Chapter 10

Function of two variables: A function $z=f(x,y)$ is a function of two independent variables x & y if a unique dependent z is obtained from the each ordered pair (x,y) of real numbers

Continuity of a function

function $z=f(x,y)$ is continuous at $P_o(x_o,y_o)$ iff

1. $f(x_o,y_o)$ is defined
2. $\lim\limits_{(x,y)\to(x_o,y_o)} f(x,y)$ exists
3. The limiting value of a function equation to the value of a function at a point $P(x_o,y_o)$ i.e.

$$\lim_{(x,y)\to(x_o,y_o)} f(x,y)=f(x_o,y_o)$$

Partial derivatives

If $z=f(x,y)$ is a function of two independent variables then the first partial derivatives

$$\frac{\partial z}{\partial x}=f_x(x,y)=f_x=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$\frac{\partial z}{\partial y}=f_y(x,y)=f_y=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta x}$$

Exercise 10.1

Q1. Function is $f(x,y,z)=x^2ye^{2x}+(x+y-z)^2$

Find the function value at the following points:

a). $f(0,0,0)$

Sol: Given $f(x,y,z)=x^2ye^{2x}+(x+y-z)^2$

We have to find $f(0,0,0)$ i.e., $x=0,y=0,z=0$

$$f(0,0,0)=(0)^2(0)e^{2(0)}+(0+0-0)^2$$
$$f(0,0,0)=0+0$$
$$f(0,0,0)=0$$

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b). $f(1,-1,1)$

Sol: Given $f(x,y,z)=x^2ye^{2x}+(x+y-z)^2$

We have to find $f(1,-1,1)$ i.e., $x=1,y=-1,z=1$

$$f(1,-1,1)=(1)^2(-1)e^{2(1)}+(1+(-1)+1)^2$$
$$f(1,-1,1)=-1e^2+(1-1+1)^2$$
$$f(1,-1,1)=-e^2+1$$

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c). $f(-1,1,-1)$

Sol: Given $f(x,y,z)=x^2ye^{2x}+(x+y-z)^2$

We have to find $f(-1,1,-1)$ i.e.,

$x=-1,y=1,z=-1$

$$f(-1,1,-1)=(-1)^2(1)e^{2(-1)}+(-1+1-(-1))^2$$
$$f(-1,1,-1)=1e^{-2}+(-1+1+1)^2$$
$$f(-1,1,-1)=e^{-2}+1$$

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d). $\frac{\partial}{\partial x}f(x,x,x)$

Sol: Given $f(x,y,z)=x^2ye^{2x}+(x+y-z)^2$

Take partial derivative with respect to x

$$\frac{\partial}{\partial x}f(x,y,z)=\frac{\partial}{\partial x}(x^2ye^{2x})+\frac{\partial}{\partial x}(x+y-z)^2$$
$$\frac{\partial}{\partial x}f(x,y,z)=y\frac{\partial}{\partial x}(x^2e^{2x})+2(x+y+z)\frac{\partial}{\partial x}(x+y-z)$$
$$\frac{\partial}{\partial x}f(x,y,z)=y\left(e^{2x}\frac{\partial}{\partial x}x^2+x^2\frac{\partial}{\partial x}e^{2x}\right)+2(x+y-z).1$$
$$\frac{\partial}{\partial x}f(x,y,z)=y\left(2xe^{2x}+x^2e^{2x}(2)\right)+2(x+y-z)$$
$$\frac{\partial}{\partial x}f(x,y,z)=2y\left(xe^{2x}+x^2e^{2x}\right)+2(x+y-z)$$

Now we put $y=x,z=x$ to find $\frac{\partial}{\partial x}f(x,x,x)$

$$\frac{\partial}{\partial x}f(x,x,x)=2x\left(xe^{2x}+x^2e^{2x}\right)+2(x+x-x)$$
$$\frac{\partial}{\partial x}f(x,x,x)=2x^2e^{2x}(1+x)+2(x)$$
$$\frac{\partial}{\partial x}f(x,x,x)=2x^2e^{2x}(1+x)+2x$$

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e). $\frac{\partial}{\partial y}f(1,y,1)$

Sol: Given $f(x,y,z)=x^2ye^{2x}+(x+y-z)^2$

Take partial derivative with respect to y

$$\frac{\partial}{\partial y}f(x,y,z)=\frac{\partial}{\partial y}(x^2ye^{2x})+\frac{\partial}{\partial y}(x+y-z)^2$$
$$\frac{\partial}{\partial y}f(x,y,z)=x^2e^{2x}\frac{\partial}{\partial y}y+2(x+y-z)\frac{\partial}{\partial y}(x+y-z)$$
$$\frac{\partial}{\partial y}f(x,y,z)=x^2e^{2x}(1)+2(x+y-z).(0+1-0)$$
$$\frac{\partial}{\partial y}f(x,y,z)=x^2e^{2x}+2(x+y-z)$$

Now we put $x=1,z=1$ to find $\frac{\partial}{\partial y}f(1,y,1)$

$$\tfrac{\partial}{\partial y}f(1,y,1)=(1)^2e^{2(1)}+2(1+y-1)$$
$$\tfrac{\partial}{\partial y}f(1,y,1)=e^2+2y$$

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f). $\frac{\partial}{\partial z}f(1,1,z^2)$

Sol: Given $f(x,y,z)=x^2ye^{2x}+(x+y-z)^2$

Take partial derivative with respect to z

$$\frac{\partial}{\partial z}f(x,y,z)=\frac{\partial}{\partial z}(x^2ye^{2x})+\frac{\partial}{\partial z}(x+y-z)^2$$
$$\frac{\partial}{\partial z}f(x,y,z)=0+2(x+y+z)\frac{\partial}{\partial z}(x+y-z)$$
$$\frac{\partial}{\partial z}f(x,y,z)=2(x+y-z)(0+0-1)$$
$$\frac{\partial}{\partial x}f(x,y,z)=-2(x+y+z)$$

Now we put $x=1, y=1, z=z^2$ to find

$$\frac{\partial}{\partial x} f\left(1,1,z^2\right)=2\left(1+1-z^2\right)$$

$$\frac{\partial}{\partial x} f\left(1,1,z^2\right)=2\left(2-z^2\right)$$

Q2. Find domain & range of each of following functions:

a). $f(x,y)=\dfrac{1}{\sqrt{x-y}}$

Sol: Given $f(x,y)=\dfrac{1}{\sqrt{x-y}}$

For the radicand $x-y\geq 0$

For the denominator $x-y\neq 0$

Therefore domain $x-y>0$

Or $x>y$

For the Range when $x>y$

Then $f(x,y)=\dfrac{1}{\sqrt{x-y}}\geq 0$

b). $f(x,y)=\sqrt{\dfrac{y}{x}}$

Sol: Given $f(x,y)=\sqrt{\dfrac{y}{x}}$

For the radicand $y\geq 0, \quad x\geq 0$ or

$y\leq 0, x\leq 0$

For the denominator $x\neq 0$

Therefore domain $y\geq 0, \quad x>0$ or

$y\leq 0, x<0$

For the Range when $y\geq 0, \quad x>0$ or

$y\leq 0, x<0$

Then $f(x,y)=\sqrt{\dfrac{y}{x}}\geq 0$

c). $f(x,y)=e^{\frac{x+1}{y-2}}$

Sol: Given $f(x,y)=e^{\frac{x+1}{y-2}}$

For the denominator $y-2\neq 0$

Or $y\neq 2$

Therefore domain $y\neq 2$

Hence the domain

For the Range when $y\neq 2$

We know that exponential function is always positive

Then $f(x,y)=e^{\frac{x+1}{y-2}}\geq 0$

d). $f(x,y)=\dfrac{1}{\sqrt{9-x^2-y^2}}$

Sol: Given $f(x,y)=\dfrac{1}{\sqrt{9-x^2-y^2}}$

For the radicand $9-x^2-y^2\geq 0$

$9\geq x^2+y^2$

$or \quad x^2+y^2\leq 9$

For the denominator $9-x^2-y^2\neq 0$

Or $x^2+y^2\neq 9$

Therefore domain $x^2+y^2<9$

For the Range when $x^2+y^2<9$

Then $f(x,y)=\dfrac{1}{\sqrt{9-x^2-y^2}}\geq 0$

Q3. Find the partial derivative f_x and f_y of each of the following functions:

a). $f(x,y)=\sin\left(x^2\right)\cos y$

Sol: Given $f(x,y)=\sin\left(x^2\right)\cos y$

Take partial derivative with respect to x

$$f_x=\frac{\partial}{\partial x} f(x,y)=\frac{\partial}{\partial x}\left\{\sin\left(x^2\right)\cos y\right\}$$

$$f_x=\cos y\frac{\partial}{\partial x}\sin\left(x^2\right)$$

$$f_x=\cos y\cos\left(x^2\right)\frac{\partial}{\partial x}\left(x^2\right)$$

$$f_x=2x\cos y\cos\left(x^2\right)$$

Now Take partial derivative with respect to y

$$f_y=\frac{\partial}{\partial y} f(x,y)=\frac{\partial}{\partial y}\left\{\sin\left(x^2\right)\cos y\right\}$$

$$f_y=\sin\left(x^2\right)\frac{\partial}{\partial y}\cos y$$

$$f_y=\sin\left(x^2\right)\{-\sin y\}\frac{\partial}{\partial y}y$$

$$f_y=-\sin\left(x^2\right)\sin y$$

b). $f(x,y)=\sqrt{3x^2+y^4}$

Sol: Given $f(x,y)=\sqrt{3x^2+y^4}$

Take partial derivative with respect to x

$$f_x=\frac{\partial}{\partial x} f(x,y)=\frac{\partial}{\partial x}\left\{\sqrt{3x^2+y^4}\right\}$$

$$f_x=\frac{\partial}{\partial x}\left\{3x^2+y^4\right\}^{\frac{1}{2}}$$

$$f_x=\frac{1}{2}\left\{3x^2+y^4\right\}^{\frac{1}{2}-1}\frac{\partial}{\partial x}\left\{3x^2+y^4\right\}$$

$$f_x=\frac{1}{2}\left\{3x^2+y^4\right\}^{\frac{-1}{2}}\left\{3(2x)+0\right\}$$

$$f_x=\frac{6x}{2}\frac{1}{\left\{3x^2+y^4\right\}^{\frac{1}{2}}}$$

$$f_x=\frac{3x}{\sqrt{3x^2+y^4}}$$

Now Take partial derivative with respect to y

$$f_y=\frac{\partial}{\partial y} f(x,y)=\frac{\partial}{\partial y}\left\{\sqrt{3x^2+y^4}\right\}$$

$$f_y=\frac{\partial}{\partial y}\left\{3x^2+y^4\right\}^{\frac{1}{2}}$$

$$f_y=\frac{1}{2}\left\{3x^2+y^4\right\}^{\frac{1}{2}-1}\frac{\partial}{\partial y}\left\{3x^2+y^4\right\}$$

$$f_y=\frac{1}{2}\left\{3x^2+y^4\right\}^{\frac{-1}{2}}\left\{0+4y^3\right\}$$

$$f_y=\frac{4y^3}{2}\frac{1}{\left\{3x^2+y^4\right\}^{\frac{1}{2}}}$$

$$f_y=\frac{2y^3}{\sqrt{3x^2+y^4}}$$

c). $f(x,y) = xy^3 \tan^{-1} y$

Sol: Given $f(x,y) = xy^3 \tan^{-1} y$

Take partial derivative with respect to x

$$f_x = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left\{xy^3 \tan^{-1} y\right\}$$

$$f_x = y^3 \tan^{-1} y \frac{\partial}{\partial x} x$$

$$f_x = y^3 \tan^{-1} y.1$$

$$f_x = y^3 \tan^{-1} y$$

Now Take partial derivative with respect to y

$$f_y = \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left\{xy^3 \tan^{-1} y\right\}$$

$$f_y = x\frac{\partial}{\partial y}\left\{y^3 \tan^{-1} y\right\}$$

$$f_y = x\left\{y^3 \frac{\partial}{\partial y}\tan^{-1} y + \tan^{-1} y \frac{\partial}{\partial y} y^3\right\}$$

$$f_y = x\left\{y^3 \frac{1}{1+y^2} + 3y^2 \tan^{-1} y\right\}$$

d). $f(x,y) = x^3 + x^2y + xy^2 + y^3$

Sol: Given $f(x,y) = x^3 + x^2y + xy^2 + y^3$

Take partial derivative with respect to x

$$f_x = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left\{x^3 + x^2y + xy^2 + y^3\right\}$$

$$f_x = \frac{\partial}{\partial x}x^3 + y\frac{\partial}{\partial x}x^2 + y^2\frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y^3$$

$$f_x = 3x^2 + y(2x) + y^2 + 0$$

$$f_x = 3x^2 + 2xy + y^2$$

Now Take partial derivative with respect to y

$$f_y = \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left\{x^3 + x^2y + xy^2 + y^3\right\}$$

$$f_y = \frac{\partial}{\partial y}x^3 + x^2\frac{\partial}{\partial y}y + x\frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial y}y^3$$

$$f_y = 0 + x^2 + x(2y) + 3y^2$$

$$f_y = x^2 + 2xy + 3y^2$$

e). $f(x,y) = \sin^{-1} xy$

Sol: Given $f(x,y) = \sin^{-1} xy$

Take partial derivative with respect to x

$$f_x = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\sin^{-1} xy$$

$$f_x = \frac{1}{\sqrt{1-(xy)^2}}\frac{\partial}{\partial x}(xy)$$

$$f_x = \frac{1}{\sqrt{1-x^2y^2}}\left\{y\frac{\partial}{\partial x}x\right\}$$

$$f_x = \frac{1}{\sqrt{1-x^2y^2}}\{y.1\}$$

$$f_x = \frac{y}{\sqrt{1-x^2y^2}}$$

Now Take partial derivative with respect to y

$$f_y = \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\sin^{-1} xy$$

$$f_y = \frac{1}{\sqrt{1-(xy)^2}}\frac{\partial}{\partial y}(xy)$$

$$f_y = \frac{1}{\sqrt{1-x^2y^2}}\left\{x\frac{\partial}{\partial y}y\right\}$$

$$f_y = \frac{1}{\sqrt{1-x^2y^2}}\{x.1\}$$

$$f_y = \frac{x}{\sqrt{1-x^2y^2}}$$

f). $f(x,y) = x^2e^{x+y}\cos y$

Sol: Given $f(x,y) = x^2e^{x+y}\cos y$

Take partial derivative with respect to x

$$f_x = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left\{x^2e^{x+y}\cos y\right\}$$

$$f_x = \cos y\frac{\partial}{\partial x}\left\{x^2e^{x+y}\right\}$$

$$f_x = \cos y\left\{x^2\frac{\partial}{\partial x}e^{x+y} + e^{x+y}\frac{\partial}{\partial x}x^2\right\}$$

$$f_x = \cos y\left\{x^2e^{x+y}\frac{\partial}{\partial x}(x+y) + 2xe^{x+y}\right\}$$

$$f_x = \cos y\left\{x^2e^{x+y}(1+0) + 2xe^{x+y}\right\}$$

$$f_x = \cos y\left\{x^2e^{x+y} + 2xe^{x+y}\right\}$$

$$f_x = xe^{x+y}\cos y\{x+2\}$$

Now Take partial derivative with respect to y

$$f_y = \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left\{x^2e^{x+y}\cos y\right\}$$

$$f_y = x^2\frac{\partial}{\partial x}\left\{e^{x+y}\cos y\right\}$$

$$f_y = x^2\left\{\cos y\frac{\partial}{\partial y}e^{x+y} + e^{x+y}\frac{\partial}{\partial y}\cos y\right\}$$

$$f_y = x^2\left\{\cos y.e^{x+y}\frac{\partial}{\partial y}(x+y) + e^{x+y}[-\sin y]\right\}$$

$$f_y = x^2\left\{\cos y.e^{x+y}(0+1) - \sin ye^{x+y}\right\}$$

$$f_y = x^2e^{x+y}\{\cos y - \sin y\}$$

Q4. The production function z for the United states was once estimated as $z = f(x,y) = x^{0.7}y^{0.3}$ where x stands for the amount of labor and y stands for the amount of capital. Find the marginal productivity of the labor $\frac{\partial z}{\partial x}$ and of capital $\frac{\partial z}{\partial y}$

Hint: Marginal productivity is the rate at which production z changes (increases or decreases) for a unit change in labor x and capital y

Sol: Given $z = f(x,y) = x^{0.7}y^{0.3}$

Take partial derivative with respect to x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x,y) = y^{0.3}\frac{\partial}{\partial x}x^{0.7}$$

$$\frac{\partial z}{\partial x} = 0.7y^{0.3}x^{0.7-1}\frac{\partial}{\partial x}x$$

$$\frac{\partial z}{\partial x}=0.7y^{0.3}x^{-0.3}$$
$$\frac{\partial z}{\partial x}=0.7\frac{y^{0.3}}{x^{0.3}}$$

Now Taking partial derivative with respect to y

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}f(x,y)=x^{0.7}\frac{\partial}{\partial y}y^{0.3}$$
$$\frac{\partial z}{\partial y}=0.3x^{0.7}y^{0.3-1}\frac{\partial}{\partial y}y$$
$$\frac{\partial z}{\partial y}=0.3x^{0.7}y^{-0.7}$$
$$\frac{\partial z}{\partial y}=0.3\frac{x^{0.7}}{y^{0.7}}$$

Q5. A production function for Canada is: $z=x^{0.4}y^{0.6}$ where x stands for the amount of labor and y stands for the amount of capital. Find the marginal productivity of the labor $\frac{\partial z}{\partial x}$ and of capital $\frac{\partial z}{\partial y}$

Sol: Given $z=f(x,y)=x^{0.4}y^{0.6}$

Take partial derivative with respect to x

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}f(x,y)=y^{0.6}\frac{\partial}{\partial x}x^{0.4}$$
$$\frac{\partial z}{\partial x}=0.4x^{0.4-1}y^{0.6}\frac{\partial}{\partial x}x$$
$$\frac{\partial z}{\partial x}=0.4x^{-0.6}y^{0.6}$$
$$\frac{\partial z}{\partial x}=0.4\frac{y^{0.6}}{x^{0.6}}$$

Now Taking partial derivative with respect to y

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}f(x,y)=x^{0.4}\frac{\partial}{\partial y}y^{0.6}$$
$$\frac{\partial z}{\partial y}=0.6x^{0.4}y^{0.6-1}\frac{\partial}{\partial y}y$$
$$\frac{\partial z}{\partial y}=0.6x^{0.4}y^{-0.4}$$
$$\frac{\partial z}{\partial y}=0.6\frac{x^{0.4}}{y^{0.4}}$$

Homogenous function: $f(x,y)$ is said to be homogenous if $f(\lambda x,\lambda y)=\lambda^n f(x,y)$ where λ is a positive inetger

Euler's theorem: If $z=f(x,y)$ is continuously differentiable and defines homogenous function of degree n, then $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$

Exercise 10.2

Q1. Are the following functions homogeneous

a). $u=f(x,y)=\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

Sol: Given $u=f(x,y)=\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

To check the function put $x=tx, y=ty$ i.e.,

$$u=f(tx,ty)=\sin^{-1}\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}$$
$$u=f(tx,ty)=\sin^{-1}\left(\frac{t}{\sqrt{t}}\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$
$$u=f(tx,ty)=\sin^{-1}\left(\sqrt{t}\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

Thus $f(tx,ty)$ cannot be express able to $\lambda^n f(x,y)$

Hence given function is not homogenous function.

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b). $z=f(x,y)=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

Sol: Given $z=f(x,y)=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

To check the function put $x=tx, y=ty$ i.e.,

$$z=f(tx,ty)=\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}$$
$$z=f(tx,ty)=\frac{t}{\sqrt{t}}\frac{x+y}{\sqrt{x}+\sqrt{y}}$$
$$z=f(tx,ty)=\sqrt{t}\frac{x+y}{\sqrt{x}+\sqrt{y}}$$
$$z=f(tx,ty)=t^{\frac{1}{2}}f(x,y)$$

Hence given function is homogenous function of degree $\frac{1}{2}$

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c). $z=f(x,y)=x^3e^{\frac{y}{x}}-3y^2\sqrt{x^2+y^2}$

Sol: Given $z=f(x,y)=x^3e^{\frac{y}{x}}-3y^2\sqrt{x^2+y^2}$

To check the function put $x=tx, y=ty$ i.e.,

$$z=f(tx,ty)=(tx)^3e^{\frac{ty}{tx}}-3(ty)^2\sqrt{(tx)^2+(ty)^2}$$
$$z=f(tx,ty)=t^3x^3e^{\frac{y}{x}}-3t^2y^2\sqrt{t^2x^2+t^2y^2}$$
$$z=f(tx,ty)=t^3x^3e^{\frac{y}{x}}-3t^2y^2\sqrt{t^2}\sqrt{x^2+y^2}$$
$$z=f(tx,ty)=t^3x^3e^{\frac{y}{x}}-3t^2y^2t\sqrt{x^2+y^2}$$
$$z=f(tx,ty)=t^3x^3e^{\frac{y}{x}}-3t^3y^2\sqrt{x^2+y^2}$$
$$z=f(tx,ty)=t^3\left(x^3e^{\frac{y}{x}}-3y^2\sqrt{x^2+y^2}\right)$$
$$z=f(tx,ty)=t^3f(x,y)$$

Hence given function is homogenous function of degree 3

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d). $z=f(x,y)=(x^2+3y^2)^{\frac{1}{3}}$

Sol: Given $z=f(x,y)=(x^2+3y^2)^{\frac{1}{3}}$

To check the function put $x=tx, y=ty$ i.e.,

$$z=f(tx,ty)=\left((tx)^2+3(ty)^2\right)^{\frac{1}{3}}$$
$$z=f(tx,ty)=(t^2x^2+3t^2y^2)^{\frac{1}{3}}$$

$$z=f(tx,ty)=t^{\frac{2}{3}}\left(x^2+3y^2\right)^{\frac{1}{3}}$$

$$z=f(tx,ty)=t^{\frac{2}{3}}f(x,y)$$

Hence given function is homogenous function of degree $\frac{2}{3}$

Q2. Verify Euler's theorem for the following homogeneous functions:

a). $z=f(x,y)=ax^2+2hxy+by^2$

Sol: Given $z=f(x,y)=ax^2+2hxy+by^2$

Take partial derivative with respect to x

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial}{\partial x}\left(ax^2+2hxy+by^2\right)$$

$$\frac{\partial z}{\partial x}=a\frac{\partial}{\partial x}x^2+2hy\frac{\partial}{\partial x}x+b\frac{\partial}{\partial x}y^2$$

$$\frac{\partial z}{\partial x}=a(2x)+2hy(1)+b(0)$$

$$\frac{\partial z}{\partial x}=2ax+2hy$$

Now Taking partial derivative with respect to y

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}f(x,y)=\frac{\partial}{\partial y}\left(ax^2+2hxy+by^2\right)$$

$$\frac{\partial z}{\partial y}=a\frac{\partial}{\partial y}x^2+2hx\frac{\partial}{\partial y}y+b\frac{\partial}{\partial y}y^2$$

$$\frac{\partial z}{\partial y}=a(0)+2hx(1)+b(2y)$$

$$\frac{\partial z}{\partial y}=2hx+2by$$

Putting the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$

$$x(2ax+2hy)+y(2hx+2by)=nz$$

$$2ax^2+2hxy+2hxy+2by^2=nz$$

$$2ax^2+4hxy+2by^2=nz$$

$$2\left(ax^2+2hxy+by^2\right)=nz$$

$$2z=nz$$

$$\Rightarrow n=2$$

Hence given function is homogenous of degree 2

b). $z=f(x,y)=\left(x^2+xy+y^2\right)^{-1}$

Sol: Given $z=f(x,y)=\left(x^2+xy+y^2\right)^{-1}$

Take partial derivative with respect to x

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial}{\partial x}\left(x^2+xy+y^2\right)^{-1}$$

$$\frac{\partial z}{\partial x}=-1\left(x^2+xy+y^2\right)^{-2}\frac{\partial}{\partial x}\left(x^2+xy+y^2\right)$$

$$\frac{\partial z}{\partial x}=-\left(x^2+xy+y^2\right)^{-2}\left(\frac{\partial}{\partial x}x^2+y\frac{\partial}{\partial x}x+\frac{\partial}{\partial x}y^2\right)$$

$$\frac{\partial z}{\partial x}=-\left(x^2+xy+y^2\right)^{-2}(2x+y+0)$$

$$\frac{\partial z}{\partial x}=-\left(x^2+xy+y^2\right)^{-2}(2x+y)$$

Now Taking partial derivative with respect to y

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}f(x,y)=\frac{\partial}{\partial y}\left(x^2+xy+y^2\right)^{-1}$$

$$\frac{\partial z}{\partial y}=-1\left(x^2+xy+y^2\right)^{-2}\frac{\partial}{\partial y}\left(x^2+xy+y^2\right)$$

$$\frac{\partial z}{\partial y}=-\left(x^2+xy+y^2\right)^{-2}\left(\frac{\partial}{\partial y}x^2+x\frac{\partial}{\partial y}y+\frac{\partial}{\partial y}y^2\right)$$

$$\frac{\partial z}{\partial y}=-\left(x^2+xy+y^2\right)^{-2}(0+x.1+2y)$$

$$\frac{\partial z}{\partial x}=-\left(x^2+xy+y^2\right)^{-2}(x+2y)$$

Putting values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$

$$x\left(-\left(x^2+xy+y^2\right)^{-2}(2x+y)\right)$$
$$+y\left(-\left(x^2+xy+y^2\right)^{-2}(x+2y)\right)=nz$$

$$\left(x^2+xy+y^2\right)^{-2}\left\{-x(2x+y)-y(x+2y)\right\}=nz$$

$$\left(x^2+xy+y^2\right)^{-2}\left\{-2x^2-xy-xy-2y^2\right\}=nz$$

$$\left(x^2+xy+y^2\right)^{-2}\left\{-2x^2-2xy-2y^2\right\}=nz$$

$$-2\left(x^2+xy+y^2\right)^{-2}\left\{x^2+xy+y^2\right\}=nz$$

$$-2\left(x^2+xy+y^2\right)^{-2+1}=nz$$

$$-2\left(x^2+xy+y^2\right)^{-1}=nz$$

$$-2z=nz$$

$$\Rightarrow n=-2$$

Hence given function is homogenous of degree -2

c). $z=f(x,y)=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$

Sol: Given $z=f(x,y)=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$

Take partial derivative with respect to x

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial}{\partial x}\left(\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\sin^{-1}\frac{x}{y}+\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x}$$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial x}\frac{x}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\frac{y}{x}$$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\frac{1}{y}+\frac{y}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial x}x^{-1}$$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}}\frac{1}{y}-\frac{y}{\frac{x^2+y^2}{x^2}}x^{-2}$$

$$\frac{\partial z}{\partial x}=\frac{1}{\frac{\sqrt{y^2-x^2}}{y}}\frac{1}{y}-\frac{y}{\frac{x^2+y^2}{x^2}}\frac{1}{x^2}$$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}$$

Now Taking partial derivative with respect to y

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}f(x,y)=\frac{\partial}{\partial y}\left(\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\sin^{-1}\frac{x}{y}+\frac{\partial}{\partial y}\tan^{-1}\frac{y}{x}$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial y}\frac{x}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\frac{y}{x}$$

$$\frac{\partial z}{\partial y}=\frac{x}{\sqrt{1-\frac{x^2}{y^2}}}\frac{\partial}{\partial y}y^{-1}+\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}$$

$$\frac{\partial z}{\partial y}=\frac{x}{\sqrt{\frac{y^2-x^2}{y^2}}}\left(-y^{-2}\right)+\frac{1}{\frac{x^2+y^2}{x^2}}\frac{1}{x}$$

$$\frac{\partial z}{\partial y}=\frac{-x}{\frac{\sqrt{y^2-x^2}}{y}}\frac{1}{y^2}+\frac{1}{\frac{x^2+y^2}{x^2}}\frac{1}{x}$$

$$\frac{\partial z}{\partial y}=\frac{-x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}$$

Putting values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$

$$x\left(\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}\right)+y\left(\frac{-x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}\right)=nz$$

$$\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}-\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}=nz$$

$$0=nz$$

$$0.z=nz$$

$$\Rightarrow n=0$$

Hence given function is homogenous of degree 0

d). $z=f(x,y)=\dfrac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$

Sol: Given $z=f(x,y)=\dfrac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$

Take partial derivative with respect to x

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial}{\partial x}\left(\frac{x^{\frac14}+y^{\frac14}}{x^{\frac15}+y^{\frac15}}\right)$$

$$\frac{\partial z}{\partial x}=\frac{\left(x^{\frac15}+y^{\frac15}\right)\frac{\partial}{\partial x}\left(x^{\frac14}+y^{\frac14}\right)-\left(x^{\frac14}+y^{\frac14}\right)\frac{\partial}{\partial x}\left(x^{\frac15}+y^{\frac15}\right)}{\left(x^{\frac15}+y^{\frac15}\right)^2}$$

$$\frac{\partial z}{\partial x}=\frac{\left(x^{\frac15}+y^{\frac15}\right)\left(\frac14x^{\frac14-1}+0\right)-\left(x^{\frac14}+y^{\frac14}\right)\left(\frac15x^{\frac15-1}+0\right)}{\left(x^{\frac15}+y^{\frac15}\right)^2}$$

$$\frac{\partial z}{\partial x}=\frac{\frac14x^{\frac14-1}\left(x^{\frac15}+y^{\frac15}\right)-\frac15x^{\frac15-1}\left(x^{\frac14}+y^{\frac14}\right)}{\left(x^{\frac15}+y^{\frac15}\right)^2}$$

Now Taking partial derivative with respect to y

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}f(x,y)=\frac{\partial}{\partial y}\left(\frac{x^{\frac14}+y^{\frac14}}{x^{\frac15}+y^{\frac15}}\right)$$

$$\frac{\partial z}{\partial y}=\frac{\left(x^{\frac15}+y^{\frac15}\right)\frac{\partial}{\partial y}\left(x^{\frac14}+y^{\frac14}\right)-\left(x^{\frac14}+y^{\frac14}\right)\frac{\partial}{\partial y}\left(x^{\frac15}+y^{\frac15}\right)}{\left(x^{\frac15}+y^{\frac15}\right)^2}$$

$$\frac{\partial z}{\partial y}=\frac{\left(x^{\frac15}+y^{\frac15}\right)\left(0+\frac14y^{\frac14-1}\right)-\left(x^{\frac14}+y^{\frac14}\right)\left(0+\frac15y^{\frac15-1}\right)}{\left(x^{\frac15}+y^{\frac15}\right)^2}$$

$$\frac{\partial z}{\partial y}=\frac{\frac14y^{\frac14-1}\left(x^{\frac15}+y^{\frac15}\right)-\frac15y^{\frac15-1}\left(x^{\frac14}+y^{\frac14}\right)}{\left(x^{\frac15}+y^{\frac15}\right)^2}$$

Putting values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$

$$x\left(\frac{\frac14x^{\frac14-1}\left(x^{\frac15}+y^{\frac15}\right)-\frac15x^{\frac15-1}\left(x^{\frac14}+y^{\frac14}\right)}{\left(x^{\frac15}+y^{\frac15}\right)^2}\right)$$

$$+y\left(\frac{\frac14y^{\frac14-1}\left(x^{\frac15}+y^{\frac15}\right)-\frac15y^{\frac15-1}\left(x^{\frac14}+y^{\frac14}\right)}{\left(x^{\frac15}+y^{\frac15}\right)^2}\right)=nz$$

$$\frac{1}{\left(x^{\frac15}+y^{\frac15}\right)^2}\left\{\begin{array}{l}\frac14x^{\frac14-1+1}\left(x^{\frac15}+y^{\frac15}\right)-\frac15x^{\frac15-1+1}\left(x^{\frac14}+y^{\frac14}\right)\\+\frac14y^{\frac14-1+1}\left(x^{\frac15}+y^{\frac15}\right)-\frac15y^{\frac15-1+1}\left(x^{\frac14}+y^{\frac14}\right)\end{array}\right\}=nz$$

$$\frac{1}{\left(x^{\frac15}+y^{\frac15}\right)^2}\left\{\begin{array}{l}\frac14x^{\frac14}\left(x^{\frac15}+y^{\frac15}\right)-\frac15x^{\frac15}\left(x^{\frac14}+y^{\frac14}\right)\\+\frac14y^{\frac14}\left(x^{\frac15}+y^{\frac15}\right)-\frac15y^{\frac15}\left(x^{\frac14}+y^{\frac14}\right)\end{array}\right\}=nz$$

$$\frac{1}{\left(x^{\frac15}+y^{\frac15}\right)^2}\left\{\begin{array}{l}\left(\frac14x^{\frac14}+\frac14y^{\frac14}\right)\left(x^{\frac15}+y^{\frac15}\right)\\-\left(\frac15x^{\frac15}+\frac15y^{\frac15}\right)\left(x^{\frac14}+y^{\frac14}\right)\end{array}\right\}=nz$$

$$\frac{1}{\left(x^{\frac15}+y^{\frac15}\right)^2}\left\{\begin{array}{l}\frac14\left(x^{\frac14}+y^{\frac14}\right)\left(x^{\frac15}+y^{\frac15}\right)\\-\frac15\left(x^{\frac15}+y^{\frac15}\right)\left(x^{\frac14}+y^{\frac14}\right)\end{array}\right\}=nz$$

$$\frac{\left(x^{\frac14}+y^{\frac14}\right)\left(x^{\frac15}+y^{\frac15}\right)}{\left(x^{\frac15}+y^{\frac15}\right)^2}\left\{\frac14-\frac15\right\}=nz$$

$$\frac{\left(x^{\frac14}+y^{\frac14}\right)}{\left(x^{\frac15}+y^{\frac15}\right)}\left\{\frac{5-4}{20}\right\}=nz$$

$$z\left\{\frac{1}{20}\right\}=nz$$

$$\Rightarrow n=\frac{1}{20}$$

Q3. $u=f\left(\frac{y}{x}\right)$ then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Sol: Given $u=f\left(\frac{y}{x}\right)$

Take partial derivative with respect to x

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x}=f'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x}=f'\left(\frac{y}{x}\right)y\frac{\partial}{\partial x}\left(x^{-1}\right)$$

$$\frac{\partial u}{\partial x}=f'\left(\frac{y}{x}\right)y\left(-x^{-2}\right)$$

$$\frac{\partial u}{\partial x}=-\frac{y}{x^2}f'\left(\frac{y}{x}\right)$$

Now Taking partial derivative with respect to y

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y}=f'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y}=f'\left(\frac{y}{x}\right)\frac{1}{x}\frac{\partial}{\partial y}(y)$$

$$\frac{\partial u}{\partial y}=\frac{1}{x}f'\left(\frac{y}{x}\right)$$

Putting values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nz$$

$$x\left(-\frac{y}{x^2}f'\left(\frac{y}{x}\right)\right)+y\left(\frac{1}{x}f'\left(\frac{y}{x}\right)\right)=nz$$

$$-\frac{y}{x}f'\left(\frac{y}{x}\right)+\frac{y}{x}f'\left(\frac{y}{x}\right)=nz$$

$$0=nz$$

$$0.z=nz$$

$$\Rightarrow n=0$$

Hence given function is homogenous of degree 0

Q4 $z=xy\ f\left(\frac{x}{y}\right)$ then show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$

Sol: Given $z=xy\ f\left(\frac{x}{y}\right)$

Take partial derivative with respect to x

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left\{xyf\left(\frac{y}{x}\right)\right\}$$

$$\frac{\partial z}{\partial x}=\left\{f\left(\frac{y}{x}\right)\frac{\partial}{\partial x}(xy)+xy\frac{\partial}{\partial x}f\left(\frac{y}{x}\right)\right\}$$

$$\frac{\partial z}{\partial x}=\left\{f\left(\frac{y}{x}\right)\left\{y\frac{\partial}{\partial x}x\right\}+xyf'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right)\right\}$$

$$\frac{\partial z}{\partial x}=\left\{f\left(\frac{y}{x}\right)\{y.1\}+xy.\left\{f'\left(\frac{y}{x}\right)y\frac{\partial}{\partial x}\left(x^{-1}\right)\right\}\right\}$$

$$\frac{\partial z}{\partial x}=\left\{y.f\left(\frac{y}{x}\right)+xy.f'\left(\frac{y}{x}\right)y\left(-x^{-2}\right)\right\}$$

$$\frac{\partial z}{\partial x}=\left\{y.f\left(\frac{y}{x}\right)-\frac{y^2}{x}f'\left(\frac{y}{x}\right)\right\}$$

Now Taking partial derivative with respect to y

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left\{xyf\left(\frac{y}{x}\right)\right\}$$

$$\frac{\partial z}{\partial y}=\left\{f\left(\frac{y}{x}\right)\frac{\partial}{\partial y}(xy)+xy\frac{\partial}{\partial y}f\left(\frac{y}{x}\right)\right\}$$

$$\frac{\partial z}{\partial y}=\left\{f\left(\frac{y}{x}\right)\left\{x\frac{\partial}{\partial y}y\right\}+xy.f'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right)\right\}$$

$$\frac{\partial z}{\partial y}=\left\{f\left(\frac{y}{x}\right)\{x.1\}+xy.f'\left(\frac{y}{x}\right)\frac{1}{x}\frac{\partial}{\partial y}y\right\}$$

$$\frac{\partial z}{\partial y}=\left\{x.f\left(\frac{y}{x}\right)+y.f'\left(\frac{y}{x}\right)\right\}$$

Putting values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$

$$x\left(y.f\left(\frac{y}{x}\right)-\frac{y^2}{x}f'\left(\frac{y}{x}\right)\right)+y\left(x.f\left(\frac{y}{x}\right)+y.f'\left(\frac{y}{x}\right)\right)=nz$$

$$xy.f\left(\frac{y}{x}\right)-y^2f'\left(\frac{y}{x}\right)+xyf\left(\frac{y}{x}\right)+y^2f'\left(\frac{y}{x}\right)=nz$$

$$2xy.f\left(\frac{y}{x}\right)=nz$$

$$2.z=nz$$

$$\Rightarrow n=2$$

Hence given function is homogenous of degree 2

Q5a). If $u=\tan^{-1}\frac{x^2+y^2}{x+y}$ then show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin u\cos u$$

Sol: Given $u=\tan^{-1}\frac{x^2+y^2}{x+y}$

$$\tan u=\frac{x^2+y^2}{x+y}$$

Take partial derivative with respect to x

$$\frac{\partial}{\partial x}\tan u=\frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right)$$

$$\sec^2 u\frac{\partial u}{\partial x}=\frac{(x+y)\frac{\partial}{\partial x}\left(x^2+y^2\right)-\left(x^2+y^2\right)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$\frac{\partial u}{\partial x}=\frac{(x+y)(2x+0)-\left(x^2+y^2\right)(1+0)}{\sec^2 u(x+y)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2x(x+y)-\left(x^2+y^2\right)}{\sec^2 u(x+y)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2x^2+2xy-x^2-y^2}{\sec^2 u(x+y)^2}$$

$$\frac{\partial u}{\partial x}=\frac{x^2+2xy-y^2}{\sec^2 u(x+y)^2}$$

Again $\tan u=\frac{x^2+y^2}{x+y}$

Now Taking partial derivative with respect to y

$$\frac{\partial}{\partial y}\tan u = \frac{\partial}{\partial y}\frac{x^2+y^2}{x+y}$$

$$\sec^2 u\frac{\partial u}{\partial y} = \frac{(x+y)\frac{\partial}{\partial y}(x^2+y^2)-(x^2+y^2)\frac{\partial}{\partial y}(x+y)}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x+y)(0+2y)-(x^2+y^2)(0+1)}{\sec^2 u(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y(x+y)-(x^2+y^2)}{\sec^2 u(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2xy+2y^2-x^2-y^2}{\sec^2 u(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{y^2+2xy-x^2}{\sec^2 u(x+y)^2}$$

Putting values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$

$$x\left(\frac{x^2+2xy-y^2}{\sec^2 u(x+y)^2}\right)+y\left(\frac{y^2+2xy-x^2}{\sec^2 u(x+y)^2}\right)=\sin u\cos u$$

$$\frac{x^3+2x^2y-xy^2}{\sec^2 u(x+y)^2}+\frac{y^3+2xy^2-x^2y}{\sec^2 u(x+y)^2}=\sin u\cos u$$

$$\frac{x^3+2x^2y-xy^2+y^3+2xy^2-x^2y}{\sec^2 u(x+y)^2}=\sin u\cos u$$

$$\frac{x^3+2x^2y-x^2y-xy^2+2xy^2+y^3}{\sec^2 u(x+y)^2}=\sin u\cos u$$

$$\frac{x^3+x^2y+xy^2+y^3}{\sec^2 u(x+y)^2}=\sin u\cos u$$

$$\frac{x^2(x+y)+y^2(x+y)}{\sec^2 u(x+y)^2}=\sin u\cos u$$

$$\frac{(x+y)(x^2+y^2)}{\sec^2 u(x+y)^2}=\sin u\cos u$$

$$\cos^2 u\frac{(x^2+y^2)}{(x+y)}=\sin u\cos u$$

$$\cos^2 u\tan u=\sin u\cos u$$

Hence given function is satisfied the relation

b) If $u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ then show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Sol: Given $u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Or $\sin u=\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Take partial derivative with respect to x

$$\frac{\partial}{\partial x}\sin u=\frac{\partial}{\partial x}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$$

$$\cos u\frac{\partial u}{\partial x}=\frac{(\sqrt{x}+\sqrt{y})\frac{\partial}{\partial x}(x^{\frac12}-y^{\frac12})-(\sqrt{x}-\sqrt{y})\frac{\partial}{\partial x}(x^{\frac12}+y^{\frac12})}{(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial u}{\partial x}=\frac{\frac12 x^{\frac12-1}(\sqrt{x}+\sqrt{y})-\frac12 x^{\frac12-1}(\sqrt{x}-\sqrt{y})}{\cos u(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial u}{\partial x}=\frac12 x^{\frac12-1}\frac{(\sqrt{x}+\sqrt{y})-(\sqrt{x}-\sqrt{y})}{\cos u(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial u}{\partial x}=\frac12 x^{\frac12-1}\frac{\sqrt{x}+\sqrt{y}-\sqrt{x}+\sqrt{y}}{\cos u(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial u}{\partial x}=\frac12 x^{\frac12-1}\frac{2\sqrt{y}}{\cos u(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial u}{\partial x}=\frac{x^{\frac12-1}\sqrt{y}}{\cos u(\sqrt{x}+\sqrt{y})^2}$$

Now Taking partial derivative with respect to y

$$\frac{\partial}{\partial y}\sin u=\frac{\partial}{\partial y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$$

$$\cos u\frac{\partial u}{\partial y}=\frac{(\sqrt{x}+\sqrt{y})\frac{\partial}{\partial y}(x^{\frac12}-y^{\frac12})-(\sqrt{x}-\sqrt{y})\frac{\partial}{\partial y}(x^{\frac12}+y^{\frac12})}{(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial u}{\partial y}=\frac{-\frac12 y^{\frac12-1}(\sqrt{x}+\sqrt{y})-\frac12 y^{\frac12-1}(\sqrt{x}-\sqrt{y})}{\cos u(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial u}{\partial y}=\frac12 y^{\frac12-1}\frac{-\sqrt{x}-\sqrt{y}-\sqrt{x}+\sqrt{y}}{\cos u(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial u}{\partial y}=\frac12 y^{\frac12-1}\frac{-2\sqrt{x}}{\cos u(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial u}{\partial y}=\frac{-y^{\frac12-1}\sqrt{x}}{\cos u(\sqrt{x}+\sqrt{y})^2}$$

Putting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$$

$$x\left(\frac{x^{\frac12-1}\sqrt{y}}{\cos u(\sqrt{x}+\sqrt{y})^2}\right)+y\left(\frac{-y^{\frac12-1}\sqrt{x}}{\cos u(\sqrt{x}+\sqrt{y})^2}\right)=0$$

$$\frac{x^{\frac12}\sqrt{y}}{\cos u(\sqrt{x}+\sqrt{y})^2}-\frac{y^{\frac12}\sqrt{x}}{\cos u(\sqrt{x}+\sqrt{y})^2}=0$$

$$\frac{\sqrt{x}\sqrt{y}-\sqrt{x}\sqrt{y}}{\cos u(\sqrt{x}+\sqrt{y})^2}=0$$

$$0=0$$

Hence given function is satisfied the relation