 x \tan x)$

$= 2\{\sec^2 x \frac{d}{dx}\tan x + \tan x \frac{d}{dx}\sec^2 x\}$

$= 2\{\sec^2 x \cdot \sec^2 x + \tan x \cdot 2\sec x \cdot \frac{d}{dx}\sec x\}$

$= 2\{\sec^4 x + 2\tan x \cdot \sec x \cdot (\sec x \tan x)\}$

$f'''(x) = 2\sec^4 x + 4\tan^2 x \cdot \sec^2 x$

$\Rightarrow f'''(\pi/4) = 2\sec^4\tfrac{\pi}{4} + 4 \cdot \tan^2\pi/4 \sec^2\pi/4$

$= 2(\sqrt{2})^4 + 4 \cdot 1^2 \cdot (\sqrt{2})^2 = 2(4) + 4(2) = 8+8 = 16$

put all these values in (1)

$$\tan x = 1 + 2(x-\pi/4) + \cancel{4}^{2}\frac{(x-\pi/4)^2}{\cancel{2}\cdot 1} + \cancel{16}^{8}\frac{(x-\pi/4)^3}{3\cdot\cancel{2}\cdot 1} + \cdots$$

$$\tan x = 1 + 2(x-\pi/4) + 2(x-\pi/4)^2 + \frac{8}{3}(x-\tfrac{\pi}{4})^3$$

which are the req. 1st four terms

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b) $f(x) = \sqrt{x}$ $\qquad x = 4 \qquad (x_0 = 4)$

Sol From Taylor's expansion

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \cdots$$

put $x_0 = 4$

$$f(x) = f(4) + f'(4)(x-4) + f''(4)\frac{(x-4)^2}{2!} + f'''(4)\frac{(x-4)^3}{3!} + \cdots \quad 2 \to (1)$$

$\because \; f(x) = \sqrt{x} \Rightarrow \boxed{f(4) = \sqrt{4} = 2}$

Diff

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \Rightarrow \boxed{f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = 1/4}$

Diff

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}x^{-3/2}\right) = \frac{-1}{4x^{3/2}} \Rightarrow \boxed{f''(4) = \frac{-1}{4(4)^{3/2}} = \frac{-1}{4(2^2)^{3/2}} = \frac{-1}{32}}$

Diff

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}x^{-5/2}\right) = \frac{+3}{8x^{5/2}} \Rightarrow f'''(4) = \frac{3}{8(4)^{5/2}} = \frac{3}{8(2^2)^{5/2}} = \frac{3}{8(32)}$

$\boxed{f'''(4) = \frac{3}{256}}$

put all these values in (1)

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{32}\frac{(x-4)^2}{2\cdot 1} + \frac{\cancel{3}}{256}\cdot\frac{(x-4)^3}{\cancel{3}\cdot 2\cdot 1} + \cdots$$

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4) + \frac{1}{512}(x-4)^3$$

which are the req: 1st four terms

---

e) $f(x) = x + e^x$ $\qquad x = 1 \qquad (x_0 = 1)$

Sol From Taylor's expansion

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \cdots$$

put $x_0 = 1$

$$f(x) = f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2!} + f'''(1)\frac{(x-1)^3}{3!} + \cdots \quad 2 \to (1)$$

$\because f(x) = x + e^x \Rightarrow f(1) = 1 + e^1 = 1 + e$

Diff

$f'(x) = 1 + e^x \Rightarrow f'(1) = 1 + e$

Diff

$f''(x) = e^x \Rightarrow f''(1) = e^1 = e$

Diff

$f'''(x) = e^x \Rightarrow f'''(1) = e^1 = e$

put all these values in ①

$$x + e^x = \underbrace{1+e} + \underbrace{(1+e)}(x-1) + e\frac{(x-1)^2}{2!} + e\frac{(x-1)^3}{3!} + \cdots$$

$$= (1+e)[\cancel{1} + x - \cancel{1}] + e\frac{(x-1)^2}{2!} + e\frac{(x-1)^3}{3!} + \cdots$$

$$= (1+e)x + e\left[\frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots\right]$$

Proved

Note Maclaurin's expansion

$$f(x) = f(0) + f'(0)\,x + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + \cdots$$

(5) Find the Maclaurin Series expansion

a) $f(x) = \frac{1}{1+x}$

Sol From Maclaurin Series expansion

$$f(x) = f(0) + f'(0)\,x + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + \cdots \longrightarrow ①$$

$\because f(x) = \frac{1}{1+x} = (1+x)^{-1} \Rightarrow f(0) = \frac{1}{1+0} = 1$

Diff

$f'(x) = -1\cdot(1+x)^{-2}(1) = \frac{-1}{(1+x)^2} \Rightarrow f'(0) = \frac{-1}{(1+0)^2} = -1$

Diff

$f''(x) = +2(1+x)^{-3}(1) = \frac{2}{(1+x)^3} \Rightarrow f''(0) = \frac{2}{(1+0)^3} = 2$

Diff

$f'''(x) = -6(1+x)^{-4}(1) = \frac{-6}{(1+x)^4} \Rightarrow f'''(0) = \frac{-6}{(1+0)^4} = -6$

put all these values in ①

$$\frac{1}{1+x} = 1 + (-1)x + \cancel{2}\cdot\frac{x^2}{\cancel{2}\cdot 1} - \cancel{6}\cdot\frac{x^3}{\cancel{3}\cdot\cancel{2}\cdot 1} + \cdots$$

$$= 1 - x + x^2 - x^3 + \cdots$$

Note $2\sin x\cos x = \sin 2x$

b) $f(x) = \sin^2 x$

Sol From Maclaurin's theorem

$$f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + \cdots \rightarrow ①$$

$\because$ $f(x) = \sin^2 x \Rightarrow f(0) = \sin^2 0 = 0$

Diff

$f'(x) = 2\sin^1 x\cdot\cos x$

$f'(x) = \sin 2x \Rightarrow f'(0) = \sin 0 = 0$

Diff

$f''(x) = \cos 2x\cdot 2 = 2\cos 2x \Rightarrow f''(0) = 2\cos 0 = 2(1) = 2$

Diff

$f'''(x) = 2(-\sin 2x\cdot 2) = -4\sin 2x \Rightarrow f'''(0) = -4\sin 0 = 0$

Diff

$f^{iv}(x) = -4(\cos 2x\cdot 2) = -8\cos 2x \Rightarrow f^{iv}(0) = -8\cos 0 = -8$

Diff

$f^{v}(x) = -8(-\sin 2x\cdot 2) = +16\sin 2x \Rightarrow f^{v}(0) = 16\sin 0 = 0$

Diff

$f^{vi}(x) = 16(\cos 2x\cdot 2) = 32\cos 2x \Rightarrow f^{vi}(0) = 32\cos 0 = 32$

& so on

put all these values in ①

$$\sin^2 x = 0 + 0\cdot x + 2\cdot\frac{x^2}{2!} + 0\cdot\frac{x^3}{3!} - 8\cdot\frac{x^4}{4!} + 0\cdot\frac{x^5}{5!} + 32\cdot\frac{x^6}{6!} + \cdots$$

$$= 0 + 0 + \frac{\cancel{2}x^2}{\cancel{2}\cdot 1} + 0 - \frac{\cancel{8}\cdot x^4}{4\cdot 3\cdot\cancel{2}\cdot 1} + 0 + \frac{\cancel{32}^{\cancel{16}^{2}}\cdot x^6}{\cancel{6}_{3}\cdot 5\cdot\cancel{4}\cdot 3\cdot\cancel{2}\cdot 1} + \cdots$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \cdots \quad \text{Ans}$$

13°

c) $f(x) = \cosh x$

Sol from Maclaurin's expansion

$$f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + f^{iv}(0)\cdot\frac{x^4}{4!} + \cdots \rightarrow (1)$$

$\because f(x) = \cosh x \Rightarrow f(0) = \cosh 0 = 1$

Diff

$f'(x) = \sinh x \Rightarrow f'(0) = \sinh 0 = 0$

Diff

$f''(x) = \cosh x \Rightarrow f''(0) = \cosh 0 = 1$

Diff

$f'''(x) = \sinh x \Rightarrow f'''(0) = \sinh 0 = 0$

Diff

$f^{iv}(x) = \cosh x \Rightarrow f^{iv}(0) = \cosh 0 = 1$

& so on

put all these values in (1)

$$\cosh x = 1 + 0\cdot x + 1\cdot\frac{x^2}{2!} + 0\cdot\frac{x^3}{3!} + 1\cdot\frac{x^4}{4!} + \cdots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \quad \text{Ans}$$

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d) $f(x) = \ln(1-4x)$

Sol from Maclaurin's expansion

$$f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + \cdots \rightarrow (1)$$

$\because f(x) = \ln(1-4x) \Rightarrow \boxed{f(0) = \ln(1-0) = 0}$

Diff

$$f'(x) = \frac{1}{1-4x}\frac{d}{dx}(1-4x) = \frac{1}{1-4x}(-4)$$

$$f'(x) = \frac{-4}{1-4x} = -4(1-4x)^{-1} \Rightarrow \boxed{f'(0) = -4(1-0)^{-1} = -4}$$

Diff

$$f''(x) = -4\{-1(1-4x)^{-2}(-4)\} = -16(1-4x)^{-2} \Rightarrow \boxed{f''(0) = -16(1-0)^{-2} = -16}$$

Diff

$$f'''(x) = -16\{-2(1-4x)^{-3}(-4)\} = -128(1-4x)^{-3} \Rightarrow \boxed{f'''(0) = -128(1-0)^{-3} = -128}$$

& so on

put all these values in ①

$$\ln(1-4x) = 0 - 4\cdot x - 16\cdot\frac{x^2}{2!} - 128\cdot\frac{x^3}{3!} + \cdots$$

$$= -4x - \cancel{16}^{8}\cdot\frac{x^2}{\cancel{2}\cdot 1} - \cancel{128}^{64}\cdot\frac{x^3}{3\cdot\cancel{2}\cdot 1} + \cdots$$

$$= -4x - 8x^2 - \frac{64}{3}x^3 + \cdots \qquad \text{Ans.}$$

6) a) $f(x) = e^x$

Sol. from maclaurin's expansion

$$f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + \cdots \longrightarrow ①$$

$\because f(x) = e^x \Rightarrow f(0) = e^0 = 1$

Diff $f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

Diff

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$

Diff $f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1$

& so on

put all these values in ①

$$e^x = 1 + 1\cdot x + 1\cdot\frac{x^2}{2!} + 1\cdot\frac{x^3}{3!} + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

put $x = 1$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \qquad \text{(Proved)}$$

b) $\because e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \cdots$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$$

$$= 1 + 1 + .5 + 0.16667 + 0.04167 + 0.00833 + .00139 + .00020 + \cdots$$

$e = 2.71826$

which is the req: value of e correct to 4 decimal places.

The angle of intersection of two Curves

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

To find acute angle use $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 . m_2}\right|$

---

7) Find the angle of intersection between the following curves.

a) $x^2 - y^2 = a^2$ —① , $x^2 + y^2 = a^2\sqrt{2}$ —②

First we have to find Point of intersection

solve ① & ② ① + ②

$$\left.\begin{aligned} x^2 - \cancel{y^2} &= a^2 \\ x^2 + \cancel{y^2} &= a^2\sqrt{2} \end{aligned}\right\} \text{add}$$

$$2x^2 = a^2 + a^2\sqrt{2}$$

$$2x^2 = a^2(1+\sqrt{2}) \Rightarrow$$

$$x^2 = \frac{(1+\sqrt{2})a^2}{2} \Rightarrow \boxed{x = \sqrt{\frac{1+\sqrt{2}}{2}}\, a}$$

put in ①

$$\frac{(1+\sqrt{2})a^2}{2} - y^2 = a^2$$

x's by 2

$$(1+\sqrt{2})a^2 - 2y^2 = 2a^2$$

$$(1+\sqrt{2})a^2 - 2a^2 = 2y^2$$

$$(1+\sqrt{2}-2)a^2 = 2y^2$$

$$(\sqrt{2}-1)a^2 = 2y^2$$

$$\Rightarrow y^2 = \frac{\sqrt{2}-1}{2}a^2 \Rightarrow \boxed{y = \sqrt{\frac{\sqrt{2}-1}{2}}\, a}$$

Point of intersection is $(x, y) = \left(\sqrt{\frac{1+\sqrt{2}}{2}}\, a, \sqrt{\frac{\sqrt{2}-1}{2}}\, a\right)$

See →

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$\because \quad x^2-y^2=a^2$

Diff w.r.t. x

$2x-2y\dfrac{dy}{dx}=0$

$\Rightarrow \cancel{2}y\dfrac{dy}{dx}=\cancel{2}x$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{x}{y}$

slope of tangent line at Point $\left(\sqrt{\frac{1+\sqrt2}{2}}a, \sqrt{\frac{\sqrt2-1}{2}}a\right)$ is

$m_1=\dfrac{\sqrt{\frac{1+\sqrt2}{2}}\cancel{a}}{\sqrt{\frac{\sqrt2-1}{2}}\cancel{a}}=\sqrt{\dfrac{1+\sqrt2}{\sqrt2-1}}$

$=\sqrt{\dfrac{\sqrt2+1}{\sqrt2-1}\times\dfrac{\sqrt2+1}{\sqrt2+1}}$ ← M&D

$=\sqrt{\dfrac{(\sqrt2+1)^2}{(\sqrt2)^2-1^2}}=\sqrt{\dfrac{(\sqrt2+1)^2}{2-1}}=\sqrt{\dfrac{(\sqrt2+1)^2}{1}}$

$\boxed{m_1=\sqrt2+1}$

$\because \quad x^2+y^2=a^2\sqrt2$

Diff: w.r.t. x

$2x+2y\dfrac{dy}{dx}=0$

$\cancel{2}y\dfrac{dy}{dx}=-\cancel{2}x$

$\Rightarrow \dfrac{dy}{dx}=-\dfrac{x}{y}$

slope of tangent line at Point $\left(\sqrt{\frac{1+\sqrt2}{2}}a, \sqrt{\frac{\sqrt2-1}{2}}a\right)$ is

$m_2=-\dfrac{\sqrt{\frac{\sqrt2+1}{2}}\cancel{a}}{\sqrt{\frac{\sqrt2-1}{2}}\cancel{a}}=-\sqrt{\dfrac{\sqrt2+1}{\sqrt2-1}}$

$=-\sqrt{\dfrac{\sqrt2+1}{\sqrt2-1}\times\dfrac{\sqrt2+1}{\sqrt2+1}}$ ← M&D

$=-\sqrt{\dfrac{(\sqrt2+1)^2}{(\sqrt2)^2-1^2}}=-\sqrt{\dfrac{(\sqrt2+1)^2}{2-1}}=-\sqrt{\dfrac{(\sqrt2+1)^2}{1}}$

$\boxed{m_2=-(\sqrt2+1)}$

Let θ be the angle of intersection so

$\tan\theta=\left|\dfrac{m_1-m_2}{1+m_1.m_2}\right|$

$=\left|\dfrac{\sqrt2+1+(\sqrt2+1)}{1+(\sqrt2+1)[-(\sqrt2+1)]}\right|=\left|\dfrac{\sqrt2+1+\sqrt2+1}{1-(\sqrt2+1)^2}\right|$

$=\left|\dfrac{2\sqrt2+2}{1-\{(\sqrt2)^2+1^2+2\sqrt2\}}\right|=\left|\dfrac{2(\sqrt2+1)}{\cancel{1}-2-\cancel{1}-2\sqrt2}\right|=\left|\dfrac{\cancel{2(\sqrt2+1)}}{\cancel{-2(1+\sqrt2)}}\right|$

$\tan\theta=|-1|$

$\tan\theta=1 \Rightarrow \theta=\tan^{-1}(1)=\dfrac{\pi}{4}$ Ans

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b) $y^2 = ax \Rightarrow x = \frac{y^2}{a}$ —— ① , $x^3 + y^3 = 3axy$ —— ②

put value of x from ① in ②

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3\cancel{a}\left(\frac{y^2}{\cancel{a}}\right)y$$

$$\frac{y^6}{a^3} + y^3 = 3y^3$$

$$\frac{y^6}{a^3} - 2y^3 = 0$$

x by a^3

$$y^6 - 2a^3y^3 = 0 \Rightarrow y^3(y^3 - 2a^3) = 0$$

$\Rightarrow$ $y^3 = 0$ or $y^3 - 2a^3 = 0$

$\boxed{y = 0}$

put in ①

$x = \frac{0}{a} = 0$

Point of Contact $(0,0)$

$y^3 = 2a^3$ taking cube root

$(y^3)^{1/3} = (2a^3)^{1/3}$

$\boxed{y = 2^{1/3}a}$ put in ①

$$x = \frac{(2^{1/3}a)^2}{a} = \frac{2^{2/3}a^2}{\cancel{a}}$$

$\boxed{x = 2^{2/3}a}$

Point of Contact $= (2^{2/3}a, 2^{1/3}a)$

Now

$\because$ $y^2 = ax$

Diff w.r.t. x

$$\frac{d}{dx}y^2 = \frac{d}{dx}ax$$

$$2y\frac{dy}{dx} = a \cdot 1$$

$$\frac{dy}{dx} = \frac{a}{2y}$$

slope of tangent line at $(2^{2/3}a, 2^{1/3}a)$ is

$$m_1 = \frac{\cancel{a}}{2(2^{1/3}\cancel{a})} = \frac{1}{2^{1+1/3}} = \frac{1}{2^{4/3}}$$

$$\boxed{m_1 = \frac{1}{2^{4/3}}}$$

$\because$ $x^3 + y^3 = 3axy$

Diff: w.r.t. x

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left\{x\frac{dy}{dx} + y\frac{dx}{dx}\right\}$$

$\div$ by 3

$$x^2 + y^2\frac{dy}{dx} = a\left\{x\frac{dy}{dx} + y \cdot 1\right\}$$

$$x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$y^2\frac{dy}{dx} - ax\frac{dy}{dx} = ay - x^2$$

$$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

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slope of tangent line at Point $(2^{2/3}a, 2^{1/3}a)$ is

$$m_2 = \frac{a\cdot 2^{1/3}\cdot a - (2^{2/3}a)^2}{(2^{1/3}a)^2 - a\cdot 2^{2/3}a}$$

$$m_2 = \frac{2^{1/3}\cdot a^2 - 2^{4/3}a^2}{\cancel{2^{2/3}a^2} - \cancel{2^{2/3}a^2}} = \frac{a^2(2^{1/3}-2^{4/3})}{0}$$

$$\boxed{m_2 = \infty}$$

Let θ be the angle of intersection so

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right| \quad \left(\frac{\infty}{\infty}\right)$$

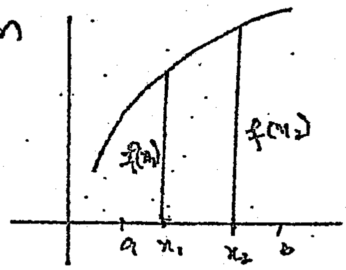
$$= \left|\frac{\cancel{m_2}\left(\frac{m_1}{m_2} - 1\right)}{\cancel{m_2}\left(\frac{1}{m_2} + m_1\right)}\right| = \left|\frac{\frac{m_1}{m_2} - 1}{\frac{1}{m_2} + m_1}\right|$$

$$= \left|\frac{\frac{\frac{1}{2^{4/3}}}{\infty} - 1}{\frac{1}{\infty} + \frac{1}{2^{4/3}}}\right| = \left|\frac{0 - 1}{0 + \frac{1}{2^{4/3}}}\right| = \frac{1}{\frac{1}{2^{4/3}}} = 1\times\frac{2^{4/3}}{1}$$

$$\tan\theta = (2^4)^{1/3} = (16)^{1/3} = \sqrt[3]{16}$$

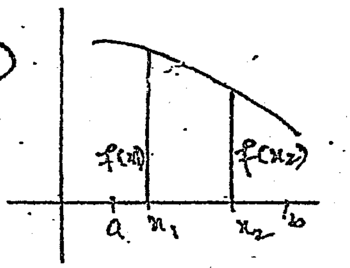
$$\Rightarrow \theta = \tan^{-1}(\sqrt[3]{16}) \quad \text{Ans}$$

Increasing and decreasing function



Let $f(x)$ be defined on an interval (a,b) and let $x_1, x_2 \in (a,b)$ then

(i) $f(x)$ is increasing on interval (a,b) if for $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$



(ii) $f(x)$ is decreasing on interval (a,b) if for $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

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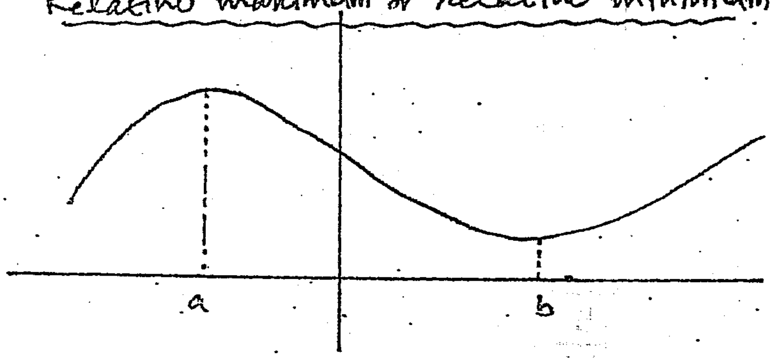
Note A differentiable function $f(x)$ is increasing on (a,b) if $f'(x)>0$ for all x such that $a<x<b$ and $f(x)$ is decreasing on (a,b) if $f'(x)<0$ for all x such that $a<x<b$

Theorem Let $f(x)$ be a function, which is continuous on $[a,b]$ and differentiable on (a,b). Then

(i) $f(x)$ is increasing on (a,b) if $f'(x)>0$ $\forall\, x\in(a,b)$

(ii) $f(x)$ is decreasing on (a,b) if $f'(x)<0$ $\forall\, x\in(a,b)$

Relative maximum or Relative minimum: The function $f(x)$ is said to have a relative maximum at a number a if $f(a)\geq f(x)$ $\forall x$ in an open interval containing a.

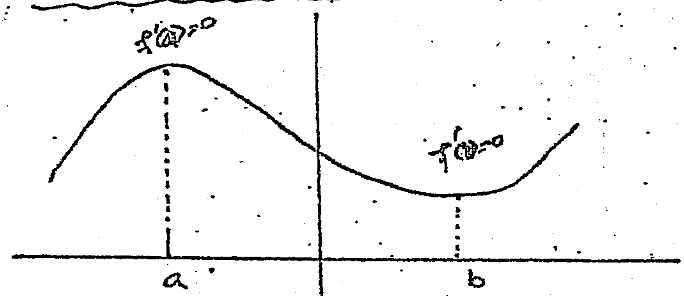


Also $f(x)$ is said to have a relative minimum at a number b if $f(x)\leq f(x)$ $\forall x$ in an open interval containing b. In general, the relative maxima and the relative minima are called relative extrema.

Critical value and critical Point: Suppose $f(x)$ is defined at a number c and either $f'(c)=0$ or $f'(c)$ does not exist. Then the number c is called Critical value & the Point $(c,f(c))$ is called Critical Point.

Note if $f(c)$ not define, then c can not be a critical value

First Derivative Rule



From graph

$\because$ $f'(x) > 0$ for $x < a$

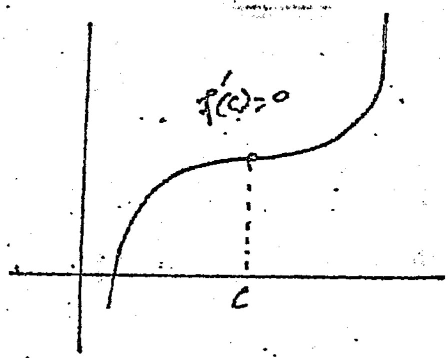
& $f'(x) < 0$ for $x > a$

So $f(x)$ have a relative maximum at number $x = a$

$\because$ $f'(x) < 0$ for $x < b$

& $f'(x) > 0$ for $x > b$

so $f(x)$ have a relative minimum at $x = b$



From graph

$\because$ $f'(x) > 0$ for $x < c$

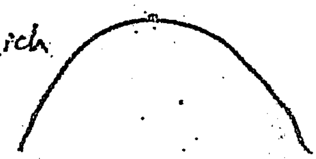
& $f'(x) > 0$ for $x > c$

so $f(x)$ have an inflection point at $x = c$

2nd derivative rule

1) At a point where $f'(x) = 0$ & $f''(x) > 0$, then the curve is concave up, such point is called minimum point, which is the lowest point of the curve

2) At a point where $f'(x) = 0$ & $f''(x) < 0$, the the curve is concave down, such a point is called maximum point, which is the highest point of the curve



3) At a point where $f'(x) = 0$, $f''(x) = 0$ & $f''(x)$ changes sign before & after that point, such a point is called inflection point.

Note: a point where $f(x)$ changes concavity is called inflection point.

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EXERCISE 3.3

1) a) A: Point where $f'(x)=0$ or undefined & before that point $f'(x)>0$ & after that point $f'(x)<0$ then $f(x)$ have a relative maximum at that point.
If before the point $f'(x)<0$ & after the point $f'(x)>0$ then $f(x)$ have a relative minimum at that point.
If $f'(x)$ does not change the sign before & after that point then such a point is called inflection point.
It is called first derivative test

b) The critical values of graph of $f(x)$ are intercepts for the graph of $f'(x)$.

c) A point where $f'(x)=0$ or undefined & $f''(x)<0$ then $f(x)$ have a relative maximum at that point & if $f''(x)>0$ then $f(x)$ have a relative minimum at that point & if $f''(x)=0$ then the 2nd derivative test fails & gives no information

d) 2nd derivative test informs us about concavity & point of inflection. The changes in concavity create inflection point.

2) a) $f(x)=x^3+3x^2+1$ —— (1)

Diff $f'(x)=3x^2+6x$ —— (2)

Diff $f''(x)=6x+6$ —— (3)

For critical values:

$f'(x) = 0$

$3x^2 + 6x = 0$

$3x(x+2) = 0 \Rightarrow x(x+2) = 0$

$\boxed{x=0}$ or $x+2=0$

$\boxed{x=-2}$

$(-\infty \quad -2 \quad 0 \quad \infty)$

Critical values are $x=0$ & $x=-2$

Now we have three intervals $(-\infty,-2)$, $(-2,0)$ & $(0,\infty)$

Let $x=-3 \in (-\infty,-2)$
put in ②
$f'(-3) = 3(-3)^2 + 6(-3)$
$= 27-18 = 9$ (+ive)
$\Rightarrow f(x)$ is increasing in $(-\infty,-2)$

Let $x=-1 \in (-2,0)$
put in ②
$f'(-1) = 3(-1)^2 + 6(-1)$
$= 3-6 = -3$
$\Rightarrow f(x)$ is decreasing in $(-2,0)$

Let $x=1 \in (0,\infty)$
put in ②
$f'(1) = 3(1)^2 + 6(1)$
$= 3+6 = 9$ (+ive)
$\Rightarrow f(x)$ is increasing in $(0,\infty)$

Hence $f(x)$ is increasing in $(-\infty,-2) \cup (0,\infty)$
& $f(x)$ is decreasing in $(-2,0)$.

Now put $x=0$ in ③

$f''(0) = 6(0)+6 = 6$ (+ive)

so $f(x)$ have a Relative minimum at $x=0$

put $x=0$ in ①

$y = f(x) = 0+0+1 = 1$

Critical Point: $(x,y) = (0,1)$ is Relative minimum

put $x=-2$ in ③

$f''(-2) = 6(-2)+6 = -12+6$
$= -6$ (-ive)

So $f(x)$ have a Relative maximum at $x=-2$

put $x=-2$ in ①

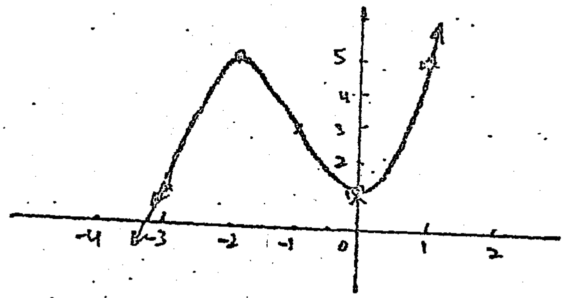
$y = (-2)^3 + 3(-2)^2 + 1$
$= -8 + 12 + 1 = 5$

Critical Point: $(-2,5)$ is Relative maximum

To draw the graph of $y = f(x) = x^3 + 3x^2 + 1$ construct the table

x	-3	-2	-1	0	1
y	1	5	3	1	5

14^0



b) Given function is

$$f(x) = x^3 + 35x^2 - 125x - 9375 \quad \text{——— (1)}$$

Diff

$$f'(x) = 3x^2 + 70x - 125.1 - 0 \quad \text{——— (2)}$$

Diff

$$f''(x) = 6x + 70 \quad \text{——— (3)}$$

For Critical values

$$f'(x) = 0$$

$$3x^2 + 70x - 125 = 0$$

$$3x^2 + 75x - 5x - 125 = 0$$

$$3x(x + 25) - 5(x + 25) = 0$$

$$(x+25)(3x-5) = 0$$

$-\infty$ $\quad$ -25 $\quad$ $5/3$ $\quad$ ∞

$x + 25 = 0$ or $3x - 5 = 0$

$\boxed{x = -25}$ $\quad$ $3x = 5$ or $\boxed{x = 5/3}$

which are the required critical values.

Now we three intervals $(-\infty, -25), (-25, 5/3)$ & $(5/3, \infty)$

Let $x = -26 \in (-\infty, -25)$
put in (2)
$f'(x) = 3(-26)^2 + 70(-26) - 125$
$= 83$ (+ive)
$f(x)$ is increasing in $(-\infty, -25)$

Let $x = 0 \in (-25, 5/3)$
put in (2)
$f'(0) = 3(0) + 70(0) - 125$
$= -125$ (-ive)
$\Rightarrow f(x)$ is decreasing in $(-25, 5/3)$

Let $x = 2 \in (\frac{5}{3}, \infty)$
put in (1)
$f'(2) = 3(2)^2 + 70(2) - 125$
$= 27$ (+ive)
$\Rightarrow f(x)$ is increasing in $(\frac{5}{3}, \infty)$

Hence $f(x)$ is increasing in $(-\infty, -25) \cup (\frac{5}{3}, \infty)$
& $f(x)$ is decreasing in $(-25, 5/3)$

put $x = -25$ in ③

$f''(-25) = 6(-25) + 70 = -150 + 70$
$= -80$ (-ive)

so $f(x)$ have a relative maximum at $x = -25$

put $x = -25$ in ①

$y = f(-25) = (-25)^3 + 35(-25)^2 - 125(-25) - 9375$
$= -15625 + 21875 + 3125 - 9375$
$y = 0$

Critical Point: $(x, y) = (-25, 0)$ is relative maximum

put $x = 5/3$ in ③

$f''(5/3) = 6(\frac{5}{3}) + 70 = 80$ (+ive)

so $f(x)$ have a relative minimum at $x = 5/3$

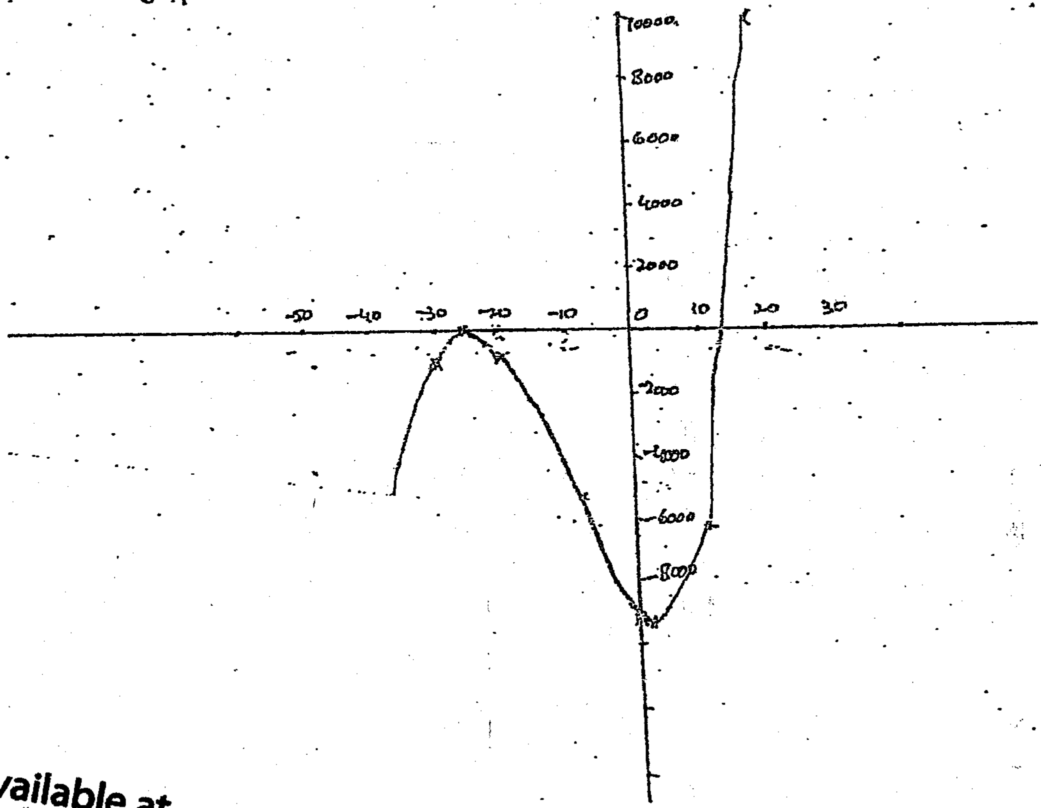
put $x = 5/3$ in ①

$y = f(5/3) = (5/3)^3 + 35(\frac{5}{3})^2 - 125(5/3) - 9375$
$= \frac{125}{27} + \frac{35 \times 25}{9} - \frac{625}{3} - 9375$
$y = \frac{-256000}{27} = -9481.48$

Critical Point: $(x, y) = (\frac{5}{3}, -9481.48)$ is relative minimum

To draw the graph of $y = x^3 + 35x^2 - 125x - 9375$, we construct the table.

x	-40	-30	-25	-20	-10	0	5/3	10	20
y	-12375	-1125	0	-875	-5625	-9375	-9481	-6125	10125



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3) Find the critical values of the following function

a) $f(x) = 2x^3 - 3x^2 - 72x + 15$

Diff w.r.t. x

$f'(x) = 6x^2 - 6x - 72$

For critical values

$f'(x) = 0$

$6x^2 - 6x - 72 = 0$

$\div$ ing by 6

$x^2 - x - 12 = 0$

$x^2 - 4x + 3x - 12 = 0$

$x(x-4) + 3(x-4) = 0$

$(x-4)(x+3) = 0$

$x - 4 = 0$ or $x + 3 = 0$

$\boxed{x = 4}$ or $\boxed{x = -3}$

which are the req: critical values

b) $f(x) = \frac{1}{3}x^3 - x^2 - 15x + 6$

Diff: w.r.t. x

$f'(x) = \frac{1}{\cancel{3}}(\cancel{3}x^2) - 2x - 15 \cdot 1 + 0$

$f'(x) = x^2 - 2x - 15$

For critical values

$f'(x) = 0$

$x^2 - 2x - 15 = 0$

$x^2 - 5x + 3x - 15 = 0$

$x(x-5) + 3(x-5) = 0$

$(x-5)(x+3) = 0$

$x - 5 = 0$ or $x + 3 = 0$

$\boxed{x = 5}$ $\boxed{x = -3}$

which are the req: critical values.

c) $f(x) = 6x^{2/3} - 4x$

Diff w.r.t. x

$f'(x) = \overset{2}{\cancel{6}}(\frac{2}{\cancel{3}}x^{2/3 - 1}) - 4 \cdot 1 = 4x^{-1/3} - 4$

$= 4\left[\frac{1}{x^{1/3}} - 1\right] = 4\left(\frac{1 - x^{1/3}}{x^{1/3}}\right)$

$f'(x) = 4\left[\frac{1 - x^{1/3}}{x^{1/3}}\right]$

$\because f'(x) = 0$ when $1 - x^{1/3} = 0$

$1 = x^{1/3}$

cubing B. sides

$1 = x$ or $\boxed{x = 1}$

$\because f'(x)$ undefined when $x^{1/3} = 0$

cubing B. sides

$\boxed{x = 0}$

Critical values are 0 & 1

d) $f(x) = 3x^{4/3} - 12x^{1/3}$

Diff

$f'(x) = \cancel{3}\left(\frac{4}{\cancel{3}}x^{\frac{4}{3}-1}\right) - \cancel{12}\left(\frac{\overset{4}{}}{\cancel{3}}x^{1/3 - 1}\right)$

$= 4 \cdot x^{1/3} - 4x^{-2/3} = 4\left[x^{1/3} - \frac{1}{x^{2/3}}\right]$

$= 4\left(\frac{x^{1/3 + 2/3} - 1}{x^{2/3}}\right) = 4\left(\frac{x^{3/3} - 1}{x^{2/3}}\right)$

$f'(x) = \frac{4(x-1)}{x^{2/3}}$

$\because f'(x) = 0$ when $x - 1 = 0$

$\boxed{x = 1}$

& $\because f'(x)$ undefined when $x^{2/3} = 0$

$\Rightarrow \boxed{x = 0}$

Critical values are 0 & 1.

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4) a) $f(x) = (x^3 - 3x + 1)^7$ —— ① at $x = 1$, $x = -1$

Diff

$f'(x) = 7(x^3-3x+1)^{7-1} \cdot \frac{d}{dx}(x^3-3x+1)$

$= 7(x^3-3x+1)^6 [3x^2-3] = 7(x^3-3x+1) \cdot 3(x^2-1)$

$f'(x) = 21(x^3-3x+1)^6(x^2-1)$ —— ②

At $x = 1$ (By using 1st derivative test)

let $x = .9$ put in ②

$f'(.5) = 21(.9^3 - 3(.9)+1)^6 (.9^2-1)$

$= (+ive)(.81 - 1)$

$= (+ive)(-ive) = -ive$

$x = 1.1$ put in ②

$f'(1.5) = 21(1.1^3 - 3(1.\ )+1)^6 (1.1^2-1)$

$= +ive\ (1.21 - 1)$

$= +ive$

$\because f'(x)$ changes sign from $-ive$ to $+ive$ so $f(x)$ has a relative minimum at $x = 1$

At $x = -1$

let $x = -1.1$ put in ②

$f'(-1.5) = 21[(-1.1)^3 - 3(-1.1)+1]^6 ((-1.1)^2-1)$

$= +ive\ (+1.21 - 1)$

$= +ive$

let $x = -.9$ put in ②

$f'(-.5) = 21[(-.9)^3 - 3(-.9)+1][(-.9)^2-1]$

$= +ive\ (.81 - 1)$

$= +ive\ (-ive) = -ive$

$\because f'(x)$ changes sign from $+ive$ to $-ive$

so $f(x)$ has a relative maximum at $x = -1$

---

b) $f(x) = (x^4 - 4x + 2)^5$ —— ① $(x = 1$

Diff

$f'(x) = 5(x^4-4x+2)^4 \frac{d}{dx}(x^4-4x+2)$

$= 5(x^4-4x+2)^4 (4x^3 - 4)$

$= 5(x^4-4x+2)^4 \cdot 4 \cdot (x^3 - 1)$

$f'(x) = 20(x^4 - 4x+2)^4 (x^3-1)$ —— ②

At $x=1$ (By 1st derivative test)

Let $x = .9$ put in ②

$F'(.5) = 20[(.9)^4 - 4(.9)+2]^4 [(.9)^3-1]$

$= +ive\ (.729-1)$

$= -ive$

Let $x = 1.1$ put in ②

$f'(1.5) = 20[(1.1)^4 - 4(1.1)+2]^4 [(1.1)^3-1]$

$= +ive\ (1.331-1)$

$= +ive$

$\because f'(x)$ changes sign from $-ive$ to $+ive$ so $f(x)$ has a relative minimum at $x=1$

c) $f(x) = (x^2-4)^4 (x^2-1)^3$ —— ① $x = 1, 2$

Diff

$f'(x) = (x^2-4)^4 \frac{d}{dx}(x^2-1)^3 + (x^2-1)^3 \frac{d}{dx}(x^2-4)^4$ (Product Rule)

$= (x^2-4)^4 . 3(x^2-1)^2 (2x-0) + (x^2-1)^3 . 4 . (x^2-4)^3 (2x-0)$

$= 6x(x^2-4)^4 (x^2-1)^2 + 8x(x^2-1)^3 (x^2-4)^3$

$= (x^2-4)^3 (x^2-1)^2 [6x(x^2-4) + 8x(x^2-1)]$

$= (x^2-4)^3 (x^2-1)^2 [6x^3 - 24x + 8x^3 - 8x]$

$f'(x) = (x^2-4)^3 (x^2-1)^2 [14x^3 - 32x]$ —— ②

At $x=1$ (By 1st Derivative test)

Let $x = .9$ put in ②

$f'(.5) = (.9^2-4)^3 (.9^2-1)^2 [14(.9)^3 - 32(.9)]$

$= (.81-4)^3 (.81-1)^2 [14 \times .729 - 32(.9)]$

$= (-ive)(+ive)[10.206 - 28.8]$

$= (-ive)(+ive)(-ive) = +ive$

$x = 1.1$ put in ②

$f'(1.5) = (1.1^2-4)^3 (1.1^2-1)^2 [14(1.1)^3 - 32(1.1)]$

$= (1.21-4)^3 (1.21-1)^2 [18.634 - 35.2]$

$= (-ive)(+ive)(-ive) = +ive$

$\because f'(x)$ does not change the sign so $f(x)$ has neither relative maximum nor relative minimum at $x=1$, But there is an inflection point at $x=1$

At $x=2$ (By 1st Derivative test)

let $x=1.9$ put in (2)

$$f'(1.5) = (1.9^2-4)^3(1.9^2-1)^2[14(1.9)^3-32(1.9)]$$
$$= (3.61-4)^3(3.61-1)^2[96.026-60.8]$$
$$= (-ive)(+ive)(+ive) = -ive$$

let $x=2.1$ put in (2)

$$f'(2.25) = (2.1^2-4)^3(2.1^2-1)^2[14(2.1)^3-32(2.1)]$$
$$= (4.41-4)^3(4.41-1)^2[129.654-67.2]$$
$$= +ive\ (+ive)(+ive) = +ive$$

$\because$ $f'(x)$ changes sign from $-ive$ to $+ive$ so $f(x)$ has Relative minimum at $x=2$

d) $f(x) = \sqrt[3]{x^3-48}$ —(1) $\boxed{x=4}$

Diff: w.r.t. x

$$f'(x) = \frac{d}{dx}(x^3-48)^{1/3}$$
$$= \frac{1}{\cancel{3}}(x^3-48)^{1/3-1}(\cancel{3}x^2-0)$$
$$= (x^3-48)^{-2/3}x^2$$
$$f'(x) = \frac{x^2}{(x^3-48)^{2/3}} \longrightarrow (2)$$

At $x=4$ (By 1st derivative test)

let $x=3.9$ put in (2)

$$f'(3.9) = \frac{(3.9)^2}{[(3.9)^3-48]^{2/3}}$$
$$= \frac{+ive}{(59.319-48)^{2/3}} = \frac{+ive}{+ive}$$
$$= +ive$$

let $x=4.1$ put in (2)

$$f'(4.1) = \frac{(4.1)^2}{(4.1^3-48)^{2/3}}$$
$$= \frac{(4.1)^2}{(68.921-48)^{2/3}} = \frac{+ive}{+ive}$$
$$= +ive$$

$\because$ $f'(x)$ does not change the sign so $f(x)$ has neither Relative maximum nor Relative minimum at $x=4$

5) Find all the relative extrema of the following (44)

a) $f(x) = x^3 - 3x^2 + 1$ —— ①

Diff

$f'(x) = 3x^2 - 6x$ —— ②

Diff

$f''(x) = 6x - 6$ —— ③

For critical values

$f'(x) = 0$

$3x^2 - 6x = 0$

$\div$ ing by 3

$x^2 - 2x = 0$

$x(x-2) = 0$

$\boxed{x=0}$ or $x-2=0$

$\Rightarrow \boxed{x=2}$

put $x=0$ in ③

$f''(0) = 6(0) - 6 = -6$ (-ive)

so $f(x)$ has a relative maximum at $x=0$.

put $x=0$ in ①

$y = f(0) = 0 - 0 + 1 = 1$

Hence $f(x)$ has a relative maximum of 1 at $x=0$

At $x=2$ put in ③

$f''(2) = 6(2) - 6 = 12 - 6 = 6$ (+ive)

$\therefore$ $f(x)$ has a relative minimum at $x=2$, put $x=2$ in ①

$y = f(2) = 2^3 - 3(2)^2 + 1$

$= 8 - 12 + 1 = -3$

Hence $f(x)$ has a relative maximum of -3 at $x=2$

b) $f(x) = x^3 + 6x^2 + 9x + 2$ —— ①

Diff

$f'(x) = 3x^2 + 12x + 9$ —— ②

Diff:

$f''(x) = 6x + 12$ —— ③

For critical values

$f'(x) = 0$

$3x^2 + 12x + 9 = 0$

$\div$ ing by 3

$x^2 + 4x + 3 = 0$

$x^2 + 3x + x + 3 = 0$

$x(x+3) + 1(x+3) = 0$

$(x+3)(x+1) = 0$

$x-3=0$ or $x+1=0$

$\boxed{x=3}$ $\boxed{x=-1}$

put in ③

$f''(-3) = 6(-3) - 12 = -18 - 12 = -6$ (-ive)

$\therefore$ $f(x)$ has a relative maximum at $x=3$, put $x=3$ in ①

$y = (-3)^3 + 6(-3)^2 + 9(-3) + 2$

$= \cancel{-27} + 54 - \cancel{27} + 2 = 2$

Hence $f(x)$ has a relative maximum of 2 at $x=-3$

Now At $x=-1$ put in ③

$f''(-1) = 6(-1) + 12 = 6$ (+ive)

$\therefore$ $f(x)$ has a relative minimum at $x=-1$, put $x=-1$ in ①

$f(-1) = (-1)^3 + 6(-1)^2 + 9(-1) + 2$

$= -1 + 6 - 9 + 2 = -2$

Hence $f(x)$ has a relative minimum of -2 at $x=-1$

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6). Given that

$$f'(x) = (x-1)^2(x-2)(x-4)(x+5)^4 \quad \text{——} (1)$$

For Critical values

$$f'(x) = 0$$

$$(x-1)^2(x-2)(x-4)(x+5)^4 = 0$$

$\Rightarrow$ $x-1=0$ or $x-2=0$ or $x-4=0$ or $x+5=0$

$\Rightarrow$ $\boxed{x=1}$ $\boxed{x=2}$ $\boxed{x=4}$ $\boxed{x=-5}$

which are the required critical values

At, $x=1$ (By 1st Derivative test)

let $x=.9$ put in (1)

$$f'(.9) = (.9-1)^2(.9-2)(.9-4)(.9+5)^4$$
$$= (+ive)(-ive)(-ive)(+ive) = +ive$$

let $x=1.1$ put in (1)

$$f'(1.1) = (1.1-1)^2(1.1-2)(1.1-4)(1.1+5)^4$$
$$= (+ive)(-ive)(-ive)(+ive)$$
$$= +ive$$

$\because$ $f'(x)$ does not changes sign

$\therefore$ $f(x)$ has neither relative maximum nor relative minimum.

At $x=2$

let $x=1.9$ put in (1)

$$f'(1.9) = (1.9-1)^2(1.9-2)(1.9-4)(1.9+5)^4$$
$$= (+ive)(-ive)(-ive)(+ive)$$
$$= +ive$$

let $x=2.1$ put in (1)

$$f'(2.1) = (2.1-1)^2(2.1-2)(2.1-4)(2.1+5)^4$$
$$= (+ive)(+ive)(-ive)(+ive)$$
$$= -ive.$$

$\because$ $f'(x)$ changes sign from +ive to −ive

$\therefore$ $f(x)$ has a relative maximum at $x=2$

At $x=4$

let $x=3.9$ put in (1)

$$f'(3.9) = (3.9-1)^2(3.9-2)(3.9-4)(3.9+5)^4$$
$$= (+ive)(+ive)(-ive)(+ive)$$
$$= -ive$$

let $x=4.1$ put in (1)

$$f'(4.1) = (4.1-1)^2(4.1-2)(4.1-4)(4.1+5)^4$$
$$= (+ive)(+ive)(+ive)(+ive)$$
$$= +ive$$

$\because f'(x)$ changes sign from -ive to +ive

$\therefore f(x)$ has a relative minimum at $x = 4$

At $x = -5$

Let $x = -5.1$ put in ①

$f'(-5.1) = (-5.1-1)^2(-5.1-2)(-5.1-4)(-5.1+5)^4$

$= +ive\ (-ive)(-ive)(+ive)$

$= +ive$

Let $x = -4.9$ put in ①

$f'(-4.9) = (-4.9-1)^2(-4.9-2)(-4.9-4)(-4.9+5)^4$

$= +ive\ (-ive)(-ive)(+ive)$

$= +ive$

$\because f'(x)$ does not change the sign

$\therefore f(x)$ has neither relative maximum nor relative minimum

b) $f'(x) = \dfrac{(2x-1)(x+3)}{(x-1)^2}$ ——— ①

$\because f'(x) = 0$ when $2x-1=0$ or $x+3=0$

$x = 1/2$ $\quad x = -3$

$\because f'(x)$ is undefined when $x-1=0 \Rightarrow x=1$

So critical values are $x = \frac{1}{2} = .5$, $x = -3$, $x = 1$

At $x = \frac{1}{2} = .5$ (By 1st derivative test)

Let $x = .4$ put in ①

$f'(.4) = \dfrac{[2(.4)-1][.4+3]}{(.4-1)^2}$

$= \dfrac{(.8-1)(.4+3)}{(-.6)^2} = \dfrac{(-ive)(+ive)}{+ive}$

$= -ive$

Let $x = .6$ put in ①

$f'(.6) = \dfrac{(2(.6)-1)(.6+3)}{(.6-1)^2}$

$= \dfrac{(1.2-1)(3.6)}{(-.4)^2} = \dfrac{+ive\ (+ive)}{+ive}$

$= +ive$

$\because f'(x)$ changes sign from -ive to +ive

$\therefore f(x)$ has a relative minimum at $x = \frac{1}{2}$

At $x=-3$

let $x=-3.1$ put in ①

$$f'(-3.1) = \frac{[2(-3.1)-1][-3.1+3]}{(-3.1-1)^2}$$

$$= \frac{(-6.2-1)(-.1)}{(-4.1)^2} = \frac{(-ive)(-ive)}{+ive}$$

$= +ive$

let $x=-2.9$ put in ①

$$f'(-2.9) = \frac{[2(-2.9)-1][-2.9+3]}{(-2.9-1)^2}$$

$$= \frac{(-5.8-1)(0.1)}{(-3.9)^2} = \frac{-ive(+ive)}{+ive}$$

$= -ive$

∵ $f'(x)$ changes sign from +ive to -ive

∴ $f(x)$ has a relative maximum at $x=-3$

At $x=1$

Let $x=.9$ put in ①

$$f'(.9) = \frac{[2(.9)-1][.9+3]}{(.9-1)^2}$$

$$= \frac{(1.8-1)(3.9)}{(-.1)^2} = \frac{(+ive)(+ive)}{+ive}$$

$= +ive$

let $x=1.1$ put in ①

$$f'(1.1) = \frac{[2(1.1)-1][1.1+3]}{(1.1-1)^2}$$

$$= \frac{(2.2-1)(1.1+3)}{(.1)^2} = \frac{+ive(+ive)}{+ive}$$

$= +ive$

∵ $f'(x)$ does not change the sign

∴ $f(x)$ has neither relative maximum nor relative minimum

7) Given equation is

$$P(x) = 30 + 108x - x^3 \quad \text{—— ①} \quad , \quad 0 \le x \le 10$$

Diff. w.r.t x

$$P'(x) = 0 + 108.1 - 3x^2$$

$$P'(x) = 108 - 3x^2 \quad \text{—— ②}$$

Diff

$$P''(x) = 0 - 6x$$

$$P''(x) = -6x \quad \text{—— ③}$$

a) To find expenditure (x) {critical values}

let $P'(x) = 0$

$108 - 3x^2 = 0$

$\Rightarrow \cancel{-}3x^2 = \cancel{-}108$

$x^2 = \frac{108}{3} = 36 \Rightarrow x = \pm 6$

$\boxed{x=6}$ put in ③ | $x = -6$ rejected

$P''(x) = -6(6) = -36$ (-ive)

So Profit is maximum at $x=6$

that is, expenditure on advertising that leads maximum Profit in thousands of dollars is $x = \$6000$

b) To find maximum profit put $x = 6$ in ①

$P(6) = 80 + 108(6) - 6^3$

$= 80 + 648 - 216 = 512$

Profit in hundred of dollars is $P(6) = \$51200$

8) $P(x) = -x^3 + 9x^2 + 120x - 400$ —— ①

Diff $P'(x) = -3x^2 + 18x + 120$ —— ②

Diff $P''(x) = -6x + 18$ —— ③

To find number of hundred thousand of dollars (critical values) that maximize Profit

let $P'(x) = 0$

$-3x^2 + 18x + 120 = 0$

$\div$ ing by -3

$x^2 - 6x - 40 = 0$

$x^2 - 10x + 4x - 40 = 0$

$x(x-10) + 4(x-10) = 0$

$(x-10)(x+4) = 0$

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$x-10=0$ or $x+4=0$

$\boxed{x=10}$ $\boxed{x=-4}$ rejected

put in (3)

$P''(10)=-6(10)+18=-42$ (-ive)

So Profit is maximum at $x=10$, that is,

Profit is maximum for tyres $x=10{,}00{,}000$.

To find Maximum Profit put $x=10$ in (1)

$$P(10) = -10^3+9(10)^2+120(10)-400$$

$$=-1000+900+1200-400=700$$

Maximum Profit in thousands of dollars = \$700,000

---

9) $$k(x) = \frac{4x}{3x^2+27} \quad \text{—— (1)}$$

Diff w.r.t. x

$$k'(x) = 4\frac{d}{dx}\left[\frac{x}{3x^2+27}\right]$$

$$=4\left\{\frac{(3x^2+27)\frac{d}{dx}x - x\frac{d}{dx}(3x^2+27)}{(3x^2+27)^2}\right\}$$

$$=4\left\{\frac{(3x^2+27)\cdot 1 - x(6x)}{(3x^2+27)^2}\right\} = 4\left\{\frac{3x^2+27-6x^2}{(3x^2+27)^2}\right\}$$

$$=4\left\{\frac{27-3x^2}{(3x^2+27)^2}\right\} = 4\cdot\frac{3(9-x^2)}{(3x^2+27)^2}$$

$$k'(x) = \frac{12(9-x^2)}{(3x^2+27)^2} \quad \text{—— (2)}$$

For Critical values

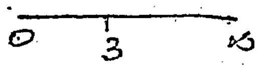
$$k'(x)=0$$

$$\frac{12(9-x^2)}{(3x^2+27)^2}=0$$

$12(9-x^2)=0$ $\quad$ ÷ing by 12

$9-x^2=0$

$x^2=9 \Rightarrow x=\pm 3$



$x=3$ $\quad$ or $\quad$ ($x=-3$ rejected)

Now we have two interval $(0,3)$ & $(3,\infty)$.

let $x=1 \in (0,3)$
put in (2)

$$K'(1)=\frac{12(9-1)}{(3(1)^2+27)^2}=\frac{12\times 8}{(3+27)^2}=+\text{ive}$$

so $k(x)$ is increasing in $(0,3)$

let $x=4 \in (3,\infty)$
put in (2)

$$K'(4)=\frac{12(9-4^2)}{(3(4)^2+27)^2}=\frac{12(9-16)}{(3\times16+27)^2}$$

$$=\frac{12(-7)}{(48+27)^2}=-\text{ive}$$

so $k(x)$ is decreasing in $(3,\infty)$

a) concentration of the drug increasing on the time interval $(0,3)$

b) concentration of the drug decreasing on the time interval $(3,\infty)$

c) Since $K'(x)$ is +ive before $x=3$ & -ive after $x=3$ so concentration is maximum at the time $x=3$

d) To find maximum concentration put $x=3$ in (1)

$$K(3)=\frac{4(3)}{(3(3)^2+27)}=\frac{12}{3\times9+27}=\frac{12}{27+27}=\frac{12}{54}=.22\%$$

or $=.0022$

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10) Total cost is the Product of the (number of gallons per hour)(the number of hours)(the cost per gallon) that develops the rule:

$$C(x) = \frac{1}{48}\left(\frac{300}{x}+2x\right)\underbrace{(32)(2.25)}$$

$$= \frac{1}{48}\left(\frac{300+2x^2}{x}\right)(72) = \frac{\cancel{2}}{\cancel{48}_{24}}\left(\frac{150+x^2}{x}\right)(\cancel{72}^{3})$$

$$\boxed{C(x) = \frac{450+3x^2}{x}} \longrightarrow ①$$

Diff w.r.t. x

$$C'(x) = \frac{d}{dx}\left[\frac{450+3x^2}{x}\right]$$

$$= \frac{x\frac{d}{dx}(450+3x^2) - (450+3x^2)\frac{d}{dx}x}{x^2} =$$

$$= \frac{x(0+6x) - (450+3x^2)\cdot 1}{x^2} = \frac{6x^2-450-3x^2}{x^2}$$

$$\boxed{C'(x) = \frac{3x^2-450}{x^2}} \longrightarrow ②$$

Diff w.r.t. x.

$$C''(x) = \frac{d}{dx}\left(\frac{3x^2-450}{x^2}\right)$$

$$= \frac{x^2\frac{d}{dx}(3x^2-450) - (3x^2-450)\frac{d}{dx}x^2}{(x^2)^2}$$

$$= \frac{x^2(6x-0) - (3x^2-450)\cdot 2x}{x^4} = \frac{\cancel{6x^3}-\cancel{6x^3}+900x}{x^4}$$

$$\boxed{C''(x) = \frac{900}{x^3}} \longrightarrow ③$$

To obtain critical values (or values of x)

let $C'(x) = 0$

$$\frac{3x^2-450}{x^2} = 0$$

$$3x^2-450 = 0 \times x^2 = 0$$

$\div$ing by 3

$$x^2-150=0 \Rightarrow x^2=150 \Rightarrow x = \pm\sqrt{150}$$

let $x = \sqrt{150}$ & $\boxed{x = -\sqrt{150} \text{ rejected.}}$

put in (3)

$$C''(x) = \frac{900}{(\sqrt{150})^3} = +ive$$

So $C(x)$ is minimum at $x = \sqrt{150} = 12.2$

To find minimum cost put $x = \sqrt{150}$ in (1)

$$C(\sqrt{150}) = \frac{450+3(\sqrt{150})^2}{\sqrt{150}} = \frac{450+3(150)}{\sqrt{150}} = \frac{450+450}{\sqrt{150}} = \frac{900}{\sqrt{150}}$$

$$C(\sqrt{150}) = \$73.5$$

END OF CH # 3

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