Chapter 9

Differential equation: An equation that involves the derivatives of dependent variables of one or more independent variables

Ordinary differential equation: An equation that involves the derivatives of dependent variables of one independent variable

Partial differential equation: An equation that involves the derivatives of dependent variables of more than one independent variables

Order of differential equation:
Order of highest order derivative

Degree of a differential equation
Power of the highest order derivative

Linear differential equation: D.E is said to be linear if the following condition holds
1. Dependent variables and its derivatives occur to the first power only
2. There is no products involving the dependent variables or its derivatives
3. There should be linear functions of dependent variables, such as trigonometric (sine) exponential etc

Exercise 9.1

Q1. Find the order, degree linear and nonlinear of each of following ordinary differential equations:

a). $\frac{dy}{dx}=x^2+y$

Sol: Given $\frac{dy}{dx}=x^2+y$

Order = 1 Degree = 1 Linear

b). $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+11y=3x$

Sol: Given $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+11y=3x$

Order = 2 Degree = 1 Linear

c). $\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^3-y=0$

Sol: Given $\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^3-y=0$

Order = 3 Degree = 1 Non-Linear

Q2. In each case, show that the indicated function is a solution of the differential equation

a). $y=e^x+e^{2x}$, $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$

Sol: Given $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0.............(1)$

$y=e^x+e^{2x}$

Differentiating both sides with respect to x

$\frac{dy}{dx}=\frac{d}{dx}e^x+\frac{d}{dx}e^{2x}$

$\frac{dy}{dx}=e^x\frac{d}{dx}(x)+e^{2x}\frac{d}{dx}(2x)$

$\frac{dy}{dx}=e^x+2e^{2x}$

Again differentiating

$\frac{d^2y}{dx^2}=\frac{d}{dx}e^x+2\frac{d}{dx}e^{2x}$

$\frac{d^2y}{dx^2}=e^x\frac{d}{dx}(x)+2e^{2x}\frac{d}{dx}(2x)$

$\frac{d^2y}{dx^2}=e^x+4e^{2x}$

Putting value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1)

$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^x+4e^{2x}-3\left(e^x+2e^{2x}\right)+2\left(e^x+e^{2x}\right)$

$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^x+4e^{2x}-3e^x-6e^{2x}+2e^x+2e^{2x}$

$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^x-3e^x+2e^x+4e^{2x}-6e^{2x}+2e^{2x}$

$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$

Hence $y=e^x+e^{2x}$ is a solution of differential equation

b). $y=x-x\ln x$, $x\frac{dy}{dx}+x-y=0$

Sol: Given $x\frac{dy}{dx}+x-y=0$

$y=x-x\ln x$

Differentiating both sides with respect to x

$\frac{dy}{dx}=\frac{d}{dx}x-\frac{d}{dx}(x\ln x)$

$\frac{dy}{dx}=1-\left(x\frac{d}{dx}\ln x+\ln x\frac{d}{dx}x\right)$

$\frac{dy}{dx}=1-\left(x\frac{1}{x}+\ln x\right)$

$\frac{dy}{dx}=1-(1+\ln x)$

$\frac{dy}{dx}=1-1-\ln x$

$\frac{dy}{dx}=-\ln x$

Putting the value of y and $\frac{dy}{dx}$ in equation (1)

$x\frac{dy}{dx}+x-y=x(-\ln x)+x-(x-x\ln x)$

$x\frac{dy}{dx}+x-y=-x\ln x+x-x+x\ln x$

$x\frac{dy}{dx}+x-y=0$

Hence $y=x-x\ln x$ is a solution of differential eq

c). $y=(x+c)e^{-x}$, $\frac{dy}{dx}+y=e^{-x}$

Sol: Given $\frac{dy}{dx}+y=e^{-x}$

$y=(x+c)e^{-x}$

Differentiating both sides with respect to x

$y=(x+c)e^{-x}$

$\frac{dy}{dx}=e^{-x}\frac{d}{dx}(x+c)+(x+c)\frac{d}{dx}e^{-x}$

$\frac{dy}{dx}=e^{-x}\left(\frac{d}{dx}x+\frac{d}{dx}c\right)+(x+c)e^{-x}\frac{d}{dx}(-x)$

$\frac{dy}{dx}=e^{-x}(1+0)-(x+c)e^{-x}$

$\frac{dy}{dx}=e^{-x}-(x+c)e^{-x}$

Putting the value of y and $\frac{dy}{dx}$ in equation (1)

$\frac{dy}{dx}+y=e^{-x}-(x+c)e^{-x}+(x+c)e^{-x}$

$\frac{dy}{dx}+y=e^{-x}$

Hence $y=(x+c)e^{-x}$ is a solution of differential eq

Q3. For each of following equations, determine whether or not it becomes linear when divided by dx or dy

a). $(x+y)dy=(x-y)dx$

Sol: Given $(x+y)dy=(x-y)dx$

Dividing it by dx on both sides

$(x+y)\frac{dy}{dx}=(x-y)\frac{dx}{dx}$

$(x+y)\frac{dy}{dx}=x-y$

$x\frac{dy}{dx}+y\frac{dy}{dx}=x-y$

It is non-linear because coefficient of $\frac{dy}{dx}$ is y

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b). $a\ dy+b\,y\sin x\ dx=0$

Sol: Given $a\ dy+b\,y\sin x\ dx=0$

Dividing it by dx on both sides

$a\frac{dy}{dx}+b\,y\sin x\frac{dx}{dx}=0$

$a\frac{dy}{dx}+b\,y\sin x=0$

It is Linear. But after dividing by dy it is non linear

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c). $3y\ dx+2x\ dy=0$

Sol: Given $3y\ dx+2x\ dy=0$

Dividing it by dx on both sides

$3y\frac{dx}{dx}+2x\frac{dy}{dx}=0$

$3y+2x\frac{dy}{dx}=0$

It is linear

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d). $e^{x}\ dy+xy^{\frac{1}{3}}dx=0$

Sol: Given $e^{x}\ dy+xy^{\frac{1}{3}}dx=0$

Dividing it by dx on both sides

$e^{x}\frac{dy}{dx}+xy^{\frac{1}{3}}\frac{dx}{dx}=0$

$e^{x}\frac{dy}{dx}+xy^{\frac{1}{3}}=0$

It is non linear

Q4. In each case, use the initial condition and the general solution of the differential equation to determine a particular solution:

a). $xy=c$ $\quad y(2)=1$

Sol: Given $xy=c.....(1)$ $\quad y(2)=1$

Here $x=2,$ $\quad y=1$ putting in equation (1)

$(2)(1)=c$

$c=2$

Thus equation (1) becomes

$xy=2$

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b). $y=x-x\ln x+c$ $\quad y(1)=2$

Sol: Given $y=x-x\ln x+c$........................(1)

With $y(1)=2$ $\quad \Rightarrow x=1, y=2$ put in (1)

$2=1-(1)\ln(1)+c$

$2=1-1(0)+c$

$2=1+c$

$2-1=c$

$c=1$

Thus equation (1) becomes

$y=x-x\ln x+1$

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c) $\sin(xy)+y=c,$ $\quad y\left(\frac{\pi}{4}\right)=1$

Sol: Given $\sin(xy)+y=c$........................(1)

With $y\left(\frac{\pi}{4}\right)=1$ $\quad \Rightarrow x=\frac{\pi}{4}, y=1$ put in (1)

$\sin\left(\frac{\pi}{4}.1\right)+1=c$

$\frac{\sqrt{2}}{2}+1=c$

Thus equation (1) becomes

$\sin(xy)+y=\frac{\sqrt{2}}{2}+1$

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d). $\frac{y^2}{x}=\frac{x^2}{2}+c,$ $\quad y(1)=1$

Sol: Given $\frac{y^2}{x}=\frac{x^2}{2}+c$..........................(1)

With $y(1)=1$ $\quad \Rightarrow x=1, y=1$ put in (1)

$\frac{1^2}{1}=\frac{1^2}{2}+c$

$1=\frac{1}{2}+c$

$c=1-\frac{1}{2}$

$c=\frac{1}{2}$

Thus equation (1) becomes

$\frac{y^2}{x}=\frac{x^2}{2}+\frac{1}{2}$

Q5. Solve the following initial value problems:

a). $\frac{dy}{dx}=\cos x, \quad y(0)=1$

Sol: Given $\frac{dy}{dx}=\cos x$

With $y(0)=1 \quad x=0, y=1$

We have $\frac{dy}{dx}=\cos x$

Separation of variables

$dy=\cos x\ dx$

Integrating both sides

$\int dy=\int \cos x\ dx$

$y=\sin x+c$.....................................(1)

Putting the value of x and y

$1=\sin(0)+c$

$1=0+c$

$1=c$

Thus equation (1) becomes

$y=\sin x+1$

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b). $\frac{dy}{dx}=x^2, \quad y(0)=1$

Sol: Given $\frac{dy}{dx}=x^2$

With $y(0)=1 \quad x=0, y=1$

We have $\frac{dy}{dx}=x^2$

Separation of variables

$dy=x^2\ dx$

Integrating both sides

$\int dy=\int x^2\ dx$

$y=\frac{x^3}{3}+c$.....................................(1)

Putting the value of x and y

$1=\frac{0^3}{3}+c$

$1=0+c$

$1=c$

Thus equation (1) becomes

$y=\frac{x^3}{3}+1$

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c). $\frac{dy}{dx}=2xy^2, \quad y(3)=-1$

Sol: Given $\frac{dy}{dx}=2xy^2$

$\frac{dy}{dx}=2xy^2, \quad y(3)=-1$

With $y(3)=-1 \quad x=3, y=-1$

We have $\frac{dy}{dx}=2xy^2$

Separation of variables

$\frac{dy}{y^2}=2x\ dx$

$y^{-2}dy=2x\ dx$

Integrating both sides

$\int y^{-2}dy=2\int x\ dx$

$\frac{y^{-1}}{-1}=2\left(\frac{x^2}{2}\right)+c$

$-\frac{1}{y}=x^2+c$.....................................(1)

Putting the value of x and y

$-\frac{1}{-1}=3^2+c$

$1=9+c$

$1-9=c$

$c=-8$

Thus equation (1) becomes

$-\frac{1}{y}=x^2-8$

$\frac{1}{y}=8-x^2$

Taking reciprocal

$y=\frac{1}{8-x^2}$

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d). $\frac{dy}{dx}+y=y^2, \quad y(0)=\frac{1}{2}$

Sol: Given $\frac{dy}{dx}+y=y^2$

$\frac{dy}{dx}+y=y^2, \quad y(0)=\frac{1}{2}$

With $y(0)=\frac{1}{2} \quad x=0, y=\frac{1}{2}$

We have $\frac{dy}{dx}+y=y^2$

Separation of variables

$\frac{dy}{dx}=y^2-y$

$\frac{dy}{y^2-y}=dx$

$\frac{dy}{y(y-1)}=dx$

Integrating both sides

$\int\frac{dy}{y(y-1)}=\int dx$

Take $\frac{1}{y(y-1)}=\frac{A}{y}+\frac{B}{y-1}$...................(1)

Multiply each fraction by $y(y-1)$ we get

$y(y-1)\frac{1}{y(y-1)}=y(y-1)\frac{A}{y}+y(y-1)\frac{B}{y-1}$

$1=A(y-1)+By$........................(2)

Put $y=0$ in equation (2) we get

$1=A(0-1)+B(0)$

$1=-A$

$A=-1$

Put $y=1$ in equation (2) we get

$1=A(1-1)+B(1)$

$1=A(0)+B \qquad \Rightarrow B=1$

Putting the value of A and B in equation (1) we get

$\frac{1}{y(y-1)}=\frac{-1}{y}+\frac{1}{y-1}$

Thus integral becomes

$\int\frac{dy}{y(y-1)}=\int dx$

$-\int\frac{dy}{y}+\int\frac{dy}{y-1}=\int dx$

$-\ln|y|+\ln|y-1|=x+c$

$\ln\left|\frac{y-1}{y}\right|=x+c$

Putting the value of x and y

$\ln\left|\frac{\frac{1}{2}-1}{\frac{1}{2}}\right|=0+c$

$\ln\left|\frac{-\frac{1}{2}}{\frac{1}{2}}\right|=c$

$\ln|-1|=c$

$c=0$

Thus equation (1) becomes

$\ln\left(\frac{y-1}{y}\right)=x+0$

$\frac{y-1}{y}=e^{x}$

$y-1=ye^{x}$

$y-ye^{x}=1$

$y\left(1-e^{x}\right)=1$

$y=\frac{1}{1-e^{x}}$

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e). $\quad y\frac{dy}{dx}+xy^2-x=0, \qquad y(0)=0$

Sol: Given $y\frac{dy}{dx}+xy^2-x=0$

With $y(0)=0 \qquad x=0, y=0$

We have $y\frac{dy}{dx}+xy^2-x=0$ ....(1)

Separation of variables

$y\frac{dy}{dx}=x-xy^2$

$y\frac{dy}{dx}=x\left(1-y^2\right)$

$\frac{y}{1-y^2}dy=x\,dx$

Integrating both sides

$\int\frac{y}{1-y^2}dy=\int x\,dx$

$\frac{-1}{2}\int\frac{-2y}{1-y^2}dy=\int x\,dx$

$\frac{-1}{2}\ln\left|1-y^2\right|=\frac{x^2}{2}+c$

Multiply by -2

$\ln\left|1-y^2\right|=-x^2-2c$ ....(1)

Putting the value of x and y

$\ln\left|1-0^2\right|=-0^2-2c$

$\ln(1)=-2c$

$c=0$

Thus equation (1) becomes

$\ln\left|1-y^2\right|=-x^2$

$1-y^2=e^{-x^2}$

$y^2=1-e^{-x^2}$

$y=\sqrt{1-e^{-x^2}}$

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f). $\quad 2\frac{dy}{dx}=4xe^{-x}, \quad y(0)=42$

Sol: Given $2\frac{dy}{dx}=4xe^{-x}$

Or $\qquad \frac{dy}{dx}=2xe^{-x}$

$2\frac{dy}{dx}=4xe^{-x}, \quad y(0)=42$

With $y(0)=42 \qquad x=0, y=42$

We have $\frac{dy}{dx}=2xe^{-x}$

Separation of variables

$dy=2xe^{-x}\,dx$

Integrating both sides

$\int dy=2\int xe^{-x}\,dx$.....................................(1)

Integration by parts

$y=2\left[x\int e^{-x}dx-\int\frac{d}{dx}(x)\left(\int e^{-x}dx\right)dx\right]$

$y=2\left[x\left(-e^{-x}\right)-\int\left(-e^{-x}\right)dx\right]$

$y=-2xe^{-x}+2\int e^{-x}dx$

$y=-2xe^{-x}+2\left(-e^{-x}\right)+c$

$y=-2xe^{-x}-2e^{-x}+c$

Putting the value of x and y

$42=-2(0)e^{-0}-2e^{-0}+c$

$42=0-2(1)+c$

$42+2=c$

$c=44$

Thus equation (1) becomes

$y=-2xe^{-x}-2e^{-x}+44$

Solution of D.E by Separation of variables

solution of DE is not possible by direct integration, So we try to rearrange the DE to be solved such a way that all terms involving dependent variable appear on one side of the equation and all the terms involving independent variables appear on the other side

Homogeneous function

A function is said to be homogenous of degree n

$f(\lambda x,\lambda y)=\lambda^n f(x,y)$

Homogeneous differential equation

The DE $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ is said to be homogenous DE if it defines a homogenous function of degree zero

Solution of HDE HDE can be reduced to separable form by assuming $y=ux$

Orthogonal trajectories The two families of curves $F(x,y,c_1)$ and $G(x,y,c_2)$ are perpendicular at a point of intersection if and only if their tangents are perpendicular at the point of intersection

Exercise 9.2

Q1. Find the general solution of the following differential equations:

a). $\frac{2x\frac{dy}{dx}-2y}{x^2}=0$

Sol: Given $\frac{2x\frac{dy}{dx}-2y}{x^2}=0$

$2x\frac{dy}{dx}-2y=0.x^2$

$2x\frac{dy}{dx}-2y=0$

Dividing both sides by 2

$x\frac{dy}{dx}-y=0$

Separation of variables

$x\frac{dy}{dx}=y$

$\frac{dy}{y}=\frac{dx}{x}$

$\ln y=\ln x+\ln c$

$\ln y=\ln cx$

$y=cx$

b). $\frac{dy}{x}+y\ dx=2\ dx$

Sol: Given $\frac{dy}{x}+y\ dx=2\ dx$

$\frac{dy}{x}=2\ dx-y\,dx$

$\frac{dy}{x}=(2-y)\,dx$

Separation of variables

$\frac{dy}{2-y}=x\,dx$

Integrating both sides

$-\int\frac{-dy}{2-y}=\int x\,dx$

$-\ln(2-y)=\frac{x^2}{2}+c$

$\ln(2-y)=-\frac{x^2}{2}-c$

$2-y=e^{-\frac{x^2}{2}-c}$

$y=2-e^{-\frac{x^2}{2}-c}$

c). $\left(\frac{dy}{dx}\right)^2=1-y^2$

Sol: Given $\left(\frac{dy}{dx}\right)^2=1-y^2$

Taking square root on both sides

$\sqrt{\left(\frac{dy}{dx}\right)^2}=\sqrt{1-y^2}$

$\frac{dy}{dx}=\sqrt{1-y^2}$

Separation of variables

$\frac{dy}{\sqrt{1-y^2}}=dx$

Integrating both sides

$\int\frac{dy}{\sqrt{1-y^2}}=\int dx$

$\sin^{-1}(y)=x+c$

$y=\sin(x+c)$

d). $e^x\frac{dy}{dx}+y^2=0$

Sol: Given $e^x\frac{dy}{dx}+y^2=0$

$e^x\frac{dy}{dx}=-y^2$

Separation of variables

$\frac{dy}{y^2}=-\frac{dx}{e^x}$

$y^{-2}dy=-e^{-x}dx$

Integrating both sides

$\int y^{-2}dy=\int -e^{-x}dx$

$\frac{y^{-1}}{-1}=-\frac{e^{-x}}{-1}+c$

$-\frac{1}{y}=e^{-x}+c$

Taking reciprocal

$-y=\frac{1}{e^{-x}+c}$

$y=\frac{-1}{e^{-x}+c}$

e). $\sqrt{1-x^2}\ dy=\sqrt{1-y^2}\ dx$

Sol: Given $\sqrt{1-x^2}\ dy=\sqrt{1-y^2}\ dx$

Separation of variables

$\frac{dy}{\sqrt{1-y^2}}=\frac{dx}{\sqrt{1-x^2}}$

Integrating both sides

$\int\frac{dy}{\sqrt{1-y^2}}=\int\frac{dx}{\sqrt{1-x^2}}$

$\sin^{-1}(y)=\sin^{-1}(x)+c$

$y=\sin\left(\sin^{-1}(x)+c\right)$

f). $\operatorname{cosec}^2x\ dy+\sec y\ dx=0$

Sol: Given $\operatorname{cosec}^2x\ dy+\sec y\ dx=0$

$\operatorname{cosec}^2x\ dy=-\sec y\ dx$

Separation of variables

$\frac{dy}{\sec y}=-\frac{dx}{\operatorname{cosec}^2x}$

$\cos y\ dy=-\sin^2 x\ dx$

$\cos y\ dy=-\left(\frac{1-\cos 2x}{2}\right)dx \quad \therefore \sin^2 x=\frac{1-\cos 2x}{2}$

$\cos y\ dy=\frac{-1}{2}dx+\frac{1}{2}\cos 2x\ dx$

Integrating both sides

$\int\cos y\ dy=\frac{-1}{2}\int dx+\frac{1}{2}\int\cos 2x\ dx$

$\sin y=\frac{-x}{2}+\frac{1}{2}\frac{\sin 2x}{\frac{d}{dx}(2x)}+c$

$\sin y=\frac{-x}{2}+\frac{\sin 2x}{4}+c$

$y=\sin\left(\frac{-x}{2}+\frac{\sin 2x}{4}+c\right)$

Q2. Reduce the following differential equations in separable form and then solve:

a). $y'=(y+x)^2$

Sol: Given $\frac{dy}{dx}=(y+x)^2$ ....(1)

Let $x+y=u$

Differentiating both sides

$\frac{d}{dx}x+\frac{d}{dx}y=\frac{d}{dx}u$

$1+\frac{dy}{dx}=\frac{du}{dx}$

$\frac{dy}{dx}=\frac{du}{dx}-1$

Putting the values in equation (1) we get

$\frac{du}{dx}-1=u^2$

$\frac{du}{dx}=u^2+1$

Separation of variables

$\frac{du}{u^2+1}=dx$

Integrating both sides

$\int\frac{du}{u^2+1}=\int dx$

$\tan^{-1}(u)=x+c$

$u=\tan(x+c)$

Putting the value of u

$x+y=\tan(x+c)$

$y=\tan(x+c)-x$

b). $y'=\tan(x+y)-1$

Sol: Given $y'=\tan(x+y)-1$

Let $x+y=u$

Differentiating both sides

$\frac{d}{dx}x+\frac{d}{dx}y=\frac{d}{dx}u$

$1+\frac{dy}{dx}=\frac{du}{dx}$

$\frac{dy}{dx}=\frac{du}{dx}-1$

Putting the values in equation (1) we get

$\frac{du}{dx}-1=\tan(u)-1$

$\frac{du}{dx}=\tan(u)$

Separation of variables

$\frac{du}{\tan(u)}=dx$

$\frac{\cos u}{\sin u}du=dx$

Integrating both sides

$\ln(\sin u)=x+c$

$\sin u=e^{x+c}$

Putting the value of u

$\sin(x+y)=e^{x+c}$

c). $y'=\left(e^x-1\right)e^{-y}$

Sol: Given $y'=\left(e^x-1\right)e^{-y}$

$\frac{dy}{dx}=\left(e^x-1\right)e^{-y}$

Separation of variables

$\frac{dy}{e^{-y}}=\left(e^x-1\right)dx$

$e^y dy=\left(e^x-1\right)dx$

Integrating both sides

$\int e^y dy=\int\left(e^x-1\right)dx$

$e^y=e^x-x+c$

d). $y\frac{dy}{dx}+xy^2-x=0$

Sol: Given $y\frac{dy}{dx}+xy^2-x=0$

$y\frac{dy}{dx}=x-xy^2$

$y\frac{dy}{dx}=x\left(1-y^2\right)$

Separation of variables

$\frac{ydy}{1-y^2}=x\,dx$

Integrating both sides

$\int\frac{y\,dy}{1-y^2}=\int xdx$

$$\frac{-1}{2}\int\frac{-2y\,dy}{1-y^2}=\int xdx$$

$$\frac{-1}{2}\ln\left(1-y^2\right)=\frac{x^2}{2}+c$$

$$\ln\left(1-y^2\right)=-x^2-2c$$

$$1-y^2=e^{-x^2-2c}$$

$$y^2=1-e^{-x^2-2c}$$

$$y=\sqrt{1-e^{-x^2-2c}}$$

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e). $\frac{dy}{dx}=\frac{y}{x}+\frac{x^2}{y^2}$

Sol: Given $\frac{dy}{dx}=\frac{y}{x}+\frac{x^2}{y^2}$

$$\frac{dy}{dx}=\frac{y}{x}+\frac{1}{\frac{y^2}{x^2}}\text{...}(1)$$

Let $\frac{y}{x}=u \qquad \Rightarrow y=ux$

Differentiating both sides

$$\frac{d}{dx}y=x\frac{d}{dx}u+u\frac{d}{dx}x$$

$$\frac{dy}{dx}=x\frac{du}{dx}+u$$

Putting the values in equation (1) we get

$$x\frac{du}{dx}+u=u+\frac{1}{u^2}$$

$$x\frac{du}{dx}=\frac{1}{u^2}$$

Separation of variables

$$u^2du=\frac{dx}{x}$$

Integrating both sides

$$\int u^2du=\int\frac{dx}{x}$$

$$\frac{u^3}{3}=\ln|x|+\ln c$$

$$u^3=3\ln cx$$

Putting the value of u

$$\left(\frac{y}{x}\right)^3=3\ln\left(cx\right)$$

$$\frac{y^3}{x^3}=3\ln\left(cx\right)$$

$$y^3=3x^3\ln\left(cx\right)$$

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f). $\left(y-3\right)dy=\left(x^3+1\right)dx$

Sol: Given $\left(y-3\right)dy=\left(x^3+1\right)dx$

Integrating both sides

$$\int\left(y-3\right)dy=\int\left(x^3+1\right)dx$$

$$\frac{y^2}{2}-3y=\frac{x^4}{4}+x+c \qquad \times by\,4$$

$$2y^2-12y=x^4+4x+4c$$

Q3. Solve the following homogeneous differential equations:

a). $\frac{dy}{dx}=\frac{x+y}{x-y}$

Sol: Given $\frac{dy}{dx}=\frac{x+y}{x-y}\text{.........................}(1)$

Let $\frac{y}{x}=u \qquad \Rightarrow y=ux$

Differentiating both sides

$$\frac{d}{dx}y=x\frac{d}{dx}u+u\frac{d}{dx}x$$

$$\frac{dy}{dx}=x\frac{du}{dx}+u$$

Putting the values in equation (1) we get

$$x\frac{du}{dx}+u=\frac{x+ux}{x-ux}$$

$$x\frac{du}{dx}+u=\frac{x\left(1+u\right)}{x\left(1-u\right)}$$

$$x\frac{du}{dx}=\frac{1+u}{1-u}-u$$

$$x\frac{du}{dx}=\frac{1+u-u\left(1-u\right)}{1-u}$$

$$x\frac{du}{dx}=\frac{1+u-u+u^2}{1-u}$$

$$x\frac{du}{dx}=\frac{1+u^2}{1-u}$$

Separation of variables

$$\frac{1-u}{1+u^2}du=\frac{dx}{x}$$

Integrating both sides

$$\int\frac{1-u}{1+u^2}du=\int\frac{dx}{x}$$

$$\int\frac{1}{1+u^2}du-\int\frac{u}{1+u^2}du=\int\frac{dx}{x}$$

$$\int\frac{1}{1+u^2}du-\frac{1}{2}\int\frac{2u}{1+u^2}du=\int\frac{dx}{x}$$

$$\tan^{-1}\left(u\right)-\frac{1}{2}\ln\left(1+u^2\right)=\ln\left(x\right)+\ln\left(c\right)$$

$$\tan^{-1}\left(u\right)=\ln\left(cx\right)+\frac{1}{2}\ln\left(1+u^2\right)$$

$$\tan^{-1}\left(u\right)=\ln\left(cx\right)+\ln\sqrt{1+u^2}$$

$$\tan^{-1}\left(u\right)=\ln\left(cx\sqrt{1+u^2}\right)$$

Putting the value of u

$$\tan^{-1}\left(\frac{y}{x}\right)=\ln\left(cx\sqrt{1+\left(\frac{y}{x}\right)^2}\right)$$

---

b). $\frac{dy}{dx}=\frac{xy-y^2}{x^2}$

Sol: Given $\frac{dy}{dx}=\frac{xy-y^2}{x^2}\text{.......................}(1)$

Let $\frac{y}{x}=u \qquad \Rightarrow y=ux$

Differentiating both sides

$$\frac{d}{dx}y=x\frac{d}{dx}u+u\frac{d}{dx}x$$

$$\frac{dy}{dx}=x\frac{du}{dx}+u$$

Putting the values in equation (1) we get

$$x\frac{du}{dx}+u=\frac{ux^2-u^2x^2}{x^2}$$

$$x\frac{du}{dx}+u=\frac{x^2\left(u-u^2\right)}{x^2}$$

$$x\frac{du}{dx}=u-u^2$$

$$x\frac{du}{dx}=-u^2$$

Separation of variables

$$-\frac{1}{u^2}du=\frac{dx}{x}$$

$$-u^{-2}du=\frac{dx}{x}$$

Integrating both sides

$$-\int u^{-2}du=\int\frac{dx}{x}$$

$$-\frac{u^{-1}}{-1}=\ln cx$$

$$\frac{1}{u}=\ln cx$$

Putting the value of u

$$\frac{1}{\frac{y}{x}}=\ln cx$$

$$\frac{x}{y}=\ln cx$$

---

c). $\quad \frac{dy}{dx}=\frac{x^2+3y^2}{2xy}$

Sol: Given $\frac{dy}{dx}=\frac{x^2+3y^2}{2xy}$ ....(1)

Let $\frac{y}{x}=u \qquad \Rightarrow y=ux$

Differentiating both sides

$$\frac{d}{dx}y=x\frac{d}{dx}u+u\frac{d}{dx}x$$

$$\frac{dy}{dx}=x\frac{du}{dx}+u$$

Putting the values in equation (1) we get

$$x\frac{du}{dx}+u=\frac{x^2+3u^2x^2}{2x^2u}$$

$$x\frac{du}{dx}+u=\frac{x^2\left(1+3u^2\right)}{2x^2u}$$

$$x\frac{du}{dx}=\frac{1+3u^2}{2u}-u$$

$$x\frac{du}{dx}=\frac{1+3u^2-2u^2}{2u}$$

$$x\frac{du}{dx}=\frac{1+u^2}{2u}$$

Separation of variables

$$\frac{2u}{1+u^2}du=\frac{dx}{x}$$

Integrating both sides

$$\int\frac{2u}{1+u^2}du=\int\frac{dx}{x}$$

$$\ln\left(1+u^2\right)=\ln\left(x\right)+\ln\left(c\right)$$

$$\ln\left(1+u^2\right)=\ln\left(cx\right)$$

$$1+u^2=cx$$

Putting the value of u

$$1+\left(\frac{y}{x}\right)^2=cx$$

$$1+\frac{y^2}{x^2}=cx$$

$$\frac{x^2+y^2}{x^2}=cx$$

$$x^2+y^2=cx^3$$

---

d). $\quad \frac{dy}{dx}=\frac{\sqrt{x^2-y^2}+y}{x}$

Sol: Given $\frac{dy}{dx}=\frac{\sqrt{x^2-y^2}+y}{x}$ ....(1)

Let $\frac{y}{x}=u \qquad \Rightarrow y=ux$

Differentiating both sides

$$\frac{d}{dx}y=x\frac{d}{dx}u+u\frac{d}{dx}x$$

$$\frac{dy}{dx}=x\frac{du}{dx}+u$$

Putting the values in equation (1) we get

$$x\frac{du}{dx}+u=\frac{\sqrt{x^2-u^2x^2}+ux}{x}$$

$$x\frac{du}{dx}+u=\frac{\sqrt{x^2\left(1-u^2\right)}+ux}{x}$$

$$x\frac{du}{dx}+u=\frac{x\sqrt{1-u^2}+ux}{x}$$

$$x\frac{du}{dx}+u=\frac{x\left(\sqrt{1-u^2}+u\right)}{x}$$

$$x\frac{du}{dx}=\sqrt{1-u^2}+u-u$$

$$x\frac{du}{dx}=\sqrt{1-u^2}$$

Separation of variables

$$\frac{du}{\sqrt{1-u^2}}=\frac{dx}{x}$$

Integrating both sides

$$\int \frac{du}{\sqrt{1-u^2}} = \int \frac{dx}{x}$$
$$\sin^{-1}(u) = \ln(x) + \ln(c)$$
$$\sin^{-1}(u) = \ln(cx)$$
$$u = \sin(\ln(cx))$$
Putting the value of u
$$\tfrac{y}{x} = \sin(\ln(cx))$$
$$y = x\sin(\ln(cx))$$

---

e). $\dfrac{dy}{dx} = \dfrac{xy+y^2}{x^2+xy+y^2}$

Sol: Given $\dfrac{dy}{dx} = \dfrac{xy+y^2}{x^2+xy+y^2}$ ....(1)

Let $\dfrac{y}{x} = u \qquad \Rightarrow y = ux$

Differentiating both sides
$$\frac{d}{dx}y = x\frac{d}{dx}u + u\frac{d}{dx}x$$
$$\frac{dy}{dx} = x\frac{du}{dx} + u$$
Putting the values in equation (1) we get
$$x\frac{du}{dx} + u = \frac{ux^2 + u^2x^2}{x^2 + ux^2 + u^2x^2}$$
$$x\frac{du}{dx} + u = \frac{x^2(u+u^2)}{x^2(1+u+u^2)}$$
$$x\frac{du}{dx} = \frac{u+u^2}{1+u+u^2} - u$$
$$x\frac{du}{dx} = \frac{u+u^2-u(1+u+u^2)}{1+u+u^2}$$
$$x\frac{du}{dx} = \frac{u+u^2-u-u^2-u^3}{1+u+u^2}$$
$$x\frac{du}{dx} = \frac{-u^3}{1+u+u^2}$$
Separation of variables
$$\frac{1+u+u^2}{u^3}du = -\frac{dx}{x}$$
$$\left(\frac{1}{u^3} + \frac{u}{u^3} + \frac{u^2}{u^3}\right)du = -\frac{dx}{x}$$
$$\left(u^{-3} + u^{-2} + \frac{1}{u}\right)du = -\frac{dx}{x}$$
Integrating both sides
$$\int\left(u^{-3} + u^{-2} + \frac{1}{u}\right)du = -\int\frac{dx}{x}$$
$$\frac{u^{-2}}{-2} + \frac{u^{-1}}{-1} + \ln u = -\ln cx$$
$$\frac{-1}{2u^2} - \frac{1}{u} = -\ln cx - \ln u$$
$$\frac{-1}{2u^2} - \frac{1}{u} = -\ln cxu$$
Putting the value of u
$$\frac{-1}{2\left(\frac{y}{x}\right)^2} - \frac{1}{\left(\frac{y}{x}\right)} = -\ln cy$$
$$\frac{-x^2}{2y^2} - \frac{x}{y} = -\ln cy \qquad \times \text{ by } 2y^2$$
$$-x^2 - 2xy = -2y^2 \ln cy$$

Q4. Reduce the differential equations in the standard form of homogeneous form and then solve:

a). $x\dfrac{dy}{dx} = y + \sqrt{x^2+y^2} \qquad y(4) = 3$

Sol: Given $x\dfrac{dy}{dx} = y + \sqrt{x^2+y^2}$............(1)

With $y(4) = 3 \qquad \Rightarrow x = 4, y = 3$

Let $\dfrac{y}{x} = u \qquad \Rightarrow y = ux$

Differentiating both sides
$$\frac{d}{dx}y = x\frac{d}{dx}u + u\frac{d}{dx}x$$
$$\frac{dy}{dx} = x\frac{du}{dx} + u$$
Putting the values in equation (1) we get
$$x\left(x\frac{du}{dx} + u\right) = ux + \sqrt{x^2 + u^2x^2}$$
$$x\left(x\frac{du}{dx} + u\right) = ux + x\sqrt{1+u^2}$$
$$x\left(x\frac{du}{dx} + u\right) = x\left(u + \sqrt{1+u^2}\right)$$
$$x\frac{du}{dx} + u = u + \sqrt{1+u^2}$$
$$x\frac{du}{dx} = \sqrt{1+u^2}$$
Separation of variables
$$\frac{du}{\sqrt{1+u^2}} = \frac{dx}{x}$$
Integrating both sides
$$\int\frac{du}{\sqrt{1+u^2}} = \int\frac{dx}{x}$$
$$\ln\left|u + \sqrt{1+u^2}\right| = \ln x + \ln c$$
$$\ln\left|u + \sqrt{1+u^2}\right| = \ln cx$$
$$u + \sqrt{1+u^2} = cx$$
Putting the value of u
$$\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = cx$$
$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = cx$$
$$\frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}} = cx$$
$$\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} = cx$$
$$y + \sqrt{x^2+y^2} = cx^2 \text{ ....(2)}$$

Putting $x=4, y=3$ in equation (2) we get

$$3+\sqrt{4^2+3^3}=c(4)^2$$

$$3+\sqrt{16+9}=16c$$

$$3+\sqrt{25}=16c$$

$$3+5=16c$$

$$8=16c$$

$$c=\frac{8}{16}=\frac{1}{2}$$

Putting the value of c in equation (2) we get

$$y+\sqrt{x^2+y^2}=\frac{x^2}{2}$$

$$2y+2\sqrt{x^2+y^2}=x^2$$

---

b). $\left(x^4+y^4\right)dx=2x^3y\,dy \qquad y(1)=0$

Sol: Given $\left(x^4+y^4\right)dx=2x^3y\,dy$...........(1)

With $y(1)=0, \qquad x=1, y=0$

Let $\frac{y}{x}=u \qquad \Rightarrow y=ux$

Differentiating both sides

$$\frac{d}{dx}y=x\frac{d}{dx}u+u\frac{d}{dx}x$$

$$\frac{dy}{dx}=x\frac{du}{dx}+u$$

Putting the values in equation (1) we get

$$\left(x^4+y^4\right)dx=2x^3y\,dy$$

$$\left(x^4+y^4\right)=2x^3y\frac{dy}{dx}$$

$$x^4+u^4x^4=2x^4u\left(x\frac{du}{dx}+u\right)$$

$$x^4\left(1+u^4\right)=2x^4u\left(x\frac{du}{dx}+u\right)$$

$$1+u^4=2u\left(x\frac{du}{dx}+u\right)$$

$$1+u^4=2ux\frac{du}{dx}+2u^2$$

$$1-2u^2+u^4=2ux\frac{du}{dx}$$

$$\left(1-u^2\right)^2=2u\,x\frac{du}{dx}$$

Separation of variables

$$\frac{dx}{x}=\frac{2u\,du}{\left(1-u^2\right)^2}$$

Integrating both sides

$$\int\frac{dx}{x}=\int\frac{2u\,du}{\left(1-u^2\right)^2}$$

$$\int\frac{dx}{x}=-\int\frac{-2u\,du}{\left(1-u^2\right)^2}$$

$$\int\frac{dx}{x}=-\int\left(1-u^2\right)^{-2}(-2u)\,du$$

$$\ln x=-\frac{\left(1-u^2\right)^{-1}}{-1}+c$$

$$\ln x=\frac{1}{1-u^2}+c$$

Putting the value of u

$$\ln x=\frac{1}{1-\left(\frac{y}{x}\right)^2}+c$$

$$\ln x=\frac{1}{1-\frac{y^2}{x^2}}+c$$

$$\ln x=\frac{1}{\frac{x^2-y^2}{x^2}}+c$$

$$\ln x=\frac{x^2}{x^2-y^2}+c......................................(2)$$

Putting $x=1, y=0$ in equation (2) we get

$$\ln(1)=\frac{1^2}{1^2-0^2}+c$$

$$0=1+c$$

$$c=-1$$

Putting the value of c in equation (2) we get

$$\ln x=\frac{x^2}{x^2-y^2}-1$$

Q5. The slope of family of curve at a point $P(x,y)$ is $\frac{y-1}{1-x}$ Determine the equation of the curve that passes through the point $P(4,-3)$

Sol: Given slope i.e., $\frac{dy}{dx}=\frac{y-1}{1-x}$

Separation of variables

$$\frac{dy}{y-1}=\frac{dx}{1-x}$$

Integrating both sides

$$\int\frac{dy}{y-1}=\int\frac{dx}{1-x}$$

$$\int\frac{dy}{y-1}=-\int\frac{-dx}{1-x}$$

$$\ln(y-1)=-\ln(1-x)+\ln c$$

$$\ln(y-1)+\ln(1-x)=\ln c$$

$$\ln(y-1)(1-x)=\ln c$$

$$(y-1)(1-x)=c....................................(1)$$

Curve passes through point $P(4,-3)$ I.e.,

$x=4, y=-3$ we get

$$(-3-1)(1-4)=c$$

$$(-4)(-3)=c$$

$$c=12$$

Putting the value of c in equation (1) we get

$$(y-1)(1-x)=12$$

$$y-1=\frac{12}{1-x}$$

$$y=\frac{12}{1-x}+1$$
$$y=\frac{12+1-x}{1-x}$$
$$y=\frac{13-x}{1-x}$$
$$y=\frac{x-13}{x-1}$$

Q6. Find the solution curve of the differential equation $xy\frac{dy}{dx}=3y^2+x^2$ which passes through the point $P(-1,2)$

Sol: Given $xy\frac{dy}{dx}=3y^2+x^2$ $\quad\div$ by xy

$$\frac{dy}{dx}=3\left(\frac{y}{x}\right)+\left(\frac{x}{y}\right)\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let $\frac{y}{x}=u$ $\quad\Rightarrow y=ux$

Differentiating both sides

$$\frac{d}{dx}y=x\frac{d}{dx}u+u\frac{d}{dx}x$$
$$\frac{dy}{dx}=x\frac{du}{dx}+u$$

Putting the values in equation (1) we get

$$x\frac{du}{dx}+u=3u+\frac{1}{u}$$
$$x\frac{du}{dx}=3u-u+\frac{1}{u}$$
$$x\frac{du}{dx}=2u+\frac{1}{u}$$
$$x\frac{du}{dx}=\frac{2u^2+1}{u}$$

Separation of variables

$$\frac{u\,du}{2u^2+1}=\frac{dx}{x}$$

Integrating both sides

$$\int\frac{u\,du}{2u^2+1}=\int\frac{dx}{x}$$
$$\frac{1}{4}\int\frac{4u\,du}{2u^2+1}=\int\frac{dx}{x}$$
$$\frac{1}{4}\ln\left(2u^2+1\right)=\ln x+\ln c$$
$$\ln\left(2u^2+1\right)^{\frac{1}{4}}=\ln c\,x$$
$$\left(2u^2+1\right)^{\frac{1}{4}}=c\,x$$

Putting the value of u we get

$$\left(2\left(\tfrac{y}{x}\right)^2+1\right)^{\frac{1}{4}}=c\,x$$
$$\left(\tfrac{2y^2}{x^2}+1\right)^{\frac{1}{4}}=c\,x\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Curve passes through point $P(-1,2)$ I.e., $x=-1, y=2$ we get

$$\left(\tfrac{2(2)^2}{(-1)^2}+1\right)^{\frac{1}{4}}=c(-1)$$
$$\left(2(4)+1\right)^{\frac{1}{4}}=-c$$
$$(9)^{\frac{1}{4}}=-c$$
$$c=-9^{\frac{1}{4}}$$

Putting the value of c in equation (2) we get

$$\left(\tfrac{2y^2}{x^2}+1\right)^{\frac{1}{4}}=-9^{\frac{1}{4}}x$$
$$\left(\tfrac{2y^2}{x^2}+1\right)=-9x^4$$

Q7. Determine the particular solution $y=f(t)$ of the homogeneous differential equation $t^2y'=y^2+2ty$ with initial condition $y(1)=2$

Sol: Given $t^2y'=y^2+2ty\ldots\ldots\ldots\ldots\ldots\ldots(1)$

Let $y=ut$ $\quad\Rightarrow u=\frac{y}{t}$

Differentiating both sides

$$\frac{d}{dt}y=t\frac{d}{dt}u+u\frac{d}{dt}t$$
$$\frac{dy}{dt}=t\frac{du}{dt}+u$$

Putting the values in equation (1) we get

$$t^2\left(t\frac{du}{dt}+u\right)=u^2t^2+2t^2u$$
$$t^2\left(t\frac{du}{dt}+u\right)=t^2\left(u^2+2u\right)$$
$$t\frac{du}{dt}+u=u^2+2u$$
$$t\frac{du}{dt}=u^2+2u-u$$
$$t\frac{du}{dt}=u^2+u$$

Separation of variables

$$\frac{du}{u^2+u}=\frac{dt}{t}$$

Integrating both sides

$$\int\frac{du}{u^2+u}=\int\frac{dt}{t}$$

Take $\frac{1}{u^2+u}=\frac{1}{u(u+1)}$ Using partial fraction

$$\frac{1}{u(u+1)}=\frac{A}{u}+\frac{B}{u+1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Multiply each fraction by $u(u+1)$ we get

$$1=A(u+1)+Bu\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Put $u=0$ in equation (3) we get

$$1=A(0+1)+B(0)$$
$$1=A$$

Put $u=-1$ in equation (3) we get

$1 = A(-1+1) + B(-1)$

$B = -1$

Putting the values of A and B in Equation (2) we get

$$\frac{1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1}$$

Thus integral becomes

$$\int \frac{du}{u(u+1)} = \int \frac{dt}{t}$$

$$\int \frac{du}{u} - \int \frac{du}{u+1} = \int \frac{dt}{t}$$

$$\ln u - \ln(u+1) = \ln t + \ln c$$

$$\ln \frac{u}{u+1} = \ln ct$$

$$\frac{u}{u+1} = ct$$

Putting the value of u we get

$$\frac{\frac{y}{t}}{\frac{y}{t}+1} = ct$$

$$\frac{\frac{y}{t}}{\frac{y+t}{t}} = ct$$

$$\frac{y}{y+t} = ct \text{.......................................}(4)$$

Initial condition $y(1) = 2$ I.e., $t = 1, y = 2$ we get

$$\frac{2}{2+1} = c(1)$$

$$\frac{2}{3} = c$$

Putting the value of c in equation (4) we get

$$\frac{y}{y+t} = \frac{2}{3}t$$

Q8. A particle moves along the x-axis so that any point is equal to half its abscissa minus three times the time. At a time $t = 2, x = -4$ determine the motion of a particle along the x-axis

Sol: half its abscissa minus three times the time

$$\frac{dx}{dt} = \frac{x}{2} - 3t$$

$$\frac{dx}{dt} = \frac{x-6t}{2} \text{..}(1)$$

Let $x - 6t = u$

Differentiating both sides

$$\frac{dx}{dt} - 6 = \frac{du}{dt}$$

$$\frac{dx}{dt} = \frac{du}{dt} + 6$$

Putting the values in equation (1) we get

$$\frac{du}{dt} + 6 = \frac{u}{2}$$

$$\frac{du}{dt} = \frac{u}{2} - 6$$

$$\frac{du}{dt} = \frac{u-12}{2}$$

Separation of variables

$$\frac{du}{u-12} = \frac{dt}{2}$$

Integrating both sides

$$\int \frac{du}{u-12} = \int \frac{dt}{2}$$

$$\ln(u-12) = \frac{t}{2} + k$$

Putting the value of u we get

$$\ln(x - 6t - 12) = \tfrac{t}{2} + k$$

$$x - 6t - 12 = e^{\frac{t}{2}+k}$$

$$x = 6t + 12 + ce^{\frac{t}{2}} \text{..}(2)$$

At a time $t = 2, x = -4$ we get

$$-4 = 6(2) + 12 + ce^{1}$$

$$-4 - 24 = ce$$

$$c = -28e^{-1}$$

Putting the value of c in equation (2) we get

$$x = 6t + 12 - 28e^{-1}e^{\frac{t}{2}}$$

$$x = 6t + 12 - 28e^{\frac{t}{2}-1}$$

Q9. Rate of consumption of oil (billions of barrels) is given by $\frac{dx}{dt} = 1.2e^{0.04t}$ where $t = 0$ correspond to 1990. At this rate how much oil will be used in 8 $(t = 8)$ years.

Sol: Given $\frac{dx}{dt} = 1.2e^{0.04t}$ ....(1)

Separating the variables

$$dx = 1.2e^{0.04t}dt$$

Integrating both sides

$$\int dx = 1.2\int e^{0.04t}dt$$

$$x = \frac{1.2e^{0.04t}}{0.04} + c$$

$$x = 30e^{0.04t} + c \text{...}(2)$$

When $t = 0$ the oil consumption was $x = 0$ gives

$$0 = 30e^{0.04(0)} + c$$

$$0 = 30 + c$$

$$c = -30$$

Thus equation (2) becomes

$$x = 30e^{0.04t} - 30$$

When $t = 8$ years the oil consumption will be

$$x = 30e^{0.04(8)} - 30$$

$$x = 11.3138 \ \text{billions of barrels}$$

Q10. Rate of infection of a disease (in people per month) is given by $\frac{dI}{dt} = \frac{100t}{t^2+1}$ where t is the time in month since the disease broke out. Find the total number of infected people over the first four month of the disease.

Sol: Given $\frac{dI}{dt} = \frac{100t}{t^2+1}$ ....(1)

Separation of variables

$$dI = \frac{100t}{t^2+1}dt$$

Taking integrals

$$\int dI = 50\int \frac{2t}{t^2+1}dt$$

$I = 50\ln\left(t^2+1\right)+c$.................................(2)

At time $t=0$ the number of infected people was

$I=0$

$0 = 50\ln\left(0^2+1\right)+c$

$0 = 50\ln(1)+c$

$0 = 50(0)+c$

$c=0$

Putting the value of c in equation (2) we get

$I = 50\ln\left(t^2+1\right)$

The number of infected people for the first month

$I = 50\ln\left((1)^2+1\right)$

$I = 50\ln(1+1)$

$I = 50\ln(2)$

$I = 34.65$

number of infected people for the second month

$I = 50\ln\left((2)^2+1\right)$

$I = 50\ln(4+1)$

$I = 50\ln(5)$

$I = 80.47$

The number of infected people for the third month

$I = 50\ln\left((3)^2+1\right)$

$I = 50\ln(9+1)$

$I = 50\ln(10)$

$I = 115.13$

number of infected people for the fourth month

$I = 50\ln\left((4)^2+1\right)$

$I = 50\ln(16+1)$

$I = 50\ln(17)$

$I = 141.66$

Q11. rate of reaction of a drug is given by $\frac{dR}{dx} = 2x^2e^{-x}$ where x is the number of hours since the drug was administered. Find total reaction to the drug form $x=1$ to $x=6$

Sol: Given $\frac{dR}{dx} = 2x^2e^{-x}$.......................(1)

Separation of variables

$dR = 2x^2e^{-x}dx$

Integrating both sides

$\int dR = 2\int x^2e^{-x}dx$

$R = 2\left[x^2\int e^{-x}dx - \int\left(\frac{d}{dx}x^2\right)\left(\int e^{-x}dx\right)dx\right]$

$R = 2x^2\frac{e^{-x}}{-1} - 2\int 2x\left(\frac{e^{-x}}{-1}\right)dx$

$R = -2x^2e^{-x} + 4\int xe^{-x}dx$

$R = -2x^2e^{-x} + 4\left[x\int e^{-x}dx - \int\left(\frac{d}{dx}x\right)\left(\int e^{-x}dx\right)dx\right]$

$R = -2x^2e^{-x} + 4x\frac{e^{-x}}{-1} - 4\int 1.\frac{e^{-x}}{-1}dx$

$R = -2x^2e^{-x} - 4xe^{-x} + 4\int e^{-x}dx$

$R = -2x^2e^{-x} - 4xe^{-x} - 4e^{-x} + c$....................(2)

At time $x=0$ the reaction of drug was $R=0$

$0 = -2(0)^0e^{-(0)} - 4(0)e^{-(0)} - 4e^{-(0)} + c$

$0 = 0 - 0 - 4 + c$

$c = 4$

Thus equation (2) becomes

$R(x) = -2x^2e^{-x} - 4xe^{-x} - 4e^{-x} + 4$

The reaction of drug when time $x=1$

$R(1) = -2(1)^2e^{-(1)} - 4(1)e^{-(1)} - 4e^{-(1)} + 4$

$R(1) = 0.321 = 32.1\%$

The reaction of drug when time $x=2$

$R(2) = -2(2)^2e^{-(2)} - 4(2)e^{-(2)} - 4e^{-(2)} + 4$

$R(2) = 1.293 = 129.3\%$

The reaction of drug when time $x=3$

$R(3) = -2(3)^2e^{-(3)} - 4(3)e^{-(3)} - 4e^{-(3)} + 4$

$R(3) = 2.307 = 230.7\%$

The reaction of drug when time $x=4$

$R(4) = -2(4)^2e^{-(4)} - 4(4)e^{-(4)} - 4e^{-(4)} + 4$

$R(4) = 3.0475 = 304.75\%$

The reaction of drug when time $x=5$

$R(5) = -2(5)^2e^{-(5)} - 4(5)e^{-(5)} - 4e^{-(5)} + 4$

$R(5) = 3.5013 = 350.13\%$

The reaction of drug when time $x=6$

$R(6) = -2(6)^2e^{-(6)} - 4(6)e^{-(6)} - 4e^{-(6)} + 4$

$R(6) = 3.7521 = 375.21\%$

Q12. Rate of increasing of the number of cellular phone subscribers (in millions) since services began, was given by: $\frac{ds}{dx} = 0.38x + 0.04$ where x is the number of years since 1998, when the services started. There were 0.25 million subscribers in year 1998 $(x=0)$ Find a function that gives number of subscribers for the year 2004.

Sol: Given $\frac{ds}{dx} = 0.38x + 0.04$...............(1)

Separation of variables

$ds = (0.38x + 0.04)dx$ Integrating both sides

$\int ds = \int(0.38x + 0.04)dx$

$s = 0.38\frac{x^2}{2} + 0.04x + c$

$s = 0.19x^2 + 0.04x + c$...............................(2)

When service started $x=0$ subscribers $s=0.25$ millions

$0.25 = 0.19(0)^2 + 0.04(0) + c$

$c = 0.25$

Thus equation (2) becomes

$s = 0.19x^2 + 0.04x + 0.25$

From 1998 to 2004, $x = 6$ years

$s = 0.19(6)^2 + 0.04(6) + 0.25$

$s = 0.19(36) + 0.24 + 0.25$

$s = 7.33\ millions$

Q13. Determine the equation of the orthogonal trajectories of the following families of curves.

a). $y = cx^3$

Sol: Given $y = cx^3$..............................(1)

$\Rightarrow c = \frac{y}{x^3}$

Differentiating equation (1) we get

$\frac{dy}{dx} = c\frac{d}{dx}x^3$

$\frac{dy}{dx} = 3cx^2\frac{d}{dx}x$

$m_1 = \frac{dy}{dx} = 3cx^2$

For orthogonal trajectories $m_2 = \frac{-1}{m_1}$

$m_2 = \frac{-1}{3cx^2}$

$m_2 = \frac{-1}{3\left(\frac{y}{x^3}\right)x^2}$

$m_2 = \frac{-1}{3\frac{y}{x}}$

$\frac{dy}{dx} = \frac{-x}{3y}$

Separation of variables

$3ydy = -xdx$

Integrating both sides

$3\int ydy = -\int xdx$

$3\frac{y^2}{2} = -\frac{x^2}{2} + k$

$3y^2 + x^2 = 2k$ $\quad k$ is constant

---

b). $xy = c$

Sol: Given $xy = c$..............................(1)

$\Rightarrow c = xy$

Differentiating equation (1) we get

$\frac{d}{dx}(xy) = \frac{d}{dx}c$

$x\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = \frac{-y}{x}$

$m_1 = -\frac{y}{x}$

For orthogonal trajectories $m_2 = \frac{-1}{m_1}$

$m_2 = \frac{-1}{\left(\frac{-y}{x}\right)}$

$m_2 = \frac{x}{y}$

$\frac{dy}{dx} = \frac{x}{y}$

Separation of variables

$ydy = xdx$

Integrating both sides

$\int ydy = \int xdx$

$\frac{y^2}{2} = \frac{x^2}{2} + k$

$y^2 - x^2 = 2k$ $\quad k$ is constant

---

c). $y = cxe^x$

Sol: Given $y = cxe^x$..............................(1)

$\Rightarrow c = \frac{y}{xe^x}$

Differentiating equation (1) we get

$\frac{dy}{dx} = c\frac{d}{dx}\left(xe^x\right)$

$\frac{dy}{dx} = c\left\{e^x\frac{d}{dx}x + x\frac{d}{dx}e^x\right\}$

$m_1 = \frac{dy}{dx} = c\left\{e^x + xe^x\right\}$

For orthogonal trajectories $m_2 = \frac{-1}{m_1}$

$m_2 = \frac{-1}{c\left\{e^x + xe^x\right\}}$

$m_2 = \frac{-1}{\left(\frac{y}{xe^x}\right)e^x\{1+x\}}$

$\frac{dy}{dx} = \frac{-x}{y(1+x)}$

Separation of variables

$ydy = -\frac{x}{1+x}dx$

Integrating both sides

$\int ydy = -\int\frac{x}{1+x}dx$

$\frac{y^2}{2} = -\int\frac{1+x-1}{1+x}dx$

$y^2 = -2\int\frac{1+x}{1+x}dx + 2\int\frac{1}{1+x}dx$

$y^2 = -2\int 1.dx + 2\int\frac{1}{1+x}dx$

$y^2 = -2x + 2\ln(1+x) + k$ $\quad k$ is constant

---

d). $y^2 = x^2 + c$

Sol: Given $y^2 = x^2 + c$...........................(1)

$\Rightarrow c = y^2 - x^2$

Differentiating equation (1) we get

$\frac{d}{dx}y^2 = \frac{d}{dx}x^2 + \frac{d}{dx}c$

$$2y\frac{dy}{dx}=2x+0$$
$$\frac{dy}{dx}=\frac{2x}{2y}$$
$$m_1=\frac{dy}{dx}=\frac{x}{y}$$
For orthogonal trajectories $m_2=\frac{-1}{m_1}$
$$m_2=\frac{-1}{\frac{x}{y}}$$
$$m_2=\frac{-y}{x}$$
$$\frac{dy}{dx}=\frac{-y}{x}$$
Separation of variables
$$\frac{dy}{y}=\frac{-dx}{x}$$
Integrating both sides
$$\int\frac{dy}{y}=-\int\frac{dx}{x}$$
$$\ln y=-\ln x+\ln k \qquad k \text{ is constant}$$
$$\ln y+\ln x=\ln k$$
$$\ln xy=\ln k$$
$$xy=k$$

e). $y=c\sin 2x$

Sol: Given $y=c\sin 2x$..........................(1)
$$\Rightarrow c=\frac{y}{\sin 2x}$$
Differentiating equation (1) we get
$$\frac{dy}{dx}=c\frac{d}{dx}\sin 2x$$
$$\frac{dy}{dx}=c\cos 2x\frac{d}{dx}2x$$
$$m_1=\frac{dy}{dx}=2c\cos 2x$$
For orthogonal trajectories $m_2=\frac{-1}{m_1}$
$$m_2=\frac{-1}{2c\cos 2x}$$
$$m_2=\frac{-1}{2\left(\frac{y}{\sin 2x}\right)\cos 2x}$$
$$m_2=\frac{-\sin 2x}{2y\cos 2x}$$
$$\frac{dy}{dx}=\frac{-1}{2y}\frac{\sin 2x}{\cos 2x}$$
Separation of variables
$$ydy=\frac{-1}{2}\frac{\sin 2x}{\cos 2x}dx$$
Integrating both sides
$$\int ydy=\frac{-1}{2}\int\frac{\sin 2x}{\cos 2x}dx$$
$$\frac{y^2}{2}=\frac{-1}{2\times 2}\int\frac{2\sin 2x}{\cos 2x}dx$$
$$\frac{y^2}{2}=\frac{-1}{4}\ln(\cos 2x)+k \qquad k \text{ is constant}$$

f). $e^x\cos y=c$

Sol: Given $e^x\cos y=c$..........................(1)
$$\Rightarrow c=e^x\cos y$$
Differentiating equation (1) we get
$$\frac{d}{dx}\left(e^x\cos y\right)=\frac{d}{dx}c$$
$$e^x\frac{d}{dx}\cos y+\cos y\frac{d}{dx}e^x=0$$
$$-e^x\sin y\frac{dy}{dx}+e^x\cos y=0$$
$$\frac{dy}{dx}=\frac{-e^x\cos y}{-e^x\sin y}$$
$$m_1=\frac{dy}{dx}=\frac{\cos y}{\sin y}$$
For orthogonal trajectories $m_2=\frac{-1}{m_1}$
$$m_2=\frac{-1}{\frac{\cos y}{\sin y}}$$
$$\frac{dy}{dx}=\frac{-\sin y}{\cos y}$$
Separation of variables
$$\frac{\cos y}{\sin y}dy=-dx$$
Integrating both sides
$$\int\frac{\cos y}{\sin y}dy=-\int dx$$
$$\ln\left(\sin y\right)=-x+k \qquad k \text{ is constant}$$
$$\sin y=e^{-x+k}$$

g). $y=\sqrt{x+c}$

Sol: Given $y=\sqrt{x+c}$..........................(1)

Squaring both sides
$$y^2=\left(\sqrt{x+c}\right)^2$$
$$y^2=x+c$$
$$c=y^2-x$$
Differentiating equation (1) we get
$$\frac{dy}{dx}=\frac{d}{dx}(x+c)^{\frac{1}{2}}$$
$$\frac{dy}{dx}=\frac{1}{2}(x+c)^{\frac{1}{2}-1}\frac{d}{dx}(x+c)$$
$$m_1=\frac{dy}{dx}=\frac{1}{2}(x+c)^{\frac{-1}{2}}$$
For orthogonal trajectories $m_2=\frac{-1}{m_1}$
$$m_2=\frac{-1}{3cx^2}$$
$$m_2=\frac{-2}{(x+c)^{\frac{-1}{2}}}$$
$$\frac{dy}{dx}=-2(x+c)^{\frac{1}{2}}$$
$$\frac{dy}{dx}=-2\left(x+y^2-x\right)^{\frac{1}{2}}$$

$\frac{dy}{dx}=-2\left(y^{2}\right)^{\frac{1}{2}}$

$\frac{dy}{dx}=-2y$

Separation of variables

$\frac{dy}{y}=-2dx$

Integrating both sides

$\int\frac{dy}{y}=-2\int dx$

$\ln y=-2x+k$ $\qquad k$ *is* constant

$\ln y=-2x+k$

$y=e^{-2x+k}$

---

h). $\quad y=x^{2}+c$

Sol: Given $y=x^{2}+c$..............................(1)

$\Rightarrow c=y-x^{2}$

Differentiating equation (1) we get

$\frac{dy}{dx}=\frac{d}{dx}x^{2}+\frac{d}{dx}c$

$\frac{dy}{dx}=2x+0$

$m_1=\frac{dy}{dx}=2x$

For orthogonal trajectories $m_2=\frac{-1}{m_1}$

$m_2=\frac{dy}{dx}=\frac{-1}{2x}$

Separation of variables

$dy=\frac{-1}{2}\frac{dx}{x}$

Integrating both sides

$\int dy=\frac{-1}{2}\int\frac{dx}{x}$

$y=-\frac{1}{2}\ln x+\ln k$ $\qquad k$ *is* constant

$y=\ln\left(k.x^{\frac{-1}{2}}\right)$

$e^{y}=k.x^{\frac{-1}{2}}$

i). $\quad e^{x}\sin y=c$

Sol: Given $e^{x}\sin y=c$...........................(1)

$\Rightarrow c=e^{x}\sin y$

Differentiating equation (1) we get

$\frac{d}{dx}\left(e^{x}\sin y\right)=\frac{d}{dx}c$

$e^{x}\frac{d}{dx}\sin y+\sin y\frac{d}{dx}e^{x}=0$

$e^{x}\cos y\frac{dy}{dx}+e^{x}\sin y=0$

$\frac{dy}{dx}=\frac{-e^{x}\sin y}{e^{x}\cos y}$

$m_1=\frac{dy}{dx}=-\frac{\sin y}{\cos y}$

For orthogonal trajectories $m_2=\frac{-1}{m_1}$

$m_2=\frac{-1}{-\frac{\sin y}{\cos y}}$ $\qquad\Rightarrow\frac{dy}{dx}=\frac{\cos y}{\sin y}$

Separation of variables

$\frac{\sin y}{\cos y}dy=dx$

Integrating both sides

$-\int\frac{-\sin y}{\cos y}dy=\int dx$

$-\ln\left(\cos y\right)=x+k$ $\qquad k$ *is* constant

---

j). $\quad \cos x\cosh y=c$

Sol: Given $\cos x\cosh y=c$....................(1)

Differentiating equation (1) we get

$\frac{d}{dx}\left(\cos x\cosh y\right)=\frac{d}{dx}c$

$\cos x\frac{d}{dx}\cosh y+\cosh y\frac{d}{dx}\cos x=0$

$\cos x\sinh y\frac{dy}{dx}-\sin x\cosh y=0$

$m_1=\frac{dy}{dx}=\frac{\sin x\cosh y}{\cos x\sinh y}$

For orthogonal trajectories $m_2=\frac{-1}{m_1}$

$m_2=\frac{-1}{\frac{\sin x}{\cos x}\frac{\cosh y}{\sinh y}}$

$m_2=\frac{dy}{dx}=-\frac{\cos x}{\sin x}\frac{\sinh y}{\cosh y}$

Separation of variables

$\frac{\cosh y}{\sinh y}dy=-\frac{\cos x}{\sin x}dx$

Integrating both sides

$\int\frac{\cosh y}{\sinh y}dy=-\int\frac{\cos x}{\sin x}dx$

$\ln\left(\sinh y\right)=-\ln\left(\sin x\right)+\ln k$ $\qquad k$ *is* constant

$\ln\left(\sinh y\right)+\ln\left(\sin x\right)=\ln k$

$\ln\left(\sinh y\sin x\right)=\ln k$

$\sinh y\sin x=k$

---

k). $\quad e^{x}\left(x\cos y-y\sin y\right)=c$

Solution: $e^{x}\left(x\cos y-y\sin y\right)=c$ ....(1)

Differentiating equation (1) we get

$\frac{d}{dx}e^{x}\left(x\cos y-y\sin y\right)=\frac{d}{dx}c$

$\left(x\cos y-y\sin y\right)\frac{d}{dx}e^{x}+e^{x}\frac{d}{dx}\left(x\cos y-y\sin y\right)=0$

$\left(x\cos y-y\sin y\right)e^{x}+e^{x}\left(x\frac{d}{dx}\cos y+\cos y\frac{d}{dx}x-y\frac{d}{dx}\sin y+\sin y\frac{d}{dx}y\right)=0$

$\left(x\cos y-y\sin y\right)e^{x}+e^{x}\left(-x\sin y\frac{dy}{dx}+\cos y-y\cos y\frac{dy}{dx}+\sin y\frac{dy}{dx}\right)=0$

$\left(x\cos y-y\sin y+\cos y\right)e^{x}+e^{x}\left(-x\sin y-y\cos y+\sin y\right)\frac{dy}{dx}=0$

$e^{x}\left(-x\sin y-y\cos y+\sin y\right)\frac{dy}{dx}=-\left(x\cos y-y\sin y+\cos y\right)e^{x}$

$\frac{dy}{dx}=\frac{-x\cos y+y\sin y-\cos y}{-x\sin y-y\cos y+\sin y}$

For orthogonal trajectories $m_2=\frac{-1}{m_1}$

$m_2=\frac{dy}{dx}=-\frac{-x\sin y-y\cos y+\sin y}{-x\cos y+y\sin y-\cos y}$

Separation of variables

Integrating both sides

Wrong question